LAZY COP NUMBER AND OTHER RELATED GRAPH PARAMETERS

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LAZY COP NUMBER AND OTHER RELATED GRAPH PARAMETERS

ABSTRACT

In this thesis, we focus on graph parameters in the game of Cops and Robbers and the burning number of graphs. The game of cops and robbers is a two-player game played on a finite connected undirected graph G. The first player occupies some vertices with a set of cops, and the second player occupies a vertex with a single robber. The cops move first, followed by the robber. After that, the players move alternately. On the cops' turn, each of the cops may remain stationary or move to an adjacent vertex. On the robber's turn, he may remain stationary or move to an adjacent vertex. A round of the game is a cop move together with the subsequent robber move. The cops win if after a finite number of rounds, one of them can move to catch the robber, that is, the cop and the robber occupy the same vertex. The robber wins if he can evade the cops indefinitely. The cop number is the main graph parameter in the game of cops and robbers. In this thesis, we investigate the cop number and lazy cop number of a graph G, the minimum order of graphs for small value of cop number and the capture time. Our results focused on a variant of the game, the lazy cops and robbers, where at most one cop moves in any round. Burning a graph is a process defined on the vertex set of a simple finite graph. Initially, at time step $t = 0$, all vertices are unburned. At the beginning of every time step $t \geq 1$, an unburned vertex is chosen to burn (if such a vertex is available). Thereafter, if a vertex is burned in time step $t - 1$, then in time step t, each of its unburned neighbours becomes burned. A burned vertex will remain burned throughout the process. The process ends when all vertices are burned. The burning number of a graph G, denoted by $b(G)$, is the minimum number of time steps required to burn a graph. In this thesis, we give a survey on some known results of burning number of certain graphs and present the bounds on the burning number of the

generalized Petersen graphs.

Keywords: cop number, lazy cop number, minimum order, burning number, generalized Petersen graphs.

LAZY COP NUMBER DAN PARAMETER-PARAMETER GRAF LAIN YANG BERKAITAN

ABSTRAK

Dalam tesis ini, kami menumpukan pada parameter graf dalam permainan polis dan perompak serta nombor pembakaran graf. Permainan polis dan perompak ialah sesuatu permainan yang melibatkan dua pemain yang dimainkan pada satu graf berikat yang terhingga dan tidak berarah G. Pemain pertama meletakkan satu set polis pada beberapa mercu manakala pemain kedua meduduki satu perompak pada satu mercu. Polis-polis bergerak dahulu dan diikuti dengan perompak. Selepas itu, kedua-dua pemain bergerak secara alternatif. Pada giliran polis, setiap polis boleh pegun atau bergerak ke mercu bersebelahan. Pada giliran perompak, dia boleh pegun atau bergerak ke mercu bersebelahan. Satu pusingan pada permainan ini ialah satu giliran polis diikuti dengan satu giliran perompak. Polis menang sekiranya selepas beberapa pusingan, salah satu polis boleh bergerak untuk menangkap perompak tersebut, maksudnya, polis tersebut menduduki mercu yang sama dengan perompak. Perompak menang sekiranya dia boleh mengelakkan polis selama-lamanya. *Cop number* ialah parameter graf utama dalam permainan polis dan perompak. Dalam tesis ini, kami mengkaji *cop number* dan *lazy cop number* satu graf G, bilangan mercu minimum pada graf yang *cop number*-nya kecil dan masa penangkapan. Keputusan kami menumpukan pada variasi permainan ini, polis malas dan perompak, di mana maksimum satu polis bergerak pada setiap pusingan. Membakar satu graf ialah satu proses yang didefinisikan pada set mercu satu graf mudah dan terhingga. Pada mulanya, pada langkah masa $t = 0$, semua mercu tidak terbakar. Pada permulaan sesuatu langkah masa $t \geq 1$, satu mercu tidak terbakar dipilih untuk dibakarkan (sekiranya mercu tidak terbakar wujud). Selepas itu, sekiranya sesuatu mercu dibakarkan pada langkah masa

$t - 1$, pada langkah masa t, setiap jiran mercu tersebut yang tidak terbakar dibakarkan. Sesuatu mercu yang terbakar akan kekal terbakar sepanjang proses ini. Proses ini tamat apabila semua mercu telah dibakarkan. Nombor pembakaran satu graf G, dilambangkan $b(G)$, ialah nombor minimum langkah pembakaran yang diperlukan untuk membakarkan graf tersebut. Dalam tesis ini, kami memberikan satu kajian mengenai keputusan yang sudah diketahui pada sesetengah graf dan menunjukkan sempadan nombor pembakaran graf Petersen umum.

Kata kunci: *cop number*, *lazy cop number*, bilangan mercu minimum, nombor pembakaran, graf Petersen umum.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS AND ABBREVIATIONS

$b(G)$	:	burning number of G.
$c(G)$	:	cop number of G.
$c_L(G)$	:	lazy cop number of G.
$\mathrm{capt}(G)$	:	capture time of the game of cops and robbers in G.
$deg_G(u)$	:	degree of a vertex u in G.
$\mathrm{diam}(G)$	:	diameter of G.
$\mathrm{dist}_G(u,v)$ / $d_G(u,v)$	:	distance of u and v in G.
$E(G)$	:	edge set of G.
$\overline{G}$	:	complement of G.
$G(V,E)$	:	a graph with vertex set $V(G)$ and edge set $E(G)$.
$G\square H$	:	Cartesian product of G and H.
$K_{m,n}$	:	complete bipartitite graph.
K_n	:	complete graph with n vertices.
M_k^l	:	minimum order of a connected k-lazy cop-win graph.
M_k	:	minimum order of a connected k-cop-win graph.
m_k^l	:	minimum order of G with $c_L(G) \geq k$.
m_k	:	minimum order of G with $c(G) \geq k$.
$N_s^G[u]$	:	s-th closed neighbourhood of u in G.
$N(u)$	:	neighbour of u.
$N[u]$	:	closed neighbour set of a vertex u.
$P(5,2)$	:	Petersen graph.
$P(n,k)$	:	generalized Petersen graph.
Q_n	:	hypercube of dimension n.
$Q(n,m)$	:	generalized hypercube.
$\mathrm{rad}(G)$	:	radius of G.
$V(G)$	:	vertex set of G.
$\delta(G)$	:	minimum degree of G.
$\Delta(G)$	:	maximum degree of G.
$\gamma(G)$	:	domination number of G.

CHAPTER 1: INTRODUCTION

What do the perennial childhood game such as *hide-and-seek* as well as the computer game *Pac-Man* and the military pursuit of a target have in common? Since time immemorial, regardless of locality, man has indulged in thrills that give him that giddy adrenaline rush, and the aforementioned activities are in actual fact, of the same genre as "pursuits and escapes". As we grow older, the fascination for different sorts of games become more sophisticated and complex.

The mathematical puzzles of pursuit and escapes are of great interest today among mathematicians and computer scientists. This is because graph searching is a fast developing area of study within graph theory. The popular game of "Cops and Robbers" plays on graphs and has a number of motivating applications, besides providing interesting mathematical questions. There are many search problems on a network that can be formulated as some variant of the game of cops and robbers, such as searching for a lost person in a network of caves or a virus in a computer network, mostly modifying the network used. The cop number of a graph can be thought of as a measure of the ease of searching the graph. Besides, networks that require a smaller number of cops may be viewed as more secure than those where many cops are needed.

Recently, Kramer et al. (2014) studied the spread of emotional contagion in Facebook. They highlighted the fact that the underlying network is an essential factor such that emotional states are contagious via emotional contagion. Netizens are experiencing the same emotions without their awareness, moreover, without direct interaction between people and in complete absence of nonverbal cues. Hence, agents in the network spread the contagion to their friends or followers, and the contagion propagates over time. So, if the goal was to minimize the time for the contagion to reach the entire network, then which agents (and in which order) would you target with the contagion?

Graph burning was recently introduced as a simple model of spreading social influence, see (Bessy et al., 2017; Bonato et al., 2014; Bonato et al., 2016; Roshanbin, 2016). This process is inspired by contact processes on graphs such as graph bootstrap percolation, and graph searching paradigms such as Firefighter (Barghi & Winkler, 2015; Finbow & MacGillivray, 2009). The main parameter in graph burning is the burning number. The burning number measures the speed of the spread of contagion in a graph; the lower the burning number, the faster the contagion spreads. We can use graphs to model the structure of social networks in real life.

In this thesis, we focus on graph parameters in the game of cops and robbers and the burning of a graph. In the upcoming sections in this chapter, we shall introduce the rules of the game of cops and robbers and burning of graphs followed by some basic definitions and notations of graphs.

1.1 How to play the games?

Here, we give some rules of the game of cops and robbers and how to burn a graph.

1.1.1 The game of cops and robbers

The game of cops and robbers is a two-player game played on a finite connected undirected graph. It was independently introduced by Quilliot (1978) and Nowakowski and Winkler (1983). The first player occupies some vertices with some number of cops (multiple cops may occupy a single vertex) and the second player occupies a vertex with a single robber. After that they move alternatively along the edges of the graph. On the cops' turn, each of the cops may remain stationary or move to an adjacent vertex. On the robber's turn, he may remain stationary or move to an adjacent vertex. A *round* of the game is a cop move together with the subsequent robber move. The cops win if after a finite number of rounds, one of them can move to *catch* (or *capture*) the robber, that is, the cop and the robber occupy the same vertex. The robber wins if he can evade being caught indefinitely. A *cop winning strategy* refers to a set of instructions for the cops, if followed, guarantees that the cops can win any game played on G, regardless of how the robber moves throughout the game. Similarly, a *robber winning strategy* is a set of instructions for the robber, if followed, to evade capture indefinitely. The game of cops and robbers is often called a *vertex-pursuit game* on graphs. In this game, both the cops and robber play *optimal strategies* (that is, the cop is trying to make the game as short as possible while the robber is avoiding capture as long as possible).

The game of cops and robbers is a game of perfect information. That is, each player is aware of all the movements of the other player. A detailed survey on some graph parameters of the game of cops and robbers is presented in Chapter 2.

1.1.2 Burning a graph

As mentioned, graph burning is a discrete-time process that can be used to model the spread of social contagion in social networks. It was introduced in (Bonato et al., 2014; Bonato, Janssen, & Roshanbin, 2016; Roshanbin, 2016).

This process is defined on the vertex set of a simple finite graph. Throughout the process, each vertex is either *burned* or *unburned*. Initially at time step $t = 0$, all vertices are unburned. At the beginning of every time step $t \geq 1$, an unburned vertex is chosen to burn (if such a vertex is available). After that, if a vertex was burned in time step $t - 1$, then in time step t, each of its unburned neighbours becomes burned. A burned vertex will remain burned throughout the process. The process ends when all vertices are burned, in which case we say the graph is *burned*.

The main study in graph burning is the *burning number* of a graph G, denoted as $b(G)$, which is the minimum number of time steps needed to burn the graph G. We shall further discuss some known results on burning number of certain graphs in Chapter 2.

1.2 Definitions and Notation

In this section, the basic definitions and notation which will be frequently referred throughout this thesis are presented. For standard terms and definitions not included here, the reader is referred to (Chartrand & Lesniak, 1996; West, 2001; Wilson, 1996).

As a number of asymptotic results will be presented, we give some corresponding notation. Let f and g be functions whose domain is some subset of $\mathbb{R}$. We write $f \in O(g)$ if the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and is finite. And by writing $f = O(g)$, we mean there is a constant $c > 0$ (not depending on x) such that for all $x > N$, $f(x) \leq cg(x)$.

We write $f = \Omega(g)$ if $g = O(f)$ and $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then $f = o(g)$ (or $g = \omega(f)$). So if $f = o(1)$, then f tends to 0.

If x is a real number, then $1 + x \leq e^x$. We shall sometimes write e^x as $\exp(x)$, especially when x is a complicated expression.

A *graph* G is an ordered pair (V, E), where the vertex set $V = V(G)$ is a finite non-empty set, and the edge set $E = E(G)$ is a family of unordered pairs of elements from V. The elements in $V(G)$ (respectively $E(G)$) are called *vertices* (respectively *edges*). Two vertices $u, v \in V(G)$ are *adjacent* if $\{u, v\} \in E(G)$, and we say there is an edge uv joining them. Two vertices u and v are said to be *incident* with the edge uv, that is, u and v are *endpoints* of uv. Similarly, two distinct edges e and f are *adjacent* if they are incident to a common vertex, otherwise they are *non-adjacent*. A set of pairwise non-adjacent vertices is called an *independent set of vertices*.

The cardinality $|V(G)|$ is the *order* of G, while $|E(G)|$ is its *size*. Let u be a vertex. Its *neighbour set of* u, defined as $N(u)$, is the set of vertices adjacent and not equal to u (also called *neighbours* of u). The *closed neighbour set of* u, written as $N[u]$, is the set $N(u) \cup \{u\}$.

A *subgraph* $G' = (V', E')$ of a graph $G = (V, E)$ is a graph where $V' \subseteq V$ and $E' \subseteq E$. Let $S \subseteq V(G)$. An *induced subgraph* of G is the graph induced by the set of vertices S; that is, the graph with vertices in the set S, with two vertices are adjacent if and only if they are adjacent in G. A subgraph S is a *spanning subgraph* of G if $V(S) = V(G)$. We write $G - S$ to be the subgraph induced by $V(G) \setminus S$. Particularly, if $S = \{x\}$, then we write $G - x$. If H is an induced subgraph of G, then we may write $G - H$ for $G - V(H)$.

The *degree* of a vertex $u \in V(G)$, written as $deg_G(u)$, is the cardinal $|N(u)|$. The minimum degree of G, denoted as $\delta(G)$, is the degree of the vertex with the least number of neighbours adjacent to it. Similarly, the maximum degree of G, denoted as $\Delta(G)$, is the degree of the vertex with the greatest number of neighbours adjacent to it. A graph is *k-regular* if each vertex of the graph has degree k.

A *path* is a sequence of vertices such that each vertex is adjacent to the next vertex in the sequence; the *length of a path* is the number of its edges. A path of order n is denoted by P_n. $P[x, y]$ is a path with endpoints x and y and $P(x, y) = P[x, y] - \{x, y\}$. A *cycle* C_n is a sequence of n vertices such that each vertex is adjacent to the next vertex in the sequence of modulo n. Note that in a cycle the number of vertices and edges are equal. The *length of a cycle* is the number of edges in the cycle. The *girth* of a graph is the length of a shortest cycle contained in the graph. A connected graph without any cycles is called a *tree*. A tree is typically denoted by T. It is straightforward that a tree with n vertices has $n - 1$ edges. In fact, every minimal connected graph is a tree. It is also known that a graph T is a tree if and only if every pair of the vertices in T are connected by a unique path.

A graph G is said to be *connected* if for any two vertices u and v in G, there exists a path from u to v; if there is no such path, then G is said to be *disconnected*. Further, if a graph is disconnected, then it is the disjoint union of several connected graphs called the *connected components* of the graph. Therefore, a graph is connected if and only if it has only one component; it is disconnected if and only if it has more than one component. A connected component consisting of a single vertex is called an *isolated vertex*. A graph G is said to be *k-connected* (or *k-vertex-connected*) if it has more than k vertices and the result of deleting any set of fewer than k vertices is a connected graph. A *cut vertex* is one whose deletion results in a disconnected graph.

A *homomorphism* f from G to H is a function $f : V(G) \to V(H)$ that preserves edges; that is, if $xy \in E(G)$, then $f(x)f(y) \in E(H)$. We shall simply write $f : G \to H$ to refer a homomorphism from G to H. An *isomorphism* from G to H is a bijection $f : G \to H$ such that $f(x)f(y) \in E(H)$ if and only if $xy \in E(G)$. If there is an isomorphism from G to H, we say that G and H are *isomorphic*, written as $G \cong H$. An *automorphism* of G is an isomorphism from G to itself. A graph G is *vertex-transitive* if for all pairs of vertices u and v of G, there is an automorphism f of G, so that $f(u) = v$.

The *distance* between u and v where $\{u, v\} \in V(G)$, denoted by $\text{dist}_G(u, v)$ (or $d_G(u, v)$), is the length of a shortest path connecting u and v (and 0 if $u = v$). We write $\text{dist}(u, v)$ (or $d(u, v)$) if the graph in question is clear. The *eccentricity* of a vertex u in graph G is defined as $\max\{\text{dist}(v, u) : v \in V(G)\}$. The *radius* of G, denoted as $\text{rad}(G)$, is the minimum eccentricity over the set of all vertices in G. The *diameter* of G, denoted by $\text{diam}(G)$, is the maximum eccentricity over the set of all vertices in G, equivalently $\text{diam}(G)=\max\{\text{dist}(v, u) : u, v \in V(G)\}$.

The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ and edge set $\{uv : uv \notin E(G), u, v \in V(G)\}$. A *complete graph* is a graph whose vertices are pairwise adjacent. It is denoted as K_n if it has n vertices. The complement of a complete graph with n vertices is denoted as $\overline{K_n}$ where $E(\overline{K_n}) = \emptyset$. A *bipartite graph* is a graph whose vertex set can be partitioned into two independent sets called *partite sets* V_1 and V_2, such that each edge joins a vertex of V_1 to a vertex of V_2. A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the two partite sets have m and n vertices respectively, then the complete bipartite graph is denoted as $K_{m,n}$.

In a graph G, a set $S \subseteq G$ of vertices is a *dominating set* of G if every vertex in $G \setminus S$ has at least one neighbour in S. The *domination number* of G, written as $\gamma(G)$, is the minimum cardinality of a dominating set. Since placing a cop on each element of a dominating set of a graph G ensures a win for the cops in at most two rounds, we have the obvious bound of $c(G) \leq \gamma(G)$.

Given two graphs G and H, their *Cartesian product* $G \square H$ is a graph with vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \square H$ if and only if either

(i) $u_1 = u_2$ and v_1 is adjacent to v_2 in H, or

(ii) $v_1 = v_2$ and u_1 is adjacent to u_2 in G.

A *hypercube* of dimension n, written as Q_n, is the graph with vertex set $\{0, 1\}^n$ where two vertices are adjacent if and only if they differ in exactly one coordinate. See Figures 1.1 and 1.2.

Let n and k be two integers such that $1 \leq k \leq n - 1$. The *generalized Petersen graph* $P(n, k)$ is the graph with vertex set $\{u_i, v_i : i = 0, 1, \ldots, n - 1\}$ and edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, 1, \ldots, n - 1$ with subscripts reduced modulo $n\}$. The classical Petersen graph $P(5, 2)$ is depicted in Figure 1.3.

For ease of reading, we shall define some other specific graphs and terminologies in the relevant chapters.

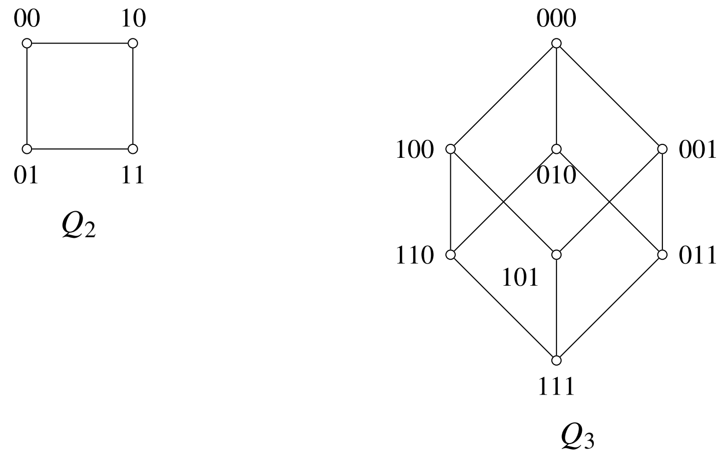


Figure 1.1: Q_2 **and** Q_3

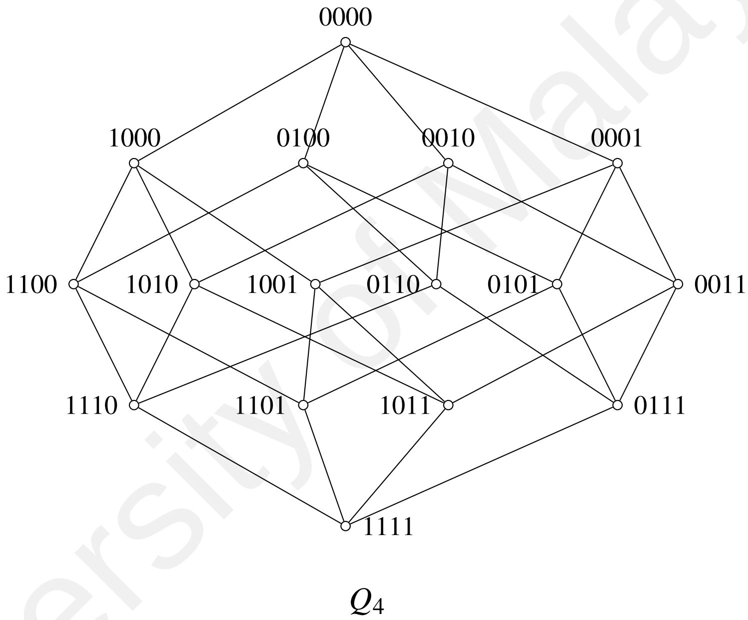


Figure 1.2: Q_4

1.3 Chapter overview

Many of the results in this thesis appeared in (**Sim, K. A.** et al., 2016, 2017, 2018).

In Chapter 2, we define and give a survey on some graph parameters related to the game of cops and robbers. We will also present some variants to the standard game of cops and robbers and consider some fundamental facts about cop number and the well-known Meyniel's conjecture. We also provide some known results of cop number for basic graphs. Then, we focus on the game of lazy cops and robbers. Moreover, we present the

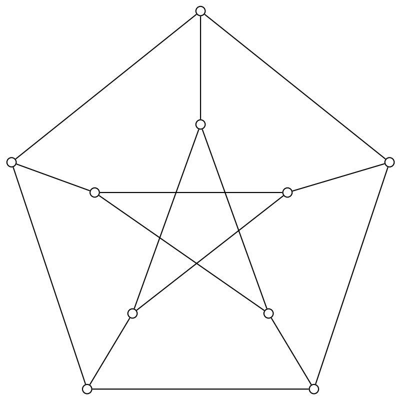


Figure 1.3: The Petersen graph $P(5,2)$

corresponding cop number for the game of lazy cops and robbers. We then investigate the minimum order of k-cop-win graphs, and provide the results for $k = 1, 2, 3$. We finish Chapter 2 by defining burning number of graphs, and showing the burning number of some specific graphs such as paths, cycles, and complete bipartite graphs. The asymptotic results on the burning number of Cartesian grids, toroidal grids, and hypercubes are also presented.

Chapter 3 represents the work done and published in (**Sim, K. A.** et al., 2017). We present asymptotic bounds on the lazy cop number for generalized hypercubes $Q(n,m)$. We also find the exact lazy cop number for the case when $n = 2$.

In Chapter 4, we find the minimum order of graphs which has lazy cop number 4. We also determine that the Petersen graph $P(5,2)$ is the unique connected graph on 10 vertices with maximum degree ≤ 3 which has lazy cop number less than 4.

Chapter 5 represents the work done and published in (**Sim, K. A.** et al., 2018). We study the burning number of the generalized Petersen graph $P(n,k)$. We show that for any fixed positive integer k, $\lim_{n\to\infty} \frac{b(P(n,k))}{\sqrt{\frac{n}{k}}} = 1$. Furthermore, we give tight bounds for $b(P(n,1))$, $b(P(n,2))$ and $b(P(n,3))$.

In Chapter 6, we present some miscellaneous results related to the graph parameters. Some of the results are published in (**Sim, K. A.** et al., 2016). These results may stand on its own, which may be useful in future. We also give some open problems and discuss some future work, concluding from the current papers.

CHAPTER 2: LITERATURE REVIEW

2.1 Graph parameters related to the game of cops and robbers

In this section, we discuss some graph parameters related to the game of cops and robbers: cop number, minimum order of graphs of small value of cop number, and capture time. We give some known results on these parameters.

