RESIDUALLY FINITE PROPERTIES OF GROUPS

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KUALA LUMPUR

2018

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DISSERTATION SUBMITTED IN FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

INSTITUTE OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
UNIVERSITY OF MALAYA
KUALA LUMPUR

2018

UNIVERSITI MALAYA

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RESIDUALLY FINITE PROPERTIES OF GROUPS

ABSTRACT

In this thesis, we shall study two stronger forms of residual finiteness, namely cyclic subgroup separability and weak potency in various generalized free products and HNN extensions. Among our results, we shall show that the generalized free products and HNN extensions where the amalgamated or associated subgroups are finite, or central, or infinite cyclic, or they are direct products of an infinite cyclic subgroup with a finite subgroup, or they are finite extensions of central subgroups, are again cyclic subgroup separable or weakly potent respectively. In order to prove our results, we shall prove a criterion each for the weak potency of generalized free products and HNN extensions, but we shall use previously established criterions for cyclic subgroup separability. Finally, we shall extend our results to tree products and fundamental groups of graphs of groups.

Keywords: Residually Finite, Weak Potency, Generalized Free Products, HNN Extensions, Fundamental Groups of Graphs of Groups.

SIFAT-SIFAT SISA TERHINGGA BAGI KUMPULAN

ABSTRAK

Di dalam tesis ini, kami mengkaji sifat-sifat yang lebih kuat dari sisa terhingga, yang dikenali sebagai kebolehpisahan subkumpulan kitaran dan poten lemah dalam pelbagai hasil darab teritlak dan perluasan HNN. Antara hasil kami, kami akan menunjukkan bahawa hasil darab teritlak dan perluasan HNN yang mana subkumpulan-subkumpulan gabungan atau berkait adalah terhingga, atau pusat, atau kitaran takterhingga, atau hasil darab langsung antara subkumpulan kitaran takterhingga dan subkumpulan terhingga, atau pemanjangan terhingga bagi subkumpulan pusat, adalah masing-masing sekali lagi subkumpulan kitaran terpisahkan atau poten lemah. Untuk membuktikan hasil kami, kami akan membuktikan kriteria bagi poten lemah untuk hasil darab bebas teritlak dan perluasan HNN, dan kami akan menggunakan kriteria yang telah dibina sebelum ini bagi subkumpulan kitaran terpisahkan. Akhir sekali, kami akan memanjangkan hasil kami ke hasil darab pokok dan kumpulan asasi bagi graf kumpulan.

Kata Kunci: Sisa Terhingga, Poten Lemah, Hasil Darab Bebas Teritlak, Perluasan HNN, Kumpulan Asasi Bagi Graf Kumpulan..

ACKNOWLEDGEMENTS

I would like to convey my deepest gratitude first and foremost to my supervisors, Prof. Dr. Wong Peng Choon and Dr. Wong Kok Bin for their guidance, support and constant patience which helped me in completing this study. Furthermore, the knowledge, understanding and dedication that both of them has shown me has truly inspired me to work hard on completing my Master's degree.

Secondly, I would like to thank the University of Malaya for the scholarship, which has been partially supported me for my Master's degree. I would also like to thank all the staff of the Institute of Mathematical Sciences for the support they have given me throughout my study.

Finally, I would like to thank my family for their unconditional love and support. And also to all my friends, especially those in the postgraduate room, thank you for all your helps and supports throughout this journey.

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CHAPTER 1: INTRODUCTION

1.1 General Introduction

The aim of this thesis is to study two stronger residually finite properties, namely cyclic subgroup separability and weak potency in generalized free products, tree products, HNN extensions and fundamental groups of graphs of groups. These two properties, though interesting by themselves, had played important roles in the determination of the residual finiteness and conjugacy separability of certain generalized free products and HNN extensions where the amalgamated or associated subgroups are cyclic (Baumslag & Solitar 1962; Kim 1993a, 1993b, 2004; Kim & Tang 1995; Wong et al. 2010; Zhou & Kim 2013)

Briefly, we say that a group G is cyclic subgroup separable if for any element g not in a cyclic subgroup H, there exists a finite image $\bar{G}$ of G such that $\bar{g}$ is not in $\bar{H}$, where $\bar{g}$ and $\bar{H}$ are the images of g and H in $\bar{G}$ respectively. We call a group G weakly potent if for any element g of infinite order and for any positive integer n, we can find an integer r and a finite image $\bar{G}$ of G such that $\bar{g}$ has order exactly rn in $\bar{G}$.

Residually finite groups was first introduced by Philip Hall in 1955. Then at Philip Hall's suggestion, Baumslag began the first systematic study of these groups. In one of his early papers, Baumslag (1963) studied the residual finiteness of generalized free products of two finitely generated nilpotent groups amalgamating various subgroups (Baumslag, 1963). One of the main tools in his proofs was the concept of *compatible filters*. When the amalgamated subgroups are cyclic, these *compatible filters* contain ideas which will lead to the concept of cyclic subgroup separability and potency.

The weak potency concept was introduced by Evans (1974) with the name of *regular quotient* to show that there exists a class of residually finite groups which is closed under the operation of forming generalized free products with single cyclic subgroup amalgamated (Evans, 1974). Potency was defined by Allenby and Tang (1981) in their investigation of the residual finiteness of some one-relator groups with torsion (Allenby & Tang, 1981). Realising that only a weaker form of potency is needed in the proofs, Tang (1995) independently introduced the concept of weak potency and used it in the proof of conjugacy separability of generalized free products with cyclic amalgamation of free-by-finite or nilpotent-by-finite groups with unique root property for elements of infinite order (Tang, 1995).

Now we give a brief outline of our chapters.

In this chapter (Chapter 1), a general introduction on generalized free products, tree products, HNN extensions and fundamental groups of graphs of groups will be given. All the definitions of the various group properties which are studied in this thesis are also included.

Next, in Chapter 2, we study the cyclic subgroup separability of generalized free products of cyclic subgroup separable groups and subgroup separable groups. First, we state a criterion by which we will use to prove the cyclic subgroup separability of generalized free products amalgamating various types of subgroups. Some of our results shall also be extended to tree products of finitely many groups.

In Chapter 3, we study the cyclic subgroup separability of certain HNN extensions. Again we state a criterion for the cyclic subgroup separability of HNN extensions, then we apply it to certain HNN extensions with various associated subgroups.

In Chapter 4, we investigate the weak potency of generalized free products of weakly potent groups. Here, we first prove a criterion for the weak potency of certain generalized free products. Then we apply it to certain generalized free products amalgamating various subgroups. Finally, as in Chapter 2, some of our results shall be extended to tree products of finitely many groups.

We continue to study about weak potency in Chapter 5. We study the weak potency of certain HNN extensions of weakly potent groups. Again we first prove a criterion for the weak potency of HNN extensions of weakly potent group. Then we apply it to certain HNN extensions of weakly potent groups with various associated subgroups.

Finally, in Chapter 6, we will extend some of our results in the preceding chapters by proving that certain fundamental groups of graphs of groups are cyclic subgroup separable.

1.2 Generalized Free Products

Now we describe the generalized free products of groups. The concept of generalized free product (or sometimes called as free product with amalgamations) was introduced by Schreier in 1926 (Lyndon & Schupp, 2001).

Let A and B be groups given by presentations, say $A = \langle S \mid D \rangle$ and $B = \langle T \mid E \rangle$. Here we assume that $S \cap T = \phi$. Also suppose that $M = \langle P \mid Q \rangle$. Let $\sigma : M \to A$ and $\theta : M \to B$ be monomorphisms. Conveniently, we let $H = \sigma(M) \subseteq A$ and $K = \theta(M) \subseteq B$. Then these subgroups are isomorphic via $\varphi = \theta \circ \sigma^{-1} : H \to K$. Thus we have the following presentation, which is commonly used:

$$G = \langle S \cup T \mid D \cup E, h = \varphi(h), \forall h \in P \rangle = A *_{H \cong K} B = A *_H B.$$

Let $g \in G$. The element g is called reduced if $g = a_1 b_1 a_2 b_2 \ldots a_n b_n$ where each $a_i \in A \backslash H$ and each $b_i \in B \backslash H$ for all $i = 1, \ldots, n$. We denote the length of the reduced element g by $\|g\|$ and defined it as:

$$\|g\| = \begin{cases} 0, & \text{if } g \in H. \\ 1, & \text{if } g \in A \cup B. \\ n, & \text{otherwise.} \end{cases}$$

If each cyclic permutation of g is reduced, then g is said to be cyclically reduced. Note also that if g is not cyclically reduced, then it is conjugate to a cyclically reduced element of G or to an element of A or of B (Lyndon & Schupp, 2001, Chapter 4.2, p. 178).

1.3 Tree Products

Now suppose that we extend the generalized free product of two groups to finitely many groups. This type of generalized free products are called tree products. Tree products were first introduced by Karras and Solitar (1970).

A description of tree products was given by Kim and Tang (1998) as follows:

> " Let T be a tree. To each vertex v of T, assign a group G_v. To each edge e of T, assign a group G_e together with monomorphisms α_e, β_e embedding G_e into the two vertex groups at the end of the edge e. Then the tree product G is defined to be the group generated by the generators and relations of the vertex groups together with the additional relations $\alpha_e(g_e) = \beta_e(g_e)$ for each $g_e \in G_e$." (Kim & Tang, 1998, p. 323)

For simplicity, we can say that G is a tree products of the (vertex) groups $G_1, \cdots, G_n$, for $n \geq 2$, amalgamating the (edge) subgroups $H_{ij} \leq G_i$ and $H_{ji} \leq G_j$ and denoted by

$$G = \langle G_1, \cdots, G_n \mid H_{ij} = H_{ji} \rangle.$$

1.4 HNN Extensions

The next group structure is called the Higman-Neumann-Neumann extension or HNN extension for short and was introduced by Graham Higman, Bernhard Neumann, and Hanna Neumann in 1949. The construction of HNN extensions with the generalized free product are quite similar and parallel. But instead of constructing from two groups, HNN extensions are constructed from one group such that, the group contains isomorphic associated subgroups (Higman, Neumann, & Neuman, 1949). In very loose language, generalized free product might be called the "disconnected case", while HNN extension is called the "connected case".

Let A be a group with presentation $A = \langle S \mid D \rangle$ and a pair of isomorphic subgroups H and K. HNN extensions can be regarded as a larger group containing A in which the subgroups H and K are isomorphic via conjugation. Thus, the HNN extension G of a group A with a stable letter t, and with associated subgroups H, K which are isomorphic via $\varphi : H \to K$ that is defined such that $t^{-1}ht = \varphi(h)$ for all $h \in H$, has presentation

$$G = \langle S, t \mid D, t^{-1}ht = \varphi(h), \forall h \in H \rangle.$$

Conveniently, we shall use the following presentation, which is also commonly used.

$$G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$$

where A is called the base group and $\varphi : H \to K$ is the isomorphism.

Let $g \in G$. The element g is said to be reduced if $g = a_0 t^{\varepsilon_1} a_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} a_n$ with $\varepsilon_i = \pm 1$ such that there is no consecutive terms $t^{-1}a_it$ if $a_i \in H$, or ta_it^{-1} if $a_i \in K$. We denote the reduced length of g as $\|g\|$ and defined it as follows:

$$\|g\| = \begin{cases} 0, & \text{if } g = a_0 \in A. \\ n, & \text{otherwise.} \end{cases}$$

Similar to as the generalized free products of group, if each cyclic permutations of g is also reduced, then g is said to be cyclically reduced . If g is not cyclically reduced, then it is conjugate to a cyclically reduced element of G or to an element of A (Lyndon & Schupp, 2001, Chapter 4.2, p. 178).

1.5 Fundamental Group of Graphs of Groups

Kim (2004) had described the fundamental group of a graph Γ of groups as follows:

> " Let $\Gamma = (V, E)$ be a graph where V is a set of vertices and E is a set of edges. To each vertex v in V, we assign a group G_v. To each edge e in E, we assign a group G_e together with monomorphisms α_e and β_e embedding G_e into the two vertex groups at the end of the edge e. Then for a maximal tree T of Γ, the fundamental group of the graph Γ of groups G_v amalgamating the edge subgroups G_e is defined to be the group generated by the generators and relations of the vertex groups and additional generators t_e for each $e \in E$ together with additional relations $t_e^{-1}(g_e\alpha_e)t_e = g_e\beta_e$ for each $g_e \in G_e$ where $t_e = 1$ if e is an edge of T. Each of the subgroups $G_e\alpha_e$ and $G_e\beta_e$ is called edge subgroup in its containing vertex group. It is well-known that the fundamental group of a graph of groups is independent from the choice of the maximal tree (Serre, 1980). In particular, if the graph Γ is a tree, then the fundamental group of Γ of groups G_v is called a tree product of the G_v." (Kim, 2004, p. 914)

Let Γ be a finite graph. The fundamental group G of the graph Γ of groups G_v can be obtained by first successively performing a free product with amalgamation for each edge in the maximal tree T. That is by taking tree product A of the G_v groups according to T and then taking HNN extensions $G = \langle A, t_i, \ldots, t_n \mid t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, n \rangle$, where H_i and K_i are in the vertex groups G_v. Thus, fundamental groups of graphs of groups are generalizations of amalgamated free products and HNN extensions of groups (Kim, 2004).

1.6 Cyclic Subgroup Separability

We shall give a brief history of cyclic subgroup separability of groups.

In 1968, Stebe introduced the concept of cyclic subgroup separability and used it to prove that the class of knot groups are residually finite (Stebe, 1968). In that same year, Dyer (1968) showed the residual finiteness of generalized free products of two polycyclic-by-finite groups amalgamating a cyclic subgroup. In that paper, she extended the result of Baumslag (1967) to cyclic subgroup separability (Dyer, 1968). In 1992, Kim has shown the cyclic subgroup separability, and hence the residual finiteness, of polygonal products of polycyclic-by-finite groups amalgamating central subgroups (Kim, 1992). Then, Kim (1993b), and Kim and Tang (1999) gave characterizations for the cyclic subgroup separability of HNN extensions of cyclic subgroup separable groups with cyclic associated subgroups.

In a paper published in 2004, Kim showed the cyclic subgroup separability and residual finiteness of fundamental groups of graphs of groups amalgamating infinite cyclic edge subgroups (Kim, 2004). Next, Wong and Wong (2007) proved the cyclic subgroup separability of polygonal products of certain subgroup separable groups amalgamating finitely generated normal subgroups (Wong & Wong, 2007). Recently, they showed the cyclic subgroup separability of HNN extensions of a non-cyclic and subgroup separable base group associating normal infinite cyclic subgroups (Wong & Wong, 2012).

1.7 Weak Potency

As we have stated above, Evans (1974) had established the concept of weak potency with the name *regular quotient* and showed the weak potency of free groups and finitely generated torsion-free nilpotent groups. Evans also used weak potency to show the cyclic subgroup separability of certain generalized free products. In 1981, Allenby and Tang introduced the concept of *potency*, to derive the residual finiteness of the generalized free product amalgamating a cyclic subgroup (Allenby & Tang, 1981). Later, weak potency was properly and independently defined by Tang (Tang, 1995).

