1 Elementary Definitions and Observations

1.1 3-Homogeneous Simplicial Complexes

Let V be a finite set of cardinality at least 3. A *3-homogeneous simplicial complex*, or simply a *complex*, on V is a non empty family K of subsets of V, each of which is called a *simplex*, such that

1. any 1-subset of V is a simplex,
2. any non empty subset of a simplex is a simplex, and
3. any simplex is contained in a simplex consisting of 3 points.

The set V is sometimes called the *underlying set* of the complex K.

The simplexes consisting of 1,2, and 3 points are called, respectively, the *vertexes*, the *edges*, and the *triangles* of K. In the sequel, we let $V(K)$ denote the set of vertexes of K, $E(K)$ the set of edges of K, and $max(K)$ the set of triangles of K. Without loss of generality, the elements of V and $V(K)$ are considered as identical. In addition, we shall identify a simplex with the set of vertexes that determines the simplex whenever the situation is more convenient.

A subfamily J of K which is a complex whose underlying set is contained in the underlying set V of K is said to be a *subcomplex* of K, denoted $J \preceq K$. Clearly, given any two complexes J and K, $J \preceq K$ if and only if $max(J) \subseteq max(K)$. A subcomplex J of a complex K such that $J \neq K$ is said to be *proper*. In this case, we write $J \prec K$.

For geometrical definitions of complexes, refer to [1, 9, 16, 18].

Example 1 *Let V be a finite set of no less than 3 points. A very natural complex on V is given by the family of all 3-subsets of V and their non empty subsets. Indeed, it is the largest complex on V.*

Example 2 *Let $V = \{1, 2, 3, 4\}$. Then each of the following families is a complex on V:*

[1] $K_1 = \{123, 124, 12, 13, 14, 23, 24, 1, 2, 3, 4\}$,
$K_2 = \{123, 124, 134, 12, 13, 14, 23, 24, 34, 1, 2, 3, 4\}$, *and*
$K_3 = \{123, 124, 134, 234, 12, 13, 14, 23, 24, 34, 1, 2, 3, 4\}$.
Note that $K_1 \preceq K_2 \preceq K_3$.

Example 3 *Let* $V = \{1, 2, 3, 4, 5\}$. *A complex on* V *is* $\{123, 345, 12, 13, 23, 34, 35, 45, 1, 2, 3, 4, 5\}$.

1.2 Connectedness and Hyper Connectedness

In this section, we introduce two related notions: connectedness and hyper connectedness. It can be seen easily that hyper connected complexes form a proper subcollection of connected complexes.

1.2.1 Connectedness

A *path* in a complex K is a sequence $x_1, x_2, \cdots, x_n$ of no fewer than two distinct elements in $V(K)$ such that $x_i \neq x_j$ for all $i \neq j$, except possibly $x_1 = x_n$, and $x_k x_{k+1} \in E(K)$ for $k = 1, 2, \cdots, n-1$.

A complex K is said to be *connected* if for all $u, v \in V(K)$ such that $u \neq v$, there is a path along the edges of K initiated by u and terminated by v. We may see immediately that the complex K is connected if and only if given any two distinct $\delta, \delta' \in max(K)$, there is a sequence $\delta = \delta_1, \delta_2, \cdots, \delta_s = \delta'$ in $max(K)$ such that $\delta_k \frown \delta_{k+1} \neq \emptyset$ for $k = 1, 2, \cdots, s-1$. A *component* of K is a maximal connected subcomplex of K. Refer to [1, 9, 16, 18].

1.2.2 Hyper Connectedness

A complex K is said to be *hyper connected* if given any $\delta, \delta' \in max(K)$, then either

1. $\delta = \delta'$, or

[1]For simplicity, we have written, say 123, for $\{1, 2, 3\}$. Unless stated otherwise, we shall adopt this shorthand convention in the sequel.

2. there is a finite sequence $\delta = \delta_1, \delta_2, \cdots, \delta_s = \delta'$ in $max(K)$ of which any two consecutive elements in the sequence, that is, δ_i and δ_{i+1} for $i = 1, 2, \cdots, s-1$, share an edge.