2.1.1 Cop number

The main focus of study in the game of cops and robbers is the *cop number*.

Definition 2.1.1. *(Aigner & Fromme, 1984) For a graph G, cop number, denoted as $c(G)$, is the minimum number of cops needed for the cops to capture the robber in G.*

The followings are some fundamental results for cop number. Computing the cop number is **NP**-hard, see (Fomin et al., 2008, 2010). Aigner and Fromme (1984) showed early results as Theorem 2.1.2.

Theorem 2.1.2. *(Aigner & Fromme, 1984) Let G be a graph with minimum degree $\delta(G) \geq k$ which contains no 3- or 4-cycles. Then $c(G) \geq k$.*

Hence, we have Theorem 2.1.3.

Theorem 2.1.3. *(Aigner & Fromme, 1984) If G has girth at least 5, then $c(G) \geq \delta(G)$.*

By referring to the Petersen graph $P(5,2)$ in Figure 1.3, it is straightforward that $c(P(5,2)) \leq 3$, by placing cops at the bottom two vertices of the middle 5-cycle and a vertex at the top of the outer 5-cycle. Theorem 2.1.3 implies that $c(P(5,2)) \geq 3$. Hence $c(P(5,2)) = 3$.

In the generalized Petersen graphs, Ball et al. (2017) proved that the cop number of every generalized Petersen graph $P(n,k)$ is at most 4.

A *planar graph* is a graph that can be drawn in the plane without edge crossings. A beautiful early result (Theorem 2.1.4) for planar graph was presented.

Theorem 2.1.4. *(Aigner & Fromme, 1984) If G is a planar graph, then $c(G) \leq 3$.*

The idea of the proof of Theorem 2.1.4 is to increase the cop territory; that is, a set S of vertices such that if the robber moved to S, then they would be caught. If the territory can always be increased, the number of vertices the robber can move to without being caught is eventually reduced to the empty set, and so the robber is captured. For better understanding on the proof of Theorem 2.1.4, see (Aigner & Fromme, 1984) and (Bonato & Nowakowski, 2011, p. 100-104).

A path P in a graph G is *isometric* if for all vertices x, y in P, their distance in P is the same as their distance in G; that is, $d_P(x, y) = d_G(x, y)$.

For a fixed integer $k \geq 1$, an induced subgraph H of G is *k-guardable* if, after finitely many moves, k cops can move only in the vertices of H in such a way that if the robber moves into H at round t, then he will be captured at round $t + 1$. We say that the k cops *guards H*. For example, a complete graph K_n is 1-guardable. Theorem 2.1.4 gives rise to *Isometric Path Lemma*.

Theorem 2.1.5. *[Isometric Path Lemma] If P is an isometric path in G, then P is 1-guardable.*

In an isometric path, one cop can patrol effectively and ensure no robber can ever escape it without being captured. See (Aigner & Fromme, 1984) and (Bonato & Nowakowski, 2011, p. 17,18).

Graph products give us interesting ways of forming new graphs from old ones. Cop number of products of graphs was first considered by Tošić (1988) for Cartesian products.

Theorem 2.1.6. *(Tošić, 1988) For graphs G and H,*

$$c(G\Box H) \leq c(G) + c(H).$$

More generally, for graphs $G_1, G_2, \ldots, G_k$, we have that

$$c\left(\Box_{i=1}^{k} G_i\right) \leq \sum_{i=1}^{k} c(G_i).$$

It was established in (Maamoun & Meyniel, 1987) that the cop number of a Cartesian product of k trees is $\lceil \frac{k+1}{2} \rceil$.

Theorem 2.1.7. *(Maamoun & Meyniel, 1987) If $T_1, T_2, \ldots, T_k$ are trees, then*

$$c\left(\Box_{i=1}^{k} T_i\right) = \left\lceil \frac{k+1}{2} \right\rceil.$$

In particular, we have $c(Q_n) = \lceil \frac{n+1}{2} \rceil$ since Q_n may be viewed as the n-fold Cartesian product of K_2. Neufeld and Nowakowski (1998) then determined the cop numbers of the Cartesian products of cycles and trees in the following theorems.

Theorem 2.1.8. *(Neufeld & Nowakowski, 1998) Let $C_1, C_2, \ldots, C_k$ be cycles, each with length of at least 4. Then*

$$c\left(\Box_{i=1}^{k} C_i\right) = k + 1.$$

Theorem 2.1.9. *(Neufeld & Nowakowski, 1998) Let $C_1, C_2, \ldots, C_k$ be cycles each of length at least 4 and let $G = \Box_{i=1}^{k} C_i$. Let $T_1, T_2, \ldots, T_j$ be trees and let $H = \Box_{i=1}^{j} T_i$ Then*

$$c\left(G\Box H\right) = k + \left\lceil \frac{j+1}{2} \right\rceil.$$

The most famous unsolved question on the cop number is the *Meyniel's conjecture*, mentioned by Frankl (1987). Meyniel's conjecture is a very challenging problem in the game of cops and robbers. Meyniel's conjecture states that if G is a connected graph of order n, then

$$c(G) = O(\sqrt{n}).$$

For n a positive integer, let G be a graph of order n. In the earlier time, Frankl (1987) proved that

$$c(G) \le (1 + o(1))\, n \frac{\log\log n}{\log n}.$$

However, this is far from the conjecture. After more than 20 years, Chiniforooshan (2008) showed that

$$c(G) = O\left(\frac{n}{\log n}\right).$$

There has been recent progress by Scott and Sudakov (2011) and Lu and Peng (2012). However, the conjecture is still wide open. Independently, they proved that the following theorem.

Theorem 2.1.10. *(Lu & Peng, 2012; Scott & Sudakov, 2011) For a graph G with n vertices,*

$$c(G) \le O\left(\frac{n}{2^{(1-o(1))\sqrt{log_2(n)}}}\right).$$

Up to date, Scott and Sudakov (2011) and Lu and Peng (2012) contributed current best effort in approaching Meyniel's conjecture but it is still far from proving Meyniel's conjecture. Solving the *soft Meyniel's conjecture*, which states that for a fixed constant $w > 0$,

$$c(G) = O(n^{1-w}),$$

would be a significant breakthrough. However, the conjecture still remains open.

We note that the proofs of Theorem 2.1.10 in (Lu & Peng, 2012; Scott & Sudakov, 2011) used the greedy approach. In addition, all of the proofs used probabilistic method, which represents a new and interesting approach to proving the conjecture.

Besides, Lu and Peng (2012) also proved Theorem 2.1.11.

Theorem 2.1.11. *(Lu & Peng, 2012) If G is a graph on n vertices with diameter two, then*

$$c(G) \leq 2\sqrt{n} - 1.$$

2.1.2 Minimum order of graphs

For a fixed positive integer k, we say a graph G is *k-cop-win* if $c(G) = k$. In the special case $k = 1$, G is said to be a *cop-win* graph. For example, a path or a tree is a cop-win (Aigner & Fromme, 1984) graph and the Petersen graph is 3-cop-win (Baird et al., 2014). In a tree T, the vertex occupied by a cop C partitions the tree into 2 components and each time C moves along the unique path toward the robber R, the component R occupied is reduced by at least one vertex.

We define M_k to be the minimum order of a connected k-cop-win graph and m_k to be the minimum order of a connected graph G satisfying $c(G) \geq k$. Note that m_k are monotonically increasing, and $m_k \leq M_k$. To date, the exact values of these parameters are only known for first three values of k. Baird et al. (2014) showed that $m_1 = M_1 = 1, m_2 = M_2 = 4$ and $m_3 = M_3 = 10$. Moreover, they proved that the Petersen graph is the unique 3-cop-win graph with order 10, see Theorem 2.1.14.

Theorem 2.1.12 presented the relationship between m_k and Meyniel's conjecture on the asymptotic maximum value of the cop number of a connected graph.

Theorem 2.1.12. *(Baird et al., 2014)*

1. *For any positive integer* k*,*$m_k \in O(k^2)$*.*
2. *Meyniel's conjecture is equivalent to the property that* $m_k \in \Omega(k^2)$*, for all* $k \in \mathbb{N}$*.*

Hence, if Meyniel's conjecture holds, then Theorem 2.1.12 implies that

$$m_k = \Theta(k^2).$$

Theorem 2.1.13. *(Baird et al., 2014) If* G *is a graph on at most 9 vertices, then* $c(G) \leq 2$*.*

They also proved that $m_3 = 10$ and that this value is attained uniquely by the Petersen graph, as Theorem 2.1.14.

Theorem 2.1.14. *(Baird et al., 2014) The Petersen graph is the unique isomorphism type of graphs on 10 vertices that are 3-cop-win.*

A vertex u is a *corner* if there is some vertex v such that $N[u] \subseteq N[v]$. A graph is *dismantlable* if some sequence of deleting corners results in the graph K_1. This is equivalent in saying that a graph G is dismantlable if we can label the vertices with positive integers $1, 2, \ldots, n$ in such a way that for each $i < n$, the vertex i is a corner in the subgraph induced by $\{i, i+1, \ldots, n\}$. We call this ordering of $V(G)$ a *cop-win ordering*. A graph is cop-win if and only if it is dismantlable (Nowakowski & Winkler, 1983).

Besides proving Theorems 2.1.13 and 2.1.14 mathematically, Baird et al. (2014) also used a computer search to calculate the cop number of every connected graph on 10 or fewer vertices. They performed this categorization by checking for cop-win orderings in (Nowakowski & Winkler, 1983) and using an algorithm provided in (Bonato et al., 2010), see Table 2.1.

In Table 2.1, for a positive integer n, $g(n)$ is the number of non-isomorphic (not necessarily connected) graphs of order n, and $g_c(n)$ is the number of non-isomorphic connected graphs of order n. Then, $f_k(n)$ is the number of non-isomorphic connected k-cop-win graphs of order n. Clearly, $f_k(n) \leq g_c(n) \leq g(n)$.

Table 2.1: Cop number of small order graphs

order n	$g(n)$	$g_c(n)$	$f_1(n)$	$f_2(n)$	$f_3(n)$
1	1	1	1	0	0
2	2	1	1	0	0
3	4	2	2	0	0
4	11	6	5	1	0
5	34	21	16	5	0
6	156	112	68	44	0
7	1044	853	403	450	0
8	12346	11117	3791	7326	0
9	274668	261080	65561	195519	0
10	12005168	11716571	2258313	9458257	1

Among these graphs there is only one graph G of order 10 that requires 3 cops to win. Moreover, since it is 3-cop win, then G has to be the Petersen graph.

Recently, Hosseini (2018) showed Theorem 2.1.15.

Theorem 2.1.15. *(Hosseini, 2018) The values M_k are strictly increasing.*

Theorem 2.1.15 implies that $M_{k-1} < M_k$ for every $k \geq 2$. In other words, the minimum order of a graph that requires k cops to capture the robber is increasing in k. Following this, Hosseini (2018) also proved Corollary 2.1.16.

Corollary 2.1.16. *(Hosseini, 2018) If G is a graph on M_k vertices with $c(G) = k$, then G is 2-connected. Moreover, $c(G\backslash v) = c(G) - 1$ for every vertex $v \in V(G)$.*

2.1.3 Capture time

A cop winning strategy may not be the fastest strategy for the cop to capture a robber in general. For example, consider a path P_n with n vertices labelled from left to right by $1, 2, \ldots, n$. Consider a cop winning strategy such that a cop moves from left to right. Using this strategy, the cop requires $n - 1$ moves to catch the robber (whose best move is to be placed initially at n and remains stationary throughout the game). However, if the cop initially occupy a vertex at the center (or almost center) of the path, the cop can win in at most $\lfloor \frac{n}{2} \rfloor$ rounds regardless of the initial position of robber.

Similarly, in a cycle C_n with vertices $1, 2, \ldots, n$, if initially we place 2 cops at vertices 1 and $\lceil \frac{n+1}{2} \rceil$ respectively, then the cops can win in at most $\lfloor \frac{m-1}{2} \rfloor$ rounds where $m = \lceil \frac{n+1}{2} \rceil$. Each of the two cops will just have to move closer to the robber in each round. The number of rounds in this cop winning strategy is clearly less than that if two cops occupy a same vertex initially, which is another cop winning strategy on C_n.

If k cops play on a graph G with $k \geq c(G)$, assuming optimal play by the robber, the *k-capture time*, denoted as $\text{capt}_k(G)$, is defined to be the minimum number of rounds (not including the initial round) until the capture is achieved by cops among all possible cop winning strategies over G. In the case $k = c(G)$, we simply write $\text{capt}(G)$ and refer to this as the *capture time* of G.

Bonato et al. (2009) studied the capture time in cop-win graphs and admits a cop strategy by induction that capture the robber in $O(n)$ rounds.

Theorem 2.1.17. *(Bonato et al., 2009) If G is a cop-win graph of order $n \geq 5$, then* $\text{capt}(G) \leq n - 3$.

By considering small order cop-win graphs of order n, the bound was improved to $\text{capt}(G) \leq n - 4$ for $n \geq 7$ in (Gavenčiak, 2010). It was first noted in (Berarducci & Intrigila, 1993) that for any constant $k \geq 2$, if G is k-cop-win, then its capture time is $O(n^{k+1})$. Bounds on the capture time with 3 cops playing on a planar graphs were proved by Bonato et al. (2017).

Theorem 2.1.18. *(Bonato et al., 2017) If G is a planar graph of order n, then*

$$\mathrm{capt}_3(G) \leq (\mathrm{diam}(G) + 1)n = O(n^2).$$

Futhermore, the $O(n^2)$ bound can be improved to the following linear bound.

Theorem 2.1.19. *(Pisantechakool & Tan, 2016) If G is a planar graph of order n, then* $\mathrm{capt}_3(G) \leq 2n$.

If there are many cops playing on a planar graph, then Theorem 2.1.20 follows.

Theorem 2.1.20. *(Bonato et al., 2017) If G is a planar graph of order n and* $k \geq 12\sqrt{n}$, *then*

$$\mathrm{capt}_k(G) \leq 6\,\mathrm{rad}(G) \log n.$$

It was shown in (Mehrabian, 2011) that if G is the Cartesian product of two trees, then $\mathrm{capt}(G) = \left\lfloor \frac{diam(G)}{2} \right\rfloor$. Hence, the 2-capture time of a $m \times n$ grid (Cartesian product of P_m and P_n) is $\lfloor \frac{m+n}{2} \rfloor - 1$.

The capture time of the hypercubes Q_n were investigated in (Bonato et al., 2013).

Theorem 2.1.21. *(Bonato et al., 2013) Let* $n \geq 1$ *be an integer, we have that*

$$\mathrm{capt}(Q_n) = \Theta(n \ln n).$$

Bonato et al. (2013) derived the asymptotic order of the capture time of the hypercube. They established an upper bound on the capture time of the hypercube by using a cop winning strategy which is similar to the one described in (Maamoun & Meyniel, 1987). By assuming the robber move randomly throughout the game, they used probabilistic method to show that the robber has a strategy to survive long enough to achieve the lower bound.

2.2 Variants of the game of cops and robbers

In the game of cops and robbers, the usual setting (as described in Section 1.1) is a discrete-time two-person game consisting of a set of cops whose goal is the capture of the robber and a robber who is trying to evade capture. Variations allow for players to possess only imperfect information, utilize only certain types of movements, allowing the players to move at various speeds, or meet specified conditions to win the game. Many variants of the game of cops and robbers have been studied. See (Bonato & Nowakowski, 2011) for an extensive surveys.

Some examples include the settings in which cops are chasing an invisible or a drunk robber (a robber who performs a random walk) (Kehagias & Prałat, 2012; Kehagias et al., 2013, 2014) and the game where the cops and robber move at different speeds (Fomin et al., 2010; Alon & Mehrabian, 2015; Chalopin et al., 2011) or on the directed graphs (Frieze et al., 2012) . Recently, a new variant called Zombies and Survivors has been introduced in (Bonato, Mitsche, et al., 2016; Fitzpatrick et al., 2016) where the zombies (analogous to the cops) must move closer to a survivor (analogous to the robber) in each round and the survivor evades capture. The corresponding cop numbers and capture times have been studied in these papers.

In this thesis, we are interested in a variant introduced by Offner and Ojakian (2014), where at most one cop moves in any round. It is called the game of *Lazy Cops and Robbers* and the *lazy cop number* is the minimum number of cops required to catch the robber in this setting. We write $c_L(G)$ for the lazy cop number of a graph G.

We give some known results of the lazy cop number. It is clear that $c_L(P_n) = c(P_n) = 1$ and $c_L(C_n) = c(C_n) = 2$. Offner and Ojakian (2014) were interested in Lazy Cops and

Robber played on the hypercube Q_n and they proved the following asymptotic bounds:

$$2^{\lfloor \sqrt{n}/20 \rfloor} \leq c_L(Q_n) = O\left(\frac{2^n \ln n}{n^{3/2}}\right).$$

The lower bound was later improved by Bal et al. (2015), by using probabilistic method coupled with a potential function argument. They showed that for every $\varepsilon > 0$,

$$c_L(Q_n) = \Omega\left(\frac{2^n}{n^{5/2+\varepsilon}}\right).$$

They also studied the game of lazy cops and robbers on random graphs and graphs on surfaces (Bal et al., 2016). Sullivan et al. (2016b) showed that $c_L(P_n \square C_m) = 2$, $c_L(K_n \square T) = 2$, $c_L(K_n \square C_m) = 3$ for $n, m \geq 3$ and $c_L(C_n \square C_n) \leq 2\left\lceil \frac{n}{3} \right\rceil$ for $n \geq 4$.

Recent work of Gao and Yang (2017) gave a non-trivial example of a planar graph G such that $c_L(G) \geq 4$ (in contrast to the upper bound of 3 given in Theorem 2.1.4).

A graph G satisfying $c_L(G) = k$ is *k-lazy cop-win*. As for finding the minimum order G with $c_L(G) = k$, we define M_k^l to be the minimum order of a connected k-lazy cop-win graph and define m_k^l to be the the minimum order of a connected graph G with $c_L(G) \geq k$. It is easy to see that $m_1^l = M_1^l = 1$. For $k = 2$, we must have $m_2^l = M_2^l = 4$ since the only connected graphs with three vertices are P_3 and C_3, both are 1-lazy cop-win graphs and the fact that $c_L(C_4) = 2$.

Sullivan et al. (2016a) proved that for the game of lazy cops and robbers, $K_3 \square K_3$ is the unique 3-lazy cop-win graph on nine vertices. In addition, all other graphs on 9 or fewer vertices have lazy cop number at most two. Hence $m_3^l = M_3^l = 9$. They also showed that $c_L(K_n \square K_n) = n$.

To the best of our knowledge, to date, there is no result on the capture time of the lazy cops and robbers that has been published.

2.3 Burning number of graphs

Suppose a graph G is burned in m time steps in a burning process. For $1 \leq i \leq m$, we denote the vertex we choose to burn at the beginning of time step i by x_i. The sequence $(x_1, x_2, \ldots, x_m)$ is called a *burning sequence* for G. Each x_i is called a *burning source* of G. The *burning number* of a graph G, denoted by $b(G)$, is the length of a shortest burning sequence for G. Determining $b(G)$ for general graphs is a non-trivial problem. It is known that computing the burning number of a graph is **NP**-complete (Bessy et al., 2017).

It is straightforward to see that $b(K_n) = 2$. For paths and cycles, Bonato, Janssen, and Roshanbin (2016) determined their burning numbers exactly.

Theorem 2.3.1. *(Bonato, Janssen, & Roshanbin, 2016, Theorem 9 and Corollary 10) Let P_n be a path with n vertices and C_n be a cycle with n vertices. Then*

$$b(P_n) = \left\lceil n^{1/2} \right\rceil = b(C_n).$$

They also investigated the sum and product of the burning number of a graph and its complement. In particular, they proved that for a graph G of order $n \geq 2$, the bounds $4 \leq b(G) + b(\overline{G}) \leq n + 2$ hold, and for $n \geq 6$, $b(G)b(\overline{G}) \leq 2n$, with the equality is achieved by complete graphs.

For general graphs, they showed the following.

Lemma 2.3.2. *(Bonato, Janssen, & Roshanbin, 2016) For any graph G with radius r and diameter d,*

$$\left\lceil (d+1)^{1/2} \right\rceil \leq b(G) \leq r + 1.$$

In the same paper, they also gave an upper bound on the burning number of any connected graph G of order n, showing that $b(n) \leq 2\sqrt{n} - 1$. Later, this upper bound was improved to roughly $\frac{\sqrt{6}}{2}\sqrt{n}$ by Land and Lu (2016).

It was conjectured in (Bonato, Janssen, & Roshanbin, 2016) that $b(G) \leq \lceil \sqrt{n} \rceil$ for any connected graph G of order n. Very recently, Bonato and Lidbetter (2017) verified this conjecture for spider graphs, which are trees with exactly one vertex of degree at least 3.

Bessy et al. (2018) also determined that for a tree T of order n with n_2 vertices of degree 2, and $n_{\geq 3}$ vertices of degree at least 3, $b(T) \leq \left\lceil \sqrt{n + n_2 + \frac{1}{4}} + \frac{1}{2} \right\rceil$ and $b(T) \leq \lceil \sqrt{n} \rceil + n_{\geq 3}$.

Several other results on burning number of graphs have also been studied recently. Mitsche et al. (n.d.) investigated the burning number of graph products and also focused on the probabilistic aspects of the burning number. Mitsche et al. (2017, n.d.) and Roshanbin (2016) provided an asymptotic results on graph products. By an $m \times n$ toroidal grid we mean the Cartesian product of C_m and P_n.

Theorem 2.3.3. *(Mitsche et al., 2017; Roshanbin, 2016) Let G be an $m \times n$ grid or toroidal grid with $1 \leq m \leq n$, where $m = m(n)$ is a function of n. Then*

$$b(G) = \begin{cases} \Theta(\sqrt{n}), & \text{if } m = O(\sqrt{n}), \\ \\ (1 + o(1)) \left(\frac{3}{2}\right)^{\frac{1}{3}} (mn)^{\frac{1}{3}}, & \text{if } m = \omega(\sqrt{n}). \end{cases}$$

Mitsche et al. (n.d.) also showed that

$$\lim_{n \to \infty} \frac{b(Q_n)}{\frac{n}{2}} = 1,$$

where Q_n is the n-dimensional hypercube. In fact, they proved a stronger result, which is the following inequalities

$$\frac{n}{2} + 1 - \sqrt{n \log n} < b(Q_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1.$$

The burning number of the hypercube Q_n is asymptotically $\frac{n}{2}$ (Mitsche et al., n.d.), but the exact value of $b(Q_n)$ is still unknown. The concept of burning number is still new and little is known about the burning number of many graph classes. We provide the burning number of generalized Petersen graphs in Chapter 5.

CHAPTER 3: LAZY COPS AND ROBBERS ON GENERALIZED HYPERCUBES

3.1 Introduction

In this chapter, we shall determine the the lazy cop number of the generalized hypercube $Q(n,m)$. First, we define the generalized hypercube $Q(n,m)$.

Let m,n be positive integers, and define $[m] := \{0,1,2,\ldots,m\}$. Then $Q(n,m)$ is the graph with vertex set

$$V(Q(n,m)) = \{(a_1,a_2,\ldots,a_n) \ : \ a_i \in [m]\},$$

and two vertices in $Q(n,m)$ are adjacent if and only if they differ in exactly one coordinate. That is,

$$E(Q(n,m)) = \left\{\{(a_1,a_2,\ldots,a_n),(b_1,b_2,\ldots,b_n)\} \ : \ a_j \neq b_j \ \text{ for some } j \text{ and } a_i = b_i \ \text{ for } i \neq j\right\}.$$

We note that $Q(n,1)$ is the hypercube Q_n.

We remark that many properties of generalized hypercubes have been studied. For example, Duh et al. (1996) computed best containers, wide diameter and fault diameter of generalized hypercubes. Mollard (1991) gave two new characterizations of the Hamming graphs. Nakano (1993) studied linear layouts of generalized hypercubes and presented the exact or nearly exact values of the bisection width, the cut width and the total edge length of them. Some of these properties are important in the study of interconnection network (see, for example, (Bhuyan & Agrawal, 1984)).