Weak potency is a strong form of residual finiteness in the sense that a weakly potent torsion-free group is residually finite. Tang (1995), and Kim and Tang (1995) used weak potency to determine the conjugacy separability of certain generalized free products of conjugacy separable groups (Kim & Tang, 1995; Tang, 1995). Since then, weak potency has been used in establishing the residual finiteness and conjugacy separability in generalized free products, tree products, polygonal products, one-relator groups and fundamental groups of graphs of groups (see (Allenby, 1981; Allenby & Tang, 1981; Kim & Tang, 1995; Tang, 1995; Wong & Tang, 1998; Wong & Wong, 2014)).

1.8 Definitions and Notations

Standard notations will be used in this thesis. In addition, we shall use the following, for any group G:

- $N \lhd_f G$ means N is a normal subgroup of finite index in G.
- $Z(G)$ denotes the centre of G.
- If G is a generalized free product or HNN extension and $g \in G$, then $\|g\|$ denotes the usual reduced length of g.
- For $h, k \in G$, $h \sim_G k$ means h is conjugate to k in G.
- The term π_c will denote cyclic subgroup separable.

Definition 1.1. Let G be a group and H be a subgroup of G. G is called *H–separable* if, for each $g \in G \backslash H$, there exists $N \lhd_f G$ such that $g \notin HN$ (or $gH \cap N = \phi$). If $H = \{1\}$, then G is called *residually finite*. If G is H–separable for every finitely generated subgroup H, then G is called *subgroup separable*. If G is H–separable for every finitely generated subgroup H of $Z(G)$, then G is called *central subgroup separable*. If G is $\langle x \rangle$–separable for every cyclic subgroup $\langle x \rangle$ of G, then G is called *cyclic subgroup separable*.

Definition 1.2. (Tang, 1995) A group G is called *weakly $\langle x \rangle$–potent*, briefly, $\langle x \rangle$–wpot, if for an element x of infinite order in G, we can find a positive integer r with the property that, for each positive integer n, there exists $M_n \lhd_f G$ such that xM_n has order exactly rn in the finite group G/M_n. A group G is called *weakly potent* if G is $\langle x \rangle$–wpot for all elements of infinite order $x \in G$. G is called *potent* if for any element of infinite order $x \in G$ and every positive integer n, there exists $M_n \lhd_f G$ such that xM_n has order exactly n in G/M_n.

Remark 1.3. We note here that the subgroup M_n in Definition 1.2 depends on n. For simplicity, when there is no confusion, we shall write M instead of M_n.

From the above definitions, finitely generated torsion-free weakly potent groups are residually finite. Furthermore, every subgroup separable group is also cyclic subgroup separable and residually finite. Free groups, polycyclic groups, finitely generated nilpotent groups and their finite extensions are known to be weakly potent for elements of infinite order and subgroup separable (hence cyclic subgroup separable and residually finite) (see Evans, 1974; Tang, 1995; Wong & Wong, 2014). On the other hand, there are infinite groups with elements of finite order that are weakly potent but not residually finite. For example, let $G = Z(p^\infty) \times \langle h \rangle$ where $\langle h \rangle$ is an infinite cyclic group and $Z(p^\infty)$ is the Prüfer group. Then G is weakly potent for elements of infinite order but G is not residually finite.

CHAPTER 2: CYCLIC SUBGROUP SEPARABILITY OF GENERALIZED FREE PRODUCTS

2.1 Introduction

In this chapter, we shall investigate the cyclic subgroup separability of certain generalized free products of cyclic subgroup separable groups and subgroup separable groups.

First, we state a criterion (Theorem 2.1) for generalized free products of cyclic subgroup separable groups to be again cyclic subgroup separable. We then apply the criterion to generalized free products $G = A *_H B$ where (i) $H \leq Z(A) \cap Z(B)$ is finitely generated (Theorem 2.4), (ii) $H = \langle h \rangle \times D$ where $\langle h \rangle$ is infinite cyclic and D is finite (Theorem 2.9) and finally, (iii) H is a finite extension of a central subgroup (Theorem 2.14). Furthermore, we shall extend Theorem 2.9 to tree products of finitely many groups in Theorem 2.18.

2.2 Preliminaries

Kim (1993a) has proved a criterion (Theorem 2.1 below) for the cyclic subgroup separability of generalized free products. In this thesis, we shall use this criterion to prove our results.

Theorem 2.1. *Let $G = A *_H B$ be a generalized free product. Suppose that,*

(a) A and B are both π_c and H–separable; and

(b) for each $R \lhd_f H$, there exist $M_A \lhd_f A$ and $M_B \lhd_f B$ such that $M_A \cap H = M_B \cap H \subseteq R$,

Then G is π_c.

The following lemma is known by many researchers in this area.

Lemma 2.2. *Let $G = A *_H B$ be a generalized free product where A and B are finite groups. Then G is free-by-finite, and hence is weakly potent and subgroup separable* (Karras & Solitar, 1970; Evans, 1974; Tang, 1995).

2.3 Generalized Free Products Amalgamating Various Subgroups

For this section, we prove that certain generalized free products amalgamating various subgroups are π_c. Note that Kim has proved the cyclic subgroup separability of generalized free products amalgamating finite subgroup and amalgamating infinite cyclic subgroup (see (Kim, 1993a)).

The following result on generalized free products amalgamating central subgroup (Theorem 2.4 below) has been proved by Wong and Tang (1998).

Lemma 2.3. *(Wong & Tang 1998) Let A be subgroup separable and let $H \leq Z(A)$ be finitely generated. Then, for each $R \lhd_f H$, there exists $N \lhd_f A$ such that $N \cap H = R$.*

Theorem 2.4. *Let $G = A *_H B$ where A and B are subgroup separable. Suppose that $H \leq Z(A) \cap Z(B)$ is finitely generated. Then G is π_c.*

Proof. Since subgroup separable groups are π_c and H–separable, we just need to show (b) in Theorem 2.1. By Lemma 2.3, for each $R \lhd_f H$, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = R = N \cap H$. Therefore G is π_c by Theorem 2.1. ■

Corollary 2.5. *Suppose that A and B are finitely generated abelian groups. Then $G = A *_H B$ is π_c.*

Kim and Tang (2013) have shown the conjugacy separability of certain generalized free products amalgamating a subgroup of the form $\langle h \rangle \times D$ where D is a central subgroup of the factor groups. Recently, Zhou and Kim (2013), showed the subgroup separability of certain generalized free products amalgamating this type of subgroup.

In this thesis, we shall use the criterion (Theorem 2.1 above) to show that certain generalized free products amalgamating this type of subgroup, where D is finite, are π_c.

Lemma 2.6. *(Kim & Tang, 2013) Let A be a group with subgroup $H = \langle h \rangle \times D$ such that $|h| = \infty$ and D is finite. If A is $\langle h \rangle$–separable, then A is H–separable.*

Proof. Let $x \in A \backslash H$. Then $x \neq h^m d$ for all $d \in D$ and for all $m \in \mathbb{Z}$. Thus we have $xd^{-1} \notin \langle h \rangle$ for all $d \in D$. Since A is $\langle h \rangle$–separable, there exists $N_d \lhd_f A$ such that $xd^{-1} \notin \langle h \rangle N_d$ for each $d \in D$. Let

$$N = \bigcap_{d \in D} N_d.$$

Then $N \lhd_f A$ and, for all $d \in D$, we have $xd^{-1} \notin \langle h \rangle N$. Suppose that $x \in HN$. Let $x = h^m dn$, where $m \in \mathbb{Z}$, $d \in D$ and $n \in N$. Then $x = h^m dnd^{-1}d = h^m n_0 d$ where $n_0 = dnd^{-1} \in N$ for $N \lhd_f A$. Hence $xd^{-1} = h^m n_0 \in \langle h \rangle N$, a contradiction. Therefore $x \notin HN$ and A is H–separable. ■

Remark 2.7. Let A and B be groups such that $\langle h \rangle$ is an infinite cyclic subgroup of A and of B. If A and B are both $\langle h \rangle$–wpot, then there exist positive integers r_1, r_2 with the property that for each positive integer n, we have $P_n \lhd_f A$ and $Q_n \lhd_f B$ such that $P_n \cap \langle h \rangle = \langle h^{r_1 n} \rangle$ and $Q_n \cap \langle h \rangle = \langle h^{r_2 n} \rangle$.

Lemma 2.8. *Let $G = A *_H B$ where $H = \langle h \rangle \times D$ such that $|h| = \infty$ and D is finite. Suppose that A and B are $\langle h \rangle$–separable and $\langle h \rangle$–wpot. Then for each $R \lhd_f H$, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq R$.*

Proof. Suppose we are given an $R \lhd_f H = \langle h \rangle \times D$. Since A and B are $\langle h \rangle$–separable, $\langle h \rangle \cap D = 1$ and D is finite, there exist $M_1 \lhd_f A$ and $N_1 \lhd_f B$ such that

$$M_1 \langle h \rangle \cap D = 1 = N_1 \langle h \rangle \cap D.$$

Let $R \cap M_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $R \cap N_1 \cap \langle h \rangle = \langle h^{s_2} \rangle$ for some integers $s_1, s_2 > 0$. By Remark 2.7, let $M_2 \lhd_f A$ and $N_2 \lhd_f B$ be such that $M_2 \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2} \rangle$ and $N_2 \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2} \rangle$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Then $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2} \rangle = N \cap \langle h \rangle$. Furthermore, $M\langle h \rangle \cap D = 1 = N\langle h \rangle \cap D$. Hence we have,

$$M \cap H = M \cap \big(\langle h \rangle \times D\big) = M \cap \langle h \rangle = N \cap \langle h \rangle = N \cap \big(\langle h \rangle \times D\big) = N \cap H.$$

Thus, we have $M \cap H = N \cap H \subseteq R$. ■

Theorem 2.9. *Let $G = A *_H B$, where $H = \langle h \rangle \times D$ such that $|h| = \infty$ and D is finite. Suppose that A and B are π_c and $\langle h \rangle$–wpot. Then G is π_c.*

Proof. Since A and B are π_c, hence they are $\langle h \rangle$–separable. Thus, A and B are H–separable by Lemma 2.6. By Lemma 2.8, for each $R \lhd_f H$, we have $N_A \lhd_f A$ and $N_B \lhd_f B$ where $N_A \cap H = N_B \cap H \subseteq R$. Therefore, G is π_c by Theorem 2.1. ■

Corollary 2.10. *Let $G = A *_H B$, where $H = \langle h \rangle \times D$ such that $|h| = \infty$ and D is finite. Suppose A and B are finite extensions of a finitely generated nilpotent group. Then G is π_c.*

Zhou et al. (2010) have shown the conjugacy separability of generalized free products of polycyclic-by-finite groups amalgamating finite extensions of central subgroups. Here, we shall show that certain generalized free products amalgamating finite extensions of central subgroup is π_c.

Lemma 2.11. *(Lim, 2012) Let A be a group and $N \lhd A$. If A is N–separable, then A/N is residually finite.*

Lemma 2.12. *Let A be a group and H be a subgroup of A. Suppose that there exists $C \leq H$ such that $C \lhd A$ is finitely generated with $|H : C| < \infty$. If A is C–separable, then A is H–separable.*

Proof. Let $x \in A\backslash H$. Now we form $\bar{A} = A/C$. Thus $\bar{x} \notin \bar{H} = H/C$. Since A is C–separable, then $\bar{A}$ is residually finite by Lemma 2.11. Since $\bar{H}$ is finite, there exists $\bar{L} \lhd_f \bar{A}$ such that $\bar{L} \cap \bar{x}\bar{H} = \phi$. Let L be the preimage of $\bar{L}$ in A, then $L \lhd_f A$ and $x \notin HL$. ■

Lemma 2.13. *Let A be a group and H be a subgroup of A. Suppose that there exists $R \leq H$ such that $R \lhd A$ is finitely generated with $|H : R| < \infty$. If A is R–separable, then there exists $N \lhd_f A$ such that $N \cap H = R$.*

Proof. Suppose we are given such R. Since A is R–separable, then by Lemma 2.11, $\bar{A} = A/R$ is residually finite. Since $\bar{H} = H/R$ is finite, there is an $\bar{N} \lhd_f \bar{A}$ such that $\bar{N} \cap \bar{H} = \bar{1}$. Let N be the preimage of $\bar{N}$ in A. Then $N \lhd_f A$ and $N \cap H = R$. ■

Theorem 2.14. *Let $G = A *_H B$ where A and B are π_c and central subgroup separable. Suppose that there exists $C \leq H$ such that $C \subseteq Z(A) \cap Z(B)$ is finitely generated with $|H : C| < \infty$. Then G is π_c.*

Proof. We shall use Theorem 2.1. Note that A and B are C–separable for A, B are central subgroup separable. By Lemma 2.12, A and B are H–separable. Now let $R \lhd_f H$ be given. Let $R_C = R \cap C$. Then $R_C \lhd_f C$ and since C is finitely generated, we have R_C is finitely generated. Also note that $|H : R_C| < \infty$. Since $R_C \subset C \subseteq Z(A) \cap Z(B)$, then $R_C \lhd A$ and $R_C \lhd B$. Furthermore, A and B are R_C–separable. Thus, by Lemma 2.13, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H = R_C \subset R$. Therefore, by Theorem 2.1, G is π_c. ■

Corollary 2.15. *Let $G = A *_H B$ where A, B are free-by-finite or polycyclic-by-finite groups. Suppose that there exists $C \leq H$ such that $C \subseteq Z(A) \cap Z(B)$ is finitely generated with $|H : C| < \infty$. Then G is π_c.*

2.4 Tree Products

We shall extend Theorem 2.9 to tree products of finitely many groups.

Lemma 2.16. *Let $G = A *_H B$ where $H = \langle h \rangle \times D$ such that $|h| = \infty$ and D is finite. Suppose that A and B are both $\langle h \rangle$–separable and $\langle h \rangle$–wpot. Let $\langle k \rangle$ be any infinite cyclic subgroup of B such that B is $\langle k \rangle$–wpot. Then G is $\langle k \rangle$–wpot.*

Proof. Since A and B are $\langle h \rangle$–separable, $\langle h \rangle \cap D = 1$ and D is finite, there exist $M_1 \lhd_f A$ and $N_1 \lhd_f B$ such that $M_1 \langle h \rangle \cap D = 1 = N_1 \langle h \rangle \cap D$. Let $M_1 \cap \langle h \rangle = \langle h^{\alpha_1} \rangle$ and $N_1 \cap \langle h \rangle = \langle h^{\alpha_2} \rangle$ for some integers $\alpha_1, \alpha_2 > 0$. By Remark 2.7, there exist positive integer r_1, r_2 such that for each positive integer n_0, there exist $M_2 \lhd_f A$ and $N_2 \lhd_f B$ such that $M_2 \cap \langle h \rangle = \langle h^{r_1 n_0} \rangle$ and $N_2 \cap \langle h \rangle = \langle h^{r_1 n_0} \rangle$. Let $N_1 \cap N_2 \cap \langle k \rangle = \langle k^s \rangle$ for some $s > 0$. Since B is $\langle k \rangle$–wpot, there exists a positive integer r with the property that for each positive integer n, we have $N_3 \lhd_f B$ such that $N_3 \cap \langle k \rangle = \langle k^{rsn} \rangle$. Let $N_1 \cap N_2 \cap N_3 \cap \langle h \rangle = \langle h^{r_1 n_0 q} \rangle$ for some $q > 0$.

