OPTIMAL CONTROL OF INFORMATION IN SOCIAL NETWORK USING PONTRYAGIN'S MAXIMUM/MINIMUM PRINCIPLE

VISNU VARATAN ARUMUGHAM

FACULTY OF ENGINEERING

UNIVERSITY OF MALAYA

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VISNU VARATAN ARUMUGHAM

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OPTIMAL CONTROL OF INFORMATION IN SOCIAL NETWORK USING PONTRYAGIN MAXIMUM/MINIMUM PRINCIPLE

ABSTRACT

This study is to formulate an optimal control problem to maximize the spread of information's on the constant budget. Advertisement in a social media is the control signals which attempt to convert ignorants and stiflers into spreaders in this study. We show the existence of a solution to the optimal control problem when the campaigning incurs non-linear costs under the isoperimetric budget constraint. The solution employs Pontryagin's Minimum Principle and a modified version of forward backward sweep technique for numerical computation to accommodate the isoperimetric budget constraint. The techniques developed in this paper are general and can be applied to similar optimal control problems in other areas. We have allowed the spreading rate of the information epidemic to vary over the campaign duration to model practical situations when the interest level of the population in the subject of the campaign changes with time. The shape of the optimal control signal is studied for different model parameters and spreading rate profiles. We have also studied the variation of the optimal campaigning costs with respect to various model parameters. Results indicate that, for some model parameters, significant improvements can be achieved by the optimal strategy compared to the static control strategy. The static strategy respects the same budget constraint as the optimal strategy and has a constant value throughout the campaign horizon. This work finds application in election and social awareness campaigns, product advertising, movie promotion and crowdfunding campaigns.

Keywords: Pontryagin's Minimum principle, forward backward sweep, cost constraints, optimal control, ignorant, stiflers, spreaders.

KAWALAN MAKLUMAT OPTIMAL DALAM JARINGAN SOSIAL MENGGUNAKAN PONTRYAGIN MAXIMUM / MINIMUM PRINSIP

ABSTRAK

Kajian ini adalah untuk merumuskan masalah kawalan optimum untuk memaksimumkan penyebaran maklumat mengenai belanjawan malar. Iklan dalam media sosial adalah isyarat kawalan yang cuba mengubah orang yang tidak tahu dan menipu kepada penyebar dalam kajian ini. Kami menunjukkan kewujudan penyelesaian kepada masalah kawalan optimum apabila kempen berkemungkinan tidak ada linear di bawah kekangan anggaran isoperimetrik. Penyelesaian ini menggunakan Prinsip Minimum Pontryagin dan versi modul maju ke belakang ke atas untuk pengiraan berangka untuk menampung kekangan anggaran isoperimetrik. Teknik-teknik yang dibangunkan dalam makalah ini adalah umum dan boleh digunakan untuk masalah kawalan optimum yang sama di kawasan lain. Kami telah membenarkan kadar penyebaran wabak maklumat untuk mengubah tempoh kempen untuk memodelkan situasi praktikal apabila tahap minat penduduk dalam subjek kempen berubah dengan masa. Bentuk isyarat kawalan optimum dikaji untuk parameter model yang berbeza dan profil kadar penyebaran. Kami juga mengkaji variasi kos kempen yang optimum berkenaan dengan pelbagai parameter model. Keputusan menunjukkan bahawa, untuk beberapa parameter model, penambahbaikan ketara dapat dicapai oleh strategi optimum berbanding dengan strategi kawalan statik. Strategi statik menghormati kekangan belanjawan yang sama sebagai strategi optimum dan mempunyai nilai tetap sepanjang hala kempen. Kerja ini menemui aplikasi dalam kempen kesedaran dan pemilihan sosial, pengiklanan produk, promosi filem dan kempen crowdfunding.

Kata Kunci: kawalan optimum, penyebar, kawalan kos, prinsip pontryagin’s minimum

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LIST OF SYMBOLS AND ABBREVIATIONS

ODE	-	Ordinary Differential Equation
BVP	-	Boundary Value Problem
SIR/SIS	-	S-susceptible, I-Infected and R-recovered/removed

CHAPTER 1 : INTRODUCTION

1.1 Research Background

Optimal control by using Pontryagin's maximum/minimum principle is the best practice to control the information spreading rate and also to increase or decrease the spreading rate. Besides that, will find the best way to control with the presence of constraints. The Pontryagin's principle can maximize the spreading rate of an information's by changing the stifles into spreaders. Information model are used to contagion process for various purposes such as in spreading fashions, trends, and election manifestos in a population. The population is divided into three categories, stifler, spreader and ignorant. The stiflers are people who have stopped spreading the information's while the spreaders for those who spreading the information's and ignorant are people who does not have the information's yet.

1.2 Problem Statement

Most of the times an advertisement's or manifestos or any kind of information's are less spread to a population, some will ignore or stopped spreading the information's. It will give negative impacts on cost and time. The cause of stiflers and ignorant is, they not really exposed to the information area or they simply don't want to spread the information's. Besides that, in product marketing such as newly designed model or newly launched movie may decrease with times after it launched or released. At the same time, during elections the amount of interests will increase over the time, so during the first 2 weeks before the elections, all the information's and manifestos will be less spread.

1.3 Aim of Investigation

The aim of this research is to formulate a model that can increase the spreaders rate by changing the ignorant and stiflers into spreaders with a fixed budget. The model will be formulate using Pontryagin's minimum/maximum principle. This can minimize the number of stiflers and ignorants at the end of the campaign.

1.4 Scope of Work

<u>The parameters that used in this paper:</u>

Symbol Definition
$i(t)$ fraction of ignorants in the population at time t
$s(t)$ fraction of spreaders in the population at time t
$r(t)$ fraction of stiflers in the population at time t
$\beta_1(t)$ per contact message spreading rate at time t
γ_1 per contact recovery rate
k number of other individuals an individual is in contact at any given time
$\beta(t) = k\beta_1(t)$ spreading rate at time t
$\gamma = k\gamma_1$ recovery rate
T campaign deadline
$u(t)$ control at time t (*e.g.* rate at which advertisements are put across in mass media)
u_{max} maximum allowed control, $0 \leq u(t) \leq u_{max}$
$c(u(t))$ instantaneous cost incurred due to application of control
B budget
$b(t)$ (cumulative) resource spent during $[0\ t]$

CHAPTER 2 : LITERATURE REVIEW

The literature review summarizes all the basic ideas of the elements involved in this project. All the elements are used in implementing the control to maximize the spreaders rate and to formulate using pontryagin’s minimum/maximum principle for control optimally the information spread rate with constant budget. The crucial parts are towards the end in which a suitable formula is modelled for the project and a suitable optimal control is designed to run it for the advertisement’s or during any promotions.
Pontyragin’s Minimum and maximum principle is used in the optimal control theory to find the suitable and best possible control for a dynamical system from one state to another. Usually will be used in the presence of constraints for the input control or states. Initially this Pontryagin’s principle used to maximize the terminal speed of rocket. As time goes and needs, it also used to minimize the performance index.