There is another equivalent definition:

Proposition 1 *Let K be a complex. Then K is hyper connected if and only if*

1. *K consists of exactly one triangle, or*

2. *given any proper subcomplex J of K, namely $J \prec K$, the subcomplex J' such that $max(J') = max(K) \backslash max(J)$ possesses an edge in common with J.*

Proof. Suppose K is hyper connected. If K possesses only one triangle, then the conclusion is trivial. Suppose K consists of no less than two triangles. Let $\delta \in J$ and $\delta' \in J'$. Since K is hyper connected, there is a sequence $\delta = \delta_1, \delta_2, \cdots, \delta_s = \delta'$ in $max(K)$ of which any two consecutive elements in the sequence share an edge. Let j be the largest integer such that $\delta_j \in J$, so that $\delta_{j+1} \in J'$. It then follows that J and J' share the edge which is common to both δ_j and δ_{j+1}. More explicitly, $\delta_j \frown \delta_{j+1}$ forms an edge in both J and J'.

Conversely, let $\delta, \delta' \in max(K)$. Note that if we let $max(J) = \{\delta\}$, then δ and J' have an edge in common by hypothesis. This implies that there is an element $\delta_2 \in max(J')$ such that $\delta \frown \delta_2 \in E(K)$. If $max(K) = \{\delta, \delta_2\}$, we're done. Otherwise, we let $max(J) = \{\delta, \delta_2\}$ and proceed analogously. Since $max(K)$ is finite, it can be seen easily that there must be a sequence $\delta = \delta_1, \delta_2, \cdots, \delta_s = \delta'$ in $max(K)$ of which any two consecutive elements in the sequence share an edge, showing that K is hyper connected.

q.e.d.

Obviously, a hyper connected complex is connected, but the converse may be false. The complexes illustrated in example 1 and example 2 are both hyper connected. The complex in example 3 is connected, but not hyper connected.

Two interesting families of hyper connected complexes are *pseudosurfaces* and *surfaces*. See [1, 9, 18].

A subcomplex J of a complex K may or may not be hyper connected. The subcomplex J is said to be a *hyper component* of K if J is hyper connected, and the union of J with any $\delta \in max(K)\backslash max(J)$ and the vertexes and edges contained in δ yields a subcomplex of K which is no longer hyper connected. In other words, J is a hyper component of K if and only if J is a maximal hyper connected subcomplex of K.

Throughout, we shall let $h(K)$ denote the number of hyper components of K. Clearly, $h(K) = 1$ if and only if K is hyper connected.

Example 4 *Let K be the complex given in example 3. We may check easily that $h(K) = 2$.*

1.3 The Incidence Quotients of Complexes

Suppose there are two complexes K and L. One question immediately arises: how do we recognise whether K and L are identical despite the labelling of their vertexes?

We say that the complexes K and L are identical, or more formally, *isomorphic*, denoted $K \simeq L$, if there is a bijective correspondence f, which is known as an *isomorphism*, mapping $V(K)$ onto $V(L)$ such that $xy \cdots z \in K$ if and only if $f(x)f(y) \cdots f(z) \in L$ where $x, y, \cdots, z \in V(K)$. The complex L is then called an *isomorphic image* of K, and vice versa. To emphasize the isomorphism f, we write $K \overset{f}{\simeq} L$. Refer to [1, 9, 16].

We may now rephrase the question: how do we determine whether $K \simeq L$? To partially resolve the question, we proceed to implant some algebraic structures into the complexes.

Let K be a complex. The *group of 1-chains* $C_1(K)$ of K is defined as the additive free abelian group on $V(K)$. The *group of 3-chains* $C_3(K)$ of K is similarly defined on $max(K)$. Next, we let the *incidence operator* D_3^2 be the homomorphism from $C_3(K)$ into $C_1(K)$ such that

$$D_3^2(\delta) = x + y + z$$

in which x, y, and z are the vertexes of δ for each $\delta \in max(K)$.

For computational convenience, we may sometimes write $\alpha \equiv \beta(D_3^2(C_3(K$

))), or simply $\alpha \equiv \beta$ whenever the complex K involved is clear, to denote that $\alpha - \beta \in D_3^2(C_3(K))$ for all $\alpha, \beta \in C_1(K)$. In this case, we say that α is *congruent* to β. Otherwise, α and β are said to be not congruent, denoted $\alpha \not\equiv \beta$.

The *incidence quotient* $Q(K)$ of K is simply the factor group of $C_1(K)$ over the set of images under D_3^2, namely

$$Q(K) = C_1(K)/D_3^2(C_3(K)).$$

Note that since the incidence operator D_3^2 is a linear map, we may regard D_3^2 as an *incidence matrix*. Refer to [8, 20]. The incidence quotient $Q(K)$ of a given complex K can be interpreted as a two-step homology group of the complex. See [5]. For general background on homology groups, see [1, 9, 13, 16, 18].