Now we choose $n_0 = r_2 \alpha_1 \alpha_2 q$ for M_2 and $n_0 = r_1 \alpha_1 \alpha_2$ for N_2 (we can choose n_0 because for each n_0, there always exists a normal subgroup of finite index that depends on which n_0 we chose). Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2 \cap N_3$. Then $M \lhd_f A$, $N \lhd_f B$ and we have $M \cap \langle h \rangle = \langle h^{\alpha_1 \alpha_2 r_1 r_2 q} \rangle = N \cap \langle h \rangle$ and $N \cap \langle k \rangle = \langle k^{rsn} \rangle$. Since $M \cap \big(\langle h \rangle \times D\big) = M \cap \langle h \rangle$ and $N \cap \big(\langle h \rangle \times D\big) = N \cap \langle h \rangle$, we have $M \cap H = N \cap H$.

Now we form $\bar{G} = A/M *_{\bar{H}} B/N$, where $\bar{H} = \langle \bar{h} \rangle \times \bar{D}$ (since $\langle \bar{h} \rangle \cap \bar{D} = 1$). Clearly $\bar{G}$ is a homomorphic image of G. Note that in $\bar{G}$, $|\bar{k}| = rsn$. Since $\bar{G}$ is residually finite by Lemma 2.2, there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{k}, \bar{k}^2, \ldots, \bar{k}^{rsn-1} \notin \bar{L}$. Clearly $\bar{G}/\bar{L}$ is a finite group in which $\bar{k}\bar{L}$ has order exactly rsn. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ such that $|kL| = rsn$ in the finite group G/L. Therefore G is $\langle k \rangle$–wpot. ■

We need an additional condition that G is $\langle h_{ij} \rangle$–separable for each i, j in order to extend Lemma 2.16 to tree products.

Lemma 2.17. *Let $G = \langle G_1, \ldots, G_n \mid H_{ij} = H_{ji} \rangle$ be a tree product of the groups $G_1, \ldots, G_n$ amalgamating the subgroups $H_{ij} \leq G_i$ and $H_{ji} \leq G_j$ where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Suppose that each $H_{ij} = \langle h_{ij} \rangle \times D_{ij}$ such that $|h_{ij}| = \infty$ and D_{ij} is finite. Suppose, furthermore, G is $\langle h_{ij} \rangle$–separable and each G_i is $\langle h_{ij} \rangle$–wpot for each i, j. Let $\langle k \rangle$ be any infinite cyclic subgroup of G_r such that G_r is $\langle k \rangle$–wpot, where $1 \leq r \leq n$. Then G is $\langle k \rangle$–wpot.*

Proof. First, note that each G_i is $\langle h_{ij} \rangle$–separable for each i, j since we assume G to be $\langle h_{ij} \rangle$–separable for each i, j. Now we prove by induction on n. The case $n = 2$ will follow from Lemma 2.16. Now for the case $n \geq 3$, we can find an extremal vertex, say G_n, of the tree product G, which is joined to a unique vertex, say G_{n-1}. The subgroup of G that is generated by $G_1, \ldots, G_{n-1}$ is just their tree product. Let this subgroup be denoted by G'. Then we write

$$G = \langle G', G_n \mid H_{(n-1)n} = H_{n(n-1)} \rangle = G' *_H G_n,$$

where $H = H_{(n-1)n} = H_{n(n-1)}$. This implies that G is a generalized free product of two groups G' and G_n with amalgamated subgroup $H = \langle h \rangle \times D$. Thus, by induction, G' is $\langle h_{(n-1)n} \rangle$–wpot. By our assumption, G_n is $\langle h_{n(n-1)} \rangle$–wpot. Furthermore, since G is $\langle h_{ij} \rangle$–separable for each i, j, this implies G' is $\langle h_{(n-1)n} \rangle$–separable and G_n is $\langle h_{n(n-1)} \rangle$–separable.

Suppose $\langle k \rangle < G'$. By induction, G' is $\langle k \rangle$–wpot. Then by Lemma 2.16, G is $\langle k \rangle$–wpot.

Suppose $\langle k \rangle < G_n$. By our assumption, G_n is $\langle k \rangle$–wpot. Then again, by Lemma 2.16, G is $\langle k \rangle$–wpot. ∎

Now, we can extend Theorem 2.9 to tree product as follows.

Theorem 2.18. *Let $G = \langle G_1, \ldots, G_n \mid H_{ij} = H_{ji} \rangle$ be a tree product of the groups $G_1, \ldots, G_n$ amalgamating the subgroups $H_{ij} \leq G_i$ and $H_{ji} \leq G_j$ where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Suppose that each $H_{ij} = \langle h_{ij} \rangle \times D_{ij}$ where $|h_{ij}| = \infty$ and D_{ij} is finite. Furthermore, suppose that each G_i is π_c and $\langle h_{ij} \rangle$–wpot. Then G is π_c.*

Proof. The proof is by induction on n. For the case $n = 2$, our result follows from Theorem 2.9. For $n \geq 3$, let $G = G' *_H G_n$ as in the proof of Lemma 2.17. By induction, G' is π_c. By our assumption, G_i is $\langle h_{ij} \rangle$–wpot for each $i \neq n$. In particular, G_{n-1} is $\langle h_{(n-1)n} \rangle$–wpot. Hence, by Lemma 2.17, G' is $\langle h_{(n-1)n} \rangle$–wpot. On the other hand, by our assumption, G_n is π_c and $\langle h_{n(n-1)} \rangle$–wpot. Therefore, G is π_c by Theorem 2.9. ∎

Corollary 2.19. *Let $G = \langle G_1, \ldots, G_n \mid H_{ij} = H_{ji} \rangle$ be a tree product of finite extensions of finitely generated nilpotent groups $G_1, \ldots, G_n$ amalgamating the subgroups $H_{ij} \leq G_i$ and $H_{ji} \leq G_j$ where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Suppose that each $H_{ij} = \langle h_{ij} \rangle \times D_{ij}$ where $|h_{ij}| = \infty$ and D_{ij} is finite. Then G is π_c.*

CHAPTER 3: CYCLIC SUBGROUP SEPARABILITY OF HNN EXTENSIONS

3.1 Introduction

In this chapter, we shall study the cyclic subgroup separability of HNN extensions of cyclic subgroup separable group and subgroup separable group. It has been shown by Kim and Tang (1999) that the HNN extension $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where A is cyclic subgroup separable and $H = \langle h \rangle$, $K = \langle k \rangle$ are infinite cyclic, is again cyclic subgroup separable if and only if A is quasi-regular at $\{h, k\}$ (Kim & Tang, 1999). We note that the Baumslag-Solitar group $BS(2,3) = \langle a, t \mid t^{-1}a^2t = a^3 \rangle$ is an example of an HNN extension which is not cyclic subgroup separable (Baumslag & Solitar, 1962).

First, we state a criterion (Theorem 3.1 below) for the cyclic subgroup separability of HNN extensions of cyclic subgroup separable group. We then apply the criterion to HNN extensions $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where (i) $H = \langle h \rangle$, $K = \langle k \rangle$ or $H = \langle h \rangle \times D$, $K = \langle k \rangle \times E$ where $\langle h \rangle$, $\langle k \rangle$ are infinite cyclic and D, E are finite (Theorem 3.3, Theorem 3.4, Theorem 3.7 and Theorem 3.8), (ii) H, K are finite extensions of central subgroups (Theorem 3.11 and Theorem 3.14) and finally (iii) H, K are finitely generated normal subgroups (Theorem 3.19).

As in the previous chapter, the term π_c shall be used in place of cyclic subgroup separable.

3.2 Preliminaries

The following theorem (Theorem 3.1) has been proved by Wong and Gan (1999).

Theorem 3.1. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where A is π_c. Suppose that*

(a) A is H–separable and K–separable; and

(b) for each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\varphi(N \cap H) = N \cap K$.

Then G is π_c.

3.3 HNN Extensions with Various Associated Subgroups

Note that Kim has proved that the HNN extensions of cyclic subgroup separable base group having finite associated subgroups are π_c (Kim, 1993b). Here, we start by applying the criterion to HNN extensions with infinite cyclic associated subgroups.

Lemma 3.2. *Let A be a group and $\langle h\rangle, \langle k\rangle$ be isomorphic infinite cyclic subgroups of A. Suppose that $\varphi : \langle h\rangle \to \langle k\rangle$ is an isomorphism such that $\varphi(h) = k$. If*

(i) $h \sim_A k$; or

(ii) $h^m = k^{\pm m}$ for some $m > 0$ and A is $\langle h\rangle$–wpot, $\langle k\rangle$–wpot,

then for each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\varphi(N \cap \langle h\rangle) = N \cap \langle k\rangle$.

Proof. Let $M \lhd_f A$ be given.

(i) Suppose that $M \cap \langle h\rangle = \langle h^\epsilon\rangle$ for some integer $\epsilon > 0$. Since $h = aka^{-1}$ for some $a \in A$, we have,

$$\begin{aligned} M \cap \langle k\rangle &= M \cap \langle a^{-1}ha\rangle \\ &= a^{-1}Ma \cap a^{-1}\langle h\rangle a \\ &= a^{-1}(M \cap \langle h\rangle)a \\ &= a^{-1}\langle h^\epsilon\rangle a \\ &= \langle (a^{-1}ha)^\epsilon\rangle \\ &= \langle k^\epsilon\rangle. \end{aligned}$$

Let $N = M$. Then we will have $N \lhd_f A$ and $\varphi\big(N \cap \langle h\rangle\big) = N \cap \langle k\rangle$.

(ii) Suppose that $M \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $M \cap \langle k \rangle = \langle k^{s_2} \rangle$ for some integers $s_1, s_2 > 0$. By Remark 2.7, for A is $\langle h \rangle$–wpot and $\langle k \rangle$–wpot, there exist positive integers r_1, r_2 such that for each positive integer n, there exist $M_1 \lhd_f A$ and $M_2 \lhd_f A$ such that $M_1 \cap \langle h \rangle = \langle h^{r_1 n} \rangle$ and $M_2 \cap \langle k \rangle = \langle k^{r_2 n} \rangle$. Choose $n = r_2 s_1 s_2 m$ for M_1 and $n = r_1 s_1 s_2 m$ for M_2 for some $m > 0$. Thus we have $M_1 \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2 m} \rangle$ and $M_2 \cap \langle k \rangle = \langle k^{r_1 r_2 s_1 s_2 m} \rangle$. Let $N = M \cap M_1 \cap M_2$. Then $N \lhd_f A$ and we have

$$\begin{aligned} N \cap \langle h \rangle &= M \cap M_1 \cap M_2 \cap \langle h \rangle \\ &= M_2 \cap \langle h^{r_1 r_2 s_1 s_2 m} \rangle \\ &= M_2 \cap \langle k^{r_1 r_2 s_1 s_2 m} \rangle \\ &= \langle k^{r_1 r_2 s_1 s_2 m} \rangle = \langle h^{r_1 r_2 s_1 s_2 m} \rangle \end{aligned}$$

and

$$\begin{aligned} N \cap \langle k \rangle &= M \cap M_1 \cap M_2 \cap \langle k \rangle \\ &= M_1 \cap \langle k^{r_1 r_2 s_1 s_2 m} \rangle \\ &= M_1 \cap \langle h^{r_1 r_2 s_1 s_2 m} \rangle \\ &= \langle h^{r_1 r_2 s_1 s_2 m} \rangle = \langle k^{r_1 r_2 s_1 s_2 m} \rangle. \end{aligned}$$

Hence, we have $\varphi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. ■

Theorem 3.3. *Let $G = \langle A, t \mid t^{-1} H t = K, \varphi \rangle$ where $H = \langle h \rangle$, $K = \langle k \rangle$ are infinite cyclic subgroups of A. Suppose that A is π_c. If $h \sim_A k$, then G is π_c.*

Proof. We shall prove by using Theorem 3.1. Note that A is $\langle h \rangle$–separable and $\langle k \rangle$–separable for A is π_c. By Lemma 3.2(i), for each $M \lhd_f A$, we have $N \lhd_f A$ such that $N \subseteq M$ and $\varphi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. Thus G is π_c by Theorem 3.1. ■

Theorem 3.4. *Let $G = \langle A, t \,|\, t^{-1}ht = k, \varphi\rangle$ where $H = \langle h\rangle$, $K = \langle k\rangle$ are infinite cyclic subgroups such that $\langle h\rangle \cap \langle k\rangle \neq 1$. Suppose that A is π_c, $\langle h\rangle$–wpot and $\langle k\rangle$–wpot. Then G is π_c if and only if $h^m = k^{\pm m}$ for some $m > 0$.*

Proof. Suppose that G is π_c. Since $\langle h\rangle \cap \langle k\rangle \neq 1$, let $h^m = k^s$, for some $m, s > 0$. Since G is π_c, there exists $L \lhd_f G$ such that $h^i \notin L\langle h^m\rangle$ for all $1 \leq i < m$. Let $L \cap \langle h\rangle = \langle h^{\epsilon m}\rangle$ for some $\epsilon > 0$. By the definition of φ, we have $L \cap \langle k\rangle = \langle k^{\epsilon m}\rangle$. Hence we have $h^{\epsilon m} = k^{\epsilon s} \in L \cap \langle k\rangle = \langle k^{\epsilon m}\rangle$. Thus, $m|s$. Similarly we can show that $s|m$. Therefore, $h^m = k^{\pm m}$ for some $m > 0$.

For the converse, we shall use Theorem 3.1. Note that A is $\langle h\rangle$–separable and $\langle k\rangle$–separable for A is π_c. By Lemma 3.2(ii), for any given $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\varphi(N \cap \langle h\rangle) = N \cap \langle k\rangle$. Hence G is π_c by Theorem 3.1. ∎

Next we consider the following HNN extensions.

Remark 3.5. Let $G = \langle A, t \,|\, t^{-1}Ht = K, \varphi\rangle$. Suppose that $H = \langle h\rangle \times D$, $K = \langle k\rangle \times E$ such that $|h| = \infty$, $|k| = \infty$ and D, E are finite subgroups. Suppose that $\varphi : H \to K$ is defined such that $\varphi(\langle h\rangle) = \langle k\rangle$ and $\varphi(D) = E$. Hence note that

- $\langle h\rangle$ and $\langle k\rangle$ are isomorphic via φ; and
- D and E are isomorphic via φ.

Hence if $x \in H$, $x = h^\alpha d$ where $\alpha \in \mathbb{Z}$, $d \in D$ and $\varphi(d) = e$, for some $e \in E$, then $\varphi(x) = \varphi(h^\alpha d) = t^{-1}(h^\alpha d)t = (t^{-1}h^\alpha t)(t^{-1}dt) = k^\alpha e$.

Lemma 3.6. *Let A be a group with subgroups $H = \langle h\rangle \times D$, $K = \langle k\rangle \times E$ such that $|h| = \infty$, $|k| = \infty$ and D, E are finite subgroups. Suppose that A is $\langle h\rangle$–separable, $\langle k\rangle$–separable and $\varphi : H \to K$ is an isomorphism where $\varphi(\langle h\rangle) = \langle k\rangle$ and $\varphi(D) = E$. If*

(i) $h \sim_A k$; or

(ii) $h^m = k^{\pm m}$ for some $m > 0$ and A is $\langle h\rangle$–wpot, $\langle k\rangle$–wpot,

then for each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\varphi(N \cap H) = N \cap K$.