2.1 Optimal Control Strategies depending on the interest level for the spread of rumor.

This article is about controlling the spread of rumor and false information’s in social media. Here pontryagin’s maximum principle is used. Besides that, adapted optimal control is used to investigate the effect of controls using isoperimetric constraints. There are three controls under isoperimetric constraints. The rumor model with three strategies of control, for preventing the spreading of rumor, deleting information of rumor and punishing spreaders. To prevent the spread of rumor effectively, we investigate how and when controls should be applied. This investigation is important because each control has an optimal time point of application and a different characteristic. We consider the optimal control problem and analyze it via.Pontryagin’s Maximum Principle to see the contents. We find the optimal strategy to prevent the spread of rumor by using numerical simulation. When the amount of controls is limited, three controls applications is analyzed respectively. Since costs, times and so on must be taken into account for implementing the strategy or the policy, the perspective is required.

2.2 Optimal control of an epidemic through educational campaign

Simplified SIR model is described here and due to the cost constraint and the total time for this campaign has been reduced. There is two scenario that will undergo in this paper. First scenario is where the campaign is oriented to reduce the infection rate by stimulating susceptible to have a protective behaviour. The second scenario will have the campaign oriented to increase the removal rate. This is done by stimulating the infected to remove itself from the infected class. The tool used to determine the optimal strategy here is Pontryagin Maximum Principle. At the end of the optimal outbreak, optimality is measured by minimizing the total number of infected class.

In this model, here have two problem that has been overcome. First, the model must be mathematically tractable and must be intuitively plausible. For the first requirement we assume, for mathematical simplicity, that this reduction (increase) is bounded below (above)and the campaigns cost are linear on the controls. With respect to the second requirement, the model is designed the campaign effects by reducing the rate at which the disease is contracted from an average individual during the campaign. For an example, during a flu outbreak one starts a campaign orienting susceptible to avoid the virus contact by assuming some protective behavior such as washing hands, avoiding close environments. This campaign affects the probability of a susceptible contracting the virus to decrease. The same reasoning applied to a campaign oriented to the infected such as stimulating quarantine, will be modelled to increase the rate at which an average individual leaves the infective rate.

2.3 Optimal control of epidemics in metapopulations

This paper used combination of optimal control method together with epidemiological theory for metapopulations. The objective of this paper is to minimize the discounted number of individuals who are infected during the course of the epidemic. Here they used susceptible-infected-susceptible (SIS) compartmental model. Besides that, here come up with optimal control model under a constraint budget and under the fixed budget constraints with quarantine. Optimizing the control under the limited budget for some of the time when the combined number of infected individuals exceeds the availability of the drugs for treatment. The optimization is approached is adopted based on the Hamilton method in Pontryagin principle. It acts as a device to minimize the objective function subject to the epidemiological dynamics of the model and to the economic constraints.

2.4 Optimal control of a delayed HIV infection model with Immune response using an efficient Numerical Method.

The paper presents a delay-differential equation model with optimal control which describes the interactions between virus (HIV), human immunodeficiency, CD4+ T cells, and cell-mediated immune response. Both the intracellular delay and the treatment are incorporated into the model in order to improve the therapies to cure the HIV infection. The efficiency of drug treatment in inhibiting viral production and preventing new infections is represented by optimal controls. Existence for the optimal control pair is established by using Pontryagin’s maximum principle to characterize these optimal controls, and the optimality system is derived. This paper proposes a new algorithm based on the forward and backward difference approximation for the numerical method. There is two reason for this work. Firstly, a delay mathematical model with two controls that describe HIV infection of CD4+ T cells during therapy is proposed. For current world there is no effective therapy for HIV infection and the cost of treatment is very expensive. Thus, an optimal therapy in order to minimize the cost of treatment, improve immune response and to reduce the viral load is developed. Secondly, efficient numerical method based on optimal control is proposed too in this paper to identify the best treatment strategy of HIV infection in order to prevent viral production and to block new infection by using drug therapy with minimum side effects. The numerical results show an increase in the uninfected CD4+ T-cell count after five days of therapy and optimal treatment strategies reduce viral load.

Step 1: for $i = -m, \ldots, 0$, do :

$$x_i = x_0,\ y_i = y_0,\ v_i = v_0,\ z_i = z_0,\ u_1^i = 0,\ u_2^i = 0.$$

end for

for $i = n, \ldots, n+m$, do :

$$\lambda_1^i = 0, \lambda_2^i = 0, \lambda_3^i = 0, \lambda_4^i = 0.$$

end for

Step 2: for $i = 0, \ldots, n-1$, do :

$$
\begin{aligned}
x_{i+1} &= x_i + h[s - dx_i - (1 - u_1^i)kv_ix_i],\\
y_{i+1} &= y_i + h[k(1 - u_1^i)e^{-\delta\tau}v_{i-m}x_{i-m} - \delta y_i - py_iz_i],\\
v_{i+1} &= v_i + h[(1 - u_2^i)N\delta y_i - \mu v_i],\\
z_{i+1} &= z_i + h(cy_iz_i - bz_i),\\
\lambda_1^{n-i-1} &= \lambda_1^{n-i} - h\{1 + \lambda_1^{n-i}[d - (1 - u_1^i)kv_{i+1}]\\
&\qquad + \chi_{[0,t_f-\tau]}(t_{n-i})\lambda_2^{n-i+m}(u_1^{i+m} - 1)ke^{-\delta\tau}v_{i+1}\},\\
\lambda_2^{n-i-1} &= \lambda_2^{n-i} - h[\lambda_2^{n-i}\delta - \lambda_3^{n-i}(1 - u_2^i)N\delta - cz_{i+1}\lambda_4^{n-i} + pz_{i+1}\lambda_2^{n-i}],\\
\lambda_3^{n-i-1} &= \lambda_3^{n-i} - h[\lambda_1^{n-i}(1 - u_1^i)kx_{i+1} + \mu\lambda_3^{n-i}\\
&\qquad + \chi_{[0,t_f-\tau]}(t_{n-i})\lambda_2^{n+m-i}(u_1^{i+m} - 1)ke^{-\delta\tau}x_{i+1}],\\
\lambda_4^{n-i-1} &= \lambda_4^{n-i} - h[1 + p\lambda_2^{n-i}y_{i+1} + \lambda_4^{n-i}(b - cy_{i+1})],\\
R_1^{i+1} &= (k/A_1)(\lambda_2^{n-i-1}e^{-\delta\tau}v_{i-m+1}x_{i-m+1} - \lambda_1^{n-i-1}v_{i+1}x_{i+1}),\\
R_2^{i+1} &= (1/A_2)\lambda_3^{n-i-1}N\delta y_{i+1},\\
u_1^{i+1} &= \min(1, \max(R_1^{i+1}, 0)),\\
u_2^{i+1} &= \min(1, \max(R_2^{i+1}, 0)).
\end{aligned}
$$

end for

Step 3: for $i = 1, \ldots, n$, write

$$x^*(t_i) = x_i,\ y^*(t_i) = y_i^*,\ v^*(t_i) = v_i,\ z^*(t_i) = z_i,\ u_1^*(t_i) = u_1^i,\ u_2^*(t_i) = u_2^i.$$

end for

Figure 2-1 Algorithm 1 forward and backward sweep

2.5 SIS/SIR Model

This model is referring to S-susceptible, I-Infected and R-recovered/removed diseases status. Below are the simplest SIS Model:

$$\frac{dS}{dt} = -\beta SI + \alpha I,$$

$$\frac{dI}{dt} = \beta SI - \alpha I.$$

BSI is an average infected individual makes contact sufficient to infect *BN* others per unit time. *S/N* is the probability that a given individual that each infected individual comes in contact with is susceptible. Thus, each infected individual cause infection per unit time *(BN)(S/N)* = *BS*. Therefore, I infected individuals cause a total number of infections per unit time of *BSI.* For the αI term, α is the fraction of infected individuals who recover and re-enter the susceptible class per unit time.