For simplicity, we may identify any element of $Q(K)$ with the corresponding 1-chain. In particular, if $x \in V(K)$, then $x + D_3^2(C_3(K))$ is written as x whenever it is more convenient. Furthermore, the symbol 0 may represent the set of images under D_3^2, namely $D_3^2(C_3(K))$, instead of the zero 1-chain. This shall be clear from the context.

Note that $V(K)$ is then a system of generators for $Q(K)$. One of our objectives is to find a minimal system of generators, which is a subset of $V(K)$, for $Q(K)$. We shall see in later sections that we have been successful in certain cases, especially in dealing with hyper connected complexes.

Before we proceed further, we state an obvious observation.

Proposition 2 *Let K and L be two complexes. Then $K \simeq L$ implies that $D_3^2(C_3(K)) \cong D_3^2(C_3(L))$ and $Q(K) \cong Q(L)$.*

Proof. Let $K \overset{f}{\simeq} L$. Define $\mu : D_3^2(C_3(K)) \rightarrow D_3^2(C_3(L))$ such that

$$\textstyle\sum_{x \in V(K)} j_x x \overset{\mu}{\mapsto} \sum_{x \in V(L)} j_x f(x)$$

for any sequence $\{j_x\}_{x \in V(K)}$ of integers such that $\sum_{x \in V(K)} j_x x \in D_3^2(C_3(K))$. Suppose $\sum_{x \in V(K)} j_x x \in D_3^2(C_3(K))$, so that

$$\begin{aligned}\textstyle\sum_{x \in V(K)} j_x x &= D_3^2(\textstyle\sum_{\delta \in max(K)} n_\delta (x_\delta y_\delta z_\delta)) \\ &= \textstyle\sum_{\delta \in max(K)} n_\delta (x_\delta + y_\delta + z_\delta)\end{aligned}$$

for a sequence $\{n_\delta\}_{\delta \in max(K)}$ of integers, in which $\delta = x_\delta y_\delta z_\delta$ for each $\delta \in max(K)$. It follows immediately that the sum of all coefficients of a vertex $x \in V(K)$ in the above summation is given by $\sum_{x\in\delta,\delta\in max(K)} n_\delta$, which is equivalent to saying that

$$j_x = \sum_{x\in\delta,\delta\in max(K)} n_\delta.$$

Therefore,

$$\begin{aligned}\sum_{x\in V(K)} j_x f(x) &= \sum_{x\in V(K)}(\sum_{x\in\delta,\delta\in max(K)} n_\delta) f(x)\\ &= \sum_{\delta\in max(K)} n_\delta(f(x_\delta)+f(y_\delta)+f(z_\delta))\\ &= D_3^2(\sum_{\delta\in max(K)} n_\delta(f(x_\delta)f(y_\delta)f(z_\delta)))\end{aligned}$$

since f is bijective . Thus, $\sum_{x\in V(K)} j_x x \overset{\mu}{\mapsto} \sum_{x\in V(K)} j_x f(x) \in D_3^2(C_3(L))$ and μ is well-defined.

That μ is an isomorphism follows from the facts that $V(L) = \{f(x)|x \in V(K)\}$ is a basis of $C_1(L)$, and that f is invertible. Hence, $D_3^2(C_3(K)) \cong D_3^2(C_3(L))$.

We may indeed extend the domain of definition of μ so that it is an isomorphism from $C_1(K)$ onto $C_1(L)$, which then induces a canonical epimorphism from $C_1(K)$ onto $Q(L)$ whose kernel is given by $D_3^2(C_3(K))$, showing that $Q(K) \cong Q(L)$.

q.e.d.

2 The Equivalence Relation on the Set of Vertexes

2.1 The Equivalence Relation on the Set of Vertexes

To give a geometrical interpretation to the incidence quotient $Q(K)$ of a complex K, we introduce an equivalence relation on $V(K)$.

Let $x, y \in V(K)$. We say that x is *equivalent* to y, denoted $x \sim y$, if either

1. $x = y$, or

2. there is a finite sequence $x = x_1, x_2, \cdots, x_n = y$ in $V(K)$ such that to each pair of consecutive elements in the sequence, namely x_i and x_{i+1}