Proof. Let $M \lhd_f A$ be given.

(i) Note that A is $\langle h\rangle$–separable, $\langle k\rangle$–separable, $\langle h\rangle \cap D = 1 = \langle k\rangle \cap E$ and D, E are finite. Thus, there is an $M_0 \lhd_f A$ such that $M_0\langle h\rangle \cap D = 1 = M_0\langle k\rangle \cap E$. Let $N = M \cap M_0$. Then $N \lhd_f A$ and $N\langle h\rangle \cap D = 1 = N\langle k\rangle \cap E$. Thus we have $N \cap \big(\langle h\rangle \times D\big) = N \cap \langle h\rangle$ and $N \cap \big(\langle k\rangle \times E\big) = N \cap \langle k\rangle$. Suppose that $N \cap \langle h\rangle = \langle h^\epsilon\rangle$ for some integer $\epsilon > 0$. As we have shown in the proof of Lemma 3.2(i), since $h \sim_A k$, we have $N \cap \langle k\rangle = \langle k^\epsilon\rangle$. Hence $\varphi\big(N \cap \langle h\rangle\big) = N \cap \langle k\rangle$ and

$$\varphi(N \cap H) = \varphi\big(N \cap (\langle h\rangle \times D)\big) = \varphi(N \cap \langle h\rangle) = N \cap \langle k\rangle = N \cap (\langle k\rangle \times E) = N \cap K.$$

Clearly $N \subseteq M$ and $\varphi(N \cap H) = N \cap K$.

(ii) Again since A is $\langle h\rangle$–separable, $\langle k\rangle$–separable, $\langle h\rangle \cap D = 1 = \langle k\rangle \cap E$ and D, E are finite, we have $M_0 \lhd_f A$ such that $M_0\langle h\rangle \cap D = 1 = M_0\langle k\rangle \cap E$. Let $M \cap M_0 \cap \langle h\rangle = \langle h^{s_1}\rangle$ and $M \cap M_0 \cap \langle k\rangle = \langle k^{s_2}\rangle$ for some integers $s_1, s_2 > 0$. By Remark 2.7, there exist positive integers r_1, r_2 such that for each positive integer n, there exist $M_1 \lhd_f A$ and $M_2 \lhd_f A$ such that $M_1 \cap \langle h\rangle = \langle h^{r_1 n}\rangle$ and $M_2 \cap \langle k\rangle = \langle k^{r_2 n}\rangle$. Choose $n = r_2 s_1 s_2 m$ for M_1 and $n = r_1 s_1 s_2 m$ for M_2 for some $m > 0$. Thus we have $M_1 \cap \langle h\rangle = \langle h^{r_1 r_2 s_1 s_2 m}\rangle$ and $M_2 \cap \langle k\rangle = \langle k^{r_1 r_2 s_1 s_2 m}\rangle$. Let $N = M \cap M_0 \cap M_1 \cap M_2$. Then $N \lhd_f A$. As shown in the proof of Lemma 3.2(ii), we will have $N \cap \langle h\rangle = \langle h^{r_1 r_2 s_1 s_2 m}\rangle$ and $N \cap \langle k\rangle = \langle k^{r_1 r_2 s_1 s_2 m}\rangle$. This implies that

$\varphi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. Since $N \cap \big(\langle h \rangle \times D\big) = N \cap \langle h \rangle$ and $N \cap \big(\langle k \rangle \times E\big) = N \cap \langle k \rangle$, we have $\varphi(N \cap H) = N \cap K$. ■

Theorem 3.7. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ be as in Remark 3.5. Suppose that A is π_c. If $h \sim_A k$, then G is π_c.*

Proof. We shall use Theorem 3.1. Note that A is $\langle h \rangle$–separable and $\langle k \rangle$–separable for A is π_c. Hence, by Lemma 2.6, A is H–separable and K–separable. By Lemma 3.6(i), for any given $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\varphi(N \cap H) = N \cap K$. Thus by Theorem 3.1, G is π_c. ■

Theorem 3.8. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ be as in Remark 3.5. Suppose that A is π_c, $\langle h \rangle$–wpot and $\langle k \rangle$–wpot and $\langle h \rangle \cap \langle k \rangle \neq 1$. Then G is π_c if and only if $h^m = k^{\pm m}$ for some $m > 0$.*

Proof. The first part of the proof is similar to Theorem 3.4. Now, suppose that $h^m = k^{\pm m}$ for some $m > 0$. We shall use Theorem 3.1. Note that A is $\langle h \rangle$–separable and $\langle k \rangle$–separable. Hence A is H–separable and K–separable by Lemma 2.6. By Lemma 3.6(ii), for any given $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\varphi(N \cap H) = N \cap K$. Therefore G is π_c by Theorem 3.1. ■

Corollary 3.9. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ be as in Remark 3.5. Suppose that A is a polycyclic-by-finite or a free-by-finite or a finite extension of a finitely generated nilpotent group.*

(i) If $h \sim_A k$, then G is π_c;

(ii) Then G is π_c if and only if $h^m = k^{\pm m}$ for some $m > 0$ whenever $\langle h \rangle \cap \langle k \rangle \neq 1$.

Wong and Gan(1999) have shown that the HNN extensions of subgroup separable groups with central associated subgroups are π_c (Wong & Gan, 1999). Here we show that the HNN extensions of polycyclic-by-finite (or free-by-finite) groups and finite extensions of finitely generated nilpotent groups where the associated subgroups are finite extensions of central subgroups, are π_c.

First, we examine the case where $H \cap K = 1$.

Lemma 3.10. *Let A be a group with subgroups H, K such that $H \cap K = 1$. Suppose that there exist $R \leq H$ and $S \leq K$ such that $R, S \subseteq Z(A)$ are finitely generated with $|H : R| < \infty$ and $|K : S| < \infty$. If A is central subgroup separable, then there exists $N \lhd_f A$ such that $N \cap H = R$ and $N \cap K = S$.*

Proof. Suppose such R and S are given. Note that $RS \subseteq Z(A)$ is finitely generated for $R, S \subseteq Z(A)$ are finitely generated. Thus, A is RS–separable for A is central subgroup separable. Hence, by Lemma 2.11, $\bar{A} = A/RS$ is residually finite. Note that $\bar{H} = HS/RS$ and $\bar{K} = KR/RS$ are finite. Thus, there exists $\bar{N} \lhd_f \bar{A}$ such that $\bar{N} \cap \bar{H} = \bar{1}$ and $\bar{N} \cap \bar{K} = \bar{1}$. Let N be the preimage of $\bar{N}$ in A. Then $N \lhd_f A$ with $N \cap H = R$ and $N \cap K = S$. ■

Theorem 3.11. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where A is π_c and central subgroup separable with subgroups H, K such that $H \cap K = 1$. Suppose that there exist $C \leq H$, $D \leq K$ such that $C, D \subseteq Z(A)$ are finitely generated with $|H : C| < \infty$, $|K : D| < \infty$ and $\varphi(C) = D$. Then G is π_c.*

Proof. We shall prove this by using Theorem 3.1. Note that A is C–separable and D–separable. Then by Lemma 2.12, A is H–separable and K–separable. Suppose that we are given $M \lhd_f A$. Let $R = M \cap C \cap \varphi^{-1}(M \cap D)$ and $S = \varphi(M \cap C) \cap M \cap D$. Then $R \lhd_f C$, $S \lhd_f D$ and $\varphi(R) = S$. Since $|H : C| < \infty$ and $|K : D| < \infty$, this implies $R \lhd_f H$, $S \lhd_f K$. Furthermore, $R, S \subseteq Z(A)$ are finitely generated. Then by Lemma 3.10, there exists $P \lhd_f A$

such that $P \cap H = R$ and $P \cap K = S$. Let $N = M \cap P$. Then $N \lhd_f A$, $N \subset M$ and

$$\varphi(N \cap H) = \varphi(M \cap P \cap H) = \varphi(M \cap R) = \varphi(R) = S = M \cap S = M \cap P \cap K = N \cap K.$$

Therefore G is π_c. ■

Corollary 3.12. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi\rangle$ where A is a polycyclic-by-finite group or a free-by-finite group or a finite extension of a finitely generated nilpotent group with subgroups H, K such that $H \cap K = 1$. Suppose that there exist $C \leq H$, $D \leq K$ such that $C, D \subseteq Z(A)$ are finitely generated with $|H : C| < \infty$, $|K : D| < \infty$ and $\varphi(C) = D$. Then G is π_c.*

Next we examine the case when $H \cap K \neq 1$.

Lemma 3.13. *Let A be central subgroup separable with subgroups H, K where $H \cap K \neq 1$. Let $\varphi : H \to K$ be an isomorphism from H onto K. Suppose that there exists $Q \leq H \cap K$ such that $Q \subseteq Z(A)$ is finitely generated with $|H : Q| < \infty$, $|K : Q| < \infty$ and $\varphi(Q) = Q$. Then for any $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\varphi(N \cap H) = N \cap K$.*

Proof. Let $R = M \cap Q$. Then $R \lhd_f Q$. Suppose that R has index k in Q. Since Q is finitely generated, there exist only a finite number of subgroups of index k in Q. Let R_0 be the intersection of all these subgroups. Then $R_0 \subset R$ and R_0 is a characteristic subgroup of finite index in Q. Since $\varphi(Q) = Q$, we have $\varphi(R_0) = R_0$. Note that $R_0 \lhd_f H$ and $R_0 \lhd_f K$. Note also that A is R_0–separable for $R_0 \subseteq Z(A)$ is finitely generated and A is central subgroup separable. Thus, by Lemma 2.13, we have $M_1 \lhd_f A$ and $M_2 \lhd_f A$ where $M_1 \cap H = R_0 = M_2 \cap K$. Let $N = M \cap M_1 \cap M_2$. Then $N \lhd_f A$ and $N \subset M$. Furthermore, we have $N \cap H = M \cap M_1 \cap M_2 \cap H = R_0 = M \cap M_1 \cap M_2 \cap K = N \cap K$. Hence $\varphi(N \cap H) = \varphi(R_0) = R_0 = N \cap K$. ■

Theorem 3.14. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi\rangle$ where A is π_c and central subgroup separable with subgroups H, K where $H \cap K \neq 1$. Suppose there exists $Q \leq H \cap K$ such that $Q \subseteq Z(A)$ is finitely generated with $|H : Q| < \infty$, $|K : Q| < \infty$ and $\varphi(Q) = Q$. Then G is π_c.*

Proof. Note that A is Q–separable for $Q \subseteq Z(A)$ is finitely generated and A is central subgroup separable. Hence, by Lemma 2.12, A is H–separable and K–separable. By Lemma 3.13, for any $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\varphi(N \cap H) = N \cap K$. Therefore, by Theorem 3.1, G is π_c. ■

Corollary 3.15. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi\rangle$ where A is a polycyclic-by-finite group or a free-by-finite group or a finite extension of a finitely generated nilpotent group with subgroups H, K where $H \cap K \neq 1$. Suppose that there exist $C \leq H$, $D \leq K$ such that $C, D \subseteq Z(A)$ are finitely generated with $|H : C| < \infty$ and $|K : D| < \infty$. Suppose that*

(i) $CD \lhd_f H \cap K$ with $|H : CD| < \infty$, $|K : CD| < \infty$ and $\varphi(CD) = CD$; or

(ii) $C \cap D \lhd_f H$, $C \cap D \lhd_f K$ and $\varphi(C \cap D) = C \cap D$.

Then G is π_c.

Proof. (i) If $CD \lhd_f H \cap K$ and $\varphi(CD) = CD$, we let $Q = CD$. (ii) If $C \cap D \lhd_f H$, $C \cap D \lhd_f K$ and $\varphi(C \cap D) = C \cap D$, we let $Q = C \cap D$. Then the result follows from Theorem 3.14. ■

Suppose that we let $C = H$ and $D = K$ in Theorems 3.11 and Corollary 3.15(b). Thus we will have the same result as in Theorems 2 and 3 of Wong and Gan (1999). Furthermore, if A is a finitely generated abelian group, then we have the following which are actually the result of Wong and Gan (1999).

Corollary 3.16. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where A is a finitely generated abelian group. Suppose that*

(a) $H \cap K = 1$*; or*

(b) $H = K$*; or*

(c) $H \cap K \lhd_f H$, $H \cap K \lhd_f K$ *and* $\varphi(H \cap K) = H \cap K$.

Then G is π_c.

Wong and Wong (2007) have shown that certain generalized free products amalgamating finitely generated normal subgroups are π_c. Here, we shall show that certain HNN extensions associating finitely generated normal subgroups are π_c. We need the following two lemmas of Wong and Wong (2007).

Lemma 3.17. *Let H be a finitely generated group and $R \lhd_f H$. Then there exists $f_H(R) \subseteq R$ such that $f_H(R)$ is a characteristic subgroup of finite index in H.*

Lemma 3.18. *Let A be subgroup separable with $H, K \lhd A$ are finitely generated and $H \cap K = 1$. Then for each $R \lhd_f H$ and $S \lhd_f K$, there exist $f_H(R) \subseteq R$, $f_K(S) \subseteq S$ such that $f_H(R)$, $f_K(S)$ are characteristic subgroups of finite index in H, K respectively. Furthermore, there exists $N \lhd_f A$ such that $N \cap H = f_H(R)$, $N \cap K = f_K(S)$ and $NH \cap NK = N$.*

Theorem 3.19. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where A is subgroup separable with $H, K \lhd A$ are finitely generated and $H \cap K = 1$. Then G is π_c.*

Proof. We shall use Theorem 3.1. First, note that A is π_c, H–separable, and K–separable for A is subgroup separable. Next, let $M \lhd_f A$ be given. Let $R = M \cap H \cap \varphi^{-1}(M \cap K)$. Then $R \lhd_f H$. By Lemma 3.17, there exists $f_H(R) \subseteq R$ such that $f_H(R)$ is a characteristic subgroup of finite index in H. Since φ is an isomorphism, then $S = \varphi\big(f_H(R)\big) \subseteq \varphi(R)$ is a characteristic subgroup of finite index in K. Now by Lemma 3.18, there exists $N_1 \lhd_f A$

such that $N_1 \cap H = f_H(R)$, $N_1 \cap K = S$ and $N_1 H \cap N_1 K = N_1$. We also have

$$\varphi(N_1 \cap H) = \varphi(f_H(R)) = S = N_1 \cap K.$$

Let $N = M \cap N_1$. Then $N \lhd_f A$. Finally, we need to show that $\varphi(N \cap H) = N \cap K$. Now $N \cap H = M \cap N_1 \cap H = N_1 \cap H$ for $N_1 \cap H \subset R \subset M$. Also $N \cap K = M \cap N_1 \cap K = N_1 \cap K$ for $N_1 \cap K \subset \varphi(R) \subset M$. Hence $\varphi(N \cap H) = N \cap K$. Therefore G is π_c. ■

CHAPTER 4: WEAK POTENCY OF GENERALIZED FREE PRODUCTS

4.1 Introduction

In this chapter, we shall investigate the weak potency of certain generalized free products. First, we shall prove a criterion (Theorem 4.1 below) for generalized free products of weakly potent groups to be weakly potent. Then our criterion will be applied to the generalized free products $G = A *_H B$ where (i) H is finite (Theorem 4.3), (ii) $H = \langle h \rangle$ or $H = \langle h \rangle \times D$ where $\langle h \rangle$ is infinite cyclic and D is a finite subgroup (Theorem 4.4 and Theorem 4.5), and finally (iii) H is a finite extension of a central subgroup (Theorem 4.8). Furthermore, we shall extend Theorem 4.5 to tree products of finitely many groups in Theorem 4.14.