2.6 Optimal control of Epidemics Information Dissemination Over Network

Controlling the information spreading is very important and this paper come up with a concept where epidemic model is used to picturize the collective dynamics of information spreading over the network. Here, the SIR model is used where S+I+R=1.

This paper develops a model to self healing scheme in mobile network and vaccine spreading schemes. For the self-healing scheme, the effectiveness of control signalling decreases, this makes the network out of the controller region. For the vaccine spreading scheme, the controllable region shrinks due to the epidemic and the spread vaccine stimulate the state transitions of nodes so that less nodes can remain in the susceptible state. This paper also used some constraints (network cost) in the model.

$$\begin{aligned}
\text{Minimize} \quad & J = \int_{T_0}^{T_f} [NI(t)]^{\beta} + \nu \cdot u^2(t) \; dt \\
\text{Subject to} \quad & \dot{I}(t) = G_I(I(t), R(t), u(t)), \\
& \dot{R}(t) = G_R(I(t), R(t), u(t), \phi(t)), \\
& S(t) + I(t) + R(t) = 1, \\
& S(t) \geq 0, \; I(t) \geq 0, \; R(t) \geq 0,
\end{aligned}$$

Figure 2-2: optimization problem

CHAPTER 3 : METHODOLOGY

3.1 Gant Chart

The project was started on June 2018 and the progress chart is as shown in the Gantt Chart below.

ACTIVITY	JUNE	JULY	AUG	SEPT	OCT	NOV	DEC	JAN
Research project								
Literature Review								
Data collection								
Project development (Calculations)								
Project development (Matlab)								
Optimizations								
Research Report								
Submission								
Presentations								

Figure 3-1: Gantt Chart

3.2 System Model and problem formulation

The uncontrolled Maki Thompson model is used here to formulate an optimal control problem for the controlled system. The definitions of all the parameters used in this paper are listed in Table1.

Table 1: Definitions of parameters used in this paper

Symbol	**Definition**
$i(t)$	fraction of ignorants in the population at time t
$s(t)$	fraction of spreaders in the population at time t
$r(t)$	fraction of stiflers in the population at time t
$\beta_1(t)$	per contact message spreading rate at time t
γ_1	per contact recovery rate
k	number of other individuals an individual is in contact at any given time
$\gamma = k\gamma_1$	recovery rate
$\beta(t) = k\beta_1(t)$	spreading rate at time t
T	campaign deadline
$u(t)$	control at time t (*e.g.* rate at which advertisements are put across in mass media) u_{max} maximum allowed control, $0 \leq u(t) \leq u_{max}$
$c(u(t))$	instantaneous cost incurred due to application of control
B	budget
$b(t)$	(cumulative) resource spent during [0 t]

So, the problem is to minimize the ignorants at the end of the campaign in the fixed population size. By doing this, the number of individuals who aware of the spreading of the information's can be maximize.

3.2.1 Uncontrolled Maki Thompson model

By considering a system in a fixed population size, came up with this uncontrolled Maki Thompson model. At time t, the fractions of, spreaders, ignorant and stiflers in the population are represented by $s(t)$, $i(t)$, and $r(t)$ respectively, where $s(t) + i(t) + r(t) = 1$. $\beta 1(t)$ is 'Per contact message spreading rate' at time t and $\gamma 1$ is 'per contact recovery rate'.

Firstly, the rate of decrease of the fraction of ignorants in the population at time t is derived. Initially, at $t = 0$, the system starts with $s(0) = s0$, $r(0) = 0$, $i(0) = 1 - s0$, where $s0$ is the initial fraction of spreaders which acts as the seed for the epidemic. 'Per contact message spreading rate', $\beta 1(t)$ can be interpreted as follows: the information passes from a spreader to an ignorant in a small time, interval dt at time t, due to a single ignorant-spreader contact with a probability $\beta 1(t)dt$. Assumed that each member in the population communicate with an average of k others at any time. Thus, an ignorant communicate with (an average of) $ks(t)$ spreaders at time t. The message will be transferred to the ignorant with probability, $1-(1-\beta 1(t)dt)ks(t) \approx \beta 1(t)ks(t)dt$. Since the fraction of ignorants at time t is $i(t)$, thus the reduce in fraction of ignorants in small interval dt at time t is $\beta 1(t)ks(t)i(t)dt$. By defining $\beta(t)$, $\beta 1(t)k$, the Eq. (1a) is obtained. $\beta(t)$ is referred as the 'spreading rate'.

Secondly, the rate of increase of fraction of stiflers at time t *is* derived. A spreader recovers to become a stifler due to interactions and communications with other spreaders and stiflers. 'Per contact recovery rate' is interpreted as at any time t, any spreader in contact with any single spreader or stifler will automatically convert to a stifler with probability $\gamma 1dt$. Any member of the population communicates with k others at any time. Hence, a spreader might in contact with an average of $k(s(t)+r(t))$ spreaders and stiflers, increasing the probability of recovery to $1 - (1 - \gamma 1dt)k(s(t)+r(t)) \approx k(s(t) + r(t))\gamma 1dt$, in a small interval dt at time t. Since the fraction of spreaders at time t is $s(t)$, so the increase in fraction of spreaders at time t in a small interval dt is given by $s(t)k(s(t) + r(t))\gamma 1dt$. By defining γ, $\gamma 1k$, the rate of increase of stiflers in the population as $\gamma s(t)(s(t) + r(t))$ (Eq. (1c)). γ referred as the 'recovery rate' in this paper. Eq. (1b) is a consequence of Eqs. (1a) and (1c).

Thus, the evolution of the spreaders, ignorants, and stiflers in the fixed population in the uncontrolled Maki Thompson system is given by:

$$\dot{i}(t) = -\beta(t)i(t)s(t), \quad \text{Equation 1(a)}$$
$$\dot{s}(t) = \beta(t)i(t)s(t) - \gamma s(t)(s(t) + r(t)), \quad \text{Equation 1(b)}$$
$$\dot{r}(t) = \gamma s(t)(s(t) + r(t)). \quad \text{Equation 1(c)}$$

3.2.2 The controlled system

The sum of spreaders, ignorants and stiflers is equals to 0. $r(t) = 1 - i(t) - s(t)$. Thus, this system can be controlled by this function $u \in U$ which can transfers individuals from stiflers and ignorant class to the spreader class. Assumed that application of the control incurs a non-linear cost, given by $c(u(t))$ at time t. Also, the fixed budget, as mentioned in the equation 2e. The function $c(.)$ is assumed to be continuous and increasing in its argument, to maximize the number of individuals who are aware of the information by the campaign deadline $t = T$. So the reward function is $s(T) + r(T) = 1 - i(T)$. Hence the cost function (to be minimized) to be $J = i(T)$ is chosen. The optimal control problem is:

$$\min_{u \in U} J = i(T), \quad \text{Equation 2a}$$
$$\text{subject to: } \dot{i}(t) = -\beta(t)i(t)s(t) - u(t)i(t), \quad \text{Equation 2b}$$
$$\dot{s}(t) = (\beta(t) + \gamma)i(t)s(t) - \gamma s(t) + u(t)i(t) + \alpha u(t)(1 - i(t) - s(t)), \quad \text{Eqn 2c}$$
$$s(0) = s_0, \quad i(0) = 1 - s_0, \quad \text{Equation 2d}$$
$$\int_0^T c(u(t))dt = B. \quad \text{Equation 2e}$$

3.3 Solution for the optimal control problem

3.3.1- Pontryagin's Minimum principle

The solution to the Problem of equation 2 to 2e replaced by the equivalent condition using Pontryagin's Minimum principle. This helps the system of ordinary differential equations (boundary value problem (BVP)) which are important conditions for optimum. The standard forward and backward sweep method used to solve the BVPs by optimal control problems is not directly applicable and needs to be adapted to prevent from isoperimetric budget constraint.