We generalize existing methods to prove asymptotic bounds for the lazy cop number of $Q(n,m)$ and show the proofs of Theorem 3.1.1 in Section 3.2.

Theorem 3.1.1. *Let m be a positive integer and let $\varepsilon > 0$. Then for sufficiently large n, we have*

$$\frac{(m+1)^n}{n^{5/2+\varepsilon}} \leq c_L(Q(n,m)) = O\left(\frac{(m+1)^n \ln n}{n^{3/2}}\right).$$

We shall first find the exact lazy cop number for the case when $n = 2$.

Proposition 3.1.2. $c_L(Q(2,m)) = m + 1$.

Proof. Suppose there are m cops occupying vertices (x_i, y_i), $1 \leq i \leq m$. The robber can choose the vertex (u, v) as the starting position where $u \in [m] \setminus \{x_1, \ldots, x_m\}$ and $v \in [m] \setminus \{y_1, \ldots, y_m\}$. Such u and v can be found because $|\{x_1, \ldots, x_m\}| \leq m$ and $|\{y_1, \ldots, y_m\}| \leq m$. Therefore, the distance between the robber and any of the cops is 2. Suppose that on the cops' turn, a cop moves from (x_{i_0}, y_{i_0}) to (x'_{i_0}, y_{i_0}). If $x'_{i_0} \neq u$, then the robber remains at its vertex (u, v). So, after this round, the distance is still 2. If $x'_{i_0} = u$, then the robber moves from (u, v) to (u', v) where $u' \in [m] \setminus \{x_1, \ldots, x_{i_0-1}, u, x_{i_0+1}, \ldots, x_m\}$. Again, the distance is still 2. Hence, the robber can evade the cops indefinitely and $c_L(Q(2,m)) > m$.

Now, it remains to show that $c_L(Q(2,m)) \leq m + 1$. If there are $m + 1$ cops, then we can place a cop on each of the vertices in the set $S = \{(i,i) : 0 \leq i \leq m\}$ as the cops' starting positions. Note that S is a dominating set of $Q(2,m)$. So, no matter which vertex the robber chooses as the starting position, a cop will definitely catch the robber immediately. Hence, $c_L(Q(2,m)) \leq m + 1$. □

3.2 Main results on $c_L(Q(n,m))$

3.2.1 Upper bound

It is clear that by occupying the dominating set of a graph, the cops win. So the lazy cop number (and also the cop number) of a graph is bounded above by the size of the smallest dominating set of the graph. When Offner and Ojakian (2014) proved the upper bound of $c_L(Q_n)$, their strategy for the cops is to occupy the dominating set of the middle levels of Q_n and then move up or down according to the position of the robber. We shall use this idea in proving the upper bound of $c_L(Q(n,m))$. The following result bounds the size of the smallest dominating set of a graph in terms of its minimum degree.

Lemma 3.2.1. (Alon & Spencer, 2016, Theorem 1.2.2 on p. 6) *A graph with N vertices and minimum degree δ has a dominating set of size at most $N\frac{1+\ln(1+\delta)}{1+\delta}$.*

We will also need the Stirling's formula (see, for example, (Cameron, 1995, 3.6.2 on p. 31)) in our proofs:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

For a non-negative integer k, let *level k* refer to those vertices of $Q(n,m)$ with exactly k non-zero coordinates. So the number of vertices in level k is exactly $m^k\binom{n}{k}$. For any real number x, let $\lfloor x \rfloor$ be the smallest integer less than or equal to x. Note that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.

The following lemma tells us that level $\lfloor \frac{m(n+1)}{m+1} \rfloor$ has the greatest number of vertices.

Lemma 3.2.2. *Let m, n, k be integers such that $1 \le m \le n$ and $0 \le k \le n$. Let*

$$a_k = m^k\binom{n}{k}.$$

Then $a_k \le a_{k+1}$ if and only if $0 \le k \le \frac{mn-1}{m+1}$.

Proof. Note that

$$\frac{a_{k+1}}{a_k} = m\left(\frac{n-k}{k+1}\right).$$

Thus, $a_k \leq a_{k+1}$ if and only if $0 \leq k \leq \frac{mn-1}{m+1}$. $\square$

Here is a technical lemma to bound $\binom{n}{\alpha n}$ for $0 < \alpha < 1$.

Lemma 3.2.3. *Let α and β be real numbers with $0 < \alpha < 1$. Then for sufficiently large n,*

$$\binom{n}{\lfloor \alpha n + \beta \rfloor} = O\left(\frac{1}{\sqrt{n}\alpha^{\alpha n}(1-\alpha)^{(1-\alpha)n}}\right).$$

Proof. By the Stirling's formula,

$$\begin{aligned}
\binom{n}{\lfloor \alpha n + \beta \rfloor} &= O\left(\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi \lfloor \alpha n + \beta \rfloor}\left(\frac{\lfloor \alpha n + \beta \rfloor}{e}\right)^{\lfloor \alpha n + \beta \rfloor}\sqrt{2\pi(n - \lfloor \alpha n + \beta \rfloor)}\left(\frac{n - \lfloor \alpha n + \beta \rfloor}{e}\right)^{n - \lfloor \alpha n + \beta \rfloor}}\right) \\
&= O\left(\frac{n^{n+\frac{1}{2}}}{\sqrt{\alpha n + \beta - 1}\,(\alpha n + \beta - 1)^{\lfloor \alpha n + \beta \rfloor}\sqrt{n - (\alpha n + \beta)}\,(n - (\alpha n + \beta))^{n - \lfloor \alpha n + \beta \rfloor}}\right).
\end{aligned}$$

Since $\beta - 1 \leq \lfloor \alpha n + \beta \rfloor - \alpha n \leq \beta$ and $\lim_{n\to\infty}\left(1 + \frac{\beta-1}{\alpha n}\right)^{\alpha n} = \exp(\beta - 1) > 0$, we have

$$\begin{aligned}
\alpha^{\lfloor \alpha n + \beta \rfloor - \alpha n}\left(1 + \frac{\beta - 1}{\alpha n}\right)^{\lfloor \alpha n + \beta \rfloor} &= \alpha^{\lfloor \alpha n + \beta \rfloor - \alpha n}\left(1 + \frac{\beta - 1}{\alpha n}\right)^{\lfloor \alpha n + \beta \rfloor - \alpha n}\left(1 + \frac{\beta - 1}{\alpha n}\right)^{\alpha n} \\
&\geq A,
\end{aligned}$$

where A is a positive constant depending on α and β. Therefore,

$$\begin{aligned}
\sqrt{\alpha n + \beta - 1}\,(\alpha n + \beta - 1)^{\lfloor \alpha n + \beta \rfloor} &= \sqrt{n}\left(\sqrt{\alpha + \frac{\beta - 1}{n}}\right)\alpha^{\lfloor \alpha n + \beta \rfloor} n^{\lfloor \alpha n + \beta \rfloor}\left(1 + \frac{\beta - 1}{\alpha n}\right)^{\lfloor \alpha n + \beta \rfloor} \\
&\geq \left(\sqrt{n}\alpha^{\alpha n} n^{\lfloor \alpha n + \beta \rfloor}\right)\left(A\sqrt{\alpha + \frac{\beta - 1}{n}}\right) \\
&\geq \left(\sqrt{n}\alpha^{\alpha n} n^{\lfloor \alpha n + \beta \rfloor}\right)\left(A\sqrt{\frac{\alpha}{2}}\right).
\end{aligned}$$

Since $\lim_{n\to\infty}\left(1-\frac{\beta}{(1-\alpha)n}\right)^{(1-\alpha)n} = \exp(-\beta) > 0$, we have

$$
\begin{aligned}
&(1-\alpha)^{n-\lfloor \alpha n+\beta\rfloor-(1-\alpha)n}\left(1-\frac{\beta}{(1-\alpha)n}\right)^{n-\lfloor \alpha n+\beta\rfloor} \\
&= (1-\alpha)^{\alpha n-\lfloor \alpha n+\beta\rfloor}\left(1-\frac{\beta}{(1-\alpha)n}\right)^{\alpha n-\lfloor \alpha n+\beta\rfloor}\left(1-\frac{\beta}{(1-\alpha)n}\right)^{(1-\alpha)n} \\
&\geq B,
\end{aligned}
$$

where B is a positive constant depending on α and β. Therefore,

$$
\begin{aligned}
&\sqrt{n-(\alpha n+\beta)}\,(n-(\alpha n+\beta))^{n-\lfloor \alpha n+\beta\rfloor} \\
&= \sqrt{n}\left(\sqrt{1-\alpha-\frac{\beta}{n}}\right)(1-\alpha)^{n-\lfloor \alpha n+\beta\rfloor}\, n^{n-\lfloor \alpha n+\beta\rfloor}\left(1-\frac{\beta}{(1-\alpha)n}\right)^{n-\lfloor \alpha n+\beta\rfloor} \\
&\geq \left(\sqrt{n}\,(1-\alpha)^{(1-\alpha)n}\, n^{n-\lfloor \alpha n+\beta\rfloor}\right)\left(B\sqrt{1-\alpha-\frac{\beta}{n}}\right) \\
&\geq \left(\sqrt{n}\,(1-\alpha)^{(1-\alpha)n}\, n^{n-\lfloor \alpha n+\beta\rfloor}\right)\left(B\sqrt{\frac{1-\alpha}{2}}\right).
\end{aligned}
$$

Hence,

$$
\binom{n}{\lfloor \alpha n+\beta\rfloor} = O\left(\frac{1}{\sqrt{n}\alpha^{\alpha n}(1-\alpha)^{(1-\alpha)n}}\right).
$$

□

We are now ready to prove the upper bound of Theorem 3.1.1. Our proof is a generalization of (Offner & Ojakian, 2014, Theorem 5.4).

Theorem 3.2.4. *Let m be a positive integer. Then for sufficiently large n,*

$$
c_L(Q(n,m)) = O\left(\frac{(m+1)^n \ln n}{n^{3/2}}\right).
$$

Proof. The case $m = 1$ was proved in (Offner & Ojakian, 2014, Theorem 5.4). So we shall assume that $m > 1$. We describe a strategy for the cops (the same strategy used by Offner and Ojakian (2014) in proving the case of Q_n) where we position the cops so that they dominate a level and then move up or down the levels in a phalanx in order to catch the robber.

Let G_k denote the subgraph of $Q(n, m)$ induced by level k. Then G_k has $m^k \binom{n}{k}$ vertices and every vertex has degree exactly $k(m - 1)$. We claim that for $0 \leq k \leq n$, G_k has a dominating set of size $O\left(\frac{(m+1)^n \ln n}{n^{3/2}}\right)$.

By Lemma 3.2.1, G_k has a dominating set of size at most

$$m^k \binom{n}{k} \frac{1 + \ln(1 + k(m - 1))}{1 + k(m - 1)}. \tag{3.1}$$

Note that $\lim_{t\to\infty} t^{\frac{m}{m+t}} = 1$ and $\lim_{t\to\infty} t^{\frac{t}{m+t}} = \infty$. Thus,

$$\lim_{t\to\infty} \frac{t + m}{t^{t/(t+m)}} = \lim_{t\to\infty} t^{\frac{m}{m+t}} + \lim_{t\to\infty} \frac{m}{t^{t/(t+m)}} = 1.$$

So, we may choose a t_0 satisfying

(a) $t_0 > 1$;

(b) $\frac{t_0+m}{t_0^{t_0/(t_0+m)}} < m + \frac{1}{2}$.

Now, (a) implies that $\frac{m}{t_0+m} < \frac{m}{m+1}$.

Case 1. Suppose $0 \le k \le \frac{mn}{t_0+m}$. Since $\frac{1+\ln(1+x(m-1))}{1+x(m-1)}$ is a decreasing function for $x \ge 0$, we have $\frac{1+\ln(1+k(m-1))}{1+k(m-1)} \le 1$. By Lemma 3.2.2 and Lemma 3.2.3, with $\alpha = \frac{m}{t_0+m}$ and $\beta = 0$,

$$\begin{aligned}
m^k\binom{n}{k} &\le m^{\frac{mn}{t_0+m}}\binom{n}{\left\lfloor \frac{mn}{t_0+m}\right\rfloor} \\
&= O\left(m^{\frac{mn}{t_0+m}} \frac{1}{\sqrt{n}\left(\frac{m}{t_0+m}\right)^{\frac{mn}{t_0+m}}\left(1-\frac{m}{t_0+m}\right)^{\left(1-\frac{m}{t_0+m}\right)n}}\right) \\
&= O\left(\frac{(t_0+m)^n}{t_0^{\frac{t_0 n}{t_0+m}}\sqrt{n}}\right) \\
&= O\left(\left(\frac{t_0+m}{t_0^{\frac{t_0}{t_0+m}}}\right)^n \frac{1}{\sqrt{n}}\right) \\
&= O\left(\frac{\left(m+\frac{1}{2}\right)^n}{\sqrt{n}}\right),
\end{aligned}$$

where the last inequality follows from our choice of t_0. Therefore, the dominating set is of size at most $O\left(\frac{\left(m+\frac{1}{2}\right)^n}{\sqrt{n}}\right)$.

Note that if $x > 1$, then $\lim_{n\to\infty} \frac{x^n}{n} = \infty$. Now, $\frac{m+1}{m+\frac{1}{2}} = 1 + \frac{1}{2\left(m+\frac{1}{2}\right)} > 1$. Thus, $\lim_{n\to\infty} \frac{\left(\frac{m+1}{m+\frac{1}{2}}\right)^n}{n} = \infty$, and so for n sufficiently large,

$$\frac{\left(m+\frac{1}{2}\right)^n}{\sqrt{n}} < \frac{(m+1)^n}{n^{3/2}} < \frac{(m+1)^n \ln n}{n^{3/2}}.$$

Hence, the dominating set is of size at most $O\left(\frac{(m+1)^n \ln n}{n^{3/2}}\right)$.

Case 2. Suppose $\frac{mn}{t_0+m} < k \le n$. Again, since $\frac{1+\ln(1+x(m-1))}{1+x(m-1)}$ is a decreasing function in x, we have

$$\frac{1+\ln(1+k(m-1))}{1+k(m-1)} \le \frac{1+\ln\left(1+\left(\frac{mn}{t_0+m}\right)(m-1)\right)}{1+\left(\frac{mn}{t_0+m}\right)(m-1)} = O\left(\frac{\ln n}{n}\right).$$

Note that $\frac{mn}{t_0+m} < \frac{m(n+1)}{m+1} < n$ (for sufficiently large n). Therefore, by Lemma 3.2.2 and Lemma 3.2.3, with $\alpha = \beta = \frac{m}{m+1}$,

$$\begin{aligned} m^k \binom{n}{k} &\leq m^{\frac{m(n+1)}{m+1}} \binom{n}{\left\lfloor \frac{m(n+1)}{m+1} \right\rfloor} \\ &= O\left(m^{\frac{mn}{m+1}} \frac{1}{\sqrt{n} \left(\frac{m}{m+1}\right)^{\frac{mn}{m+1}} \left(1 - \frac{m}{m+1}\right)^{\left(1-\frac{m}{m+1}\right)n}} \right) \\ &= O\left(\frac{(m+1)^n}{\sqrt{n}} \right). \end{aligned}$$

Therefore, the dominating set is of size at most $O\left(\frac{(m+1)^n \ln n}{n^{3/2}}\right)$.

Our claim has been established. The cops should initially select a set of vertices D, that dominates $G_{\lfloor \frac{m(n+1)}{m+1} \rfloor}$, and place two cops on each vertex in D, coloring one red and the other blue, thus supplying enough cops to dominate any G_k. If the robber chooses an initial position in level $\lfloor \frac{m(n+1)}{m+1} \rfloor$, then he will be caught immediately. Suppose the robber chooses a vertex in some level $i > \lfloor \frac{m(n+1)}{m+1} \rfloor$. At first, the blue cops should remain in place, while the red cops rearrange themselves one by one to dominate $G_{\lfloor \frac{m(n+1)}{m+1} \rfloor + 1}$. Since the blue cops are still dominating $G_{\lfloor \frac{m(n+1)}{m+1} \rfloor}$, the robber is restricted to move only in levels greater than $\lfloor \frac{m(n+1)}{m+1} \rfloor$. After the red cops have dominated $G_{\lfloor \frac{m(n+1)}{m+1} \rfloor + 1}$, the robber will be in a level greater than $\lfloor \frac{m(n+1)}{m+1} \rfloor + 1$. Now, the red cops should remain in place while the blue cops rearrange themselves to dominate $G_{\lfloor \frac{m(n+1)}{m+1} \rfloor + 2}$. Since the red cops are still dominating $G_{\lfloor \frac{m(n+1)}{m+1} \rfloor + 1}$, the robber is restricted to move only in levels greater than $\lfloor \frac{m(n+1)}{m+1} \rfloor + 1$. Proceeding in this manner, with cops of one color dominating G_k while the cops of the other color proceed to dominate G_{k+1}, the cops will force the robber to move to higher and higher levels, until the cops eventually dominate G_n at which point the robber will be caught. Suppose instead that the robber chooses an initial position in some level $i < \lfloor \frac{m(n+1)}{m+1} \rfloor$. Similarly, with cops of one color dominating G_k while the cops of the other color proceed to dominate G_{k-1}, the cops will force the robber to move to lower and lower levels, until the cops eventually dominate G_0 at which point the robber will be caught. This completes the proof of the theorem. □

We remark here that in the proof above, if we take G_k to be the subgraph induced by levels k and $k+1$ instead, we can improve the upper bound of $c_L(Q(n,m))$ for any fixed m, but only up to a constant. The asymptotic bound remains the same.

3.2.2 Lower bound

Bal et al. (2015) introduced a potential function and used the probabilistic method to prove the lower bound of $c_L(Q_n)$. In this subsection, we shall use this idea to prove the lower bound of Theorem 3.1.1. From now on, we shall assume $m \geq 1$ and $\varepsilon > 0$ are fixed, and n is sufficiently large. Let $\alpha = \frac{m}{m+1}$ and

$$\beta = \max\left(\sqrt{\frac{(2\alpha(1+\varepsilon)+1)}{m+1}}, (1+\varepsilon)\sqrt{\frac{3}{\varepsilon}}\right).$$

It is clear that $\beta > 1$ and so for all $n \geq n_0$, there is an integer r_n satisfying

$$\alpha n - 2\beta\sqrt{n} \leq r_n \leq \alpha n - \beta\sqrt{n}. \tag{3.2}$$

Let $\rho = \rho(n)$ be the function defined by $\rho(n) = \alpha n - r_n$ for $n \geq n_0$ and $\rho(n) = 0$ for $1 \leq n < n_0$. Then for $n \geq n_0$, $\beta\sqrt{n} \leq \rho(n) \leq 2\beta\sqrt{n}$ and $\alpha n - \rho(n)$ is an integer. Furthermore, $\rho \to \infty$ and $\frac{\rho}{n} \to 0$ as $n \to \infty$.

For $1 \leq i \leq \alpha n - \rho$, let

$$\varepsilon_i = \frac{(m+1)(1+\varepsilon)}{nm - (m+1)(i+1)},$$

and

$$w_i = Am^{-i}\binom{n-2}{i}^{-1}\prod_{j=1}^{i}(1+\varepsilon_j), \qquad A = \frac{m(n-2)}{1+\varepsilon_1}. \tag{3.3}$$

For $\alpha n - \rho \le i \le n$, let

$$w_i = (n-i)\frac{w_{\alpha n-\rho}}{(1-\alpha)n+\rho}. \tag{3.4}$$

Note that for $1 \le i \le \alpha n - \rho$,

$$\varepsilon_i = \frac{(m+1)(1+\varepsilon)}{nm-(m+1)(i+1)} \le \frac{(m+1)(1+\varepsilon)}{(m+1)(\rho-1)} = o(1),$$

$w_1 = 1$, $w_2 = \frac{2}{mn}(1+o(1))$, $w_3 = O\left(\frac{1}{n^2}\right)$ and $w_n = 0$.

We remark that w_i is a modification of the weight function used in the proof of the lower bound of $c_L(Q_n)$ in (Bal et al., 2015).

Lemma 3.2.5.

$$1 < \frac{w_{\alpha n-\rho-1}}{w_{\alpha n-\rho}} < 1 + O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. Note that

$$\begin{aligned}
\frac{w_{\alpha n-\rho-1}}{w_{\alpha n-\rho}} &= m\left(\frac{n-1-(\alpha n-\rho)}{\alpha n-\rho}\right)(1+\varepsilon_{\alpha n-\rho})^{-1} \\
&= m\left(\frac{(1-\alpha)n-1+\rho}{\alpha n-\rho}\right)\left(1+\frac{(m+1)(1+\varepsilon)}{nm-(m+1)(\alpha n-\rho+1)}\right)^{-1} \\
&= \left(\frac{\alpha n-m+m\rho}{\alpha n-\rho}\right)\left(1+\frac{(m+1)(1+\varepsilon)}{(m+1)(\rho-1)}\right)^{-1} \\
&= \left(1+\frac{(m+1)\rho-m}{\alpha n-\rho}\right)\left(\frac{\rho-1}{\rho+\varepsilon}\right) \\
&= \left(1+\frac{(m+1)\rho-m}{\alpha n-\rho}\right)\left(1-\frac{1+\varepsilon}{\rho+\varepsilon}\right) \\
&= 1+\frac{(m+1)\rho-m}{\alpha n-\rho}-\frac{1+\varepsilon}{\rho+\varepsilon}-\left(\frac{(m+1)\rho-m}{\alpha n-\rho}\right)\left(\frac{1+\varepsilon}{\rho+\varepsilon}\right).
\end{aligned}$$

Since $\rho \geq \sqrt{\frac{(2\alpha(1+\varepsilon)+1)n}{m+1}}$ and $\lim_{n\to\infty} \frac{\rho}{n} = 0$, for sufficiently large n, $\frac{(m+1)\rho-m}{2(\alpha n-\rho)} \geq \frac{1+\varepsilon}{\rho+\varepsilon}$ and $\frac{1+\varepsilon}{\rho+\varepsilon} \leq \frac{1}{4}$. Thus,

$$\frac{w_{\alpha n-\rho-1}}{w_{\alpha n-\rho}} \geq 1 + \frac{(m+1)\rho - m}{4(\alpha n - \rho)} > 1.$$

Finally, as $\beta\sqrt{n} \leq \rho \leq 2\beta\sqrt{n}$, we have

$$\frac{(m+1)\rho - m}{\alpha n - \rho} + \frac{1+\varepsilon}{\rho+\varepsilon} + \left(\frac{(m+1)\rho - m}{\alpha n - \rho}\right)\left(\frac{1+\varepsilon}{\rho+\varepsilon}\right) = O\left(\frac{1}{\sqrt{n}}\right),$$

whence the lemma follows. □

Lemma 3.2.6. *For $1 \leq i \leq n$, w_i is a strictly decreasing sequence.*

Proof. For $\alpha n - \rho \leq i \leq n$, w_i decreases linearly from $w_{\alpha n-\rho}$ to $w_n = 0$. It is left to show that w_i is decreasing for $1 \leq i \leq \alpha n - \rho$. Since $\frac{n-1-x}{x}$ and $\left(1 + \frac{(m+1)(1+\varepsilon)}{nm-(m+1)(x+1)}\right)^{-1}$ are decreasing functions, we have

$$\begin{aligned}
\frac{w_{i-1}}{w_i} &= m\left(\frac{n-1-i}{i}\right)(1+\varepsilon_i)^{-1} \\
&\geq m\left(\frac{n-1-(\alpha n-\rho)}{\alpha n-\rho}\right)\left(1 + \frac{(m+1)(1+\varepsilon)}{nm-(m+1)((\alpha n-\rho)+1)}\right)^{-1} \\
&= \frac{w_{\alpha n-\rho-1}}{w_{\alpha n-\rho}} > 1,
\end{aligned}$$

where the last inequality follows from Lemma 3.2.5. Hence, w_i is strictly decreasing. □

Lemma 3.2.7.

$$w_{\alpha n-\rho} = O\left(\frac{n^{2+\frac{\varepsilon}{2}}}{(m+1)^n}\right).$$

Proof. First, note that for any integers a, b and real number t with $1 \leq a \leq b$ and $t > b+1$,

$$\sum_{i=a}^{b} \frac{1}{t-i} \leq \int_a^{b+1} \frac{1}{t-x}\, dx = \ln\left(\frac{t-a}{t-b-1}\right).$$

Therefore,

$$
\begin{aligned}
\prod_{j=1}^{\alpha n-\rho}(1+\varepsilon_j) &= \exp\left(\sum_{j=1}^{\alpha n-\rho}\ln\left(1+\varepsilon_j\right)\right)\\
&\leq \exp\left(\sum_{j=1}^{\alpha n-\rho}\varepsilon_j\right)\\
&= \exp\left(\sum_{j=1}^{\alpha n-\rho}\frac{(m+1)(1+\varepsilon)}{nm-(m+1)(j+1)}\right)\\
&= \exp\left(\frac{(m+1)(1+\varepsilon)}{m+1}\sum_{j=1}^{\alpha n-\rho}\frac{1}{\alpha n-j-1}\right)\\
&\leq \exp\left((1+\varepsilon)\ln\left(\frac{\alpha n-2}{\rho-2}\right)\right)\\
&= O\left(\left(\frac{n}{\rho}\right)^{1+\varepsilon}\right)\\
&= O\left(n^{\frac{1+\varepsilon}{2}}\right),
\end{aligned}
$$

where the last inequality follows from $\rho \geq \beta\sqrt{n}$.