4.2 The Criterion

In this section, we prove the following criterion.

Theorem 4.1. *Let $G = A *_H B$. Suppose that*

(a) A and B are H–separable;

(b) for each $R \lhd_f H$, there exist $P_A \lhd_f A$ and $P_B \lhd_f B$ such that $P_A \cap H = P_B \cap H \subseteq R$; and

(c) for any infinite order element $x \in A$ (or $x \in B$), there is a positive integer r, such that for each positive integer n, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ and $M \cap \langle x \rangle = \langle x^{rn} \rangle$ (or $N \cap \langle x \rangle = \langle x^{rn} \rangle$ if $x \in B$).

Then G is weakly potent.

Proof. Let $x \in G$ such that $|x| = \infty$.

CASE 1. Suppose that $\|x\| \leq 1$, that is $x \in A \cup B$. We may assume without loss of generality that $x \in A$. Now by (c), there is a positive integer r, such that for each positive integer n, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ and $M \cap \langle x \rangle = \langle x^{rn} \rangle$.

Now we form $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ such that $\bar{A} = A/M$, $\bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. Clearly $|\bar{x}| = rn$ in the homomorphic image $\bar{G}$ of G. Since $\bar{A}$ and $\bar{B}$ are finite, then by Lemma 2.2, $\bar{G}$ is residually finite. Thus, there is an $\bar{L} \lhd_f \bar{G}$ such that $\bar{x}, \bar{x}^2, \ldots, \bar{x}^{rn-1} \notin \bar{L}$ but $\bar{x}^{rn} \in \bar{L}$. Let L be the preimage of $\bar{L}$ in G. Thus, $L \lhd_f G$ and $|xL| = rn$ in the finite group G/L.

Case 2. Suppose that $\|x\| > 1$, that is $x \notin A \cup B$ and x is reduced.

Subcase 2.1. Suppose that x is cyclically reduced. Without loss of generality, let $x = a_1b_1a_2b_2, \ldots, a_nb_n$ where $a_i \in A\backslash H$ and $b_i \in B\backslash H$ for all $i = 1, \ldots, n$. By (a), there exist $P_1 \lhd_f A$ and $Q_1 \lhd_f B$ such that $a_i \notin P_1H$ and $b_i \notin Q_1H$ for all i. Let $R = P_1 \cap Q_1$. This implies $R \lhd_f H$. By assumption (b), we have $P_2 \lhd_f A$ and $Q_2 \lhd_f B$ where $P_2 \cap H = Q_2 \cap H \subseteq R$. Let $M = P_1 \cap P_2$ and $N = Q_1 \cap Q_2$. Then $M \lhd_f A$, $N \lhd_f B$ and $a_i \notin MH$ and $b_i \notin NH$ for all i. Now we show that $M \cap H = N \cap H$. First, note that $P_2 \cap H \subseteq R \subseteq P_1$ and $Q_2 \cap H \subseteq R \subseteq Q_1$. So we have

$$\begin{aligned} M \cap H &= P_1 \cap P_2 \cap H \\ &= P_2 \cap H \\ &= Q_2 \cap H \\ &= Q_1 \cap Q_2 \cap H \\ &= N \cap H. \end{aligned}$$

Now we form $\bar{G}$ as in Case 1. Note that $\|\bar{x}\| = \|x\|$ in $\bar{G}$ and hence, $|\bar{x}| = \infty$. Since $\bar{A}$ and $\bar{B}$ are finite, then by Lemma 2.2, $\bar{G}$ is weakly potent. Then we can find a positive integer r with the property that for each positive integer n, there exists $\bar{L} \lhd_f \bar{G}$ such that $|\bar{x}\bar{L}| = rn$ in $\bar{G}/\bar{L}$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ and xL has order exactly rn in the finite group G/L.

SUBCASE 2.2. Suppose that x is not cyclically reduced. Then x is conjugate to a cyclically reduced element x' of G. By CASE 1 or SUBCASE 2.1, there exists $L \lhd_f G$ such that $x'L$ has order exactly rn in the finite group G/L. Note that $|xL| = rn$ in the finite group G/L. Thus, the proof is now complete. ■

4.3 Generalized Free Products Amalgamating Various Subgroups

In this section, we shall apply our criterion to certain generalized free products. Theorem 4.3 is a known and useful result. For completeness, we give a proof. We begin with the following obvious lemma.

Lemma 4.2. *Let A be a residually finite group with a finite subgroup H. Then A is H–separable.*

Theorem 4.3. *(Lim, 2012) Let $G = A *_H B$ where A and B are residually finite and weakly potent. Suppose that H is finite. Then G is weakly potent.*

Proof. We shall prove by using Theorem 4.1. Note that A and B are H–separable by Lemma 4.2. This proves (a) in Theorem 4.1.

Next, suppose that we are given any $R \lhd_f H$. For A, B are residually finite and H is finite, we have $P_A \lhd_f A$ and $P_B \lhd_f B$ such that $P_A \cap H = 1 = P_B \cap H$. Thus, $P_A \cap H = P_B \cap H = 1 \subset R$. This proves (b) in Theorem 4.1.

Now we show (c) in Theorem 4.1. Let $x \in A$ such that $|x| = \infty$. Again since A, B are residually finite and H is finite, we have $M_1 \lhd_f A$ and $N_1 \lhd_f B$ such that $M_1 \cap H = 1 = N_1 \cap H$. Let $M_1 \cap \langle x \rangle = \langle x^s \rangle$ for some integer $s > 0$. By the weak potency of A, there is a positive integer r such that for each positive integer n, there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle x \rangle = \langle x^{rsn} \rangle$. Let $M = M_1 \cap M_2$ and $N = N_1$. Then $M \lhd_f A$, $N \lhd_f B$ and $M \cap \langle x \rangle = \langle x^{rsn} \rangle$. Furthermore, $M \cap H = M_1 \cap M_2 \cap H = 1 = N_2 \cap H = N \cap H$. Thus, the proof is now complete. ■

Next we prove the weak potency of generalized free products amalgamating an infinite cyclic subgroup. The following theorem appeared in (Wong et al. 2010) but there are some overlooked cases in the proof. Here we prove the theorem using Theorem 4.1.

Theorem 4.4. *Let $G = A *_H B$ where $H = \langle h \rangle$ is an infinite cyclic group. Suppose that A and B are weakly potent and $\langle h \rangle$–separable. Then G is weakly potent.*

Proof. Suppose that we are given any $R \lhd_f H$. Then $R = \langle h^k \rangle$ for some $k > 0$. Since A and B are weakly potent, by Remark 2.7, there exists a positive integer r_1 such that for each positive integer n, there exists $P_A \lhd_f A$ such that $P_A \cap \langle h \rangle = \langle h^{r_1 n} \rangle$. For this case, we choose $n = r_2 k$. Hence we have $P_A \cap \langle h \rangle = \langle h^{r_1 r_2 k} \rangle$. By Remark 2.7 also, we can find a positive integer r_2 such that for each positive integer n, there exists $P_B \lhd_f B$ such that $P_B \cap \langle h \rangle = \langle h^{r_2 n} \rangle$. For this case, we choose $n = r_1 k$. Hence we have $P_B \cap \langle h \rangle = \langle h^{r_1 r_2 k} \rangle$. Therefore, $P_A \cap \langle h \rangle = P_B \cap \langle h \rangle = \langle h^{r_1 r_2 k} \rangle \subseteq R$.

Next, let $x \in A$ such that $|x| = \infty$. By Remark 2.7, we let $M_1 \lhd_f A$ be such that $M_1 \cap \langle h \rangle = \langle h^{r_1 r_2} \rangle$. Suppose that $M_1 \cap \langle x \rangle = \langle x^s \rangle$ for some integer $s > 0$. By weak potency of A, there is a positive integer r such that for each positive integer n, there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle x \rangle = \langle x^{rsn} \rangle$. Let $M = M_1 \cap M_2$. Then $M \lhd_f A$, $M \cap \langle x \rangle = \langle x^{rsn} \rangle$ and

$$M \cap \langle h \rangle = \langle h^{r_1 r_2 t} \rangle$$

for some integer $t > 0$. By weak potency of B, again by Remark 2.7, we let $N \lhd_f B$ be such that

$$N \cap \langle h \rangle = \langle h^{r_1 r_2 t} \rangle.$$

Thus, we have $M \cap \langle h \rangle = N \cap \langle h \rangle$. Therefore, by Theorem 4.1, G is weakly potent. ■

Now we apply our criterion to generalized free products of weakly potent groups amalgamating the subgroup of the form $\langle h \rangle \times D$, where D is finite.

Theorem 4.5. *Let $G = A *_H B$, where $H = \langle h \rangle \times D$ such that $|h| = \infty$ and D is finite. Suppose that A and B are weakly potent and $\langle h \rangle$–separable. Then G is weakly potent.*

Proof. By Lemma 2.6, A and B are H–separable. By Lemma 2.8, for each $R \lhd_f H$, we have $P_A \lhd_f A$ and $P_B \lhd_f B$ where $P_A \cap H = P_B \cap H \subseteq R$.

Now we show the following. Let $x \in A$ such that $|x| = \infty$. Since A, B are $\langle h \rangle$–separable, $\langle h \rangle \cap D = 1$ and D is finite, we have $M_1 \lhd_f A$ and $N_1 \lhd_f B$ such that $M_1\langle h \rangle \cap D = 1 = N_1\langle h \rangle \cap D$. Let $M_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $N_1 \cap \langle h \rangle = \langle h^{s_2} \rangle$ for some integers $s_1, s_2 > 0$. By Remark 2.7, let $M_2 \lhd_f A$ be such that $M_2 \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2} \rangle$. Suppose that $M_1 \cap M_2 \cap \langle x \rangle = \langle x^s \rangle$ for some integer $s > 0$. For A is weakly potent, there is a positive integer r such that for each positive integer n, we have $M_3 \lhd_f A$ such that $M_3 \cap \langle x \rangle = \langle x^{rsn} \rangle$. Let $M = M_1 \cap M_2 \cap M_3$. Then $M \lhd_f A$, $M \cap \langle x \rangle = \langle x^{rsn} \rangle$ and

$$M \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2 t} \rangle$$

for some integer $t > 0$. By weak potency of B, again by Remark 2.7, we let $N_2 \lhd_f B$ be such that $N_2 \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2 t} \rangle$. Let $N = N_1 \cap N_2$. Then $N \lhd_f B$ and

$$N \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2 t} \rangle.$$

Thus, we have $M \cap \langle h \rangle = N \cap \langle h \rangle$ and also $M\langle h \rangle \cap D = 1 = N\langle h \rangle \cap D$. Since $M \cap \big(\langle h \rangle \times D\big) = M \cap \langle h \rangle$ and $N \cap \big(\langle h \rangle \times D\big) = N \cap \langle h \rangle$, we then have $M \cap H = N \cap H$. Therefore, by Theorem 4.1, G is weakly potent. ∎

Corollary 4.6. *Let $G = A *_H B$ where $H = \langle h \rangle \times D$ such that D is finite. Suppose that A and B are finite extensions of a finitely generated nilpotent groups. Then G is weakly potent.*

Proof. If $|h| < \infty$ and since D is finite, then H is finite. Thus, by Theorem 4.3, G is weakly potent. If $|h| = \infty$, then G is weakly potent by Theorem 4.5. ■

Finally, we shall apply our criterion (Theorem 4.1) to certain generalized free products of weakly potent groups amalgamating finite extensions of a central subgroup. We begin with the following lemma of Wong and Wong (2014).

Lemma 4.7. *Let C be a finitely generated abelian group and $c \in C$ where $|c| = \infty$. Then for any positive integer n, there exists a characteristic subgroup R_{ch} of finite index in C such that $R_{ch} \cap \langle c \rangle = \langle c^n \rangle$.*

Theorem 4.8. *Let $G = A *_H B$. Suppose there exists $C \leq H$ such that $C \subseteq Z(A) \cap Z(B)$ is finitely generated with $|H : C| < \infty$. Suppose that A and B are central subgroup separable and A/C, B/C are weakly potent. Then G is weakly potent.*

Proof. We shall use Theorem 4.1. Note that A and B are C–separable for A and B are central subgroup separable. Thus, by Lemma 2.12, A and B are H–separable. Now suppose that we are given any $R \lhd_f H$. Let $R_C = R \cap C$. Then $R_C \lhd_f C$. Note that $R_C \lhd A$ and $R_C \lhd B$ is finitely generated for R_C is a subgroup of finite index in the finitely generated subgroup C. Furthermore, $|H : R_C| < \infty$ since $|H : C| < \infty$. Then by Lemma 2.13, there exist $P_A \lhd_f A$ and $P_B \lhd_f B$ such that $P_A \cap H = P_B \cap H = R_C \subseteq R$.

Next we show the following. Let $x \in A$ such that $|x| = \infty$. Note that C is a finitely generated abelian group.

CASE 1. Suppose that $C \cap \langle x \rangle = 1$. Then $|xC| = \infty$. Now form $\bar{G} = A/C *_{H/C} B/C$. Clearly, $\bar{G}$ is a homomorphic image of G. Note that $\bar{A} = A/C$ and $\bar{B} = B/C$ are residually finite by Lemma 2.11. Furthermore, since they are weakly potent and H/C is finite, then by Theorem 4.3, $\bar{G}$ is weakly potent. Denote $\bar{x} = xC$. Thus, we can find a positive integer r such that for each positive integer n, there exists $\bar{L} \lhd_f \bar{G}$ such that $|\bar{x}\bar{L}| = rn$ in $\bar{G}/\bar{L}$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ and $|xL| = rn$ in G/L. This implies $L \cap \langle x \rangle = \langle x^{rn} \rangle$. Now let $M = L \cap A$ and $N = L \cap B$. This implies $M \lhd_f A$ and $N \lhd_f B$. Furthermore, we have $M \cap \langle x \rangle = L \cap A \cap \langle x \rangle = L \cap \langle x \rangle = \langle x^{rn} \rangle$ and $M \cap H = L \cap A \cap H = L \cap H = L \cap B \cap H = N \cap H$.