Denote the adjoint variables by $\lambda i(t)$, $\lambda s(t)$ and $\lambda b(t)$. At time t, $u*(t)$ represent the optimal control and, $i*(t)$, $s*(t)$, $b*(t)$ and $\lambda*i\ (t)$, $\lambda*s(t)$, $\lambda*b(t)$ are the state and adjoint variables evaluated at the optimum respectively.

The Hamiltonian for Problem (2), with (2e) replaced by the equivalent equations is given by,

$$
\begin{aligned}
H(i(t), s(t), b(t), u(t), \lambda s(t), \lambda i(t), \lambda b(t), t) = \\
&\lambda i(t)\ [\beta(t)i(t)s(t) - u(t)i(t)\ ] \\
&+ \lambda s(t)\ [(\ \beta(t) + \gamma)\ i(t)s(t) - \gamma s(t) + u(t)i(t) + \alpha u(t)\ (1 - i(t) - s(t))] \\
&+ \lambda b(t)\ [c(u(t))]
\end{aligned}
$$

State equations: $i^*(t)$, $s^*(t)$, $b^*(t)$, $u^*(t)$

Adjoints equations:

$$\lambda *i(t) =(-\partial / \partial\ i(t))H(i(t), s(t), b(t), u(t), \lambda\ i(t), \lambda\ s(t), \lambda\ b(t), t)\Big|_{i(t)=i*(t),s(t)=s*(t),b(t)=b*(t),u(t)=u*\ (t), \lambda\ i(t)= \lambda *i\ (t), \lambda\ s\ (t)= \lambda *s(t), \lambda\ b(t)= \lambda *b\ (t)}$$

$$= \lambda *i\ (t)\beta\ (t)s*(t) + \lambda *i\ (t)u*(t) - \lambda *s\ (t)\beta\ (t)s*(t) - \lambda *s\ (t)\gamma\ s*(t) - \lambda *s(t)u*(t) + \lambda *s\ (t)\alpha\ u*(t).$$

$$\lambda *s(t) = (-\partial / \partial\ s(t))H(i(t), s(t), b(t), u(t), \lambda\ i(t), \lambda\ s(t), \lambda\ b(t), t)\Big|_{i(t)=i*(t),s(t)=s*(t),b(t)=b*(t),u(t)=u*(t), \lambda\ i(t)= \lambda *i\ (t), \lambda\ s\ (t)= \lambda *s(t), \lambda\ b(t)= \lambda *b\ (t)}$$

$$= \lambda *i\ (t)\beta\ (t)i*(t) - \lambda *s\ (t)\beta\ (t)i*(t) - \lambda *s(t)\gamma\ i*(t) + \lambda *s\ (t)\gamma\ + \lambda *s\ (t)\alpha\ u*(t).$$

$$\lambda *b(t) =(-\partial / \partial\ b(t))H(i(t), s(t), b(t), u(t), \lambda\ i(t), \lambda\ s(t), \lambda\ b(t), t)\Big|_{i(t)=i*(t),s(t)=s*(t),b(t)=b*(t),u(t)=u*(t), \lambda\ i(t)= \lambda *i\ (t), \lambda\ s\ (t)= \lambda *s\ (t), \lambda\ b(t)= \lambda *b\ (t)}= 0.$$

$$(\partial / \partial\ u(t))H(i(t), s(t), b(t), u(t), \lambda\ i(t), \lambda\ s(t), \lambda\ b(t), t)\Big|_{i(t)=i*(t),s(t)=s*(t),b(t)=b*(t),u(t)=u*(t), \lambda\ i(t)= \lambda *i\ (t), \lambda\ s\ (t)= \lambda *s(t), \lambda\ b(t)= \lambda *b\ (t)}$$

$$= - \lambda *i\ (t)i*(t) + \lambda *s(t)i*(t) + \lambda *s\ (t)\alpha\ (1 - i*(t) - s*(t)) + \lambda *b(t)c'(u*(t)) = 0.$$

3.3.2- Numerical Solution and issues in computation

Here have to solve the BVP involving state and adjoint equations, to solve the optimal control problem numerically, also called the optimality system. The state equations are given by (2b), (2c), (2d) and (3) and the adjoint equations are given by in the previous page. Note that the value of the control variable has to be substitute in the above-mentioned differential equations to get a system entirely in terms of state variable and adjoint variable.

The optimality system can be solved using boundary value problem by solving techniques such as the shooting method. But found that the naive implementation of the shooting algorithm stalls before converging to a correct solution due to possibly because of inconsistency in the numerically computed gradient values. Also, due to the isoperimetric constraint, $b\ (0) = 0$ and $b(T) = B$ in (3), it is not possible to implement naive forward backward sweep algorithm. Hence, here briefly discussed the adaptation of the forward backward sweep algorithm which was used to solve the optimality system in the following.

$\lambda{*}b(t)$ is an unknown value which is constant over time, $0 \leq t \leq T$, for the optimality system, call it $\lambda cb{*}$. We have taken the approach of finding $\lambda cb{*}$ using bisection algorithm. Initialize the computation with two approximate values of $\lambda cb{*}$ (call them $\lambda cb{*}{-}high$ and $\lambda cb{*}{-}low$), one for which $b(T) < B$, and other for which $b(T) > B$. Then, refine the value of $\lambda c{*}\ b$ using bisection method till the constraint $b(T) = B$ is satisfied with desired tolerance.

The values of $\lambda cb{*}{-}low$, $\lambda cb{*}{-}high$, Bth, λth and $Nsweep$ used in all of the computations in this paper are 0, 100, 10−4,10−4 and 50 respectively. Since $\lambda cb{*}{-}low$ is small, so control computed is large, hence $b\lambda c{*}\ b{-}low\ (T)$ is large (very close to maximum allowed budget, $c(umax)T$). Similarly, $\lambda cb{*}{-}high$ is large, so control is small, hence $b\lambda c{*}\ b{-}high\ (T)$ is small (very close to 0). These values were found to be suitable to initialize the bisection method. Here have implemented this algorithm in MATLAB and have used its initial value problem solver ode45() to evaluate the differential equations. The solver uses fourth order Runge-Kutta algorithm with variable step size for computation and is capable of integrating backwards as required by the adjoint equations.

