It then follows that if $uvw \in \text{max}(K')$ is such that $v$ and $w$ are parents of terminal nodes in $F$, then $u$ can also be written as an integral multiple, that is a sum of powers of $-2$, of $x_{0,b}$. Indeed, $u \equiv ((-2)^{h_{j}u_{1}} + (-2)^{h_{j}u_{2}} + (-2)^{h_{j}u_{3}} + (-2)^{h_{j}u_{4}})x_{0,b}$ where for $i = 1, 2, 3, 4$, $u_{i} \equiv (-2)^{h_{j}u_{i}}x_{0,b}$ and $u_{i}$'s are the four children of $v$ and $w$.

If we proceed in this manner, we find that each $u \in V(K')$ is an integral multiple, namely a sum of powers of $-2$, of $x_{0,b}$.

In fact, we may find that for each $j$, $j = 1, 2, 3, \langle \rangle^{j} \equiv (\sum_{u \in P_{j}}(-2)^{h_{j}u} - \sum_{v \in Q_{j}}(-2)^{h_{j}v})x_{0,b}$ where $P_{j}$ and $Q_{j}$ are, respectively, set of odd terminal nodes and set of even terminal nodes of $T_j$, and we have assumed that $w \equiv (-2)^{h_{j}w}x_{0,b}$ for $w \in P_{j} \sim Q_{j}$.

Now, we may see readily that $|P_{1} \sim P_{2} \sim P_{3}| = p = |P|$, namely there is a one-one correspondence from $P_{1} \sim P_{2} \sim P_{3}$ onto $P$. So is the case for $Q_{1} \sim Q_{2} \sim Q_{3}$ and $Q$. It then follows that $\sum_{u \in P_{1} \sim P_{2} \sim P_{3}}(-2)^{h_{j}u} - \sum_{v \in Q_{1} \sim Q_{2} \sim Q_{3}}(-2)^{h_{j}v} = n = \sum_{j \in P}(-2)^{h_{j}} - \sum_{k \in Q}(-2)^{h_{k}}$. Refer to remark 2 and remark 3.

Next, it is clear that $P_{i} \sim P_{j} = \emptyset$ for $i \neq j$. So do $Q_{j}$'s for $j = 1, 2, 3$. Thus, $\langle \rangle^{1} + \langle \rangle^{2} + \langle \rangle^{3} \equiv 0$ implies that $x_{0,b} \equiv -(n - 1)x_{0,b}$, which is equivalent to saying that $nx_{0,b} \equiv 0$ in $Q(K)$. Also note that for any $rst \in \text{max}(K) \setminus \{x_{1,a}x_{0,a}x_{0,b}, x_{1,a}x_{0,b}x_{0,c}\}$, $a_{r} + a_{s} + a_{t} = 0$, where $a_{r}$, $a_{s}$, and $a_{t}$ are the coefficients such that $r \equiv -a_{r}x_{0,b}$, $s \equiv -a_{s}x_{0,b}$, and $t \equiv -a_{t}x_{0,b}$, which is obviously divisible by $n$. If $rst \in \{x_{1,a}x_{0,a}x_{0,b}, x_{1,a}x_{0,b}x_{0,c}\}$, we see that $a_{r} + a_{s} + a_{t} = n$ which is also divisible by $n$. It then follows from lemma 1 that $Q(K) \cong Z/|1 + (n - 1)| = Z/n$.

Let $K_{n} = K$ and we're done.

q.e.d.

4 Closures and Closed Complexes

Given a complex $K$, it is interesting to note that some 3-subsets of $V(K)$, which do not originally belong to $\text{max}(K)$, may be added to $K$ so as to increase the number of triangles with the structure of the incidence quotient $Q(K)$ being essentially preserved.
4.1 Closed Complexes

This motivates the following definitions:

Let $K$ be a complex. The induced closed complex of $K$, denoted $\overline{K}$, is defined as the complex whose set of vertexes and set of triangles are given as follow:

$$V(\overline{K}) = V(K) \text{ and }$$

$$\max(\overline{K}) = \{\delta \subseteq V(K) ||\delta|| = 3, D_3^2(\delta) \in D_3^2(C_3(K))\}.$$  

We also say that $K$ induces the complex $\overline{K}$ which is defined as above.