CASE 2. Suppose that $C \cap \langle x \rangle = \langle x^r \rangle$ for some integer $r > 0$. By Lemma 4.7, there exists a characteristic subgroup R_{ch} of finite index in C such that $R_{ch} \cap \langle x^r \rangle = \langle x^{rn} \rangle$ for any positive integer n. Hence

$$R_{ch} \cap \langle x \rangle = R_{ch} \cap C \cap \langle x \rangle = R_{ch} \cap \langle x^r \rangle = \langle x^{rn} \rangle.$$

Note that R_{ch} is a finitely generated normal subgroup of A and of B. Furthermore, $R_{ch} \lhd_f H$. Now we form $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ where $\bar{A} = A/R_{ch}$, $\bar{B} = B/R_{ch}$ and $\bar{H} = H/R_{ch}$. Note that, in $\bar{G}$, $|\bar{x}| = rn$ and A, B are R_{ch}–separable for A and B are central subgroup separable and $R_{ch} \subset Z(A) \cap Z(B)$ is finitely generated. Then by Lemma 2.11, $\bar{A}$ and $\bar{B}$ are residually finite. Then, for $\bar{H}$ is finite, $\bar{G}$ is residually finite by Theorem 3 of Baumslag (1963). Thus, there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{x}, \bar{x}^2, \ldots, \bar{x}^{rn-1} \notin \bar{L}$ but $\bar{x}^{rn} \in \bar{L}$. This implies $\bar{L} \cap \langle \bar{x} \rangle = \langle \bar{x}^{rn} \rangle$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ and $L \cap \langle x \rangle = \langle x^{rn} \rangle$. Now as in Case 1, we let $M = L \cap A$ and $N = L \cap B$. Thus, we will have $M \lhd_f A$, $N \lhd_f B$, $M \cap \langle x \rangle = \langle x^{rn} \rangle$ and $M \cap H = N \cap H$. Therefore our result now follows from Theorem 4.1. ■

Remark 4.9. Let A be a finite extension of a finitely generated nilpotent group. Then there exists $T \lhd_f A$ where T is a finitely generated nilpotent group. If there exists $N \lhd A$, then $TN/N \cong T/(T \cap N)$ is finitely generated nilpotent and TN is a normal subgroup of finite index in A. Hence, A/N is weakly potent and subgroup separable.

Thus, by Theorem 4.8, we have the following.

Corollary 4.10. *Let $G = A *_H B$ where A and B are finite extensions of a finitely generated nilpotent groups. Suppose there exists $C \leq H$ such that $C \subseteq Z(A) \cap Z(B)$ is finitely generated with $|H : C| < \infty$. Then G is weakly potent.*

4.4 Tree Products

Note that Lim (2012) and Wong et al. (2010) have proved the weak potency of the tree products of weakly potent groups amalgamating finite subgroups and amalgamating infinite cyclic subgroups respectively. In this section, we shall extend Theorem 4.5 to tree products of finitely many groups. We need the following lemma of Kim (1992).

Lemma 4.11. *Let $G = A *_H B$, where A and B are H–separable. Suppose that for each $R \lhd_f H$, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H \subseteq R$. Let K be any subgroup of B such that B is K–separable. Then G is K–separable.*

From Lemma 4.11, we can obtain the following lemma.

Lemma 4.12. *Let $G = A *_H B$ where $H = \langle h \rangle \times D$ such that $|h| = \infty$ and D is finite. Suppose that A and B are weakly potent and $\langle h \rangle$–separable. Let K be any subgroup of B such that B is K–separable. Then G is K–separable.*

Proof. By Lemma 2.6, A and B are H–separable. Since A and B are $\langle h \rangle$–separable and $\langle h \rangle$–wpot, by Lemma 2.8, for each $R \lhd_f H$, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H \subseteq R$. Thus, our result follows from Lemma 4.11. ■

We need an additional assumption that G is weakly potent in order to extend Lemma 4.12 to a tree product.

Lemma 4.13. *Let $G = \langle G_1, \ldots, G_n \mid H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, \ldots, G_n$ amalgamating the subgroups $H_{ij} \leq G_i$ and $H_{ji} \leq G_j$ where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Suppose that each $H_{ij} = \langle h_{ij} \rangle \times D_{ij}$ such that $|h_{ij}| = \infty$ and D_{ij} is finite. Suppose furthermore that each G_i is $\langle h_{ij} \rangle$–separable and G is weakly potent. Let K be any subgroup of G_r such that G_r is K–separable where $1 \leq r \leq n$. Then G is K–separable.*

Proof. First, note that each G_i is weakly potent since we assume G to be weakly potent. Now we prove by induction on n. The case $n = 2$ follows from Lemma 4.12. For the case $n \geq 3$, let $G = G' *_H G_n$ as in Lemma 2.17. Now by induction, G' is $\langle h_{(n-1)n} \rangle$–separable. Note that G_n is $\langle h_{n(n-1)} \rangle$–separable by our assumption. Furthermore, G' and G_n are weakly potent since G is weakly potent.

Suppose $K < G'$. By induction, G' is K–separable. Then by Lemma 4.12, G is K–separable.

Suppose $K < G_n$. By our assumption, G_n is K–separable. Then again, by Lemma 4.12, G is K–separable. ■

Now we can extend Theorem 4.5 to tree products as follows.

Theorem 4.14. *Let $G = \langle G_1, \ldots, G_n \,|\, H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, \ldots, G_n$ amalgamating the subgroups $H_{ij} \leq G_i$ and $H_{ji} \leq G_j$ where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Suppose that each $H_{ij} = \langle h_{ij} \rangle \times D_{ij}$ such that each $|h_{ij}| = \infty$ and D_{ij} is finite. Furthermore, suppose that each G_i is $\langle h_{ij} \rangle$–separable and weakly potent. Then G is weakly potent.*

Proof. We shall proof by using induction on n. For the case $n = 2$, our result will follows from Theorem 4.5. For the case $n \geq 3$, let $G = G' *_H G_n$ as in Lemma 2.17. By induction hypothesis, G' is weakly potent. By assumption, each G_i is $\langle h_{ij} \rangle$–separable. In particular,

G_{n-1} is $\langle h_{(n-1)n} \rangle$–separable. Then by Lemma 4.13, G' is $\langle h_{(n-1)n} \rangle$–separable. Next, by our assumption, G_n is $\langle h_{n(n-1)} \rangle$–separable and weakly potent. Therefore, by Theorem 4.5, G is weakly potent. ■

CHAPTER 5: WEAK POTENCY OF HNN EXTENSIONS

5.1 Introduction

In this chapter, we shall investigate the weak potency of HNN extensions of weakly potent groups. It has been shown by Wong et al. (2010) that the HNN extension $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where the base group A is weakly potent and $H = \langle h \rangle$, $K = \langle k \rangle$ are infinite cyclic with $h^m = k^{\pm m}$ for some $m > 0$ is again weakly potent. We will expand on this theorem.

As in the previous chapter, we first prove a criterion (Theorem 5.2 below) for the weak potency of HNN extensions of weakly potent group. Then we apply our criterion to the HNN extension $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where (i) H, K are finite (Theorem 5.3), (ii) $H = \langle h \rangle$, $K = \langle k \rangle$ or $H = \langle h \rangle \times D$, $K = \langle k \rangle \times E$ where $\langle h \rangle$, $\langle k \rangle$ are infinite cyclic and D, E are finite (Theorem 5.4, Theorem 5.5, Theorem 5.6 and Theorem 5.7) and finally (iii) H, K are finite extensions of central subgroups (Theorem 5.10).

5.2 The Criterion

In this section, we prove Theorem 5.2, which is similar to Theorem 3.2 of Wong and Wong (2014). The following lemma is known to many researchers in this area (see (Karras & Solitar, 1970; Evans, 1974; Wong, 1993).

Lemma 5.1. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where A is a finite group. Then G is free-by-finite, and hence, weakly potent and subgroup separable .*

Theorem 5.2. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi\rangle$. Suppose that*

(a) A is residually finite, H–separable and K–separable; and

(b) for each $M \lhd_f A$, there exists $P \lhd_f A$ such that $P \subseteq M$ and $\varphi(P \cap H) = P \cap K$.

Then G is weakly potent if and only if, for any $a \in A$ of infinite order, we can find a positive integer r such that for each positive integer n, there exists $N \lhd_f A$ such that $N \cap \langle a\rangle = \langle a^{rn}\rangle$ and $\varphi(N \cap H) = N \cap K$.

Proof. Suppose that G is weakly potent. Let $a \in A$ such that $|a| = \infty$. Then, we can find a positive integer r such that for each positive integer n, there exists $L \lhd_f G$ such that $L \cap \langle a\rangle = \langle a^{rn}\rangle$. Let $N = L \cap A$. Then $N \lhd_f A$ and $N \cap \langle a\rangle = L \cap A \cap \langle a\rangle = L \cap \langle a\rangle = \langle a^{rn}\rangle$. Furthermore, we have $N \cap H = L \cap A \cap H = L \cap H$ and hence $\varphi(N \cap H) = t^{-1}(L \cap H)t = t^{-1}Lt \cap t^{-1}Ht = L \cap K = L \cap A \cap K = N \cap K$.

Conversely, let $x \in G$ such that $|x| = \infty$.

Case 1. Suppose that $\|x\| = 0$, that is $x \in A$. By our assumption, there is a positive integer r such that for each positive integer n, there exists $N \lhd_f A$ such that $N \cap \langle x\rangle = \langle x^{rn}\rangle$ and $\varphi(N \cap H) = N \cap K$. Now we form $\bar{G} = \langle \bar{A}, t \mid t^{-1}\bar{H}t = \bar{K}, \bar{\varphi}\rangle$ where $\bar{A} = A/N$, $\bar{H} = HN/N$, $\bar{K} = KN/N$ and $\bar{\varphi}$ is the induced homomorphism of φ. Clearly, $|\bar{x}| = rn$ in the homomorphic image $\bar{G}$ of G. Now since $\bar{A}$ is finite, by Lemma 5.1, $\bar{G}$ is residually finite. Hence there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{x}, \bar{x}^2, \ldots, \bar{x}^{rn-1} \notin \bar{L}$ but $\bar{x}^{rn} \in \bar{L}$. Clearly, $\bar{G}/\bar{L}$ is a finite group in which $|\bar{x}\bar{L}| = rn$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ and the order of xL is exactly rn in the finite group G/L.

Case 2. Suppose that $\|x\| \geq 1$.

Subcase 2.1. Suppose that x is cyclically reduced. We may assume without loss of generality that $x = a_0t^{\varepsilon_1}a_1t^{\varepsilon_2}\ldots t^{\varepsilon_n}a_n$, where $a_i \in A$ and $n \geq 1$. By (a), there exists $M \lhd_f A$ such that $a_i \notin HM$ if $a_i \notin H$, $a_i \notin KM$ if $a_i \notin K$ and $a_i \notin M$ if $a_i \in H \cap K \backslash \{1\}$

for all $i = 1, \ldots, n$. By assumption (b), there exists $P \lhd_f A$ such that $P \subseteq M$ and $\varphi(P \cap H) = P \cap K$. Let $N = M \cap P$. Then $N \lhd_f A$ and $a_i \notin HN$ if $a_i \notin H$, $a_i \notin KN$ if $a_i \notin K$ and $a_i \notin N$ if $a_i \in H \cap K \backslash \{1\}$ for all $i = 1, \ldots, n$. Furthermore, we have

$$\varphi(N \cap H) = \varphi(M \cap P \cap H) = \varphi(P \cap H) = P \cap K = M \cap P \cap K = N \cap K.$$

Now as in Case 1, we form $\bar{G}$. Then $\bar{x} \in \bar{G}$ is cyclically reduced, $\|\bar{x}\| = \|x\|$ with $|\bar{x}| = \infty$. Since $\bar{A}$ is finite, $\bar{G}$ is weakly potent by Lemma 5.1. Thus, we can find a positive integer r such that for each positive integer n, there exists $\bar{L} \lhd_f \bar{G}$ such that $|\bar{x}\bar{L}| = rn$ in $\bar{G}/\bar{L}$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ and xL has order exactly rn in the finite group G/L.

Subcase 2.2. Suppose that x is not cyclically reduced. Then x is conjugate to a cyclically reduced element x' of G. By Case 1 or Subcase 2.1, there exists $L \lhd_f G$ such that $x'L$ has order exactly rn in the finite group G/L. Note that the order of xL is exactly rn in the finite group G/L. Thus, the proof is now complete. ■

5.3 HNN Extensions with Various Associated Subgroups

We begin by proving the following useful and known result (Theorem 5.3). For completeness, we give a proof here.

Theorem 5.3. *(Lim, 2012) Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where A is residually finite and weakly potent. Suppose that H and K are finite. Then G is weakly potent.*

Proof. Note that A is H–separable and K–separable by Lemma 4.2. Let $M \lhd_f A$ be given. Since H, K are finite and A is residually finite, there exists $M_0 \lhd_f A$ such that $M_0 \cap H = 1 = M_0 \cap K$. Let $P = M \cap M_0$. Then $P \lhd_f A$, $P \subseteq M$ and $P \cap H = 1 = P \cap K$. Hence $\varphi(P \cap H) = P \cap K$.

Now let $a \in A$ such that $|a| = \infty$. Again, since H and K are finite, we have $N_0 \lhd_f A$ such that $N_0 \cap H = 1 = N_0 \cap K$. Let $N_0 \cap \langle a \rangle = \langle a^s \rangle$ for some integer $s > 0$. Since A is weakly potent, we can find a positive integer r with the property that for each positive integer n, there exists $N_1 \lhd_f A$ such that $N_1 \cap \langle a \rangle = \langle a^{rsn} \rangle$. Let $N = N_0 \cap N_1$. Then $N \lhd_f A$, $N \cap \langle a \rangle = \langle a^{rsn} \rangle$ and $N \cap H = 1 = N \cap K$. Hence $\varphi(N \cap H) = N \cap K$. Therefore G is weakly potent by Theorem 5.2. ■

We now consider the HNN extensions $G = \langle A, t \,|\, t^{-1}Ht = K, \varphi \rangle$, where $H = \langle h \rangle$, $K = \langle k \rangle$ are infinite cyclic.

Theorem 5.4. *Let $G = \langle A, t \,|\, t^{-1}Ht = K, \varphi \rangle$ where $H = \langle h \rangle$, $K = \langle k \rangle$ are infinite cyclic. Suppose that A is residually finite, weakly potent, $\langle h \rangle$–separable and $\langle k \rangle$–separable. If $h \sim_A k$, then G is weakly potent.*

Proof. We shall prove by using Theorem 5.2. By Lemma 3.2(i), for each $M \lhd_f A$, there exists $P \lhd_f A$ such that $P \subseteq M$ and $\varphi(P \cap \langle h \rangle) = P \cap \langle k \rangle$.

Now let any $a \in A$ such that $|a| = \infty$. Since A is weakly potent, there is a positive integer r such that for each positive integer n, there exists $N \lhd_f A$ such that $N \cap \langle a \rangle = \langle a^{rn} \rangle$. Suppose that $N \cap \langle h \rangle = \langle h^{\epsilon} \rangle$ for some $\epsilon > 0$. Since $h = bkb^{-1}$ for some $b \in A$, as we have shown in the proof of Lemma 3.2(i), we have $N \cap \langle k \rangle = \langle k^{\epsilon} \rangle$. Thus, $\varphi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. Therefore, by Theorem 5.2, G is weakly potent. ■

Wong et al. (2010) have proved that HNN extensions of weakly potent groups with cyclic associated subgroups having non-trivial intersection are weakly potent (see Theorem 3.1 of Wong et al., 2010). But there are some overlooked cases in the proof. In this thesis, we shall prove the theorem using the criterion.