CHAPTER 4 RESULTS & DISCUSSIONS

4.1 Results

4.1.1- Matlab Code using Pontryagin's principle (Main code)

```
%% initial definitions
clear all;clc;
options='';%odeset(b,i,s,r,y);
tmax=4;
dicretization=0.01;
max_iter=20;
a=0.1;
% u=0.1;
b=10;
r=0.01;
umax=20;
tspan1=0:0.01:tmax;
tspan2=tmax:-0.01:0;

 u=0.1*ones(length(tspan1));
 u(1:5,1)=1;
for j=1:max_iter


[T,Y]=ode45(@(t,y) ODEsimple2(t,y,b,r,a,u(:,j),tspan1),tspan1,[0.9 1
0],options);% solving ODE to obtain y
x1=Y(:,1); %% ignorant first state
x2=Y(:,2);  %% spreaders second state
x3=Y(:,3);%% b third state
xt=T(:,1); %% states time
```

State ODE

```
x11=flipud(x1);
x22=flipud(x2);
x33=flipud(x3);
xtt=flipud(xt);
```

Flipped states

```
[T1,X]=ode45(@(t,x)
ODEcostate2(t,x,x11,x22,x33,xtt,b,r,a,u(:,j)),tspan2,[1 0
50],options);% solving ODE to obtain ylambda
cox1=flipud(X(:,1));
cox2=flipud(X(:,2));
cox3=flipud(X(:,3));
```

Costate ODE

```
for k=1:length(xt)
    %pause();
    S=((cox1(k)*x1(k)-cox2(k)*x1(k)-cox2(k)*(1-x1(k)-
x2(k)))/cox3(k))^-1;
    if (S>umax)
        S=umax;
    elseif (S<0)
        S=0;
    end
    c_p=2*u(k,j);
    SF=-(S*c_p*cox3(k))+(cox3(k))*c_p*u(k,j);
    if(SF<0)
            u(k,j+1)=umax;
    elseif(SF>0)
            u(k,j+1)=0;
    elseif (SF==0)

            u(k,j+1)=S;
    end

end

end
```

Switching functions

```
%state equations
%syms
figure(1)
plot(T,Y)
legend('x1','x2','x3')
figure(2)
plot(xt,x1)
figure(3)
plot(x2)

figure(4)
plot(u(:,max_iter))
```

Output graphs

4.1.2- States ODE code

```
function dydt = ODEsimple2(t,y,b,r,a,u,tspan1)

u=u(floor(t+1));

dydt(1)=-b*y(1)*y(2)-u*y(1);
dydt(2)=(b+r)*y(1)*y(2)-r*y(2)+u*y(1)+a*u*(1-y(1)-y(2));
dydt(3)=u^2;
dydt=dydt';

%y=lambda


end
```

4.1.3- Costates ODE

```
function dxdt = ODEcostate2(t,x,x11,x22,x33,xtt,b,r,a,u)

u=u(floor(t+1));
x11=interp1(xtt,x11,t);
x22=interp1(xtt,x22,t);
x33=interp1(xtt,x33,t);

dxdt(1)= x(1)*b*x22+x(1)*u-x(2)*b*x22-x(2)*r*x22-x(2)*u+x(2)*a*u;
dxdt(2)= x(1)*b*x11-x(2)*b*x11-x(2)*r*x11+x(2)*r+x(2)*a*u;
dxdt(3)=0;
dxdt=dxdt';


end
```

4.2 – Results (output)

4.2.1- a=`0.5; b=20; r=0.03; umax=40;`

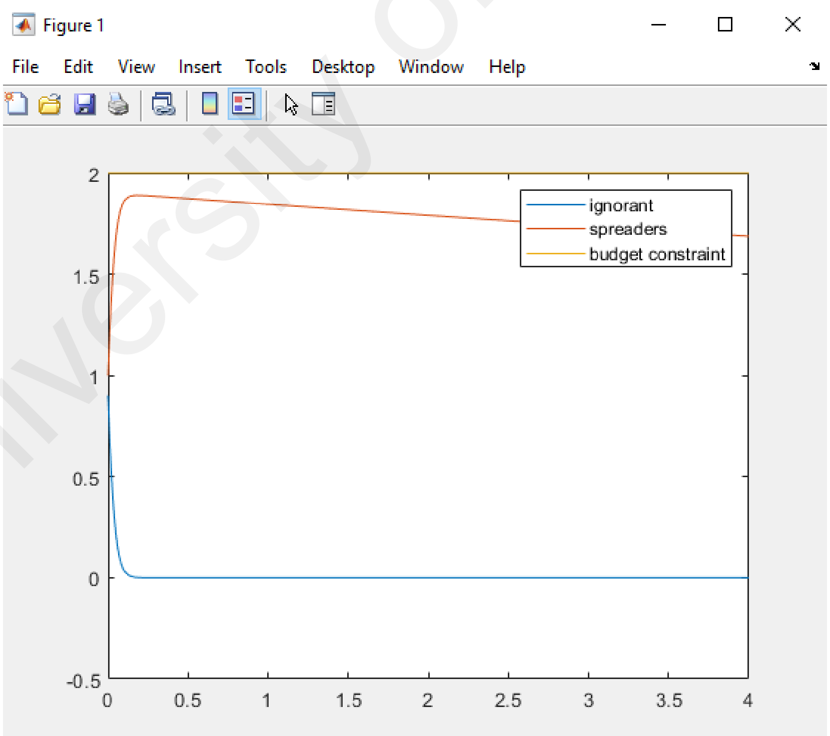

Figure 4-1 combination graph with x1,x2 and x3

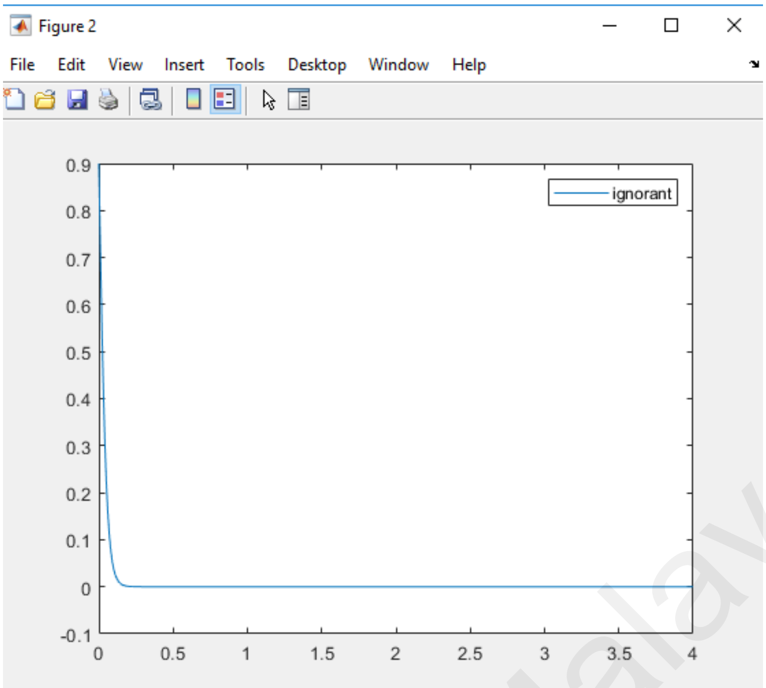

Figure 4-2 ignorant (x1)