Next, $\lim_{n\to\infty}\frac{\binom{n}{\alpha n-\rho}}{\binom{n-2}{\alpha n-\rho}} = \lim_{n\to\infty}\frac{n(n-1)}{((1-\alpha)n+\rho)((1-\alpha)n+\rho-1)} = \frac{1}{(1-\alpha)^2}$. So,

$$
\binom{n-2}{\alpha n-\rho}^{-1} = O\left(\binom{n}{\alpha n-\rho}^{-1}\right).
$$

By the Stirling's formula,

$$
\begin{aligned}
\binom{n}{\alpha n-\rho}^{-1} &= \frac{(\alpha n-\rho)!((1-\alpha)n+\rho)!}{n!}\\
&= O\left(\frac{\left(\sqrt{2\pi(\alpha n-\rho)}\left(\frac{\alpha n-\rho}{e}\right)^{\alpha n-\rho}\right)\left(\sqrt{2\pi((1-\alpha)n+\rho)}\left(\frac{(1-\alpha)n+\rho}{e}\right)^{(1-\alpha)n+\rho}\right)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}\right)\\
&= O\left(\sqrt{n}\alpha^{\alpha n-\rho}(1-\alpha)^{(1-\alpha)n+\rho}\left(1-\frac{\rho}{\alpha n}\right)^{\alpha n-\rho}\left(1+\frac{\rho}{(1-\alpha)n}\right)^{(1-\alpha)n+\rho}\right).
\end{aligned}
$$

Now,

$$\begin{aligned}
\left(1-\frac{\rho}{\alpha n}\right)^{\alpha n-\rho} &= \exp\left((\alpha n-\rho)\ln\left(1-\frac{\rho}{\alpha n}\right)\right) \\
&= \exp\left(-(\alpha n-\rho)\left(\frac{\rho}{\alpha n}+\frac{1}{2}\left(\frac{\rho}{\alpha n}\right)^2+\frac{1}{3}\left(\frac{\rho}{\alpha n}\right)^3+O\left(\left(\frac{\rho}{n}\right)^4\right)\right)\right) \\
&= \exp\left(-\rho+\frac{\rho^2}{2\alpha n}+\frac{\rho^3}{6(\alpha n)^2}+O\left(\left(\frac{\rho}{n}\right)^3\right)\right) \\
&= \exp\left(-\rho+\frac{\rho^2}{2\alpha n}+O\left(\frac{1}{\sqrt{n}}\right)\right),
\end{aligned}$$

where the last inequality follows from $\rho = O(\sqrt{n})$.

Similarly,

$$\begin{aligned}
&\left(1+\frac{\rho}{(1-\alpha)n}\right)^{(1-\alpha)n+\rho} \\
&\quad = \exp\left(((1-\alpha)n+\rho)\left(\frac{\rho}{(1-\alpha)n}-\frac{1}{2}\left(\frac{\rho}{(1-\alpha)n}\right)^2+O\left(\left(\frac{\rho}{n}\right)^3\right)\right)\right) \\
&\quad = \exp\left(\rho+\frac{\rho^2}{2(1-\alpha)n}+O\left(\frac{1}{\sqrt{n}}\right)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(1-\frac{\rho}{\alpha n}\right)^{\alpha n-\rho}\left(1+\frac{\rho}{(1-\alpha)n}\right)^{(1-\alpha)n+\rho} &= \exp\left(\frac{\rho^2}{2(1-\alpha)n}+\frac{\rho^2}{2\alpha n}+O\left(\frac{1}{\sqrt{n}}\right)\right) \\
&= \exp\left(\frac{\rho^2}{n}\left(\frac{1}{2\alpha(1-\alpha)}\right)+O\left(\frac{1}{\sqrt{n}}\right)\right) \\
&\leq \exp\left(\frac{3\beta^2}{\alpha(1-\alpha)}\right),
\end{aligned}$$

where the last inequality follows from $\rho \leq 2\beta\sqrt{n}$. Hence,

$$\begin{aligned} m^{-(\alpha n-\rho)}\binom{n-2}{\alpha n-\rho}^{-1} &= O\left(m^{-(\alpha n-\rho)}\binom{n}{\alpha n-\rho}^{-1}\right) \\ &= O\left(m^{-(\alpha n-\rho)}\sqrt{n}\alpha^{\alpha n-\rho}(1-\alpha)^{(1-\alpha)n+\rho}\right) \\ &= O\left(\frac{\sqrt{n}}{(m+1)^n}\right), \end{aligned}$$

and

$$w_{\alpha n-\rho} = O\left(n\left(\frac{\sqrt{n}}{(m+1)^n}\right)\left(n^{\frac{1+\varepsilon}{2}}\right)\right),$$

whence the lemma follows. □

Lemma 3.2.8. *For* $2 \leq i \leq \alpha n - \rho - 1$*,*

$$\frac{i}{(n-2)m}w_{i-1} + \frac{(m-1)i}{(n-2)m}w_i + \frac{(n-i-2)m}{(n-2)m}w_{i+1} \leq w_i\left(1-\frac{\varepsilon}{2n}\right).$$

Proof. Note that

$$\begin{aligned} V = \frac{i}{(n-2)m}w_{i-1} + \frac{(m-1)i}{(n-2)m}w_i + \frac{(n-i-2)m}{(n-2)m}w_{i+1} \\ = w_i\left(\frac{i}{(n-2)m}\left(\frac{w_{i-1}}{w_i}\right) + \frac{(m-1)i}{(n-2)m} + \frac{n-i-2}{n-2}\left(\frac{w_{i+1}}{w_i}\right)\right) \\ = w_i\left(\frac{n-1-i}{n-2}(1+\varepsilon_i)^{-1} + \frac{(m-1)i}{(n-2)m} + \frac{i+1}{(n-2)m}(1+\varepsilon_{i+1})\right). \end{aligned}$$

Since

$$(1+\varepsilon_i)^{-1} = 1 - \varepsilon_i + \varepsilon_i^2 - \varepsilon_i^3 + \cdots \leq 1 - \varepsilon_i + \varepsilon_i^2,$$

and

$$1 + \varepsilon_{i+1} = 1 + \varepsilon_i\left(1 + \frac{m+1}{nm-(m+1)(i+2)}\right) \leq 1 + \varepsilon_i + \varepsilon_i^2,$$

we obtain

$$
\begin{aligned}
V &\leq w_i \left(\frac{n-1-i}{n-2}(1-\varepsilon_i+\varepsilon_i^2) + \frac{(m-1)i}{(n-2)m} + \frac{i+1}{(n-2)m}(1+\varepsilon_i+\varepsilon_i^2) \right) \\
&= w_i \left(1 + \frac{m+1}{(n-2)m} - \varepsilon_i \left(\frac{n-1-i}{n-2} - \frac{i+1}{(n-2)m} \right) + \varepsilon_i^2 \left(\frac{n-1-i}{n-2} + \frac{i+1}{(n-2)m} \right) \right).
\end{aligned}
$$

From $2 \leq i \leq \alpha n - \rho - 1$ and $\alpha = \frac{m}{m+1}$, we get

$$
\begin{aligned}
\varepsilon_i^2 \left(\frac{n-1-i}{n-2} + \frac{i+1}{(n-2)m} \right) &= \left(\frac{(m+1)(1+\varepsilon)}{nm-(m+1)(i+1)} \right)^2 \left(\frac{nm-(m-1)(i+1)}{(n-2)m} \right) \\
&\leq \left(\frac{(m+1)(1+\varepsilon)}{(m+1)\rho} \right)^2 \left(\frac{nm-3(m-1)}{(n-2)m} \right) \\
&\leq \left(\frac{1+\varepsilon}{\rho} \right)^2 \left(\frac{3}{2} \right) \\
&\leq \frac{\varepsilon}{2n},
\end{aligned}
$$

where the last inequality follows provided that $\rho \geq (1+\varepsilon)\sqrt{\frac{3n}{\varepsilon}}$. Since

$$
\begin{aligned}
\varepsilon_i \left(\frac{n-1-i}{n-2} - \frac{i+1}{(n-2)m} \right) &= \frac{(m+1)(1+\varepsilon)}{nm-(m+1)(i+1)} \left(\frac{nm-(m+1)(i+1)}{(n-2)m} \right) \\
&= \frac{(m+1)(1+\varepsilon)}{(n-2)m},
\end{aligned}
$$

we have

$$
\begin{aligned}
V &\leq w_i \left(1 + \frac{m+1}{(n-2)m} - \frac{(m+1)(1+\varepsilon)}{(n-2)m} + \frac{\varepsilon}{2n} \right) \\
&= w_i \left(1 - \frac{(m+1)\varepsilon}{(n-2)m} + \frac{\varepsilon}{2n} \right) \\
&\leq w_i \left(1 - \frac{\varepsilon}{n} + \frac{\varepsilon}{2n} \right) \\
&= w_i \left(1 - \frac{\varepsilon}{2n} \right).
\end{aligned}
$$

□

We are now ready to prove the lower bound of $c_L(Q(n,m))$. Our proof is a generalization of (Bal et al., 2015, Theorem 1).

Theorem 3.2.9. *Let m be a positive integer and let $\varepsilon > 0$. Then for sufficiently large n,*

$$c_L(Q(n,m)) \geq \frac{(m+1)^n}{n^{5/2+\varepsilon}}.$$

Proof. Let N_i be the number of cops at distance i from the robber. We say that a cop at distance i from the robber has *weight* w_i, where w_i is as defined in equations (3.3) and (3.4). Let the *potential function* P be defined as

$$P = \sum_{i=1}^{n} N_i w_i.$$

Recall that $w_1 = 1$. If the cops can catch the robber on their turn, then immediately before the cops' turn we must have $P \geq 1$, since some cop must be at distance 1 from the robber. To show that the robber can evade the cops indefinitely, we shall prove that the robber can always move such that right before the cops' move,

$$P < 1. \tag{3.5}$$

Without lost of generality, all cops start at the same vertex and the robber starts at a vertex at distance n from the cops. Therefore, $P = 0$ and invariant (3.5) holds. Suppose that before the cops make their move, the potential function satisfies invariant (3.5).

Let the coordinate of the robber be $u = (x_1, \ldots, x_n)$ right before the cops' move. Suppose that on the cops' turn, a cop C^* moves to a vertex v. We may assume that C^* moves closer to u, as otherwise, the robber may remain in u to maintain invariant (3.5). Let P_1 represent the total weight of all cops at distance i from u with $2 \leq i \leq \alpha n - \rho - 1$ other than C^*. Let

P_2 represent the total weight of all cops at distance at least $\alpha n - \rho$ from u other than C^*.

The robber's strategy is to move to a vertex to maintain invariant (3.5). We shall show that such a vertex always exists by computing the expected potential function for all possible vertices the robber can move to.

We need to choose two distinct integers $i_0, i_1 \in \{1, 2, \ldots, n\}$ based on the vertex v (the vertex C^* is occupying). We shall explain how to choose i_0 and i_1 later. Assume at the moment that i_0 and i_1 have been chosen. We only allow the robber to move to a vertex that differs with u at coordinate a where $a \notin \{i_0, i_1\}$. For each $a \notin \{i_0, i_1\}$, the robber has $m = |[m] \setminus \{x_i\}|$ vertices to choose from. Thus, the robber has $(n-2)m$ possible vertices to move to.

First, we shall show that the expected value of P_1 is at most

$$P_1 \left(1 - \frac{\varepsilon}{2n}\right) \tag{3.6}$$

after the robber's move in each round.

Proof of inequality (3.6). Let C be a cop at distance i from the robber, where $2 \leq i \leq \alpha n - \rho - 1$. Before the robber's move, C has weight w_i. Let w_C represent the expected weight of C after the robber's move. Let C and u differ at coordinates $r_1, \ldots, r_i$.

Suppose $i_0, i_1 \in \{r_1, \ldots, r_i\}$. If the robber moves to a vertex that differs with u at coordinate r_j and $r_j \notin \{i_0, i_1\}$, then out of the m choices, one of the choices will reduce the distance of C by 1, whereas the other $m - 1$ choices will maintain the distance of C. If the robber moves to a vertex that differs with u at coordinate a and $a \notin \{r_1, \ldots, r_i\}$, then the distance of C will increase by 1. Therefore,

$$\begin{aligned}
w_C &= \frac{i-2}{(n-2)m} w_{i-1} + \frac{(m-1)(i-2)}{(n-2)m} w_i + \frac{(n-i)m}{(n-2)m} w_{i+1} \\
&\le \frac{i}{(n-2)m} w_{i-1} + \frac{(m-1)i}{(n-2)m} w_i + \frac{(n-i-2)m}{(n-2)m} w_{i+1},
\end{aligned}$$

where the last inequality follows from the fact that w_i is decreasing (Lemma 3.2.6). Similarly, if $|\{i_0, i_1\} \cap \{r_1, \ldots, r_i\}| = 1$, then

$$\begin{aligned}
w_C &= \frac{i-1}{(n-2)m} w_{i-1} + \frac{(m-1)(i-1)}{(n-2)m} w_i + \frac{(n-i-1)m}{(n-2)m} w_{i+1} \\
&\le \frac{i}{(n-2)m} w_{i-1} + \frac{(m-1)i}{(n-2)m} w_i + \frac{(n-i-2)m}{(n-2)m} w_{i+1}.
\end{aligned}$$

Finally, if $|\{i_0, i_1\} \cap \{r_1, \ldots, r_i\}| = 0$, then

$$w_C = \frac{i}{(n-2)m} w_{i-1} + \frac{(m-1)i}{(n-2)m} w_i + \frac{(n-i-2)m}{(n-2)m} w_{i+1}.$$

In either case, we have

$$\begin{aligned}
w_C &\le \frac{i}{(n-2)m} w_{i-1} + \frac{(m-1)i}{(n-2)m} w_i + \frac{(n-i-2)m}{(n-2)m} w_{i+1} \\
&\le w_i \left(1 - \frac{\varepsilon}{2n}\right),
\end{aligned}$$

where the last inequality follows from Lemma 3.2.8. By summing up each cop's individual contribution toward the potential, we see that after the robber's move, (3.6) holds. □

Next, we claim that the expected value of P_2 is at most

$$P_2 + \frac{\varepsilon}{4n}, \tag{3.7}$$

after the robber's move in each round.

Proof of inequality (3.7). Let C be a cop at distance i from the robber, where $\alpha n - \rho \le i \le n$. Before the robber's move, C has weight w_i. After the robber's move, if the distance between C and u decreases by 1, then the change in the weight of C is $w_{i-1} - w_i$, if the distance between C and u are the same, then there is no change in the weight of C, and if the distance between C and u increases by 1, then the change in the weight of C is $w_{i+1} - w_i$.

If $\alpha n - \rho \le i \le n$, then $w_i - w_{i+1} = \frac{w_{\alpha n - \rho}}{(1-\alpha)n+\rho} = O\left(\frac{w_{\alpha n-\rho}}{n}\right) = O\left(\frac{n^{1+\frac{\varepsilon}{2}}}{(m+1)^n}\right)$ by Lemma 3.2.7. If $\alpha n - \rho + 1 \le i \le n$, then $w_{i-1} - w_i = \frac{w_{\alpha n-\rho}}{(1-\alpha)n+\rho} = O\left(\frac{w_{\alpha n-\rho}}{n}\right) = O\left(\frac{n^{1+\frac{\varepsilon}{2}}}{(m+1)^n}\right)$. Now, let us compute the upper bound for $w_{\alpha n-\rho-1} - w_{\alpha n-\rho}$. By Lemma 3.2.5,

$$\frac{w_{\alpha n-\rho-1}}{w_{\alpha n-\rho}} < 1 + O\left(\frac{1}{\sqrt{n}}\right).$$

Therefore, $w_{\alpha n-\rho-1} - w_{\alpha n-\rho} = O\left(\frac{w_{\alpha n-\rho}}{\sqrt{n}}\right) = O\left(\frac{n^{\frac{3+\varepsilon}{2}}}{(m+1)^n}\right)$. Hence, the total change in the weight of C is at most $O\left(\frac{n^{\frac{3+\varepsilon}{2}}}{(m+1)^n}\right)$.

If the total number of cops is at most $\frac{(m+1)^n}{n^{5/2+\varepsilon}}$, then the expected change in P_2 is at most

$$O\left(\left(\frac{(m+1)^n}{n^{5/2+\varepsilon}}\right)\left(\frac{n^{\frac{3+\varepsilon}{2}}}{(m+1)^n}\right)\right),$$

and (3.7) follows. □

Now, we consider three cases based on on the vertex v (the vertex C^* is occupying).

Case 1. Suppose v and u differ at one coordinate, say i_0. Then the coordinate of C^* is $v = (x_1, \ldots, x_{i_0-1}, y_{i_0}, x_{i_0+1}, \ldots, x_n)$ where $y_{i_0} \ne x_{i_0}$. The robber cannot move to a vertex that differs with u at coordinate i_0, as otherwise, C^* will be able to catch the robber in the next move. So the robber can only move to a vertex that differs with u at coordinate i where $i \ne i_0$. Choose any $i_1 \in \{1, 2, \ldots, n\} \setminus \{i_0\}$. Note that in this scenario, C^* moves from a vertex of distance 2 to another vertex of distance 1. Recall that the robber is allowed to move to a vertex that differs with u at coordinate a where $a \notin \{i_0, i_1\}$. So, after the robber's move, (3.6) and (3.7) hold.

Since C^* was at distance 2 before its move, we have

$$P_1 + P_2 + w_2 < 1. \tag{3.8}$$

After the robber's move, C^* is at distance 2 from the robber. Combining (3.6), (3.7) and (3.8), the total expected potential is at most

$$\begin{aligned}
P_1 & \left(1 - \frac{\varepsilon}{2n}\right) + w_2 + P_2 + \frac{\varepsilon}{4n} \\
& < (1 - w_2 - P_2)\left(1 - \frac{\varepsilon}{2n}\right) + w_2 + P_2 + \frac{\varepsilon}{4n} \\
& = 1 - \frac{\varepsilon}{2n} + \frac{\varepsilon(w_2 + P_2)}{2n} + \frac{\varepsilon}{4n} \\
& = 1 - \frac{\varepsilon}{4n} + \frac{\varepsilon(w_2 + P_2)}{2n} \\
& \leq 1 - \frac{\varepsilon}{8n} \\
& < 1,
\end{aligned}$$

where the second to last inequality follows from $w_2 = O\left(\frac{1}{n}\right)$ and $P_2 = O\left(\left(\frac{(m+1)^n}{n^{5/2+\varepsilon}}\right) w_{\alpha n - \rho}\right) = O\left(\frac{1}{n^{\frac{1+\varepsilon}{2}}}\right)$. Since the expected potential is less than 1, there must be a move for the robber to maintain invariant (3.5).

Case 2. Suppose v and u differ at two coordinates, say i_0 and i_1. Then the coordinate of C^* is $v = (x_1, \ldots, x_{i_0-1}, y_{i_0}, x_{i_0+1}, \ldots, x_{i_1-1}, y_{i_1}, x_{i_1+1}, \ldots, x_n)$ where $i_0 < i_1$, $y_{i_0} \neq x_{i_0}$ and $y_{i_1} \neq x_{i_1}$. Note that in this scenario, C^* moves from a vertex of distance 3 to another vertex of distance 2. Here, there is the possibility that other cops are at distance 2 from the robber. Recall that the robber is allowed to move to a vertex that differs with u at coordinate a

where $a \notin \{i_0, i_1\}$. So, after the robber's move, (3.6) and (3.7) hold.

Since C^* was at distance 3 before its move, we have

$$P_1 + P_2 + w_3 < 1. \tag{3.9}$$

After the robber's move, C^* is at distance 3 from the robber. Combining (3.6), (3.7) and (3.9), the total expected potential is at most

$$\begin{aligned}
P_1 \left(1 - \frac{\varepsilon}{2n}\right) + w_3 + P_2 + \frac{\varepsilon}{4n} \\
&< (1 - w_3 - P_2)\left(1 - \frac{\varepsilon}{2n}\right) + w_3 + P_2 + \frac{\varepsilon}{4n} \\
&= 1 - \frac{\varepsilon}{2n} + \frac{\varepsilon(w_3 + P_2)}{2n} + \frac{\varepsilon}{4n} \\
&= 1 - \frac{\varepsilon}{4n} + \frac{\varepsilon(w_3 + P_2)}{2n} \\
&\leq 1 - \frac{\varepsilon}{8n} \\
&< 1,
\end{aligned}$$

where the second to last inequality follows from $w_3 = O\left(\frac{1}{n^2}\right)$ and $P_2 = O\left(\frac{1}{n^{\frac{1+\varepsilon}{2}}}\right)$. Again, the robber may maintain invariant (3.5).

Case 3. Suppose v and u differ at at least three coordinates. Note that in this scenario, C^* moves from a vertex of distance $s + 1$ to another vertex of distance $s \geq 3$. Here, there is the possibility that other cops are at distance 2 from the robber. Choose any $i_0, i_1 \in \{1, 2, \ldots, n\}$. Recall that the robber is allowed to move to a vertex that differs with u at coordinate a where $a \notin \{i_0, i_1\}$. So, after the robber's move, (3.6) and (3.7) hold.

Since C^* was at distance $s + 1$ before its move, we have $P_1 + P_2 + w_{s+1} < 1$. Thus,

$$P_1 + P_2 < 1 - w_{s+1} < 1. \tag{3.10}$$

Now, C^* is at distance s from the robber just before the robber's move. Let w_{C^*} represent the expected weight of C^* after the robber's move. If $3 \leq s \leq \alpha n - \rho - 1$, then $w_{C^*} \leq w_s \left(1 - \frac{\varepsilon}{2n}\right) = O\left(w_3\right) = O\left(\frac{1}{n^2}\right)$. If $\alpha n - \rho \leq s \leq n-1$, then $w_{C^*} \leq w_s + O\left(\frac{w_{\alpha n - \rho}}{\sqrt{n}}\right) = O\left(w_3\right) = O\left(\frac{1}{n^2}\right)$. Hence, the expected potential is at most

$$
\begin{aligned}
&P_1\left(1 - \frac{\varepsilon}{2n}\right) + P_2 + \frac{\varepsilon}{4n} + w_{C^*} \\
&\quad < (1 - P_2)\left(1 - \frac{\varepsilon}{2n}\right) + P_2 + \frac{\varepsilon}{4n} + O\left(\frac{1}{n^2}\right) \\
&\quad = 1 - \frac{\varepsilon}{2n} + \frac{\varepsilon P_2}{2n} + \frac{\varepsilon}{4n} + O\left(\frac{1}{n^2}\right) \\
&\quad = 1 - \frac{\varepsilon}{4n} + \frac{\varepsilon P_2}{2n} + O\left(\frac{1}{n^2}\right) \\
&\quad \leq 1 - \frac{\varepsilon}{8n} \\
&\quad < 1,
\end{aligned}
$$

where the second to last inequality follows from $P_2 = O\left(\frac{1}{n^{\frac{1+\varepsilon}{2}}}\right)$.