A complex $K$ is closed provided $\overline{K} = K$.

For completeness, we define the $\lambda$-closure $\lambda \overline{K}$ of a complex $K$, in which $\lambda$ is a positive integer, as the complex whose set of vertexes and set of triangles are given as follow:

$$V(\lambda \overline{K}) = V(K) \text{ and }$$

$$\max(\lambda \overline{K}) = \{\delta \subseteq V(K) ||\delta|| = 3, D_3^2(\lambda \delta) \in D_3^2(C_3(K))\}.$$  

It is then clear that the 1-closure $1\overline{K}$ of a complex $K$ equals its induced closed complex $\overline{K}$. For this reason, we also refer to $\overline{K}$ as the closure of $K$.

That we may regard any complex which is isomorphic to $\overline{K}$ as the closure of a complex $K$ is clear, since

**Proposition 19** If $K$ and $L$ are complexes such that $K \underset{I}{\simeq} L$, then $\overline{K} \underset{I}{\simeq} \overline{L}$.

*Proof.* Let $\mu$ be defined as in the proof of proposition 2. Then $\{\mu(\alpha_i)\}_{i \in I}$ is a basis of $D_3^2(C_3(L))$ whenever $\{\alpha_i\}_{i \in I}$ is a basis of $D_3^2(C_3(K))$ for an index set $I$.

Let $xyz \in \max(\overline{K})$, so that $x + y + z = \sum_{i \in I} n_i \alpha_i$ for a sequence $\{n_i\}_{i \in I}$ of integers. It then follows that

$$f(x) + f(y) + f(z) = \mu(x) + \mu(y) + \mu(z)$$

$$= \mu(x + y + z)$$

$$= \mu(\sum_{i \in I} n_i \alpha_i)$$

$$= \sum_{i \in I} n_i \mu(\alpha_i),$$

from which we deduce that $f(x) + f(y) + f(z) \in D_3^2(C_3(L))$, showing that if $xyz \in \max(\overline{K})$, then $f(x)f(y)f(z) \in \max(L)$.
The converse follows from the facts that \( f \) is invertible, and the inverse \( \mu^{-1} \) of \( \mu \) is also an isomorphism.

Therefore, \( K \overset{f}{\preceq} L \) implies that \( \overline{K} \overset{f}{\preceq} \overline{L} \).

q.e.d.

4.2 Combinatorial Properties of Closures

In this section, we state some combinatorial properties of closures which, in a sense, justifies the term “induced closed complexes”. Indeed, we see readily that the computation of the closure of a complex is a closure operation in the terminology of matroid theory.

Refer to [2, 21] for detailed information about closures in terms of matroid theory.

Proposition 20 Let \( K \) and \( L \) be complexes. Then

1. \( K \preceq \overline{K} \),
2. \( \overline{K} = K \), and
3. \( \overline{K} \preceq \overline{L} \) if \( K \preceq L \).

Proof of part 1. This is trivial since if \( xyz \in \max(K) \), then \( x + y + z = D_3^2(xyz) \), which is necessarily contained in \( D_3^2(C_3(K)) \).

Proof of part 2. Let \( xyz \in \max(\overline{K}) \). Since \( D_3^2(C_3(\overline{K})) = D_3^2(C_3(K)) \), a fact we shall establish in next section, that \( x + y + z \in D_3^2(C_3(\overline{K})) \) is equivalent to saying that \( x + y + z \in D_3^2(C_3(K)) \). Thus, \( \overline{K} \preceq \overline{K} \).