Theorem 5.5. *(Wong et al., 2010) Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where $H = \langle h \rangle$, $K = \langle k \rangle$ are infinite cyclic with $\langle h \rangle \cap \langle k \rangle \neq 1$. Suppose that A is residually finite, weakly potent, $\langle h \rangle$–separable and $\langle k \rangle$–separable. Then G is weakly potent if and only if $h^m = k^{\pm m}$ for some $m > 0$.*

Proof. Suppose that G is weakly potent. Since $\langle h \rangle \cap \langle k \rangle \neq 1$, let $h^m = k^p$ for some non-zero integers m, p. Then, there is a positive integer r such that for each positive integer n, we have $L \lhd_f G$ such that $\bar{h} = hL$ has order exactly rn in $\bar{G} = G/L$. Choose $n = |p||m|$. Then $|\bar{h}^m| = (r|p||m|)/|m| = r|p|$ and $|\bar{h}^p| = r|m|$. Since $\bar{h}^p$ is conjugate to $\bar{h}^m$ in $\bar{G} = G/L$, we have $r|m| = r|p|$, and therefore, $|m| = |p|$.

For the converse, we shall use Theorem 5.2. By Lemma 3.2(ii), for each $M \lhd_f A$, there exists $P \lhd_f A$ such that $P \subseteq M$ and $\varphi(P \cap \langle h \rangle) = P \cap \langle k \rangle$.

Next, we show the following. Let $a \in A$ such that $|a| = \infty$. By Remark 2.7, there exist positive integers r_1, r_2 such that for each positive integer n, there exist $M_1 \lhd_f A$ and $M_2 \lhd_f A$ such that $M_1 \cap \langle h \rangle = \langle h^{r_1 n} \rangle$ and $M_2 \cap \langle k \rangle = \langle k^{r_2 n} \rangle$. Choose $n = r_2 m$ for M_1 and $n = r_1 m$ for M_2 for some $m > 0$. Thus, we have $M_1 \cap \langle h \rangle = \langle h^{r_1 r_2 m} \rangle$ and $M_2 \cap \langle k \rangle = \langle k^{r_1 r_2 m} \rangle$. Suppose that $M_1 \cap M_2 \cap \langle a \rangle = \langle a^s \rangle$ for some integer $s > 0$. Since A is weakly potent, there is a positive integer r such that for each positive integer n, there exists $M_3 \lhd_f A$ such that $M_3 \cap \langle a \rangle = \langle a^{rsn} \rangle$. Let $N = M_1 \cap M_2 \cap M_3$. Then $N \lhd_f A$ and $N \cap \langle a \rangle = \langle a^{rsn} \rangle$. Furthermore, we have

$$\begin{aligned} N \cap \langle h \rangle &= M_1 \cap M_2 \cap M_3 \cap \langle h \rangle \\ &= M_2 \cap M_3 \cap \langle h^{r_1 r_2 m} \rangle \\ &= M_2 \cap M_3 \cap \langle k^{r_1 r_2 m} \rangle \\ &= M_3 \cap \langle k^{r_1 r_2 m} \rangle \end{aligned}$$

$$= M_3 \cap \langle h^{r_1 r_2 m} \rangle,$$

and

$$
\begin{aligned}
N \cap \langle k \rangle &= M_1 \cap M_2 \cap M_3 \cap \langle k \rangle \\
&= M_1 \cap M_3 \cap \langle k^{r_1 r_2 m} \rangle \\
&= M_1 \cap M_3 \cap \langle h^{r_1 r_2 m} \rangle \\
&= M_3 \cap \langle h^{r_1 r_2 m} \rangle.
\end{aligned}
$$

This implies $N \cap \langle h \rangle = N \cap \langle k \rangle$. Now suppose that $N \cap \langle h \rangle = \langle h^{r_1 r_2 m q} \rangle$ for some $q > 0$. Then, $N \cap \langle k \rangle = \langle h^{r_1 r_2 m q} \rangle$. Hence, $\varphi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. Therefore, by Theorem 5.2, G is weakly potent. ■

We now consider HNN extensions $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ as defined in Remark 3.5.

Theorem 5.6. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ be as in Remark 3.5. Suppose that A is residually finite, weakly potent, $\langle h \rangle$–separable and $\langle k \rangle$–separable. If $h \sim_A k$, then G is weakly potent.*

Proof. We shall prove by using Theorem 5.2. By Lemma 2.6, A is H–separable and K–separable. By Lemma 3.6(i), for each $M \lhd_f A$, there exists $P \lhd_f A$ such that $P \subseteq M$ and $\varphi(P \cap H) = P \cap K$.

Next, we show the following. Let $a \in A$ such that $|a| = \infty$. Since A is $\langle h \rangle$–separable and $\langle k \rangle$–separable, D, E are finite and $\langle h \rangle \cap D = 1 = \langle k \rangle \cap E$, there exists $M' \lhd_f A$ such that $M'\langle h \rangle \cap D = 1 = M'\langle k \rangle \cap E$. Suppose that $M' \cap \langle a \rangle = \langle a^s \rangle$ for some integer $s > 0$. Since A is weakly potent, we can find a positive integer r such that for each positive integer n, there exists $M_0 \lhd_f A$ such that $M_0 \cap \langle a \rangle = \langle a^{rsn} \rangle$. Let $N = M' \cap M_0$. Then $N \lhd_f A$,

$N \cap \langle a \rangle = \langle a^{rsn} \rangle$ and $N\langle h \rangle \cap D = 1 = N\langle k \rangle \cap E$. Suppose that $N \cap \langle h \rangle = \langle h^{\epsilon} \rangle$ for some $\epsilon > 0$. Then, as shown in the proof of Lemma 3.2(i), we have $N \cap \langle k \rangle = \langle k^{\epsilon} \rangle$. Hence, $\varphi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. Since $N \cap \big(\langle h \rangle \times D\big) = N \cap \langle h \rangle$ and $N \cap \big(\langle k \rangle \times E\big) = N \cap \langle k \rangle$, we have $\varphi(N \cap H) = N \cap K$. Therefore G is weakly potent by Theorem 5.2. ■

Theorem 5.7. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ be as in Remark 3.5. Suppose that A is residually finite, weakly potent, $\langle h \rangle$–separable, $\langle k \rangle$–separable and $\langle h \rangle \cap \langle k \rangle \neq 1$. Then G is weakly potent if and only if $h^m = k^{\pm m}$ for some $m > 0$.*

Proof. The first part of the proof is similar to that in the proof of Theorem 5.5. Now suppose that $h^m = k^{\pm m}$ for some $m > 0$. We shall use Theorem 5.2. By Lemma 2.6, A is H–separable and K–separable. By Lemma 3.6(ii), for each $M \lhd_f A$, there exists $P \lhd_f A$ such that $P \subseteq M$ and $\varphi(P \cap H) = P \cap K$.

Next, we show the following. Let $a \in A$ such that $|a| = \infty$. Since A is $\langle h \rangle$–separable and $\langle k \rangle$–separable, D, E are finite and $\langle h \rangle \cap D = 1 = \langle k \rangle \cap E$, there exists $M' \lhd_f A$ such that $M'\langle h \rangle \cap D = 1 = M'\langle k \rangle \cap E$. Now, let $M' \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $M' \cap \langle k \rangle = \langle k^{s_2} \rangle$ for some integers $s_1, s_2 > 0$. As noted in Remark 2.7, there exist positive integers r_1, r_2 such that for each positive integer n, there exist $M_1 \lhd_f A$ and $M_2 \lhd_f A$ such that $M_1 \cap \langle h \rangle = \langle h^{r_1 n} \rangle$ and $M_2 \cap \langle k \rangle = \langle k^{r_2 n} \rangle$. Choose $n = r_2 s_1 s_2 m$ for M_1 and $n = r_1 s_1 s_2 m$ for M_2 for some $m > 0$. This, we have $M_1 \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2 m} \rangle$ and $M_2 \cap \langle k \rangle = \langle k^{r_1 r_2 s_1 s_2 m} \rangle$. Suppose that $M' \cap M_1 \cap M_2 \cap \langle a \rangle = \langle a^s \rangle$ for some integer $s > 0$. By weak potency of A, there is a positive integer r such that for each positive integer n, there exists $M_3 \lhd_f A$ such that $M_3 \cap \langle a \rangle = \langle a^{rsn} \rangle$. Let $N = M' \cap M_1 \cap M_2 \cap M_3$. Then $N \lhd_f A$, $N \cap \langle a \rangle = \langle a^{rsn} \rangle$ and $N\langle h \rangle \cap D = 1 = N\langle k \rangle \cap E$. Now we show $\varphi(N \cap H) = N \cap K$. Note that we have,

$$\begin{aligned} N \cap \langle h \rangle &= M' \cap M_1 \cap M_2 \cap M_3 \cap \langle h \rangle \\ &= M_2 \cap M_3 \cap \langle h^{r_1 r_2 s_1 s_2 m} \rangle \end{aligned}$$

$$
\begin{aligned}
&= M_2 \cap M_3 \cap \langle k^{r_1 r_2 s_1 s_2 m} \rangle \\
&= M_3 \cap \langle k^{r_1 r_2 s_1 s_2 m} \rangle \\
&= M_3 \cap \langle h^{r_1 r_2 s_1 s_2 m} \rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
N \cap \langle k \rangle &= M' \cap M_1 \cap M_2 \cap M_3 \cap \langle k \rangle \\
&= M_1 \cap M_3 \cap \langle k^{r_1 r_2 s_1 s_2 m} \rangle \\
&= M_1 \cap M_3 \cap \langle h^{r_1 r_2 s_1 s_2 m} \rangle \\
&= M_3 \cap \langle h^{r_1 r_2 s_1 s_2 m} \rangle.
\end{aligned}
$$

This implies $N \cap \langle h \rangle = N \cap \langle k \rangle$. Now suppose that $N \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2 m q} \rangle$ for some integer $q > 0$. Then, $N \cap \langle k \rangle = \langle h^{r_1 r_2 s_1 s_2 m q} \rangle$. Hence, $\varphi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. Since $N \cap \big(\langle h \rangle \times D\big) = N \cap \langle h \rangle$ and $N \cap \big(\langle k \rangle \times E\big) = N \cap \langle k \rangle$, we have $\varphi(N \cap H) = N \cap K$. Therefore G is weakly potent by Theorem 5.2. ■

Corollary 5.8. *Let $G = \langle A, t \mid t^{-1} H t = K, \varphi \rangle$ be as in Remark 3.5. Suppose that A is a finitely generated nilpotent group. Then G is weakly potent if $h \sim_A k$, or if and only if $h^m = k^{\pm m}$ for some $m > 0$ whenever $\langle h \rangle \cap \langle k \rangle \neq 1$.*

Next, we shall show the weak potency of HNN extensions $G = \langle A, t \mid t^{-1} H t = K, \varphi \rangle$ where H and K are finite extensions of a central subgroup of A. We only consider the case $H \cap K \neq 1$.

Lemma 5.9. *Let A be a finite extension of a finitely generated nilpotent group T with subgroups H, K such that $H \cap K \neq 1$. Let $\varphi : H \to K$ be an isomorphism from H onto K. Suppose there exists $Q \leq H \cap K$ such that $Q \subseteq Z(A)$ is finitely generated with $|H : Q| < \infty$, $|K : Q| < \infty$ and $\varphi(Q) = Q$. Then, for any element $a \in A$ of infinite order, there exists a positive integer r such that for each positive integer n, there exists $N \lhd_f A$ such that $N \cap \langle a \rangle = \langle a^{rn} \rangle$ and $\varphi(N \cap H) = N \cap K$.*

Proof. First, note that Q is finitely generated abelian. Let $a \in A$ such that $|a| = \infty$. We divide our proof into two cases.

Case 1. Suppose that $Q \cap \langle a \rangle = \langle a^r \rangle$ for some integer $r > 0$. By Lemma 4.7, for any integer $n > 0$, there exists a characteristic subgroup R_{ch} of finite index in Q such that $R_{ch} \cap \langle a^r \rangle = \langle a^{rn} \rangle$. Hence $R_{ch} \cap \langle a \rangle = R_{ch} \cap Q \cap \langle a \rangle = R_{ch} \cap \langle a^r \rangle = \langle a^{rn} \rangle$ and $\varphi(R_{ch}) = R_{ch}$ since $\varphi(Q) = Q$. Note that $R_{ch} \lhd A$ is finitely generated and $R_{ch} \lhd_f H$, $R_{ch} \lhd_f K$. Now we form $\bar{A} = A/R_{ch}$. Then $\bar{H} = H/R_{ch}$, $\bar{K} = K/R_{ch}$ are finite and $|\bar{a}| = rn$ in $\bar{A}$. Furthermore, A is R_{ch}–separable for A is subgroup separable. Hence, by Lemma 2.11, $\bar{A}$ is residually finite. Thus, there exists $\bar{N} \lhd_f \bar{A}$ such that $\bar{a}, \bar{a}^2, \ldots, \bar{a}^{rn-1} \notin \bar{N}$ but $\bar{a}^{rn} \in \bar{N}$, and $\bar{N} \cap \bar{H} = \bar{1} = \bar{N} \cap \bar{K}$. Let N be the preimage of $\bar{N}$ in A. Then $N \lhd_f A$, $N \cap \langle a \rangle = \langle a^{rn} \rangle$ and $N \cap H = R_{ch} = N \cap K$. Hence $\varphi(N \cap H) = \varphi(R_{ch}) = R_{ch} = N \cap K$.

Case 2. Suppose that $Q \cap \langle a \rangle = 1$. We form $\bar{A} = A/Q$. Then $\bar{H} = H/Q$, $\bar{K} = K/Q$ are finite and $\bar{a} = aQ$ is of infinite order in $\bar{A}$. Note that A is Q–separable since Q is finitely generated and A is subgroup separable. Hence, by Lemma 2.11, $\bar{A}$ is residually finite. Thus, for finite subgroups $\bar{H}, \bar{K}$, there exists $\bar{M} \lhd_f \bar{A}$ such that $\bar{M} \cap \bar{H} = \bar{1} = \bar{M} \cap \bar{K}$. Let $\bar{M} \cap \langle \bar{a} \rangle = \langle \bar{a}^s \rangle$ for some integer $s > 0$. Note that $\bar{A}$ is weakly potent since $\bar{A}$ is a finite extension of the finitely generated nilpotent group $\bar{T} = TQ/Q$ (see Remark 4.9). Thus, there exists a positive integer r with the property that for each positive integer n, there exists

$\bar{M}_0 \lhd_f \bar{A}$ such that $\bar{M}_0 \cap \langle \bar{a} \rangle = \langle \bar{a}^{rsn} \rangle$. Let $\bar{N} = \bar{M} \cap \bar{M}_0$. Then $\bar{N} \lhd_f \bar{A}$, $\bar{N} \cap \bar{H} = \bar{1} = \bar{N} \cap \bar{K}$ and $\bar{N} \cap \langle \bar{a} \rangle = \langle \bar{a}^{rsn} \rangle$. Let N be the preimage of $\bar{N}$ in A. Then $N \lhd_f A$, $N \cap \langle a \rangle = \langle a^{rsn} \rangle$ and $N \cap H = Q = N \cap K$. Hence $\varphi(N \cap H) = \varphi(Q) = Q = N \cap K$. ■

Theorem 5.10. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where A is a finite extension of a finitely generated nilpotent group with subgroups H, K such that $H \cap K \neq 1$. Suppose that there exists $Q \leq H \cap K$ such that $Q \subseteq Z(A)$ is finitely generated with $|H : Q| < \infty$, $|K : Q| < \infty$ and $\varphi(Q) = Q$. Then G is weakly potent.*

Proof. Let $a \in A \setminus H$. Now, we form the group $\bar{A} = A/Q$. Then, $\bar{H} = H/Q$, $\bar{K} = K/Q$ are finite and $\bar{a} = aQ \notin \bar{H}$. Note that $\bar{A}$ is residually finite for $\bar{A}$ is a finite extension of the finitely generated nilpotent group $\bar{T} = TQ/Q$ (see Remark 4.9). Therefore, there exists $\bar{N} \lhd_f \bar{A}$ such that $\bar{N} \cap \bar{a}\bar{H} = 1$. Let N be the preimage of $\bar{N}$ in A. Then $a \notin NH$. Hence, A is H–separable. Similarly, A is K–separable. The theorem now follows from Theorem 5.2 and Lemmas 3.13 and 5.9. ■

Corollary 5.11. *Let $G = \langle A, t \mid t^{-1}Ht = K, \varphi \rangle$ where A is a finite extension of a finitely generated nilpotent group with subgroups H, K where $H \cap K \neq 1$. Suppose there exists $C \leq H$ and $D \leq K$ such that $C, D \subseteq Z(A)$ are finitely generated with $|H : C| < \infty$, $|K : D| < \infty$. Suppose that*

(i) $CD \leq H \cap K$ with $|H : CD| < \infty$, $|K : CD| < \infty$ and $\varphi(CD) = CD$; or

(ii) $C \cap D \lhd_f C$, $C \cap D \lhd_f D$ and $\varphi(C \cap D) = C \cap D$.