This completes the proof of the theorem. □

CHAPTER 4: ON THE MINIMUM ORDER OF 4-LAZY COPS-WIN GRAPHS

4.1 Introduction

Sullivan et al. (2016a) proved that the minimum order of a connected graph with lazy cop number 3 is 9 and further conjectured that for a connected graph G on n vertices with $\Delta(G) \geq n - k^2$, we must have $c_L(G) \leq k$. In this chapter, we compute the exact values for m_4^l and M_4^l and prove some related results, including the above conjecture for the case $k = 4$ (see Corollary 4.4.7).

In section 4.2, we show that $c_L(P(n, 2)) = 3$ for $P(n, 2)$ of girth at least 5 (Lemma 4.2.1). Then we prove the following Theorem 4.1.1 and Theorem 4.1.2 in section 4.3 and section 4.4 respectively.

Theorem 4.1.1. *If G is a connected graph with 10 vertices and $\Delta(G) \leq 3$, then $c_L(G) \leq 3$. Furthermore, equality holds if and only if G is the Petersen graph.*

Theorem 4.1.2. *If G is a connected graph with at most 15 vertices, then $c_L(G) \leq 3$.*

The exact values for m_4^l and M_4^l (Corollary 4.1.3) can be deduced easily from Theorem 4.1.2 and the fact that $c_L(K_4 \square K_4) = 4$ (Sullivan et al., 2016a).

Corollary 4.1.3. $m_4^l = M_4^l = 16$.

We recall that for a given vertex $v \in V(G)$, its neighbourhood $N_G(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v]$ is the set $\{v\} \cup N_G(v)$. Furthermore, for any subset $U \subseteq V(G)$, $N_G(U) = \bigcup_{u \in U} N_G(u)$ and $N_G[U] = \bigcup_{u \in U} N_G[u]$. If the graph in question is clear, we shall write $N(v)$, $N[v]$, $N(U)$ and $N[U]$. A vertex occupied by a cop or robber is also called a *position*.

For the ease of reading of the upcoming proofs, let the cops $c_i, i = 1, 2, \ldots, n$ be at our disposal to play on a graph G to capture the robber r. If e is a cop or the robber and is at position $u \in V(G)$, we shall write $N_G(e)$ instead of $N_G(u)$. Similarly, $N_G[e] = N_G[u]$.

When we say a cop c moves *one step at a time* to a vertex w, we mean that c will move towards w in all cops' turn regardless of the movement of r in each robber's turn. So c will occupy w in finite steps.

Lemma 4.1.4. (Sullivan et al., 2016a, Theorem 2.5) *Assume $G = (V, E)$ has a vertex $v \in V$ with $deg(v) = 1$; say $uv \in E$ is the unique edge incident to v. Define G' to be the graph with vertex set $V' = V - \{v\}$ and edge set $E' = E - \{uv\}$. Then $c_L(G') = c_L(G)$.*

By virtue of Lemma 4.1.4, we may ignore graphs that have a vertex of degree 1. By removing vertices of degree 1, we obtain a graph with the same lazy cop number but with smaller number of vertices.

4.2 $c_L(P(n, 2))$

The generalized Petersen graph $P(n, 2)$ is the graph with vertex set

$$V(P(n, 2)) = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$$

and edge set

$$E(P(n, 2)) = \{u_i v_i, u_i u_{i+1}, v_i v_{i+2} \ : \ 1 \le i \le n\},$$

where the subscripts are taken modulo n. Note that $P(5, 2)$ is just the Petersen graph.

Lemma 4.2.1. *For $P(n,2)$ of girth ≥ 5, we have $c_L(P(n,2)) = 3$.*

Proof. Aigner and Fromme (1984) shows that for any graph G with girth at least 5, then $c(G) \geq \delta(G)$. Since $P(n,2)$ is 3-regular and $c(G) \leq c_L(G)$, this indicates that $c_L(P(n,2)) \geq 3$.

Now, it is left to show that $c_L(P(n,2)) \leq 3$. Here, we describe a winning strategy for the cops. Suppose we have 3 cops at our disposal, say c_1, c_2 and c_3. The robber will be denoted by r. If at round t, the robber is at position u_j or v_j, we set $W_t(r) = j$. We do the same for the cops c_i. We may consider W_t as the weight of a cop or the robber at round t.

Initially we place c_1 at position u_n, c_2 at position v_1 and c_3 at position v_2 (see Figure 4.1).

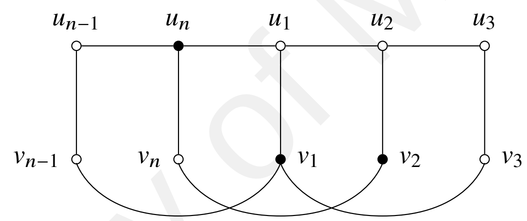


Figure 4.1: c_1 at position u_n, c_2 at position v_1 and c_3 at position v_2.

Therefore $W_1(c_1) = n$, $W_1(c_2) = 1$ and $W_1(c_3) = 2$. Note that r cannot be placed at positions $\{u_{n-1}, v_{n-1}, v_n, u_1, u_2, v_3\}$. So, initially we must have

$$\max(W_1(c_2), W_1(c_3)) = 2 < W_1(r) < n - 1 = W_1(c_1) - 1,$$

and $W_1(c_2)$ and $W_1(c_3)$ are consecutive integers.

The elements in the interval that $W_t(r)$ can lie within refers to the weight of "safe" vertices (r not being caught in round $t+1$) that r can occupy at round t. Initially, the size of the interval that $W_1(r)$ can lie within is $W_1(c_1) - 2 - \max(W_1(c_2), W_1(c_3)) = n - 4$.

Suppose that at round t, we have

$$\max\left(W_t(c_2), W_t(c_3)\right) < W_t(r) < W_t(c_1) - 1,$$

and $W_t(c_2)$ and $W_t(c_3)$ are consecutive integers. The size of the interval that $W_t(r)$ can lie within is $s = W_t(c_1) - 2 - \max\left(W_t(c_2), W_t(c_3)\right)$. Now, we shall give a strategy depending on the value of $W_t(r)$ that will reduce the size of the interval that $W_{t+1}(r)$ can lie within.

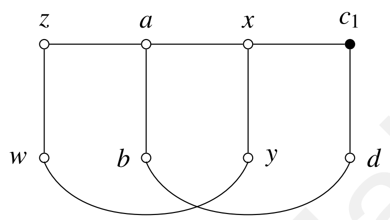


Figure 4.2: Robber is at position z or w.

Scenario 1. Suppose $W_t(r) = W_t(c_1) - 3$ (see Figure 4.2). So r is at position z or w. We move the cop c_1 to position x. At robber's turn, if r is at position z, he cannot move to a, otherwise he will be caught in the next round. Similarly, if r is at position w, he cannot move to y. Thus, at round $t + 1$, we must have $W_{t+1}(r) < W_{t+1}(c_1) - 1 = W_t(c_1) - 2$. Note that $W_{t+1}(c_2) = W_t(c_2)$, $W_{t+1}(c_3) = W_t(c_3)$ and $W_{t+1}(r) = W_t(r), W_t(r) - 1$ or $W_t(r) - 2$. So, $W_{t+1}(c_2)$ and $W_{t+1}(c_3)$ are still consecutive integers. If $W_{t+1}(r) > \max\left(W_{t+1}(c_2), W_{t+1}(c_3)\right)$, then we have achieved our objective for the size of the interval that $W_{t+1}(r)$ can lie within is $W_{t+1}(c_1) - 2 - \max\left(W_{t+1}(c_2), W_{t+1}(c_3)\right) = W_t(c_1) - 3 - \max\left(W_t(c_2), W_t(c_3)\right) = s - 1$. Recall that the size of the interval that $W_t(r)$ can lie within is s.

Suppose $W_{t+1}(r) \leq \max\left(W_{t+1}(c_2), W_{t+1}(c_3)\right)$. We may assume $W_{t+1}(c_2) = W_{t+1}(c_3) - 1$ because $W_{t+1}(c_2)$ and $W_{t+1}(c_3)$ are consecutive integers. Since $\max\left(W_t(c_2), W_t(c_3)\right) < W_t(r)$, this can only happen if $W_{t+1}(r) = W_t(r) - 1$ or $W_t(r) - 2$. If $W_{t+1}(r) = W_t(r) - 2$, then r must be at position w or f at round t (see Figure 4.3), and at his turn, he moves to the position a cop is occupying. This is absurd. If $W_{t+1}(r) = W_t(r) - 1$, then r must be at position z at round t, and at his turn, he moves to e. The robber will be caught at round $t+1$ by the cop c_3.

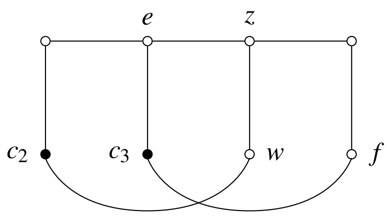


Figure 4.3: $W_t(c_2) = W_t(c_3) - 1$.

Scenario 2. Suppose $W_t(r) \neq W_t(c_1)-3$. Assume that $W_t(c_2) = W_t(c_3)-1$ (see Figure 4.3). We move the cop c_2 to position w. At the robber's turn, he cannot move to z, otherwise he will be caught at round $t+1$. So, we must have $W_t(c_3)+1 = \max(W_{t+1}(c_2), W_{t+1}(c_3)) < W_{t+1}(r)$. Since $W_t(r) < W_t(c_1) - 1$ and $W_t(r) \neq W_t(c_1) - 3$, either $W_t(r) = W_t(c_1) - 2$ or $W_t(r) < W_t(c_1)-3$. If $W_t(r) = W_t(c_1)-2$, then r is at position a or b (see Figure 4.2). If r is at a, he cannot move to x, otherwise he will be caught at round $t+1$ by the cop c_1. Similarly, if r is at b, he cannot move to d. Thus, $W_{t+1}(r) < W_{t+1}(c_1)-1 = W_t(c_1)-1$. If $W_t(r) < W_t(c_1)-3$, then $W_{t+1}(r) < W_t(c_1)-1$, for $W_{t+1}(r) \leq W_t(r)+2$. Hence, we must have $W_{t+1}(r) < W_{t+1}(c_1)-1$. We have achieved our objective for the size of the interval that $W_{t+1}(r)$ can lie within is $W_{t+1}(c_1) - 2 - \max(W_{t+1}(c_2), W_{t+1}(c_3)) = W_t(c_1) - 2 - (W_t(c_3) + 1) = s - 1$.

From Scenario 1 and 2, we see that either the robber is caught or the interval is getting smaller and smaller. This process cannot go on indefinitely. So, the robber will be caught eventually.

This completes the proof of the lemma. □

4.3 Proof of Theorem 4.1.1

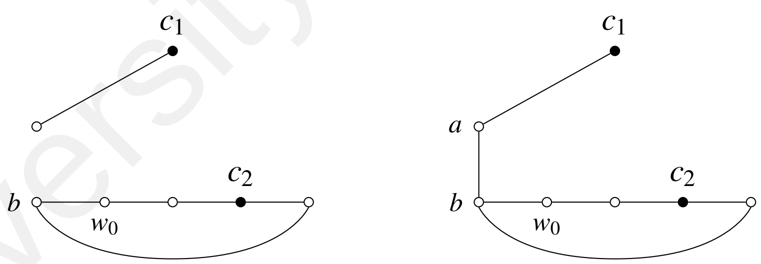
Lemma 4.3.1. *Let G be a connected graph on 10 vertices with $\Delta(G) = 3$. If $G - N[v]$ is not a 6-cycle for all $v \in V(G)$ with $deg(v) = 3$, then $c_L(G) \leq 2$.*

Proof. Let c_1 and c_2 be the two cops at our disposal to catch the robber r in G.

Case 1. Suppose there is a vertex $u_0 \in V(G)$ with $deg_G(u_0) = 3$ such that $\Delta(G - N[u_0]) \leq 2$.

Since $\Delta(G - N[u_0]) \leq 2$, every component in $G - N[u_0]$ is a path or a cycle. Initially, we place the two cops at position u_0. Then r can only be placed at a component H in $G - N[u_0]$. As long as there is a cop occupying u_0, r will have to remain in H.

- If H is a path, then we keep c_1 at u_0 and move c_2 to a vertex in H. Since $c_L(H) = 1$, r will be caught by c_2 eventually.
- Suppose H is a cycle. By the hypothesis of the lemma, H cannot be a 6-cycle. We shall assume H is a 5-cycle. The case H is a 4-cycle or a 3-cycle can be proved similarly.
  - Assume there is a vertex $w_0 \in V(H)$ with $deg_G(w_0) = 2$. Then w_0 is not adjacent to any vertices in $N[u_0]$. There are two possibilities (see Figure 4.4). We keep c_1 at u_0 and move c_2 into position as in Figure 4.4.

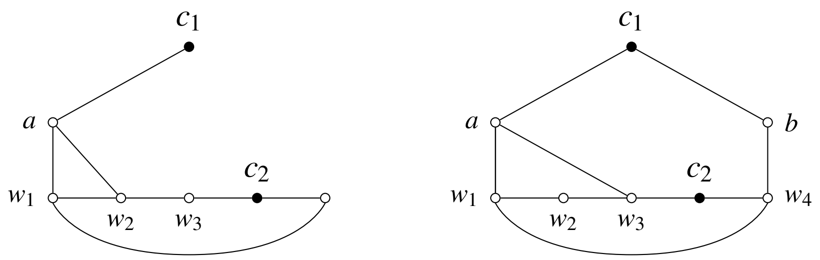


(a) (b)

Figure 4.4: Two possible graphs such that $deg_G(w_0) = 2$**.**

Since $deg_G(b) \leq 3$ and $deg_H(b) = 2$, b is not adjacent to any vertices in $N(u_0)$ (Figure 4.4 (a)) or b is adjacent to $a \in N(u_0)$ (Figure 4.4 (b)). In either case, r can only stay at positions b or w_0. In Figure 4.4 (a), we keep c_2 at his position and move c_1 to position w_0 one step at a time. In Figure 4.4 (b), we keep c_2 at his position and move c_1 to position b via a. In either case, r will be caught.

– Assume $deg_G(w) = 3$ for all $w \in V(H)$.



(a) (b)

Figure 4.5: Positions of c_1 and c_2.

Since $deg_H(w) = 2$, $N(w) \cap N(u_0) = 1$ for all $w \in V(H)$. This means there is a vertex $a \in N(u_0)$ with $|N(a) \cap V(H)| = 2$. We keep c_1 at u_0 and move c_2 into position as in Figure 4.5. Note that r can only stay at positions w_1 or w_2. In Figure 4.5 (a), we keep c_2 at his position and move c_1 to a. The robber will be caught. In Figure 4.5 (b), we move c_2 to w_3. Then r can be at positions w_1 or w_4 only. Now move c_1 to b. At robber's turn, he can only remain at w_1. In the next round, we move c_1 from b to w_4. The robber will be caught.

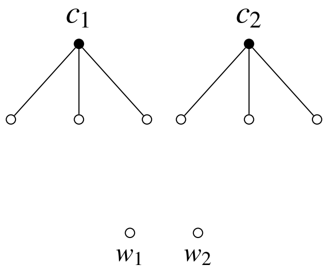
Case 2. Suppose $\Delta(G - N[u]) = 3$ for all $u \in V(G)$ with $deg_G(u) = 3$.

Pick a vertex $u_0 \in V(G)$ with $deg_G(u_0) = 3$ and pick a vertex $v_0 \in V(G - N[u_0])$ with $deg_{G-N[u_0]}(v_0) = 3$. By what we have stated in Case 2, such a v_0 can always be found.

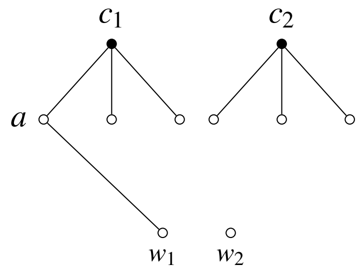
Initially we place c_1 at u_0 and c_2 at v_0. Note that $G - N[u_0] - N[v_0]$ is a disjoint union of 2 vertices or a 2-path. Let $V(G - N[u_0] - N[v_0]) = \{w_1, w_2\}$.

Suppose $G - N[u_0] - N[v_0]$ is a disjoint union of 2 vertices. We may assume r is at position w_1. Since $deg(w_1) \leq 3$, there is a c_i such that $|N(c_i) \cap N(w_1)| \leq 1$ for some $i = 1, 2$. We may assume $|N(c_1) \cap N(w_1)| \leq 1$ (see Figure 4.6).

In Figure 4.6 (a), we keep c_2 at his position and move c_1 to w_1 one step at a time. Note that r can only remain at w_1 for c_2 is occupying v_0. So, the robber will be caught. In



(a) $|N(c_1) \cap N(w_1)| = 0$



(b) $|N(c_1) \cap N(w_1)| = 1$

Figure 4.6: Two possible graphs such that $|N(c_1) \cap N(w_1)| \leq 1$.

Figure 4.6 (b), we keep c_2 at his position and move c_1 to a. The robber will also be caught.

Suppose $G - N[u_0] - N[v_0]$ is a 2-path.

(i) $|N(w_2) \cap N(c_1)| = 0$ and $|N(w_1) \cap N(c_1)| \leq 1$.

This situation is quite similar like the one in Figure 4.6 except that w_1 and w_2 are adjacent. So, we use the same cop- winning strategy, that is, keep c_2 at his position and move c_1 towards w_1. The robber will be caught.

(ii) $|N(w_2) \cap N(c_1)| = 0$ and $|N(w_1) \cap N(c_1)| = 2$.

Since $deg_G(w_1) = 3$, w_1 is not adjacent to any vertices in $N(c_2)$, i.e., $|N(w_1) \cap N(c_2)| = 0$. If $|N(w_2) \cap N(c_2)| \leq 1$, then the cops will have winning strategy similar to (i). So, we may assume $|N(w_2) \cap N(c_2)| = 2$ (see Figure 4.7). If r is at w_1, then we move c_2 to b and in the next round from b to w_2. The robber will be caught. If r is at w_2, then we move c_1 to a and in the next round from a to w_1. The robber will also be caught.

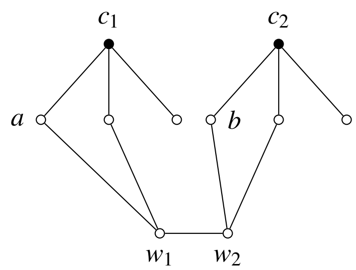


Figure 4.7: $|N(w_2) \cap N(c_2)| = 2$.

From (i) and (ii), we may assume that $|N(w_i) \cap N(c_1)| = 1 = |N(w_i) \cap N(c_2)|$ for $i = 1, 2$ (see Figure 4.8). There are two possibilities. In Figure 4.8(a), we move c_1 to z_1. The robber will be caught. Recall that w_1 and w_2 is not adjacent to $N(c_1)$ except z_1. Otherwise $|N(w_i) \cap N(c_2)| = 0$, which is similar to (i).

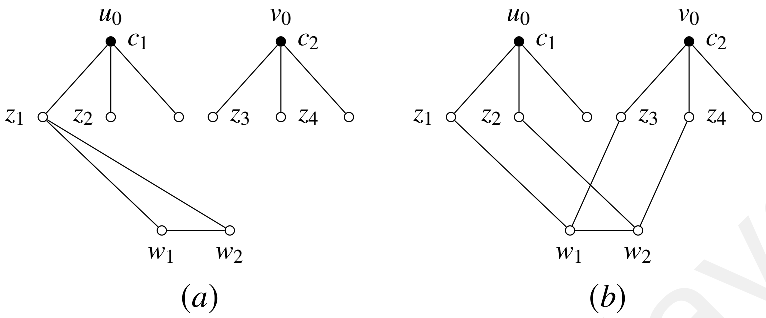


Figure 4.8: Two possible graphs such that $|N(w_2) \cap N(c_2)| = 2$.

Now, from the graph in Figure 4.8 (b), we remove $N[w_1]$ from G (see Figure 4.9).

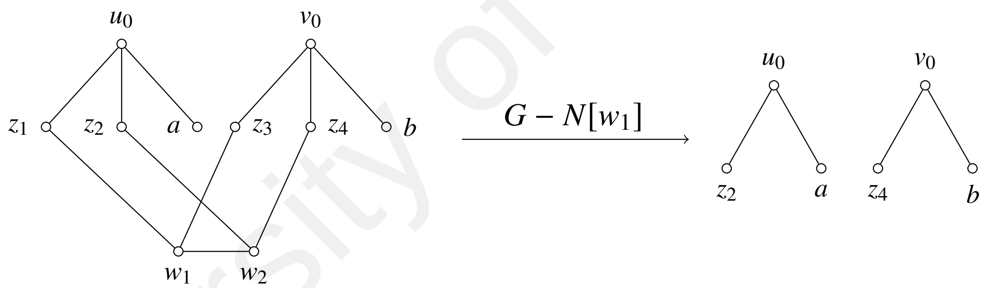


Figure 4.9: $N[w_1]$ is removed from Figure 4.8 (b).

Let $J_1 = G - N[w_1]$. From what we assume in Case 2, there is a vertex of degree 3 in J_1. Note that u_0, v_0, z_2 and z_4 are at most of degree 2 in J_1. We may assume a is of degree 3 in J_1.

- Suppose a is adjacent to vertices z_2 and b (see Figure 4.10 (a)). We move c_1 to z_1. Then r can only move to w_2 or z_2. Next, move c_2 to z_4. Then r can only move to z_2 or a. Next, move c_1 back to u_0. Then r can only move to b. Now move c_2 back to v_0. Since b is adjacent to a or z_i, the robber cannot move back to w_1 or w_2. Hence, the robber will be caught.

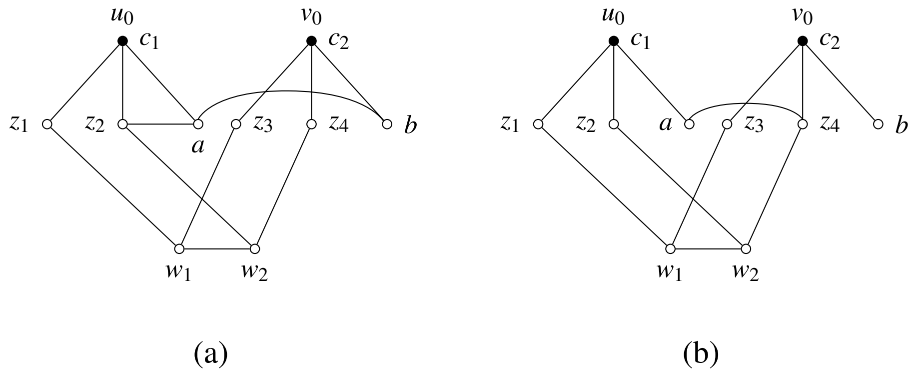


Figure 4.10: Neighbours of a.

- Suppose a is adjacent to z_4 (see Figure 4.10 (b)). Note that a cannot be adjacent to z_1 or z_3 since $deg_{J_1}(a) = 3$. It may be adjacent to z_2 or b. We move c_2 to z_3. Then r can only move to w_2 or z_4. Next, move c_1 to z_2. Then r can only move to z_4 or a. Next, move c_2 back to v_0. Then r can only move to a. Now move c_1 back to u_0. Since a is adjacent to b or z_i, the robber cannot move back to w_1 or w_2. Hence, the robber will be caught.