Hence, \( \overline{K} = K \) by part 1.

Proof of part 3. That \( K \preceq L \) implies that \( D_3^2(C_3(K)) \subseteq D_3^2(C_3(L)) \), showing that \( \overline{K} \preceq \overline{L} \).

q.e.d.

The following corollaries follow immediately. We omit the proofs.
Corollary 5 Let $K$ and $L$ be complexes such that $K \preceq L \preceq \overline{K}$. Then $\overline{K} = \overline{L}$.

Corollary 6 Let $K$ and $L$ be complexes. Then

1. $\overline{K} \sim \overline{L} \preceq \overline{K} \sim \overline{L}$ and
2. $\overline{K} \sim \overline{L} \preceq \overline{K} \sim \overline{L} = \overline{K} \sim \overline{L}$.

We may have suspected that a complex $L$ containing a complex $K$ always induces the same closed complex, that is, $\overline{L} = \overline{K}$. However, a careful analysis shows that this is false in general. In fact,

Proposition 21 If $L$ is a complex including a complex $K$ such that $L$ is not contained in $\overline{K}$, in symbols, $^2K \prec L \not\preceq \overline{K}$, then $\overline{K}$ is a proper subcomplex of $\overline{L}$, that is, $\overline{K} \prec \overline{L}$.

Proof. By part 3 of proposition 20, $\overline{K} \preceq \overline{L}$. To show that $\overline{K}$ is properly contained in $\overline{L}$, let $xyz \in \max(L) \setminus \max(\overline{K})$. It follows then that $x + y + z \in D^2_3(C_3(L)) \setminus D^2_3(C_3(K))$, implying that $xyz \notin \max(\overline{K})$.

q.e.d.

4.3 Incidence Quotients of Closed Complexes

An elementary connection between a given complex and its induced closed complex, with respect to their incidence quotients, is given by the following statement.

Proposition 22 For any complex $K$, $D^2_3(C_3(\overline{K})) = D^2_3(C_3(K))$ and $Q(\overline{K}) = Q(K)$.

\(^2\)Note that $L$ may properly contain $\overline{K}$.
Proof. Since $K \preceq \overline{K}$, $D^3_2(C_3(K)) \subseteq D^3_2(C_3(\overline{K}))$. Conversely, observe that $D^3_2(\text{max}(\overline{K}))$ generates $D^3_2(C_3(\overline{K}))$ and that if $xyz \in \text{max}(\overline{K})$, then $x + y + z \in D^3_2(C_3(K))$, showing $D^3_2(C_3(\overline{K})) \subseteq D^3_2(C_3(K))$.

That $Q(\overline{K}) = Q(K)$ follows immediately.

q.e.d.

Proposition 22 also helps explain the existence of complexes, being not isomorphic, of equal number of vertexes whose incidence quotients are isomorphic groups.

We formulate this observation as follows:

**Proposition 23** Let $K$ and $L$ be complexes having isomorphic induced closed complexes, namely $\overline{K} \simeq \overline{L}$. Then $D^3_2(C_3(K)) \cong D^3_2(C_3(L))$ and $Q(K) \cong Q(L)$.

Proof. By proposition 2, $D^3_2(C_3(\overline{K})) \cong D^3_2(C_3(\overline{L}))$ and $Q(\overline{K}) \cong Q(\overline{L})$. The rest of the statement follows from proposition 22.

q.e.d.

We proceed to show that the contents of max($\overline{K}$) of the induced closed complex $\overline{K}$ of a complex $K$ indeed restricts the structure of the incidence quotient $Q(K)$.

**Theorem 6** Let $K$ and $L$ be complexes on the same set of vertexes, that is $V(K) = V(L)$. If $L$ includes $K$, and is not contained in $\overline{K}$, in symbols, $K \prec L \nsubseteq \overline{K}$, then their incidence quotients are not isomorphic, namely $Q(L) \not\cong Q(K)$. In case $Q(K)$ is finite, the order of $Q(L)$ properly divides the order of $Q(K)$.