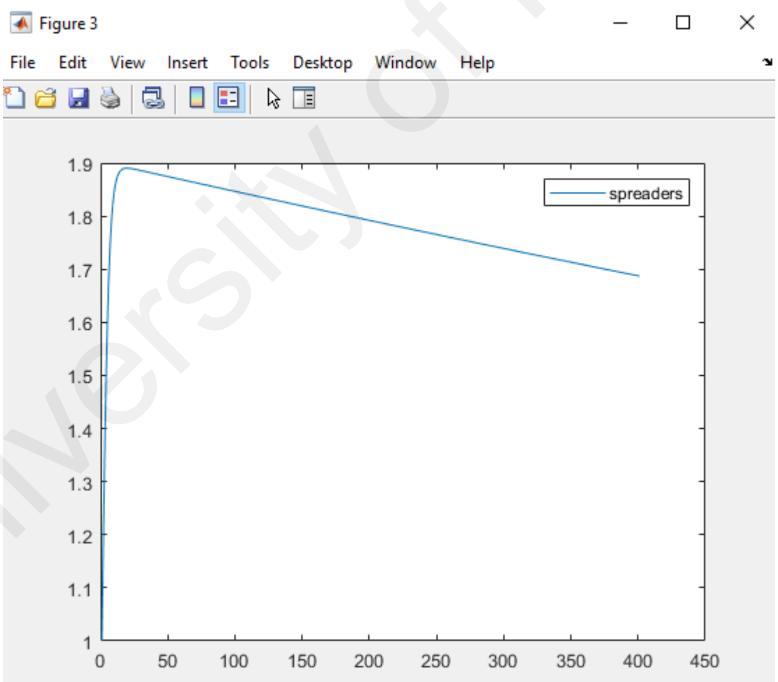

Figure 4-3 spreaders

4.2.2- `a=1.0; b=10; r=0.01; umax=30;`

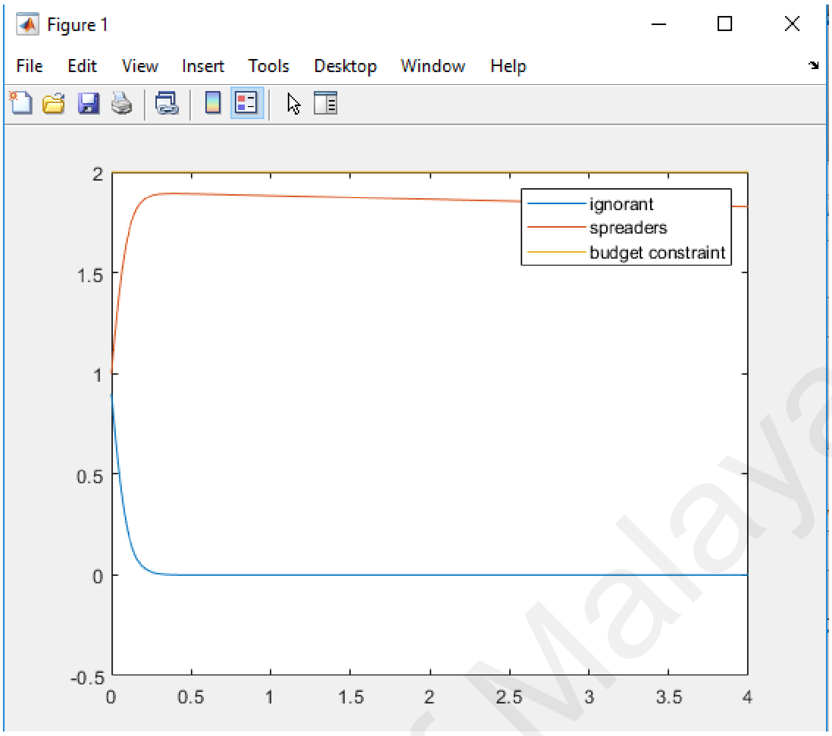

Figure 4-4 combination of ignorant and spreaders rate

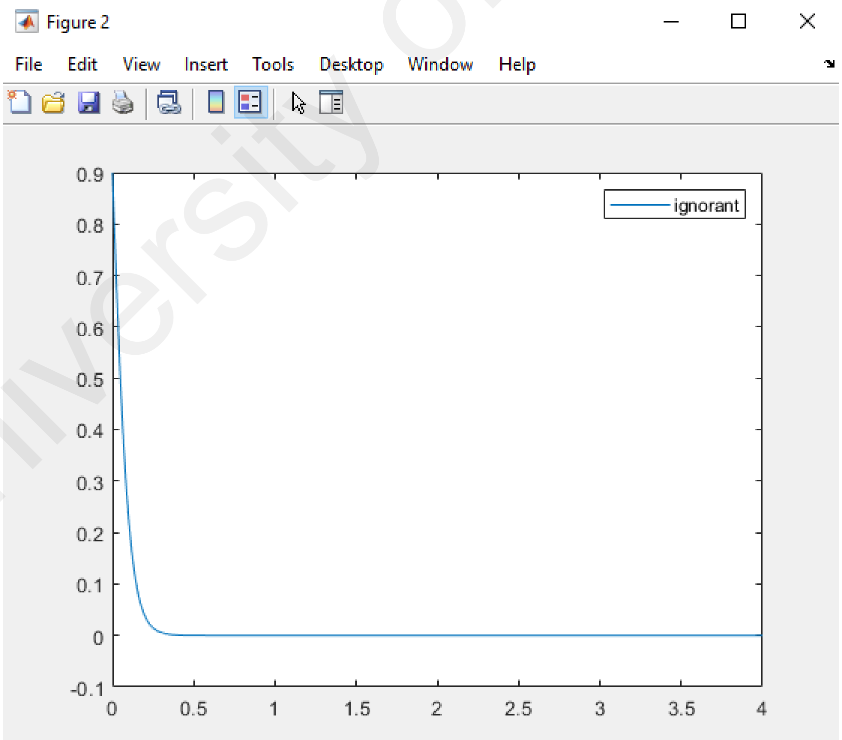

Figure 4-5 ignorant

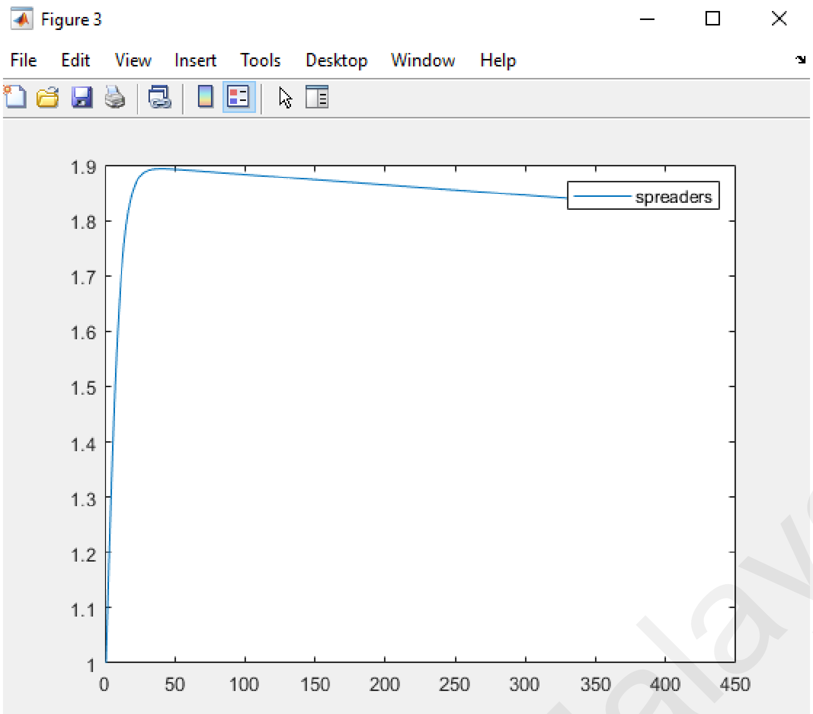

Figure 4-6 Spreaders

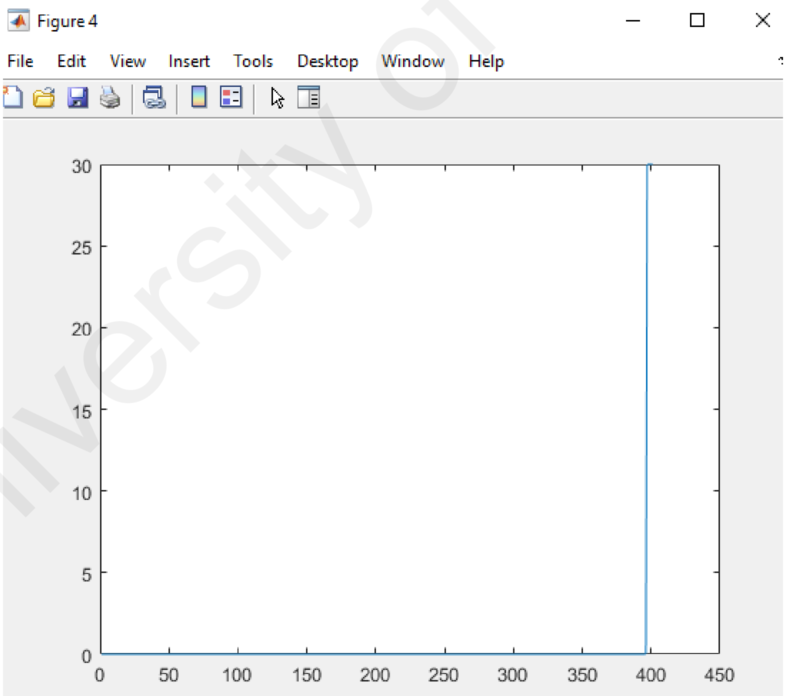

Figure 4-7 the range of optimal control

4.2.3- Optimal control with bang bang output for the maximum iterations

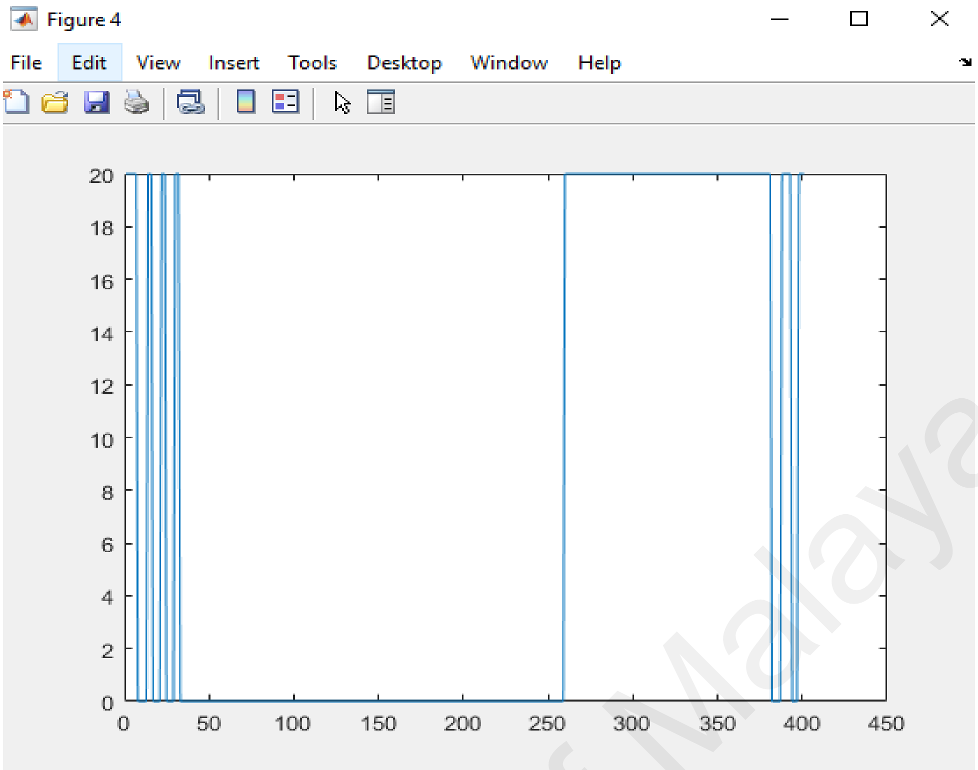

Figure 4-8: control input signal maximum iterations

4.3.3- Graphs when the budget constraints are decreased

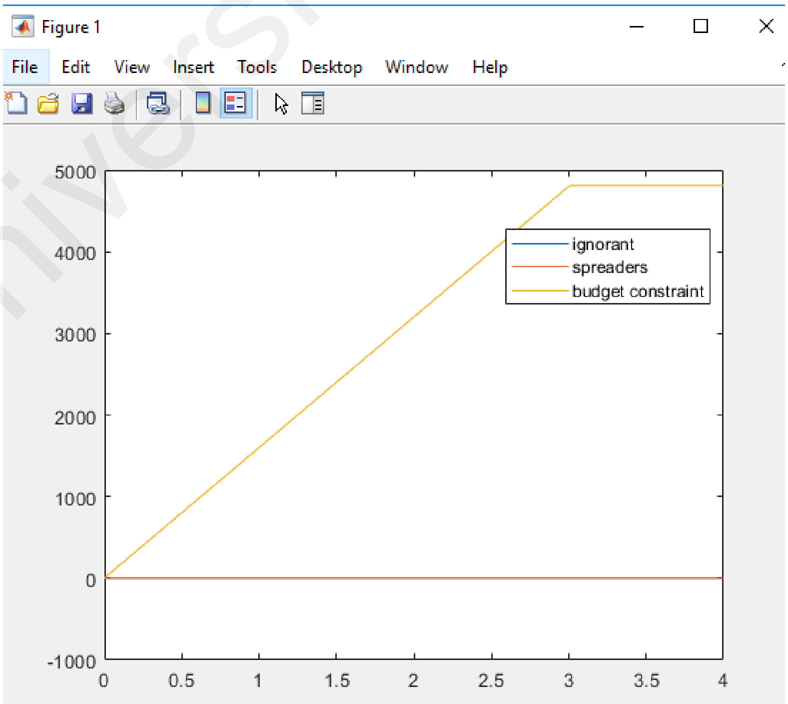

Figure 4-9: when the value of b decreased

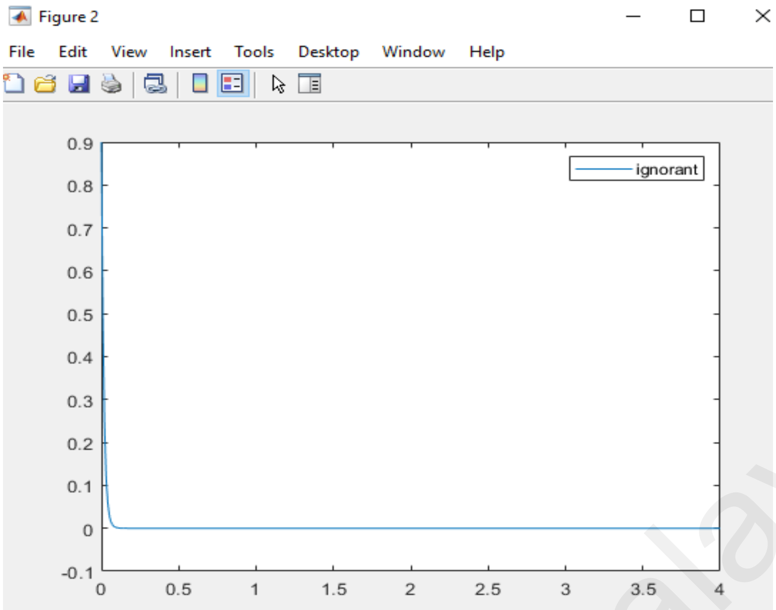

Figure 4-10: ignorant at b= 20

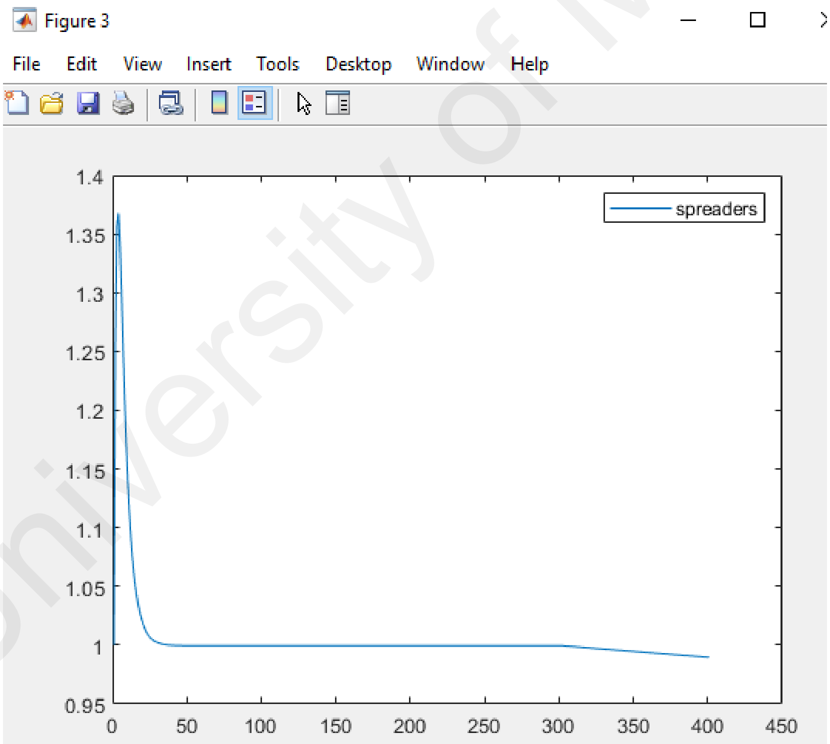

Figure 4-11: shows spreaders value

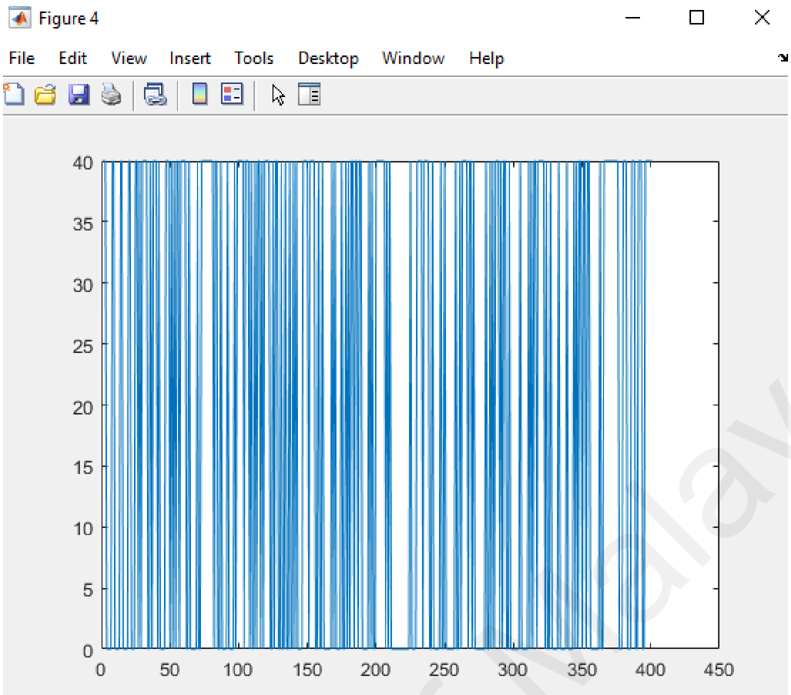

Figure 4-12: optimal control with maximum iteration

4.2 Discussion

Depending on the application, the spreading and recovery rate of the information epidemic may vary a lot. This depends on interest of people in conversing about the topic in question. Thus, we have used different parameter values to model epidemics of varying virulence. The shape of the control signals varies considerably when the values of spreading and recovery rates are changed. Later have discuss the variation in the