Then G is weakly potent.

Proof. (i) If $CD \leq H \cap K$ and $\varphi(CD) = CD$, we let $Q = CD$. (ii) If $C \cap D \lhd_f C$, $C \cap D \lhd_f D$ and $\varphi(C \cap D) = C \cap D$, we let $Q = C \cap D$. Thus, the corollary follows from Theorem 5.10. ■

CHAPTER 6: FUNDAMENTAL GROUPS OF GRAPHS OF GROUPS

6.1 Introduction

It has been shown by Kim (2004) that the fundamental group of graphs of groups $G = \langle A, t_1, \ldots, t_n \,|\, t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, n\rangle$ where each $H_i = \langle h_i \rangle$, $K_i = \langle k_i \rangle$ are infinite cyclic with $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$ are cyclic subgroup separable (see Kim (2004)). In this chapter, we shall prove the cyclic subgroup separability of fundamental groups of graphs of groups $G = \langle A, t_1, \ldots, t_n \,|\, t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, n\rangle$ where each $H_i = \langle h_i \rangle \times D_i$, $K_i = \langle k_i \rangle \times E_i$ and each $\langle h_i \rangle$, $\langle k_i \rangle$ are infinite cyclic and D_i, E_i are finite (Theorem 6.6).

6.2 Cyclic Subgroup Separability

We shall begin with the following remark.

Remark 6.1. Let $G = \langle A, t_1, \ldots, t_n \,|\, t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, n\rangle$ where each $H_i = \langle h_i \rangle \times D_i$ and $K_i = \langle k_i \rangle \times E_i$ such that $|h_i| = \infty$, $|k_i| = \infty$, D_i, E_i are finite subgroups and $\varphi_i : H_i \longrightarrow K_i$ is an isomorphism such that $\varphi_i(\langle h_i \rangle) = \langle k_i \rangle$ and $\varphi_i(D_i) = E_i$.

We define $G_j = \langle A, t_1, \ldots, t_j \,|\, t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, j\rangle$ for each $1 \leq j \leq n$. In particular, we have $G_1 = \langle A, t_1 \,|\, t_1^{-1} H_1 t_1 = K_1, \varphi_1\rangle$ and $G_n = G$. Furthermore, we note that for each $1 < j < n$,

$$\begin{aligned} G_j &= \langle A, t_1, \ldots, t_j \,|\, t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, j\rangle \\ &= \langle G_{j-1}, t_j \,|\, t_j^{-1} H_j t_j = K_j, \varphi_j\rangle. \end{aligned}$$

Hence $G_n = \langle G_{n-1}, t_n \,|\, t_n^{-1} H_n t_n = K_n, \varphi_n\rangle$. For ease of exposition, we shall write G as $G_n = \langle A, t, \ldots, t_n \,|\, t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, n\rangle$.

First, we extend Theorem 3.7 to G_n.

Theorem 6.2. *Let $G_n = \langle A, t_1, \ldots, t_n \mid t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, n\rangle$ be as in Remark 6.1. Suppose that A is π_c. If $h_i \sim_A k_i$ for each $i = 1, \ldots, n$, then G_n is π_c.*

Proof. We shall prove by induction on n. The case $n = 1$, that is, $G_1 = \langle A, t_1 \mid t_1^{-1} H_1 t_1 = K_1, \varphi_1\rangle$ follows from Theorem 3.7. Thus, G_1 is π_c. For $n \geq 2$, by the induction hypothesis, G_{n-1} is π_c. Since $h_i \sim_A k_i$ for each $i = 1, \ldots, n$, we have $h_n \sim_{G_{n-1}} k_n$. Therefore, G_n is π_c by Theorem 3.7. ∎

Before we extend Theorem 3.8 to G_n, we need Lemmas 6.3 and 6.4.

Lemma 6.3. *Let $G = \langle A, t \mid t^{-1} H t = K, \varphi\rangle$ be as in Remark 3.5. Suppose that A is $\langle h\rangle$–separable, $\langle k\rangle$–separable, $\langle h\rangle$–wpot, $\langle k\rangle$–wpot and $h^m = k^{\pm m}$ for some $m > 0$. Let any element $a \in A$ be of infinite order such that A is $\langle a\rangle$–wpot. Then G is $\langle a\rangle$–wpot.*

Proof. Since A is $\langle h\rangle$–separable, $\langle k\rangle$–separable, $\langle h\rangle \cap D = 1 = \langle k\rangle \cap E$ and D, E are both finite, there exists $M_1 \lhd_f A$ such that $M_1\langle h\rangle \cap D = 1 = M_1\langle k\rangle \cap E$. Let $M_1 \cap \langle h\rangle = \langle h^{s_1}\rangle$ and $M_1 \cap \langle k\rangle = \langle k^{s_2}\rangle$ for some integers $s_1, s_2 > 0$. Since A is $\langle h\rangle$–wpot and $\langle k\rangle$–wpot, by Remark 2.7, there exist positive integers r_1, r_2 such that for each positive integer n, there exist $M_2 \lhd_f A$ and $M_3 \lhd_f A$ such that $M_2 \cap \langle h\rangle = \langle h^{r_1 n}\rangle$ and $M_3 \cap \langle k\rangle = \langle k^{r_2 n}\rangle$. Choose $n = r_2 s_1 s_2 m$ for M_2 and $n = r_1 s_1 s_2 m$ for M_3 for some $m > 0$. Let $M_0 = M_1 \cap M_2 \cap M_3$. Then, $M_0 \lhd_f A$ and similarly as in the the proof of Lemma 3.2(ii), we have $M_0 \cap \langle h\rangle = \langle h^{r_1 r_2 s_1 s_2 m}\rangle$ and $M_0 \cap \langle k\rangle = \langle k^{r_1 r_2 s_1 s_2 m}\rangle$. Let $M_0 \cap \langle a\rangle = \langle a^s\rangle$ for some $s > 0$. Since A is $\langle a\rangle$–wpot, we can find a positive integer r such that for each positive integer n, there exists $M_4 \lhd_f A$ such that $M_4 \cap \langle a\rangle = \langle a^{rsn}\rangle$. Let $N = M_0 \cap M_4$. Then $N \lhd_f A$ with $N \cap \langle a\rangle = \langle a^{rsn}\rangle$. Furthermore, we have

$$N \cap \langle h\rangle = M_0 \cap M_4 \cap \langle h\rangle = M_4 \cap \langle h^{r_1 r_2 s_1 s_2 m}\rangle$$

and

$$N \cap \langle k \rangle = M_0 \cap M_4 \cap \langle k \rangle = M_4 \cap \langle k^{r_1 r_2 s_1 s_2 m} \rangle = M_4 \cap \langle h^{r_1 r_2 s_1 s_2 m} \rangle.$$

Hence $N \cap \langle h \rangle = N \cap \langle k \rangle$. Suppose that $N \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2 mq} \rangle$ for some $q > 0$. Then $N \cap \langle k \rangle = \langle k^{r_1 r_2 s_1 s_2 mq} \rangle$. Thus, we have $\varphi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. Since $N \cap \big(\langle h \rangle \times D\big) = N \cap \langle h \rangle$ and $N \cap \big(\langle k \rangle \times E\big) = N \cap \langle k \rangle$, we have $\varphi(N \cap H) = N \cap K$. Now we form $\bar{G} = \langle t, \bar{A} \,|\, t^{-1}\bar{h}t = \bar{k}, \bar{\varphi} \rangle$, where $\bar{A} = A/N$, $\bar{H} = \langle \bar{h} \rangle \times \bar{D}$, $\bar{K} = \langle \bar{k} \rangle \times \bar{E}$ and note that $|\bar{a}| = rsn$. By Lemma 5.1, $\bar{G}$ is residually finite since $\bar{A}$ is finite. Thus, there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{a}, \bar{a}^2, \ldots, \bar{a}^{rsn-1} \notin \bar{L}$ but $\bar{a}^{rsn} \in \bar{L}$. Let L be the preimage of $\bar{L}$ in G. Then we have $L \lhd_f G$ and $|aL| = rsn$ in G/L. Therefore G is $\langle a \rangle$–wpot. ■

We need an additional assumption that G_n is π_c in order to extend Lemma 6.3 to G_n

Lemma 6.4. *Let $G_n = \langle A, t_1, \ldots, t_n \,|\, t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, n \rangle$ be as in Remark 6.1. Suppose that A is $\langle h_i \rangle$–wpot, $\langle k_i \rangle$–wpot and $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$, for each $i = 1, \ldots, n$. Further suppose that G_n is π_c. Let $a \in A$ be any element of infinite order such that A is $\langle a \rangle$–wpot. Then G_n is $\langle a \rangle$–wpot.*

Proof. First, note that A is π_c since we assume G_n is π_c. Hence A is $\langle h_i \rangle$–separable and $\langle k_i \rangle$–separable for each $i = 1, \ldots, n$. We shall prove by induction on n. For the case $n = 1$, that is, $G_1 = \langle A, t_1 \,|\, t_1^{-1} H_1 t_1 = K_1, \varphi_1 \rangle$, the result follows from Lemma 6.3. Then G_1 is $\langle a \rangle$–wpot. Furthermore, since A is $\langle h_i \rangle$–wpot and $\langle k_i \rangle$–wpot for each $i = 1, \ldots, n$, by Lemma 6.3, G_1 is $\langle h_i \rangle$–wpot and $\langle k_i \rangle$–wpot for each $i = 1, \ldots, n$. For $n \geq 2$, by the induction hypothesis, G_{n-1} is $\langle a \rangle$–wpot, $\langle h_i \rangle$–wpot and $\langle k_i \rangle$–wpot for each $i = 1, \ldots, n$. In particular, G_{n-1} is $\langle h_n \rangle$–wpot and $\langle k_n \rangle$–wpot. Note that G_{n-1} is π_c since G_{n-1} is a subgroup of G_n and we assume G_n is π_c. In particular, G_{n-1} is $\langle h_n \rangle$–separable and $\langle k_n \rangle$–separable. Thus, by Lemma 6.3, G_n in $\langle a \rangle$–wpot. ■

Now we are ready to extend Theorem 3.8 to G_n in the following theorem.

Theorem 6.5. *Let $G_n = \langle A, t_1, \ldots, t_n \mid t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, n\rangle$ be as in Remark 6.1. Suppose that A is π_c, $\langle h_i\rangle$–wpot, $\langle k_i\rangle$–wpot and $\langle h_i\rangle \cap \langle k_i\rangle \neq 1$ for each $i = 1, \ldots, n$. Then G_n is π_c if and only if $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$, for each $i = 1, \ldots, n$.*

Proof. Suppose that G_n is π_c. Since each $G_j = \langle A, t_1, \ldots, t_j \mid t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, j\rangle$ for each $1 \leq j \leq n$ is a subgroup of G_n, then each of them must be π_c. Then by Theorem 3.8, $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$.

For the converse, we prove by induction on n. For $n = 1$, that is, $G_1 = \langle A, t_1 \mid t_1^{-1} H_1 t_1 = K_1, \varphi_1\rangle$, the result follows from Theorem 3.8. Therefore, G_1 is π_c. For $n \geq 2$, by the induction hypothesis, we assume G_{n-1} is π_c. Since A is π_c, $\langle h_i\rangle$–wpot and $\langle k_i\rangle$–wpot for each $i = 1, \ldots, n$, then by Lemma 6.4, G_{n-1} is $\langle h_i\rangle$–wpot and $\langle k_i\rangle$–wpot for each $i = 1, \ldots n$. In particular, G_{n-1} is $\langle h_n\rangle$–wpot and $\langle k_n\rangle$–wpot. Therefore, by Theorem 3.8, G_n is π_c. ■

Now from Theorem 2.18 with Theorems 6.2 and 6.5, we have the following main result.

Theorem 6.6. *Let G be a fundamental group of a graph of groups G_v, amalgamating direct product of infinite cyclic and finite edge subgroups, presented by $G = \langle A, t_1, \ldots, t_n \mid t_i^{-1} H_i t_i = K_i, \varphi_i, i = 1, \ldots, n\rangle$ where A is a tree product of groups G_v according to a maximal tree of the graph, such that $H_i = \langle h_i\rangle \times D_i$, $K_i = \langle k_i\rangle \times E_i$, $|h_i| = \infty$, $|k_i| = \infty$, and D_i, E_i are finite for each $i = 1, \ldots, n$,. Suppose that each G_v is π_c and weakly potent. Then G is π_c*

(i) if $h_i \sim_A k_i$ for each $i = 1, \ldots, n$; or

(ii) if and only if $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$ for each $i = 1, \ldots, n$.

CHAPTER 7: CONCLUSION

7.1 Conclusion

In conclusion, we have obtained several results on cyclic subgroup separability and weak potency throughout this research. First, we used previously proved criterion of cyclic subgroup separability for both generalized free products and HNN extensions to proved the cyclic subgroup separability and weak potency of generalized free products amalgamating certain subgroups and HNN extensions associating certain subgroups. We also have extended our results to cyclic subgroup separability of tree products and fundamental groups of graphs of groups.

Next, we have established several criterion in Theorems 4.1 and 5.2 which are useful for determining the weak potency of generalized free products and HNN extensions respectively. With these criterion, we have proved the weak potency of generalized free products amalgamating certain subgroups and HNN extensions associating certain subgroups.

7.2 Further Research

Note that recently Zhou and Kim have shown the abelian subgroup separability of certain generalized free products and HNN extensions (Zhou & Kim, 2017, 2018). Hence some of our results, especially Theorems 2.9, 2.14, 3.7, 3.8, 4.4, 4.5, 5.6 and 5.7 can serve as useful starting points for extension to abelian subgroup separability.

The results of cyclic subgroup separability and weak potency of tree products (Theorems 2.18 and 4.14 respectively) can be further extended to polygonal products. Finally, some further research can be done to fundamental groups of graphs of groups to be weakly potent.

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