This completes the proof of the lemma. □

Now, we are ready to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. By Lemma 4.2.1, $c_L(P(5,2)) = 3$. So, it is sufficient to show that if G is not the Petersen graph $P(5,2)$, then $c_L(G) \leq 2$. If $\Delta(G) \leq 2$, then G is a path or a cycle, and thus, $c_L(G) \leq 2$. So, we may assume that $\Delta(G) = 3$ and G is not the Petersen graph. By Lemma 4.3.1, we may further assume that there is a vertex $u_0 \in V(G)$ with $deg(u_0) = 3$ and $J = G - N[u_0]$ is a 6-cycle. Note that each vertex in $V(J)$ is adjacent to at most one vertex in $N(u_0)$. Initially we may place two cops c_1 and c_2 at u_0. Note that the robber r can only remain in J as long as a cop is occupying u_0.

Case 1. Suppose there are two vertices $a, b \in V(J)$ such that a and b are not adjacent to any vertices in $N(u_0)$. We consider three cases where (i) a is adjacent to b in J, (ii) a and

b are separated by a vertex in J or (iii) a and b are separated by two vertices in J.

(i) Suppose a is adjacent to b in J. We keep c_1 at u_0 and move c_2 into position as in Figure 4.11.

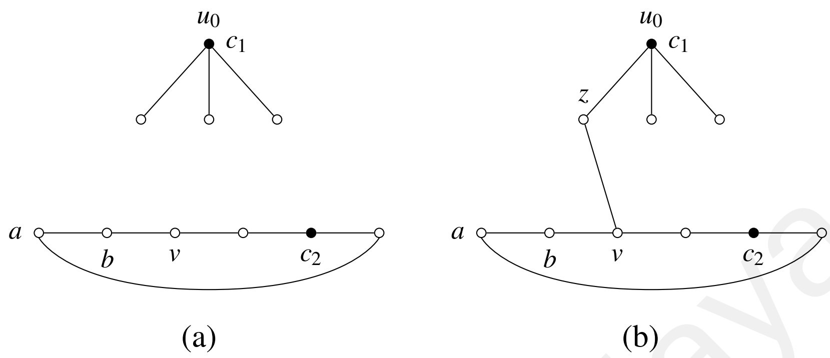


Figure 4.11: Positions of c_1 and c_2.

Note that r can only stay at a, b or v. In Figure 4.11 (a), v is not adjacent to any vertices in $N(u_0)$. So, we move c_1 towards v, one step at a time. We keep c_2 at his position. At each robber's turn, he can only remain at a, b or v. Thus he will be caught by c_1. In Figure 4.11 (b), v is adjacent to the vertex $z \in N(u_0)$. So, we move c_1 to z and then from z to v. The robber will also be caught.

(ii) Suppose a and b are separated by a vertex. We keep c_1 at u_0 and move c_2 into position as in Figure 4.12.

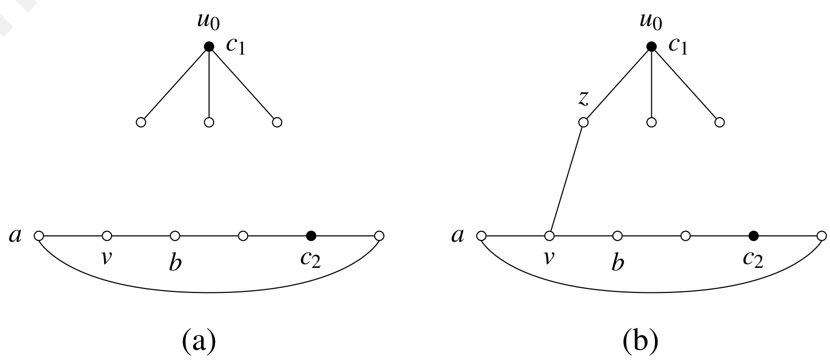


Figure 4.12: Positions of c_1 and c_2.

In Figure 4.12 (a), v is not adjacent to any vertices in $N(u_0)$. So, we move c_1 towards v, one step at a time. We keep c_2 at his position. The robber will be caught. In Figure 4.12 (b), v is adjacent to the vertex $z \in N(u_0)$. So, we move c_1 to z and then from z to v. The robber will also be caught.

(iii) Suppose a and b are separated by two vertices. By Case 1(i) and (ii), each $w_i \in V(J)\backslash\{a, b\}$ is adjacent to a vertex in $N(u_0)$. Thus there is a vertex $z \in N(u_0)$ is adjacent to two vertices w_1 and w_2 in J. There are three possibilities (see Figure 4.13). We keep c_1 at u_0 and move c_2 into position as in Figure 4.13.

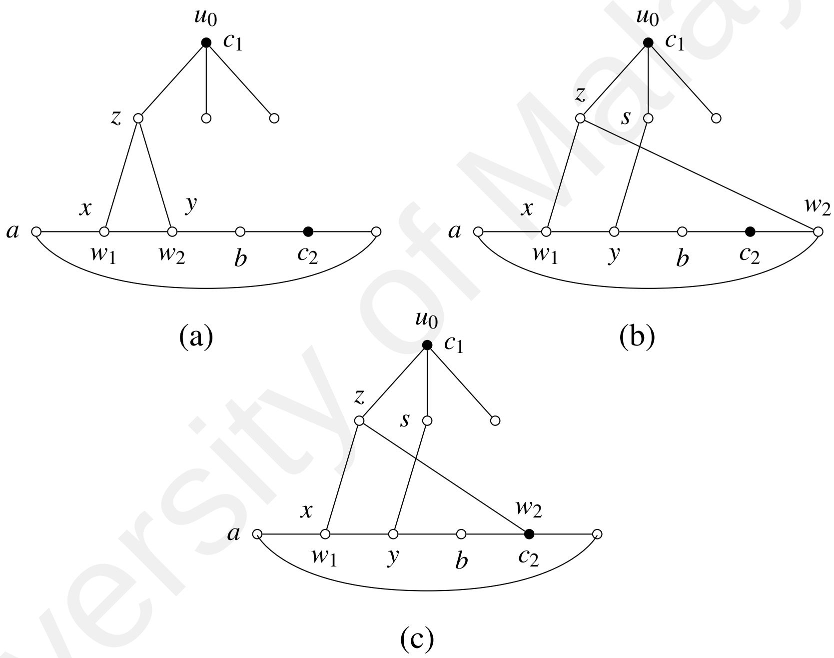


Figure 4.13: The possible graphs such that a and b are separated by two vertices.

Note that r can only stay in a, x or y. In Figure 4.13 (a), w_1 and w_2 are adjacent. We move c_1 to z and then from z to x. The robber will be caught. In Figure 4.13 (b), w_1 and w_2 are separated by one vertex in J. We move c_2 to w_2. r can only stay in x, y or b. Then, we move c_1 to s and then from s to y. The robber will be caught. In Figure 4.13 (c), w_1 and w_2 are separated by two vertices in J. We move c_1 to s, and then from s to y. Now, r can only stay in a. We move c_1 from y to x and r will be caught.

Case 2. Suppose there is a $z \in N(u_0)$ with $N(z) \cap V(J) = \{a, b\}$ such that (i) a is adjacent to b in J or (ii) a and b are separated by a vertex in J.

(i) Suppose a is adjacent to b in J. We keep c_1 at u_0 and move c_2 into position as in Figure 4.14.

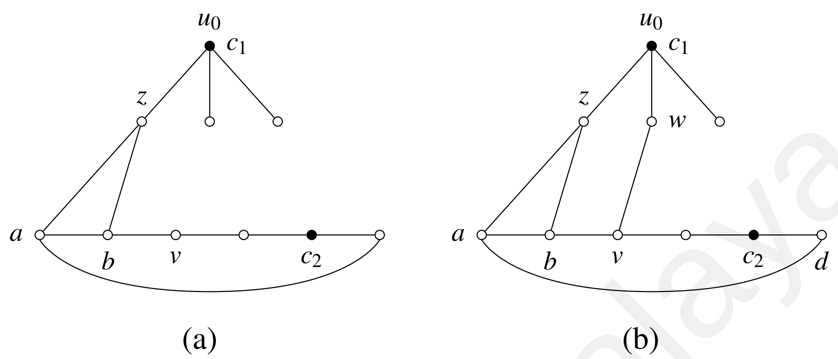


Figure 4.14: *a* and *b* are adjacent.

Note that r can only stay at a, b or v. In Figure 4.14 (a), v is not adjacent to any vertices in $N(u_0)$. So, we move c_1 to z and then from z to b. The robber will be caught. In Figure 4.14 (b), v is adjacent to the vertex $w \in N(u_0)$. So, we move c_1 to w. Note that r can only stay at a, b or z. Next, move c_2 to d and then from d to a. The robber will be caught.

(ii) Suppose a and b are separated by a vertex. We keep c_1 at u_0 and move c_2 into position as in Figure 4.15.

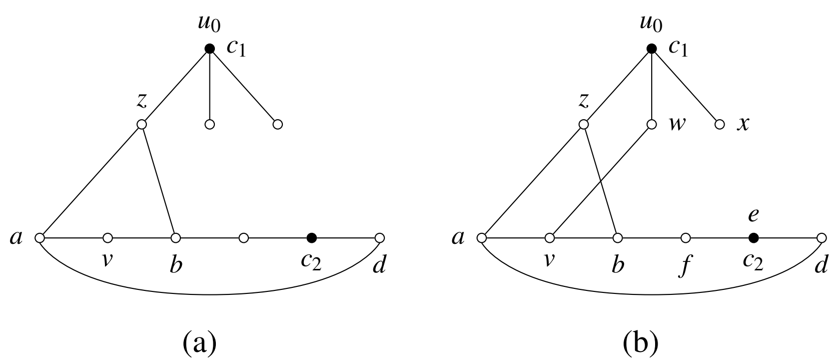


Figure 4.15: *a* and *b* are separated by one vertex.

In Figure 4.15 (a), v is not adjacent to any vertices in $N(u_0)$. So, we move c_1 to z. Note that r can only remain at v. Now move c_2 to d and then from d to a. The robber will be caught.

In Figure 4.15 (b), v is adjacent to the vertex $w \in N(u_0)$.

- Suppose w is not adjacent to any vertices in $V(J)$ except v. By Case 1, we may assume that x is adjacent to exactly 2 vertices in $\{d, e, f\}$. Therefore, $deg_G(w) = 2$. We move c_1 to z. Note that r can only remain at v or w. Next, move c_2 to f and then from f to b and from b to v. The robber will be caught.
- Suppose w is adjacent to a vertex y in $V(J)$. Note that $y \in \{d, e, f\}$. We move c_1 to z. Note that r can only remain at v or w. Next, move c_2 to y and then from y to w. The robber will be caught.

By Case 1, we must have at most one vertex in $V(J)$ not adjacent to any vertex in $N(u_0)$. Hence there exists at least two vertices $z_i \in N(u_0)$ for $i = 1, 2$ such that $|N(z_i) \cap V(J)| = 2$. By Case 2, we may assume that if there is a $z \in N(u_0)$ with $N(z) \cap V(J) = \{a, b\}$, then a and b are separated by exactly 2 vertices in J (a and b are of distance 3 in J). Since G is not the Petersen graph, we deduce that G is the unique graph isomorphic as in Figure 4.16 (a) or (b).

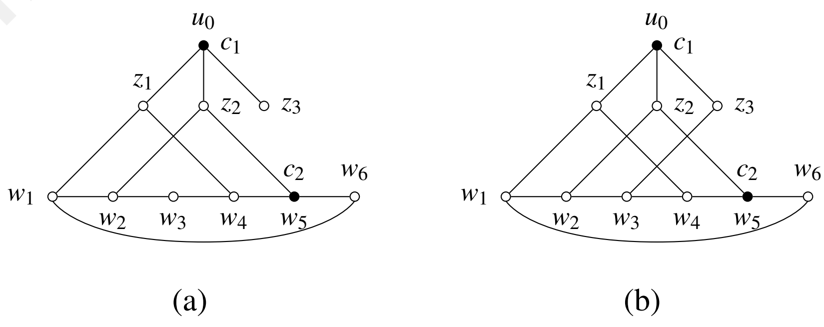


Figure 4.16: Two possible graphs of G, which is not the Petersen graph.

In Figure 4.16 (a), w_3 is not adjacent to z_3. We move c_1 to z_1. Then r can only stay at w_2 or w_3. Next, move c_2 to z_2 and then from z_2 to w_2. The robber will be caught. In Figure 4.16 (b), w_3 is adjacent to z_3. If z_3 is adjacent to w_6, then G is the Petersen graph. So we may assume that z_3 is not adjacent to w_6. We move c_1 to z_1. Then r can only stay at w_2, w_3 or z_3. Next, move c_2 to z_2. Then r can only stay at w_3 or z_3. Next, move c_2 from z_2 to w_2 and then from w_2 to w_3. The robber will be caught.

This completes the proof of the theorem. □

4.4 Proof of Theorem 4.1.2

Here, we provide some known results (Theorems 4.4.1 and 4.4.2) and prove the following lemmas which will be useful in proving Theorem 4.1.2.

Theorem 4.4.1. (Sullivan et al., 2016a, Theorem 3.1) *If G is a connected graph on at most 8 vertices, then $c_L(G) \leq 2$.*

Theorem 4.4.2. (Sullivan et al., 2016a, Theorem 2.4) *The graph $G = K_3 \square K_3$ is the unique graph on 9 vertices with $c_L(G) = 3$. All other graphs H on 9 vertices have $c_L(H) \leq 2$.*

Lemma 4.4.3. *If G is a connected graph with $\Delta(G) \leq 2$, then $c_L(G) \leq 2$.*

Proof. Since $\Delta(G) \leq 2$, G is a path or a cycle. Hence $c_L(G) \leq 2$. □

Lemma 4.4.4. *If G is a connected graph on n vertices with $\Delta(G) \leq 3$, then $c_L(G) \leq \max\left(3, \lfloor \frac{n}{4} \rfloor\right)$.*

Proof. Let $\left\lfloor \frac{n}{4} \right\rfloor = t$. We shall show that the lemma holds by using induction on t. If $t = 1$, then $n \leq 7$ and the lemma follows from Theorem 4.4.1. Assume that the lemma holds for all $1 \leq t < m$. We shall show that the lemma also holds for $n = 4m + q$ for any $0 \leq q < 4$.

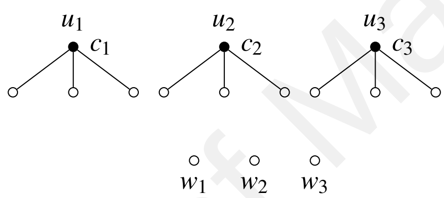
Let $u \in V(G)$ be of degree 3. If $\Delta(G - N[u]) \leq 2$, then by Lemma 4.4.3, $c_L(G - N[u]) \leq 2$. Thus, $c_L(G) \leq 3$. So, we may assume $\Delta(G - N[u]) = 3$. The number of vertices in

$G - N[u]$ is $n' = 4(m-1) + q$. If $m - 1 \geq 3$, then by induction, $c_L(G - N[u]) \leq m - 1$, and hence $c_L(G) \leq m$, the lemma holds. So, we may assume $m \leq 3$, i.e., G is a graph with at most 15 vertices. We shall show that 3 cops are enough to catch the robber. Let c_1, c_2 and c_3 denote the cops and r denotes the robber.

Let $S \subseteq V(G)$ be the set of all vertices of degree 3. A subset $M \subseteq S$ is said to be *independent* if $N[s] \cap N[s'] = \emptyset$ for all $s, s' \in M$. $M \subseteq S$ is a maximal independent set if $|M|$ is of the largest size. Note that $|M| \leq 3$, as $|V(G)| \leq 15$.

Case 1. Suppose $|M| = 3$.

Let $u_1, u_2, u_3 \in M$. Initially, we place c_i at u_i for $i = 1, 2, 3$ (see Figure 4.17).



(a)

Figure 4.17: w_1, w_2 **and** w_3 **are not adjacent.**

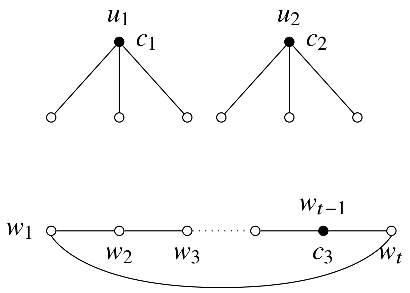
Let r be in a component J in $G - N[u_1 \cup u_2 \cup u_3]$. Let $\{w_j\} \in V(J)$ for some $j = 1, 2, 3$. Since $deg_G(w_j) \leq 3$, for any possible graph of J, $|N(J) \cap N(u_i)| \leq 1$ for some $i = 1, 2, 3$. We may assume $|N(J) \cap N(u_1)| \leq 1$. Now we move c_1 to r in J one step at a time or via the vertex of $N(J) \cap N(u_1)$ if exists. The robber r will have to remain in J as long as c_2 and c_3 are occupying u_2 and u_3, respectively. The robber will be caught.

Case 2. Suppose $|M| = 2$.

Let $u_1, u_2 \in M$. Initially we place c_1 at u_1 and c_2, c_3 at u_2. Let r be in a component J in $G - N[u_1] - N[u_2]$.

(i) Suppose J is a path or a 3-cycle. Then we keep c_1 and c_2 at their positions and use c_3 to catch the robber in J. The robber will be caught because $c_L(J) = 1$.

(ii) Suppose J is a t-cycle, $t = 4, 5, 6$ with vertex set $\{w_1, w_2, \ldots, w_t\}$ and edge set $\{w_j w_{j+1}\}$ where the subscripts are taken modulo t. We move c_3 to a vertex w_{t-1} in J as in Figure 4.18.



(a)

Figure 4.18: c_3 is placed at w_{t-1}.

Note that $|N(w_1 \cup w_2 \cup w_3) \cap N(u_i)| \leq 1$ for some $i = 1, 2$. We may assume $|N(w_1 \cup w_2 \cup w_3) \cap N(u_1)| \leq 1$. If $|N(w_1 \cup w_2 \cup w_3) \cap N(u_1)| = 0$, we move c_1 towards w_2 one step at a time. If $|N(w_1 \cup w_2 \cup w_3) \cap N(u_1)| = 1$, we move c_1 towards w_2 through the vertex of $N(w_j) \cap N(u_1)$ for some $j = 1, 2, 3$ one step at a time. We keep c_2 and c_3 at their positions all the while. The robber can only remain at w_1, w_2 or w_3. So, he will be caught.

(iii) Suppose J is a 7-cycle with vertex set $\{w_1, w_2, \ldots, w_7\}$ and edge set $\{w_j w_{j+1}\}$ where the subscripts are taken modulo 7.

Suppose there exists a vertex in the 7-cycle is not adjacent to all vertices in $N(u_1 \cup u_2)$. We may assume w_4 is not adjacent to all vertices in $N(u_1 \cup u_2)$. Then we move c_3 to w_6 in J as in Figure 4.19. Note that $|N(w_1 \cup w_2 \cup w_3) \cap N(u_i)| \leq 1$ for some $i = 1, 2$. We may assume $|N(w_1 \cup w_2 \cup w_3) \cap N(u_1)| \leq 1$. Then we move c_1 similarly as in Case 2(ii). Then, we move c_1 to w_3 if the robber is at w_4. The robber can only remain at w_1, w_2, w_3 or w_4 as c_2 and c_3 remain throughout the game. So, he will be caught.

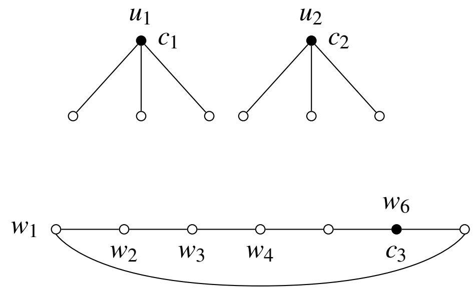


Figure 4.19: w_4 **is not adjacent to all vertices in** $N(u_1 \cup u_2)$ **and** c_3 **is moved to** w_6**.**

Now, suppose all vertices in J are adjacent to some vertices in $N(u_1 \cup u_2)$. Hence, there are at least four vertices in J are adjacent to some vertices $N(u_i)$ for some $i = 1, 2$. Following this, there exists four consecutive vertices in J such that at least three of the four vertices are adjacent to some vertices in $N(u_i)$ for some $i = 1, 2$. Without loss of generality, let $\{w_1, w_2, w_3, w_4\}$ be the consecutive four vertices such that at least three of them are adjacent to some vertices in $N(u_1)$. This indicates that $|N(w_1 \cup w_2 \cup w_3 \cup w_4) \cap N(u_2)| \leq 1$. We now move c_3 to w_6. The robber can only stay in w_1, w_2, w_3 or w_4. Then we move c_2 to r in J one step at a time or via the vertex of $N(w_1 \cup w_2 \cup w_3 \cup w_4) \cap N(u_2)$ if exists. The cops c_2 and c_3 remain throughout the game. Hence r has to remain in w_i for some $i = 1, 2, 3, 4$ and he will be caught.

Case 3. Suppose $|M| = 1$. Let $u \in M$. Then $\Delta(G - N[u]) \leq 2$. By Lemma 4.4.3, $c_L(G - N[u]) \leq 2$. Hence $c_L(G) \leq 3$.

This completes the proof. □

Lemma 4.4.5. *If G is a connected graph on n vertices with $\Delta(G) \geq n - 9$, then $c_L(G) \leq 3$.*

Proof. Let $u \in V(G)$ with $deg(u) = \Delta(G) \geq n - 9$. We have three lazy cops at our disposal, and we shall choose initially to place all the three cops at u. Let H be the component of $G - N[u]$ for which the robber is placed initially. Two of the cops at position u will be moved to H, one at a time. One cop will remain at u at all time. This restricts the robber to move only on the vertices of H throughout the game. Since $|V(H)| \leq 8$, by Theorem 4.4.1, $c_L(H) \leq 2$. Hence, 3 cops are sufficient in catching the robber. □

Lemma 4.4.6. *Let G be a connected graph with 15 vertices and there is at least one vertex of degree 4. If $G - N[u]$ is the Petersen graph for all $u \in V(G)$ with $deg_G(u) = 4$, then $\Delta(G) \geq 5$.*

Proof. It is sufficient to show that there is a vertex in $V(G)$ with degree 5. Let $u_1 \in V(G)$ with $deg_G(u_1) = 4$. Since $G - N[u_1]$ is the Petersen graph, there is a vertex u_2 in $N(u_1)$ adjacent to a vertex v_1 in $V(G - N[u_1])$. We may assume the graph is as in Figure 4.20.

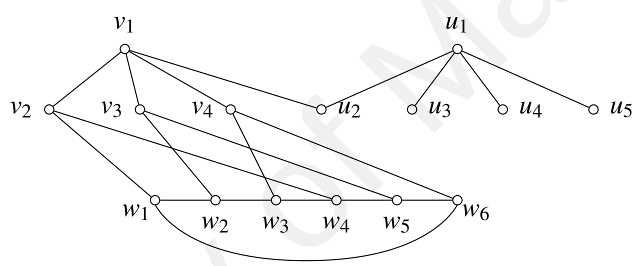


Figure 4.20: A vertex u_2 in $N(u_1)$ adjacent to a vertex v_1 in $V(G - N[u_1])$.

Now consider $G - N[v_1]$ (see Figure 4.21). Since the resulting graph must be the Petersen graph, we may assume u_3 is adjacent to w_1 and w_4, u_4 is adjacent to w_2 and w_5, and u_5 is adjacent to w_3 and w_6 (see Figure 4.22).

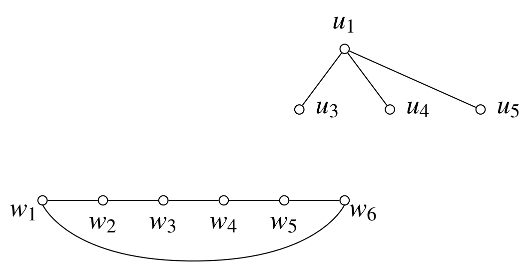


Figure 4.21: $G - N[v_1]$

Now consider $G - N[w_1]$ (see Figure 4.23). Since the resulting graph is the Petersen graph, w_4 must be adjacent to u_2. Hence, $deg_G(w_4) = 5$.