Proof. Let us first suppose that $\overline{K} \prec L$, that is, $\overline{K}$ is properly contained in $L$, so that $D^3_2(C_3(K)) \subset D^3_2(C_3(L))$. Observe that $C_1(K) = C_1(L)$, because $V(K) = V(L)$. It follows from the Third Isomorphism Theorem that

$$Q(L) = \frac{C_1(L)}{D^3_2(C_3(L))} \cong \frac{C_1(L)}{D^3_2(C_3(K))} = \frac{D^3_2(C_3(L))/D^3_2(C_3(K))}{D^3_2(C_3(L))/D^3_2(C_3(K))}$$

As in proposition 21, $\overline{K}$ may properly be contained in $L$. 

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\[ = \frac{Q(K)}{D^2_3(C_3(L))/D^2_3(C_3(K))}. \]

Now if \( Q(L) \equiv Q(K) \), then \( D^2_3(C_3(L))/D^2_3(C_3(K)) \) vanishes, which is equivalent to saying that \( D^2_3(C_3(L)) = D^2_3(C_3(K)) \). But this is impossible by the derivation of proposition 21.

Hence, it is necessary that \( Q(L) \not\equiv Q(K) \).

The conclusion when \( Q(K) \) is finite follows from Lagrange's theorem.

We drop the supposition that \( \overline{K} \prec L \). Since \( L \not\leq \overline{K} \), proposition 21 implies that \( \overline{K} \prec \overline{L} \), namely \( \overline{K} \) is a proper subcomplex of \( \overline{L} \). Hence, the preceding paragraphs can be applied to the case of \( \overline{K} \) and \( \overline{L} \).

The rest of the statement follows from proposition 22.

q.e.d.

It follows from theorem 6 that a complex \( K \) is closed if and only if \( Q(K) \not\equiv Q(L) \) in which \( L \) is a complex with \( \max(L) = \max(K) \sim \{\delta\} \) for any 3-subset \( \delta \not\in \max(K) \).

4.4 Preserved Topological Properties in Induced Closed Complexes and the Types of Triangles

Let \( K \) be a complex. It is worth observing that some geometrical properties are preserved in the induced closed complex \( \overline{K} \).

**Proposition 24** A complex \( K \) is connected if and only if its closure \( \overline{K} \) is connected. Indeed, the number of components of \( K \) is equal to that of its closure \( \overline{K} \).

**Proof.** One side of the proof is clear: since \( K \preceq \overline{K} \), any path in \( K \) is also contained in \( \overline{K} \).

To show the converse, it will suffice to suppose \( K = K_1 \sim K_2 \) in which \( K_1 \) and \( K_2 \) are the components of \( K \). By corollary 6, \( K_1 \sim \overline{K_2} \preceq \overline{K} \) since \( K_1 \sim K_2 \preceq K \), and that we may regard each \( K_i \) as a subcomplex of \( K \).

On the other hand, let \( xyz \in \max(\overline{K}) \). Then \( x + y + z \equiv 0(D^2_3(C_3(K))) \), which is equivalent to saying that
\[ x + y + z = D_3^2(\sum_{\delta \in \text{max}(K_1)} n_{\delta} \delta) + D_3^2(\sum_{\delta \in \text{max}(K_2)} n_{\delta} \delta) \]

for some sequences \( \{n_{\delta}\}_{\delta \in \text{max}(K_i)}, i = 1, 2, \) of integers.

Note that for any complex \( L \), if

\[ \sum_{x \in V(L)} n_x x \equiv 0(D_3^2(C_3(L))) \]

for a sequence \( \{n_x\}_{x \in V(L)} \) of integers, then the sum of these integers is a multiple of 3, namely

\[ \sum_{x \in V(L)} n_x \equiv 0(3). \]

This indicates that either \( x + y + z = D_3^2(\sum_{\delta \in \text{max}(K_1)} n_{\delta} \delta) \) or \( x + y + z = D_3^2(\sum_{\delta \in \text{max}(K_2)} n_{\delta} \delta) \), showing that \( xyz \) is contained entirely in exactly one of \( K_1 \) and \( K_2 \). Thus, \( K \leq K_1 \sim K_2 \).