cost function (2a) with respect to various model parameters and compare the performance of the optimal control with the static control. In this paper we have assumed the cost of application of control to be, $c(u(t)) = u2(t)$.

CHAPTER 5 CONCLUSION & FUTURE RECOMMENDATIONS

5.1 Conclusion

In this work have formulated an optimal control problem to maximize the spread of information under a fixed campaigning budget constraint. The information spread dynamics is assumed to follow the Maki Thompson rumor model, which is more suitable in this context than SIS/SIR epidemic models used in some of the previous studies. The control signal converts ignorants and stiflers into spreaders. This can be done via strategies such as advertising in mass media, publishing manifestos, door-to-door campaigns etc., depending upon the application election, product promotion, crowdfunding, social awareness campaigns, to mention a few. Assume that the general nonlinear campaigning costs and show the existence of a solution to the formulated optimal control problem.

Note that the standard Filippov/Cesari theorems are not applicable in this situation. The optimal control problem using Pontryagin's Minimum Principle and a modified version of forward backward sweep technique for numerical computation is designed, to accommodate the isoperimetric budget constraint in our formulation. The techniques developed in this paper are general and can be applied to other similar optimal control problems.

To model practical situations, such as increasing interest of people in talking about elections as polling day approaches or diminishing interest in a movie after its release, have allowed the spreading rate profile of the information epidemic to vary during the campaign duration. Have studied the shape of the optimal control signal for different model parameters and spreading rate profiles. Variations of the optimal campaigning costs with respect to various model parameters are also studied and results compared with the static campaigning strategy. In the static strategy the control is constant throughout the decision horizon and respects the same budget constraint as the optimal strategy. Have found that the optimal strategy achieves significant performance improvements compared to the static strategy for a wide range of model parameters.

5.2 Future Recommendations

The spreaders rate can be increases and the ignorant and stiflers can change to normal spreaders by using Pontryagin Maximum principle. We should increase the number of spreaders by maximizing it. Besides that, we should increase the cost constraint to develop the spreaders rate. When the recovery rate at t time is increased, then the optimal control of an spreading rate can be increased.

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