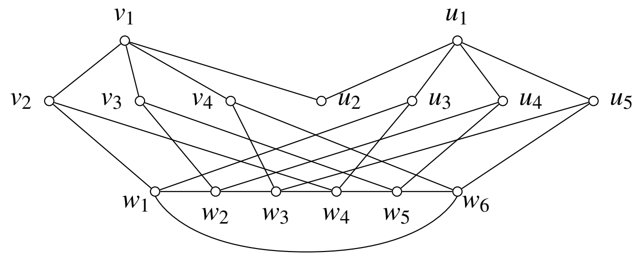


Figure 4.22: u_3 is adjacent to w_1 and w_4, u_4 is adjacent to w_2 and w_5, and u_5 is adjacent to w_3 and w_6

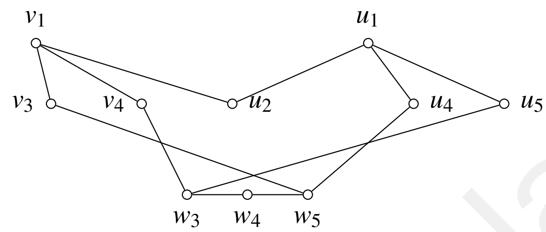


Figure 4.23: $G - N[w_1]$

□

We are now ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. We first consider the case when $|V(G)| \leq 14$. By Lemmas 4.4.3, 4.4.4 and 4.4.5, we shall only need to deal with the case when $\Delta(G) = 4$. Let u be a vertex in G with degree 4. Observe that $G - N[u]$ has at most 9 vertices and that $G - N[u]$ is not the graph $K_3 \square K_3$. So, by Theorems 4.4.1 and 4.4.2, $c_L(G - N[u]) \leq 2$, implying $c_L(G) \leq 3$.

We now assume that $|V(G)| = 15$. If $\Delta(G) \leq 3$, then by Lemmas 4.4.3 and 4.4.4, $c_L(G) \leq 3$. If $\Delta(G) \geq 6$, then by Lemma 4.4.5, $c_L(G) \leq 3$. Suppose $\Delta(G) = 5$. Let $u \in V(G)$ with $deg_G(u) = 5$. Initially, place all the three cops c_1, c_2 and c_3 at u. Then the robber r must be at one of the components in $G - N[u]$, say H. Note that r has to remain in H as long as there is a cop occupying u. If H has at most 8 vertices, then by Theorem 4.4.1, $c_L(H) \leq 2$. If H has 9 vertices and $H \not\cong K_3 \square K_3$, then by Theorem 4.4.2, $c_L(H) \leq 2$. In either case, we keep c_1 at u and use c_2 and c_3 to catch the robber in

H. Suppose $H \cong K_3 \square K_3$. There is a vertex $w \in V(H)$ with $\deg_G(w) = 5$ because G is connected. Since $K_3 \square K_3$ is vertex transitive, we may assume w is the vertex at the top left of $K_3 \square K_3$. Now keep c_1 at u and move c_2 to the center vertex of $K_3 \square K_3$. After that, move c_3 to the bottom right vertex of $K_3 \square K_3$. Note that r can only stay at w. Now move c_1 to w through the only vertex in $N(u) \cap N(w)$. The robber will be caught.

So, we are only left with the case when $\Delta(G) = 4$. Let $S \subseteq V(G)$ be the set of all vertices of degree 4. A subset $M \subseteq S$ is said to be independent if $N[w] \cap N[w'] = \emptyset$ for all $w, w' \in M$ with $w \neq w'$. $M \subseteq S$ is a maximal independent set if $|M|$ is of the largest size. Note that $|M| \leq 3$. We shall show that three cops c_1, c_2, c_3 are sufficient to catch the robber for each of the possible size of M.

Case 1. Suppose $|M| = 3$.

Let $w_1, w_2, w_3 \in M$. Place c_i at w_i for $i = 1, 2, 3$. Since $V(G) = N[w_1] \cup N[w_2] \cup N[w_3]$, the robber will be caught.

Case 2. Suppose $|M| = 2$.

Let $w_1, w_2 \in M$. Place c_1 and c_3 at w_1 and c_2 at w_2. The robber r must be at one of the components in $G - N[w_1] - N[w_2]$, say J. Since $|M| = 2$, $\Delta(J) \leq 3$. Note that r has to remain in J as long as w_1 and w_2 are occupied by cops.

(i) $\Delta(J) = 3$.

Let $a \in V(J)$ with $deg_J(a) = 3$. We keep c_1 and c_2 at w_1 and w_2, respectively, and move c_3 to a. Since $|N[w_1] \cup N[w_2] \cup N[a]| = 14$, r must be at the remaining vertex, say b. Since $deg_G(b) \leq 4$, there is a c_i $(1 \leq i \leq 3)$ such that $|N(b) \cap N(c_i)| \leq 1$. Now move c_i to b one step at a time or via the vertex in $N(b) \cap N(c_i)$ (if exists). The robber will be caught.

(ii) $\Delta(J) \leq 2$. Then J is a path or a s-cycle where $s \leq 5$.

- If J is a path or a 3-cycle, then we keep c_1 and c_2 at w_1 and w_2, respectively, and use c_3 to catch the robber in J.
- If J is a 4-cycle, then we keep c_1 and c_2 at w_1 and w_2, respectively, and move c_3 to a vertex in J. Note that r must be at the remaining vertex in J, say b (see Figure 4.24).

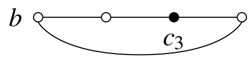


Figure 4.24: J is a 4-cycle.

Since $deg_G(b) \leq 4$, there is a c_i $(1 \leq i \leq 2)$ such that $|N(b) \cap N(c_i)| \leq 1$. Now move c_i to b one step at a time (if $|N(b) \cap N(c_i)| = 0$) or via the vertex in $N(b) \cap N(c_i)$ (if $|N(b) \cap N(c_i)| = 1$). The robber will be caught.

- If J is a 5-cycle, then we keep c_1 and c_2 at w_1 and w_2, respectively, and move c_3 to a vertex in J (see Figure 4.25).

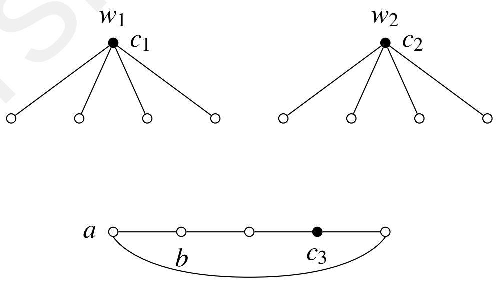


Figure 4.25: Positions of c_1, c_2 and c_3.

If there is no edge connecting a vertex in $\{a, b\}$ with a vertex in $N(c_1)$, then we move c_1 to a one step at a time. Note that r can stay at a or b only, as long as c_2 and c_3 are at their positions. So, the robber will be caught. Hence we may assume there is an edge connecting a vertex in $\{a, b\}$ with a vertex in $N(c_1)$.

Without loss of generality, we may assume a is adjacent to a vertex z in $N(c_1)$ (see Figure 4.26).

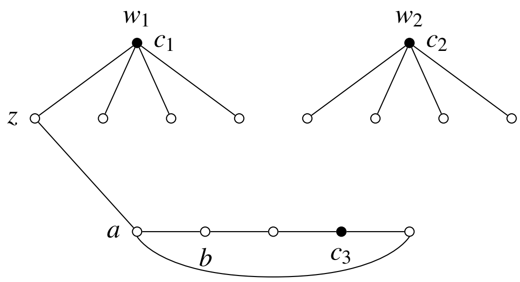


Figure 4.26: a **is adjacent to a vertex** z **in** $N(c_1)$**.**

Suppose $N(a) \cap N(c_2) = \emptyset$.

i. If $|N(b) \cap N(c_2)| \leq 1$, then keep c_1 and c_3 at their positions and move c_2 to b one step at a time (if $|N(b) \cap N(c_2)| = 0$) or via the vertex in $N(b) \cap N(c_2)$ (if $|N(b) \cap N(c_2)| = 1$). The robber will be caught.

ii. If $|N(b) \cap N(c_2)| = 2$, then $N(b) \cap N(c_1) = \emptyset$. If $|N(a) \cap N(c_1)| = 1$, move c_1 to a via z. The robber will be caught. For if $|N(a) \cap N(c_1)| = 2$ (see Figure 4.27), we move c_2 to b via x if the robber is at a and move c_1 to a via z if the robber is at b. In either case, the robber will be caught.

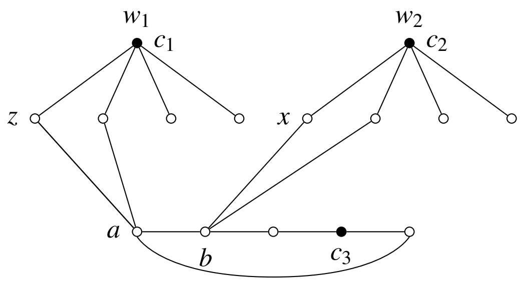


Figure 4.27: $|N(a) \cap N(c_1)| = 2$**.**

So we may assume $|N(a) \cap N(c_2)| = 1$ (see Figure 4.28).

If $|N(b) \cap N(c_2)| = 0$, then keep c_1 and c_3 at their positions and move c_2 to a via x. If $|N(b) \cap N(c_1)| = 0$, then keep c_2 and c_3 at their positions and move

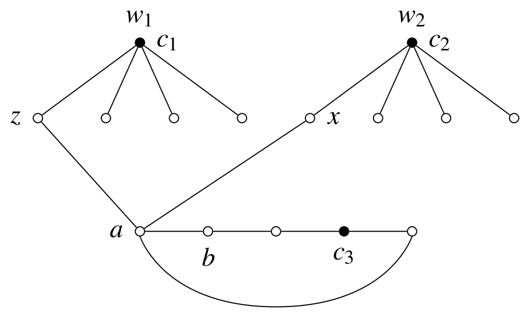


Figure 4.28: $|N(a) \cap N(c_2)| = 1$.

c_1 to a via z. In either case, the robber will be caught. So we may assume $|N(b) \cap N(c_i)| = 1$ for $i = 1, 2$. If b is adjacent to z, then keep c_2 and c_3 at their positions and move c_1 to z. The robber will be caught. So we may assume b is not adjacent to z. Similarly, we may assume b is not adjacent to x (see Figure 4.29).

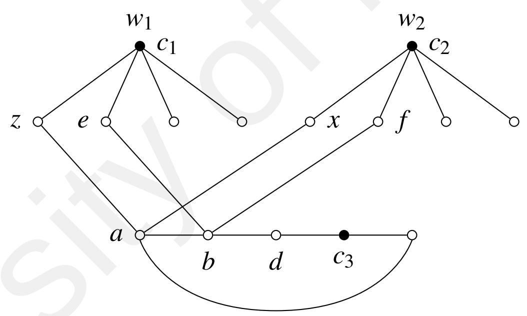


Figure 4.29: $|N(b) \cap N(c_i)| = 1$ **for** $i = 1, 2$.

Note that r can be at a or b. We shall assume r is at a. The case r is at b is similar. Move c_2 to x. Then r will have to move to b. Next, move c_1 to e. Then r will have to move to f. Now, move c_2 back to w_2. If f is not adjacent to a vertex in $N(w_1)\backslash\{e\}$, then r will be caught in the next cop's turn. If f is adjacent to a vertex in $N(w_1)\backslash\{e\}$, then r will have to move from f to a vertex in $N(w_1)\backslash\{e\}$. Now, move c_1 back to w_1. At robber's turn, if r is not at z, he will be caught in the next cop's turn. So, r must be at z and f is adjacent to z.

Now, reset the movements and assume r is at a. Move c_1 to z. Then r will have to move to b. Next, move c_2 to f. Then r will have to move to e. Now, move c_1 back to w_1. If e is not adjacent to a vertex in $N(w_2)\backslash\{f\}$, then r will be caught in the next cop's turn. If e is adjacent to a vertex in $N(w_2)\backslash\{f\}$, then r will have to move from e to a vertex in $N(w_2)\backslash\{f\}$. Now, move c_2 back to w_2. At robber's turn, if r is not at x, he will caught in the next cop's turn. So r must be at x and e is adjacent to x (see Figure 4.30).

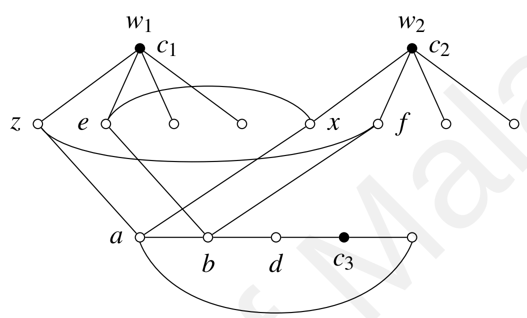


Figure 4.30: Positions of c_1, c_2 and c_3 initially.

Reset the movements and assume r is at a. Now, move c_2 to x. Then r will have to move to b. Next, move c_1 to z. Then r will have to remain at b. Move c_3 to d. The robber will be caught.

Case 3. Suppose $|M| = 1$.

Then $\Delta(G - N[u]) \leq 3$ for all $u \in V(G)$ with $deg_G(u) = 4$.

Suppose there is a vertex $w \in V(G)$ with $deg_G(w) = 4$ such that $G - N[w]$ is not connected. Place all the cops at w. The robber r must be at one of the components in $G - N[w]$, say J. If $\Delta(J) \leq 2$, then by Lemma 4.4.3, $c_L(J) \leq 2$. So, we keep one cop at w and use the other two cops to catch the robber in J. Similarly, by Theorems 4.4.1 and 4.4.2, we may assume $J = K_3 \square K_3$ or $|V(J)| = 10$. The former cannot happen because $|M| = 1$. The latter also cannot happen because $G - N[w]$ is not connected.

So, we may assume that $G - N[u]$ is connected for all $u \in V(G)$ with $deg_G(u) = 4$. If there is a vertex $v \in V(G)$ with $deg_G(v) = 4$ such that $G - N[v]$ is not the Petersen graph, then by Theorem 4.1.1, $c_L(G - N[v]) \leq 2$. Hence, we keep one cop at v and use the other two cops to catch the robber in $G - N[v]$.

Now we may assume that $G - N[u]$ is the Petersen graph for all $u \in V(G)$ with $deg_G(u) = 4$. By Lemma 4.4.6, $\Delta(G) \geq 5$, a contradiction.

Hence, $c_L(G) \leq 3$ and this completes the proof of the theorem. □

Corollary 4.4.7. *If G is a connected graph with n vertices and $\Delta(G) \geq n - 16$, then $c_L(G) \leq 4$.*

Proof. Let $u \in V(G)$ with $deg(u) = \Delta(G)$. Place all the four cops at u. The robber r must be at a component in $G - N[u]$, say H. Note that $|V(H)| \leq 15$. By Theorem 4.1.2, $c_L(H) \leq 3$. So, we keep one cop at u and use the other three cops to catch the robber in H. □

CHAPTER 5: ON THE BURNING NUMBER OF GENERALIZED PETERSEN GRAPHS

5.1 Introduction

In this chapter, we present the burning number of the *generalized Petersen graphs*. Let $n \geq 3$ and k be integers such that $1 \leq k \leq n-1$. We recall that the generalized Petersen graph $P(n,k)$ is defined to be the graph on $2n$ vertices with vertex set

$$V(P(n,k)) = \{u_i, v_i \ : \ i = 0, 1, 2, \ldots, n-1\}$$

and edge set

$$E(P(n,k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \ : \ i = 0, 1, 2, \ldots, n-1\},$$

where subscripts are taken modulo n. Let $D_1 = \{u_i : i = 0, 1, 2, \ldots, n-1\}$ and $D_2 = \{v_i : i = 0, 1, 2, \ldots, n-1\}$. The subgraph induced by D_1 is called the *outer rim* while the subgraph induced by D_2 is called the *inner rim*. A *spoke* of $P(n,k)$ is an edge of the form $u_i v_i$ for some $0 \leq i \leq n-1$.

The following are the main results of this chapter.

Theorem 5.1.1. *Let k be a fixed positive integer. Then*

$$\left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil \leq b(P(n,k)) \leq \left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor + 2.$$

In particular,

$$\lim_{n \to \infty} \frac{b(P(n,k))}{\sqrt{\frac{n}{k}}} = 1.$$

Theorem 5.1.2. *For $n \geq 3$,*

$$\lceil \sqrt{n} \rceil \leq b(P(n,1)) \leq \lceil \sqrt{n} \rceil + 1.$$

Furthermore, the bounds are tight, and if n is a square, then $b(P(n,1)) = \sqrt{n} + 1$.

Theorem 5.1.3. *For $n \geq 3$,*

$$\left\lceil \sqrt{\frac{n}{2}} \right\rceil + 1 \leq b(P(n,2)) \leq \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 2.$$

Furthermore, the bounds are tight, and if $\frac{n}{2}$ is a square, then $b(P(n,2)) = \sqrt{\frac{n}{2}} + 2$.

Theorem 5.1.4. *For $n \geq 4$,*

$$\left\lceil \sqrt{\frac{n}{3}} \right\rceil + 1 \leq b(P(n,3)) \leq \left\lceil \sqrt{\left\lfloor \frac{n}{3} \right\rfloor} \right\rceil + 3.$$

Note that $\mathrm{dist}_G(u,u) = 0$. Given a non-negative integer s, the *s-th closed neighbourhood* of a vertex u, denoted by $N_s^G[u]$, is the set of vertices whose distance from u is at most s, i.e.,

$$N_s^G[u] = \{v \in V(G) : \mathrm{dist}_G(u,v) \leq s\}.$$

Again, if the graph in question is clear, we shall write $N_s[u]$ for $N_s^G[u]$.

Let $(x_1, x_2, \ldots, x_m)$ be a burning sequence of a graph G. As in (Bonato, Janssen, & Roshanbin, 2016, Section 2), for each pair i and j, with $1 \leq i < j \leq m$, we have $\mathrm{dist}(x_i, x_j) \geq j - i$ and

$$V(G) = N_{m-1}[x_1] \cup N_{m-2}[x_2] \cup \cdots \cup N_0[x_m]. \tag{5.1}$$

In Section 5.2, we provide bounds for the burning number of $P(n,k)$ and show that $b(P(n,k))$ is asymptotically $\sqrt{\frac{n}{k}}$. In Section 5.3, we determine the exact values of $b(P(n,k))$ for $1 \leq n \leq 8$. Then, we prove Theorems 5.1.2, 5.1.3 and 5.1.4 in Section 5.4.

5.2 General case

Lemma 5.2.1. *For $n \geq 3$ and $1 \leq k < n$,*

$$b(P(n,k)) \geq \left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil.$$

Proof. Let C be a cycle with $\lfloor \frac{n}{k} \rfloor$ vertices, $V(C) = \{0, 1, 2, \ldots, \lfloor \frac{n}{k} \rfloor - 1\}$ and $E(C) = \{i(i+1) \ : \ 0, 1, \ldots, \lfloor \frac{n}{k} \rfloor - 1\}$, where the integers are taken modulo $\lfloor \frac{n}{k} \rfloor$. Recall that the outer rim and inner rim of $P(n,k)$ are $D_1 = \{u_0, u_1, \ldots, u_{n-1}\}$ and $D_2 = \{v_0, v_1, \ldots, v_{n-1}\}$, respectively.

For each $m \in \{0, 1, 2, \ldots, n-1\}$, let

$$f(m) = \begin{cases} p, & \text{if } \ m = pk + q, \ \ 0 \leq p < \lfloor \frac{n}{k} \rfloor, 0 \leq q \leq k-1; \\ \\ \lfloor \frac{n}{k} \rfloor - 1, & \text{if } \ m = \lfloor \frac{n}{k} \rfloor k + q, \ \ 0 \leq q < k-1. \end{cases} \tag{5.2}$$

Let $\varphi : V(P(n,k)) \to V(C)$ be defined by

$$\varphi(u_i) = f(i) = \varphi(v_i), \qquad \forall i \in \{0, 1, 2, \ldots, n-1\}. \tag{5.3}$$

Clearly, φ is surjective.

Let $(x_1, x_2, \ldots, x_s)$ be a burning sequence of $P(n,k)$. We construct a burning sequence for C using the map φ as follows:

(a) At the beginning of time step 1, burn $y_1 = \varphi(x_1)$;

(b) At the beginning of time step t $(2 \le t \le s)$, if $\varphi(x_t)$ is still unburned, then burn $y_t = \varphi(x_t)$, otherwise, burn any unburned vertex $y_t \in V(C)$.

Note that in (b) above, if at the beginning of time step t $(2 \le t \le s)$, no unburned vertex can be found, then $(y_1, y_2, \ldots, y_{t-1})$ is a burning sequence of C. So we may assume that such an unburned vertex can be found at the beginning of every time step. We shall show that $(y_1, y_2, \ldots, y_s)$ is a burning sequence of C. This follows from φ is surjective and the following claim.

Claim. If $z \in V(P(n,k))$ is burned at time step t_0, then its image $\varphi(z)$ in C is burned at time step $t_1 \le t_0$.

Proof. If $z = x_1$, then it is burned at time step 1. Its image $\varphi(z) = y_1$ is also burned at time step 1. The claim is true. Assume that the claim is true for a $t_0 < s$.

Suppose z is burned at time step $t_0 + 1$. If z is a burning source, then $z = x_{t_0+1}$. By (b), $\varphi(z)$ is burned at time step $t_0 + 1$ provided that $\varphi(x_{t_0+1})$ is unburned. If $\varphi(x_{t_0+1})$ is burned, then it must be burned at an earlier time step. So the claim holds.

We may assume that $z \ne x_{t_0+1}$. Note that for any two distinct vertices $w_1, w_2 \in V(P(n,k))$ such that $\varphi(w_1), \varphi(w_2) \in V(C)$ and $|\varphi(w_1) - \varphi(w_2)| \le 1$ or $|\varphi(w_1) - \varphi(w_2)| = \lfloor \frac{n}{k} \rfloor - 1$, then $\varphi(w_1) = \varphi(w_2)$ or $\varphi(w_1)$ and $\varphi(w_2)$ are adjacent in C. We shall distinguish two cases.

Case 1. Let $z = u_l$. Then it is adjacent to v_l, u_{l+1} and u_{l-1} where the subscript are taken modulo n. Furthermore, either v_l, u_{l+1} or u_{l-1} is burned at time step t_0. So by induction, $\varphi(v_l), \varphi(u_{l+1})$ or $\varphi(u_{l-1})$ is burned at time step $t_1 \le t_0$ respectively. By equations (5.2) and (5.3), $|\varphi(u_l) - \varphi(v_l)| = 0$, $|\varphi(u_l) - \varphi(u_{l-1})| \le 1$ and $|\varphi(u_l) - \varphi(u_{l+1})| \le 1$ where $l = 1, 2, \ldots, n-2$ and $|\varphi(u_0) - \varphi(u_{n-1})| = \lfloor \frac{n}{k} \rfloor - 1$. This means that $\varphi(z) = \varphi(u_l)$ is burned at time step $t_1 + 1 \le t_0 + 1$.