Now we have proven \( K = K_1 \sim K_2 \). That \( K_1 \sim K_2 = \emptyset \) can easily be seen, since \( K_1 \sim K_2 = \emptyset \). It follows that \( K_1 \) and \( K_2 \) are the components of \( K \). In other words, if \( K \) consists of two components, so does \( K \).

The case in which \( K \) is a union of more than two components follows inductively.

q.e.d.

It will soon be established that the closure \( \overline{K} \) of a hyper connected complex \( K \) is also hyper connected. Indeed, the reader may have come to the belief that the number of hyper components of a complex \( K \) is also the same as that for its closure \( \overline{K} \), namely \( h(\overline{K}) = h(K) \). Nevertheless, this is generally not true, that is, it may happen that \( h(\overline{K}) \neq h(K) \). Consider the following example.

**Example 10** Let \( K \) be a complex with

\[ V(K) = \{x_1, x_2, \ldots, x_n\} \text{ and} \]

\[ \text{max}(K) = \{x_1 x_2 x_3, x_1 x_2 x_4, x_1 x_3 x_4, x_2 x_3 x_4, \]
\[ x_4 x_5 x_6, x_4 x_5 x_7, x_4 x_6 x_7, x_5 x_6 x_7, \ldots \]
\[ x_{n-3} x_{n-2} x_{n-1}, x_{n-3} x_{n-2} x_n, x_{n-3} x_{n-1} x_n, x_{n-2} x_{n-1} x_n \} \]

in which \( n = 3m + 1 \) for a positive integer \( m \). Then the closure \( \overline{K} \) of \( K \) is the complex with

\[ V(\overline{K}) = V(K) \text{ and} \]

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\[ \text{max}(\overline{K}) = \{ \delta \subseteq V(K) ||\delta|| = 3 \}. \]

It can easily be computed that \( h(K) = m \) and \( h(\overline{K}) = 1 \), implying that \( h(\overline{K}) \neq h(K) \) provided \( m > 1 \).

We also compute that \( Q(K) \cong \mathbb{Z}/3 \).

Consider the above example. It is worthwhile to note that \( \overline{K} \) is the largest possible complex on \( V(K) \), and that this is not a coincidence. Indeed, we have the following statement.

**Proposition 25** Let \( K \) be a complex whose incidence quotient is a cyclic group of order 3, namely \( Q(K) \cong \mathbb{Z}/3 \). The closure \( \overline{K} \) of \( K \) is the largest possible complex on \( V(K) \), that is,

\[ \text{max}(\overline{K}) = \{ \delta \subseteq V(K) ||\delta|| = 3 \}. \]

**Proof.** Note that the sum of the coefficients of any 1-chain that belongs to \( D_{3}^{2}(C_{3}(K)) \), being represented in the basis \( V(K) \), must be divisible by 3. It is then readily seen that

\[ Q(K) \cong \mathbb{Z}/3 \text{ if and only if } x \equiv y \text{ for any } x, y \in V(K). \]

Indeed, \( 3x \equiv 0 \) for all \( x \in V(K) \).

We remark that the above statement holds for any complex \( L \) such that \( Q(L) \cong \mathbb{Z}/3 \).

Thus, for any triple \( x, y, z \in V(K) \), being distinct, \( x + y + z \equiv 3x \equiv 0 \), which was what to be shown.

q.e.d.

Note that this implies that for any \( n \geq 4 \), there is a unique closed complex \( K \), up to isomorphism, with \( n \) vertexes and with \( Q(K) \cong \mathbb{Z}/3 \).

This is not true in general. For instance,

for any \( n \geq 6 \), there are at least \( n - 5 \) nonisomorphic closed complexes \( K \) on \( n \) vertexes with \( Q(K) \cong \mathbb{Z}/6 \).

\((\ast 3)\)

Let \( V = \{ x_{1}, x_{2}, \ldots, x_{n} \} \) in which \( n \geq 6 \). Define a family \( W = \{ w : V \rightarrow \{0, 1\} \} \) of functions such that \( w(x_{1}) = w(x_{2}) = w(x_{3}) = 0 \) and \( w(x_{n-2}) = \)
\( w(x_{n-1}) = w(x_n) = 1 \). We proceed to show that to each \( w \in W \), there corresponds a closed complex \( K \) on \( V \) whose incidence quotient is given by \( Q(K) \cong Z/6 \).

Let \( w \in W \) be given. We define \( K \) in such a way that a 3-subset \( xyz \) of \( V \) is in \( \text{max}(K) \) if and only if \( w(x) + w(y) + w(z) \equiv 0(2) \). It is then clear that \( xyz \in \text{max}(K) \) if and only if \( w(x) = w(y) = w(z) = 0 \), or precisely one of \( w(x), w(y), \) and \( w(z) \) is 0; the others being 1. Hence, whenever \( w(x) = w(y) \) then \( x \equiv y \) (indeed, \( x \sim y \)) for all \( x, y \in V \). It follows that

\[
\text{for all } x \neq x_n, \text{ if } w(x) = 0 \text{ then } x \equiv -2x_n; \text{ otherwise } x \equiv x_n.
\]

We shall often employ this observation in the sequel without explicit reference.

The first implication of this observation is that \( 6x_n \equiv 0 \). In fact, we may check that \( K \) satisfies the hypothesis of lemma 1, and thus \( Q(K) \cong Z/6 \).

Let \( xyz \in \text{max}(\overline{K}) \setminus \text{max}(K) \). Then either \( w(x) = w(y) = w(z) = 1 \), or exactly one of \( w(x), w(y), \) and \( w(z) \) is 1; the others being 0. In the former case, \( x_n \equiv x \equiv y \equiv z \) since \( w(x) = w(y) = y(z) = 1 \). Thus, \( 3x_n \equiv x + y + z \equiv 0 \), from which we deduce that \( Z/3 \cong Q(K) \cong Z/6 \) - an obvious contradiction. The latter case can similarly be cleared up. Therefore, we conclude that \( \overline{K} = K \), namely \( K \) is closed.

Let \( W \) be given. Suppose \( K, L \) are complexes corresponding to \( w_K, w_L \in W \) respectively.

Let \( K \not\leq L \). Note that \( f(x)f(y)f(z) \in \text{max}(L) \) if and only if \( xyz \in \text{max}(K) \), which is equivalent to either \( w_K(x) = w_K(y) = w_K(z) = 0 \), or precisely one of \( w_K(x), w_K(y), \) and \( w_K(z) \) is 0; the others being 1. But \( f(x)f(y)f(z) \in \text{max}(L) \) if and only if either \( w_L(f(x)) = w_L(f(y)) = w_L(f(z)) = 0 \), or exactly one of \( w_L(f(x)), w_L(f(y)), \) and \( w_L(f(z)) \) is 0; the others being 1, from which it follows that \( \sum_{x \in V(K)} w_K(x) = \sum_{x \in V(L)} w_L(x) \).

Conversely, suppose \( \sum_{x \in V(K)} w_K(x) = \sum_{x \in V(L)} w_L(x) \). Note that there always exists a permutation \( f : V(K) \to V(L) \) such that \( w_K(x) = 1 \) if and only if \( w_L(f(x)) = 1 \) for each \( x \in V(K) \). It follows from the construction of \( K \) and \( L \) that \( xyz \in \text{max}(K) \) implies \( f(x)f(y)f(z) \in \text{max}(L) \), and vice versa, that is \( K \not\leq L \).
Hence, we may construct from \( W \) exactly \( n - 5 \) nonisomorphic closed complexes \( K \) with \( Q(K) \cong \mathbb{Z}/6 \), proving the assertion \((*3)\).

We now go back to hyper connected complexes:

**Proposition 26** If \( K \) is a hyper connected complex, so is its closure \( \overline{K} \).

**Proof.** By proposition 3, we only need to consider three cases.

Case 1. \( i(K) = 1 \).

The conclusion is direct from theorem 1, proposition 22 and proposition 25.

Case 2. \( i(K) = 2 \).

Suppose \( V(K) = [x] \sim [y] \) is a partition of \( V(K) \) into the equivalence classes of vertexes. It is then clear from the proof of part 2 of proposition 10 and that of theorem 1 that for a subset \( \delta \) of \( V(K) \) containing three points,

\[ \delta \in \text{max}(\overline{K}) \text{ if and only if exactly two points of } \delta \text{ are contained in one of the equivalence classes of vertexes.} \]

It follows that if \( x_1x_1'y_1, x_2x_2'y_2 \in \text{max}(\overline{K}) \) such that \( x_j, x_j' \in [x] \) and \( y_j \in [y] \) for \( j = 1, 2 \), then \( x_1x_1'y_2, x_1x_2'y_2 \in \text{max}(\overline{K}) \), implying that \( \overline{K} \) is hyper connected.

The case \( i(K) = 3 \) can be dealt with analogously.

q.e.d.

We need some definitions:

Let \( K \) be a complex. Recall that given \( \alpha, \beta \in C_1(K) \), \( \alpha \) and \( \beta \) are said to be congruent if \( \alpha - \beta \in D_2(C_3(K)) \). This applies in particular to the set \( V(K) \) of vertexes.

Let \( \delta, \tau \in \text{max}(K) \). Then \( \delta \) and \( \tau \) are said to be of the same type provided there exists a bijection \( \phi: \delta \rightarrow \tau \) such that \( x \equiv \phi(x) \) for all \( x \in \delta \).

We shall denote the number of the types of triangles in a complex \( K \) as \( t(K) \).

We proceed to give a more general statement.

**Theorem 7** Let \( K \) be a closed complex. Then any \( \delta, \tau \in \text{max}(K) \) are of the same type if and only if they belong to the same hyper component of \( K \). In particular, \( t(K) = h(K) \).

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Proof. Let \( x_1x_2x_3, y_1y_2y_3 \in \max(K) \) be of the same type such that \( x_i \equiv y_i \) for each \( i \). It is then clear that the sequence \( x_1x_2x_3, x_1x_2y_3, x_1y_2y_3, y_1y_2y_3 \) is in \( \max(K) \), showing \( x_1x_2x_3 \) and \( y_1y_2y_3 \) belong to the same hyper component of \( K \).

Let \( \delta, \delta' \in \max(K) \) be in the same hyper component of \( K \), say \( H \). Since \( H \) is hyper connected, there is a sequence \( \delta = \delta_1, \delta_2, \cdots, \delta_n = \delta' \) in \( \max(H) \) such that \( |\delta_i \sim \delta_{i+1}| = 2 \) for \( i = 1, 2, \cdots, n - 1 \). Since \( |\delta_i \sim \delta_{i+1}| = 2 \), it is readily derived that \( \delta_i \) and \( \delta_{i+1} \) are of the same type for each \( i \). It follows at once that \( \delta = \delta_1 \) and \( \delta' = \delta_n \) are of the same type.

Since any hyper component of \( K \) consists of exactly one type of triangles, and since any triangle of this type must belong to the hyper component, we get that \( h(K) = t(K) \).

q.e.d.

It is easily seen in general that \( t(K) \leq h(K) \) for any complex \( K \).

References