Case 2. Let $z = v_l$. It is adjacent to u_l, v_{l+k} and u_{l-k} where the subscript are taken modulo n. Either u_l, v_{l-k} or v_{l+k} is burned at time step t_0. Here, we denote $v_{-i} = v_{n-i}$ for a non-negative i. So by induction, $\varphi(u_l), \varphi(v_{l+k})$ or $\varphi(v_{l-k})$ is burned at time step $t_1 \leq t_0$ respectively. By equations (5.2) and (5.3), we have $|\varphi(v_l) - \varphi(u_l)| = 0$,

$$|\varphi(v_l) - \varphi(v_{l-k})| = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1, & \text{if } l = 0, 1, 2, \ldots, k-1; \\ 1, & \text{if } l = k, k+1, \ldots, \lfloor \frac{n}{k} \rfloor k - 1; \\ 0, & \text{if } l = \lfloor \frac{n}{k} \rfloor k, \lfloor \frac{n}{k} \rfloor k + 1, \ldots, n-1. \end{cases}$$

and

$$|\varphi(v_l) - \varphi(v_{l+k})| = \begin{cases} 1, & \text{if } l = 0, 1, 2, \ldots, \left(\lfloor \frac{n}{k} \rfloor - 1\right) k - 1; \\ 0, & \text{if } l = \left(\lfloor \frac{n}{k} \rfloor - 1\right) k, \left(\lfloor \frac{n}{k} \rfloor - 1\right) k + 1, \ldots, n-1-k; \\ \lfloor \frac{n}{k} \rfloor - 1, & \text{if } l = n-k, n-k+1, \ldots, n-1. \end{cases}$$

This means that $\varphi(z) = \varphi(v_l)$ is burned at time step $t_1 + 1 \leq t_0 + 1$.

This completes the proof of the claim.

□

Therefore, given any burning sequence of $P(n, k)$, we can construct a burning sequence for C with shorter or the same length. Hence $b(P(n, k)) \geq b(C) = \left\lceil \sqrt{\lfloor \frac{n}{k} \rfloor} \right\rceil$, where the last equality follows from Theorem 2.3.1.

□

Lemma 5.2.2. *For $n \geq 3$ and $1 \leq k < n$,*

$$b(P(n, k)) \leq \left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor + 2.$$

Proof. Recall that the outer rim and inner rim of $P(n,k)$ are $D_1 = \{u_0, u_1, \ldots, u_{n-1}\}$ and $D_2 = \{v_0, v_1, \ldots, v_{n-1}\}$, respectively. Let $r = \left\lfloor \frac{n}{k} \right\rfloor$. We shall construct a burning sequence for $P(n,k)$ of length at most $\lceil \sqrt{r} \rceil + \left\lfloor \frac{k}{2} \right\rfloor + 2$. Note that a subgraph G induced by the vertices $v_0, v_k, v_{2k}, \ldots, v_{(r-1)k}$ in $P(n,k)$ is a path or cycle of order r. By Theorem 2.3.1, $b(G) = \lceil \sqrt{r} \rceil$. So there is a burning sequence $(x_1, x_2, \ldots, x_{\lceil \sqrt{r} \rceil})$ of G. We shall take $x_1, x_2, \ldots, x_{\lceil \sqrt{r} \rceil}$ as the first part of our burning sequence for $P(n,k)$.

Note that at time step $\lceil \sqrt{r} \rceil$, all $v_0, v_k, v_{2k}, \ldots, v_{(r-1)k}$ are burned. If at time step $\lceil \sqrt{r} \rceil$, u_{rk} is unburned, then we set $x_{\lceil \sqrt{r} \rceil + 1} = u_{rk}$. Otherwise, we set $x_{\lceil \sqrt{r} \rceil + 1}$ to be any unburned vertex. Since u_{ik} is adjacent to v_{ik} for $0 \leq i \leq (r-1)$, at time step $\lceil \sqrt{r} \rceil + 1$, all $u_0, u_k, u_{2k}, \ldots, u_{(r-1)k}, u_{rk}$ are burned. Furthermore, at most $k-1$ vertices are unburned in the path $u_{ik}u_{ik+1}u_{ik+2} \cdots u_{(i+1)k}$ in the outer rim (see Figure 5.1).

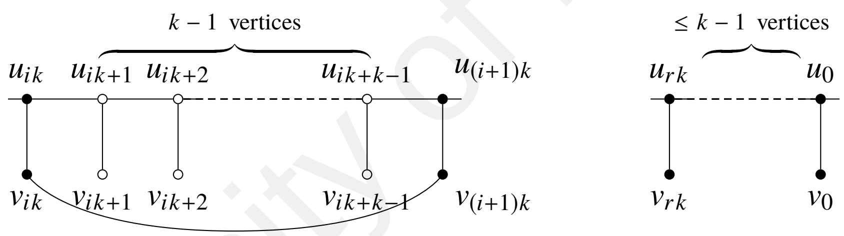


Figure 5.1: Filled vertices are burned whereas empty vertices are unburned.

Now, for $j \geq \lceil \sqrt{r} \rceil + 2$, we can choose x_j to be any unburned vertex. Note that at time step $\lceil \sqrt{r} \rceil + 1 + \left\lfloor \frac{k}{2} \right\rfloor$, all the vertices in the outer rim are burned. Since u_i and v_i are adjacent, at time step $\lceil \sqrt{r} \rceil + 2 + \left\lfloor \frac{k}{2} \right\rfloor$, all vertices in the inner rim are also burned. Hence the lemma follows. □

Proof of Theorem 5.1.1. By Lemmas 5.2.1 and 5.2.2, we have

$$\left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil \leq b(P(n,k)) \leq \left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor + 2.$$

By noting that $\lim_{n\to\infty} \frac{\left\lceil \sqrt{\lfloor \frac{n}{k} \rfloor} \right\rceil}{\sqrt{\frac{n}{k}}} = 1$ and $\lim_{n\to\infty} \frac{\lfloor \frac{k}{2} \rfloor + 2}{\sqrt{\frac{n}{k}}} = 0$, we conclude

$$\lim_{n\to\infty} \frac{b(P(n,k))}{\sqrt{\frac{n}{k}}} = 1.$$

□

5.3 Case $1 \le n \le 8$

We shall give the exact burning numbers for the case $1 \le n \le 8$ in this section. Note that $P(n,k)$ is isomorphic to $P(n, n-k)$. So we may assume that $1 \le k \le \lfloor \frac{n}{2} \rfloor$. Recall that the s-th closed neighbourhood of a vertex $x \in V(P(n,k))$ is

$$N_s[x] = \{y \in V(P(n,k)) : \ \mathrm{dist}(y,x) \le s\},$$

and the outer rim and inner rim of $P(n,k)$ are $D_1 = \{u_0, u_1, \dots, u_{n-1}\}$ and $D_2 = \{v_0, v_1, \dots, v_{n-1}\}$, respectively.

Proposition 5.3.1. *Let* $3 \le n \le 8$ *and* $1 \le k \le \lfloor \frac{n}{2} \rfloor$. *Then,*

$$b(P(n,k)) = \begin{cases} 3, & \text{if } 3 \le n \le 6 \text{ or } n{=}7,\ k \ne 1, \\ \\ 4, & \text{if } n{=}8 \text{ or } n = 7, k = 1. \end{cases}$$

Proof. Since each vertex $x \in V(P(n,k))$ is of degree 3, $|N_0[x]| = 1$, $|N_1[x]| \le 4$ and $|N_2[x]| \le 10$.

Let $3 \le n \le 7$. If (x_1, x_2) is a burning sequence of $P(n,k)$, then by equation (5.1),

$$2n \le |N_1[x_1]| + |N_0[x_2]| \le 4 + 1 = 5,$$

implying that $n < 3$, which is a contradiction. Hence $b(P(n,k)) \geq 3$. Similarly, if (x_1, x_2, x_3) is a burning sequence of $P(8,k)$, then

$$16 \leq |N_2[x_1]| + |N_1[x_2]| + |N_0[x_3]| \leq 10 + 4 + 1 = 15,$$

again is a contradiction. Hence $b(P(8,k)) \geq 4$.

Note that for each $x \in V(P(7,1))$, $|N_2[x]| = 8$. So if (x_1, x_2, x_3) is a burning sequence of $P(7,1)$, then

$$14 \leq |N_2[x_1]| + |N_1[x_2]| + |N_0[x_3]| \leq 8 + 4 + 1 = 13,$$

which is a contradiction. Hence $b(P(7,1)) \geq 4$.

Now, the proposition can be verified easily from the burning sequences in the following table (see also Figure 5.2).

Table 5.1: Burning sequences

Burning sequence	Graph
(u_0, v_1, v_2)	$P(3,1), P(4,1), P(4,2)$
(u_0, v_3, u_3)	$P(5,1), P(6,1), P(6,2)$
(u_0, u_2, v_4)	$P(5,2), P(6,3)$
(u_0, u_2, u_4, v_4)	$P(7,1)$
(u_0, u_3, v_4)	$P(7,2)$
(v_0, v_2, u_5)	$P(7,3)$
(u_0, u_2, v_4, u_4)	$P(8,1), P(8,2), P(8,3), P(8,4)$

□

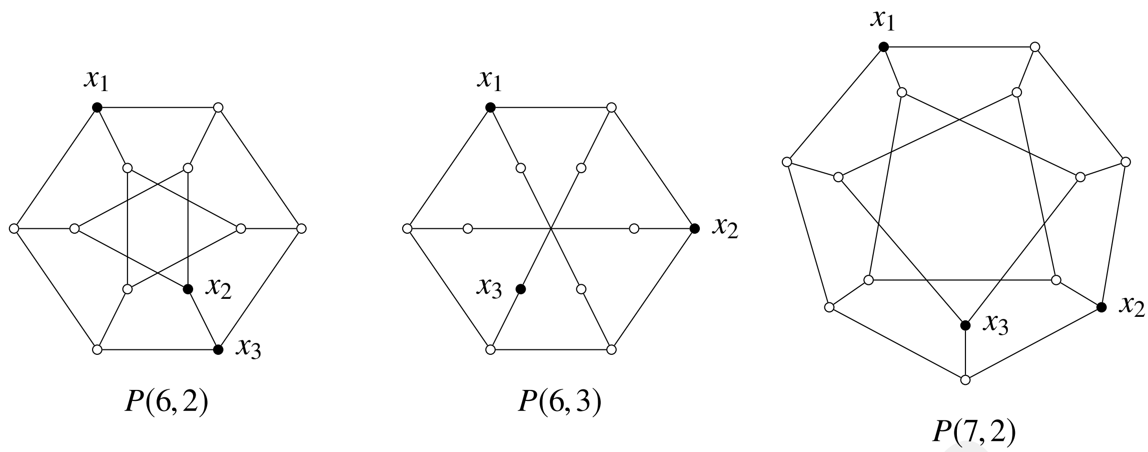


Figure 5.2: Burning sequences

5.4 Case $1 \le k \le 3$

5.4.1 Proof of Theorem 5.1.2

Note that for each $x \in V(P(n,1))$, $|N_m[x]| \le 4m$ for $m \ge 1$ and $|N_0[x]| = 1$. So if $(x_1, x_2, \ldots, x_l)$ is a burning sequence of $P(n,1)$, then by equation (5.1),

$$2n \le |N_0[x_l]| + \sum_{i=1}^{l-1} |N_{l-i}[x_i]| \le 1 + \sum_{i=1}^{l-1} 4(l-i) = 2l^2 - 2l + 1.$$

Since $l \ge 1$, by completing the square, we conclude that $l \ge \frac{2+\sqrt{4-8(1-2n)}}{4} = \frac{1}{2} + \sqrt{n - \frac{1}{4}} > \sqrt{n}$. Hence $b(P(n,1)) \ge \lceil \sqrt{n}\, \rceil$, and if n is a square, then $b(P(n,1)) \ge \lceil \sqrt{n}\, \rceil + 1$.

The subgraph C induced by the vertices in the outer rim $D_1 = \{u_0, u_1, \ldots, u_{n-1}\}$ is a cycle of length n. By Theorem 2.3.1, $b(C) = \lceil \sqrt{n}\, \rceil$. So C has a burning sequence $(y_1, y_2, \ldots, y_{\lceil \sqrt{n}\, \rceil})$. We shall take $y_1, y_2, \ldots, y_{\lceil \sqrt{n} \rceil}$ as the first part of our burning sequence for $P(n,1)$. Note that at time step $\lceil \sqrt{n} \rceil$, all the vertices in the outer rim are burned. Choose any unburned vertex z in the inner rim. Let $y_{\lceil \sqrt{n} \rceil + 1} = z$. Since $u_i v_i$ are adjacent for $1 \le i \le n-1$, at time step $\lceil \sqrt{n} \rceil + 1$ all vertices in the inner rim are also burned. Hence $b(P(n,1)) \le \lceil \sqrt{n}\, \rceil + 1$, and if n is a square, then $b(P(n,1)) = \sqrt{n} + 1$.

Finally, by Proposition 5.3.1, $b(P(5,1)) = 3 = \left\lceil \sqrt{5} \right\rceil$. So the bounds are tight. This completes the proof of Theorem 5.1.2. $\square$

5.4.2 Proof of Theorem 5.1.3

We shall first define an isomorphic graph of $P(n,2)$, say $H(n)$. Let

$$W_1 = \left\{ s_i, s'_i, t_i, t'_i : i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\};$$
$$W_2 = \left\{ t_i t_{i+1}, t'_i t'_{i+1}, s_i s'_{i+1}, s_j t_j, s'_j t'_j, s_j s'_j : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

If n is even, then let

$$V(H(n)) = W_1;$$
$$E(H(n)) = W_2 \cup \left\{ t_1 t_{\frac{n}{2}}, t'_1 t'_{\frac{n}{2}}, s_{\frac{n}{2}} s'_1 \right\}. \tag{5.4}$$

If n is odd, then let

$$V(H(n)) = W_1 \cup \{s_0, t_0\};$$
$$E(H(n)) = W_2 \cup \left\{ s_0 s_{\frac{n-1}{2}}, s_0 s'_1, s_0 t_0, t_0 t_1, t_0 t'_{\frac{n-1}{2}}, t_{\frac{n-1}{2}} t'_1 \right\}. \tag{5.5}$$

We now show that $P(n,2)$ is isomorphic to $H(n)$ (see Figures 5.3 and 5.4). Define $\phi : V(P(n,2)) \to V(H(n))$ as follows: Let $\phi(u_i) = s'_{\frac{i}{2}+1}$ if i is even and $i \neq n-1$; $\phi(u_i) = s_{\frac{i-1}{2}+1}$ if i is odd; $\phi(u_{n-1}) = s_0$ if $n-1$ is even. Let $\phi(v_i) = t'_{\frac{i}{2}+1}$ if i is even and $i \neq n-1$; $\phi(v_i) = t_{\frac{i-1}{2}+1}$ if i is odd; $\phi(v_{n-1}) = t_0$ if $n-1$ is even. Note that the subgraph induced by all the vertices s_i, s'_i in $H(n)$ is isomorphic to the outer rim in $P(n,2)$, and the subgraph induced by all the vertices t_i, t'_i in $H(n)$ is isomorphic to the inner rim in $P(n,2)$. Furthermore, $s_i t_i, s'_i t'_i$ are the spokes in $P(n,2)$. So $P(n,2)$ is isomorphic to $H(n)$.

Let $T_1 = \{t_i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, $T_2 = \{t'_i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, and *level* $L_i = \{s_i, s'_i, t_i, t'_i\}$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

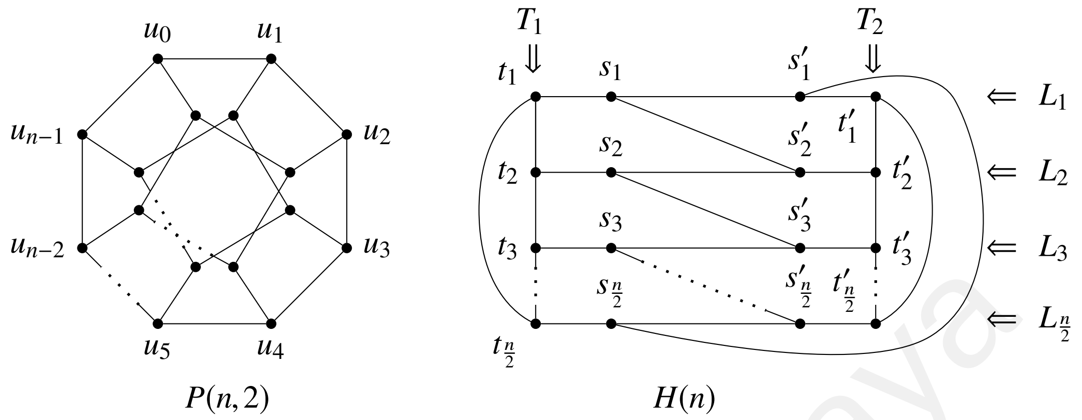


Figure 5.3: $H(n)$ **is isomorphic to** $P(n,2)$ **where** n **is even.**

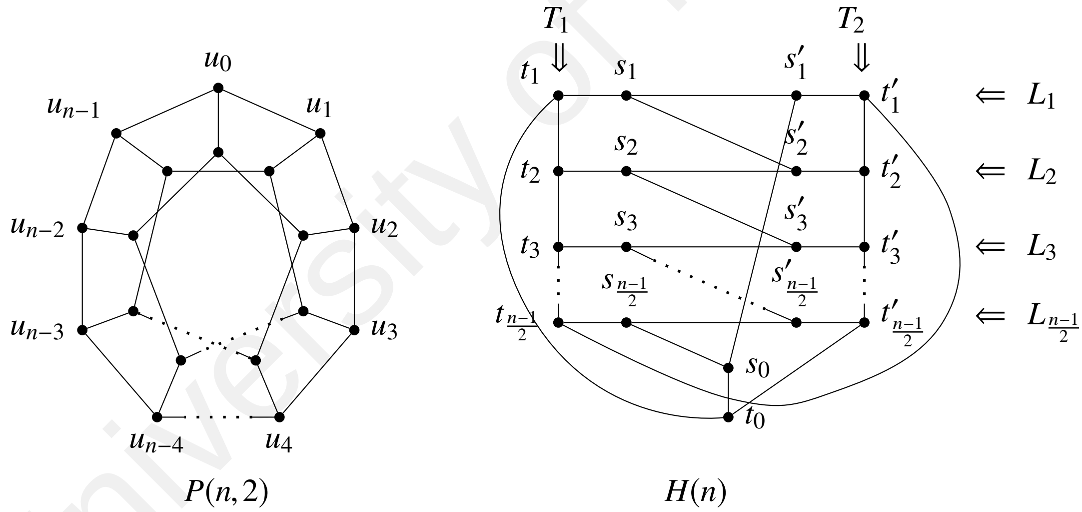


Figure 5.4: $H(n)$ **is isomorphic to** $P(n,2)$ **where** n **is odd.**

Lemma 5.4.1. *For* $n \geq 3$*,*

$$b(P(n,2)) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 1.$$

Furthermore, if $\frac{n}{2}$ *is a square, then* $b(P(n,2)) \geq \sqrt{\frac{n}{2}} + 2$*.*

Proof. Note that if $x \notin T_1 \cup T_2$, then $|N_0[x]| = 1$, $|N_1[x]| \leq 4$, $|N_2[x]| \leq 10$, $|N_3[x]| \leq 16$, $|N_4[x]| \leq 22$, $|N_5[x]| \leq 30$ and $|N_r[x]| \leq 30 + 8(r - 5)$ for $r \geq 6$ (see Figure 5.5). After 5 steps, a maximum of 8 vertices are newly burned in each following step.

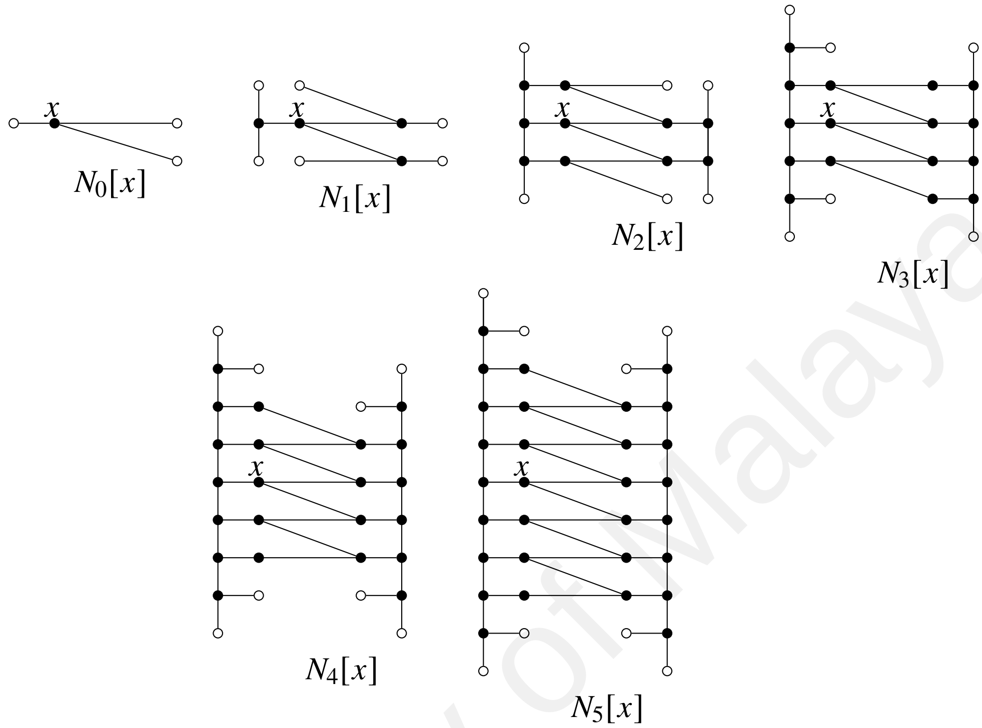


Figure 5.5: Spreading of fire from $x \notin T_1 \cup T_2$. Filled vertices are burned whereas empty vertices are unburned.

If $x \in T_1 \cup T_2$, then $|N_0[x]| = 1$, $|N_1[x]| \leq 4$, $|N_2[x]| \leq 10$, $|N_3[x]| \leq 18$ and $|N_r[x]| \leq 18 + 8(r - 3)$ for $r \geq 4$ (see Figure 5.6). After 4 steps, a maximum of 8 vertices are newly burned in each following step.

In either case, we have $|N_0[x]| = 1$, $|N_1[x]| \leq 4$, $|N_2[x]| \leq 10$, $|N_3[x]| \leq 18$ and $|N_r[x]| \leq 18 + 8(r - 3) = 8r - 6$ for $r \geq 4$.

By Proposition 5.3.1, $b(P(n, 2)) = 3 = \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 1$ for $3 \leq n \leq 7$ and

$$b(P(8, 2)) = 4 = \sqrt{\frac{n}{2}} + 2.$$

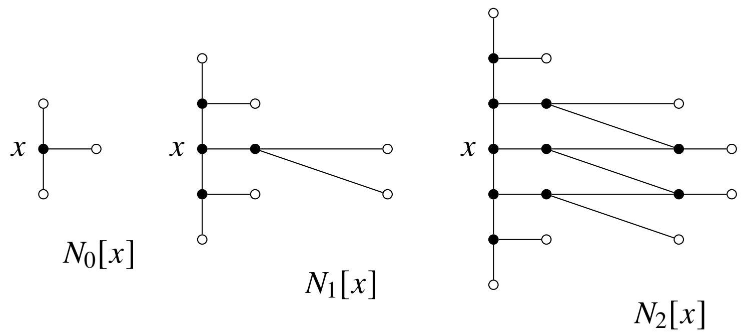


Figure 5.6: Spreading of fire from $x \in T_1 \cup T_2$. Filled vertices are burned whereas empty vertices are unburned.

Hence the lemma holds for $3 \le n \le 8$. So we may assume $n \ge 9$. Suppose $9 \le n \le 16$, then $\lceil \sqrt{\frac{n}{2}} \rceil \le 3$. If $P(n,2)$ has a burning sequence of length 3, say (x_1, x_2, x_3), then by equation (5.1), $18 \le 2n \le \sum_{i=1}^{3} |N_{3-i}[x_i]| \le 1 + 4 + 10 = 15$, a contradiction. Suppose $17 \le n \le 32$, then $\lceil \sqrt{\frac{n}{2}} \rceil \le 4$. If $P(n,2)$ has a burning sequence of length 4, say (x_1, x_2, x_3, x_4), then $34 \le 2n \le \sum_{i=1}^{4} |N_{4-i}[x_i]| \le 1 + 4 + 10 + 18 = 33$, a contradiction. So $b(P(n,2)) \ge \lceil \sqrt{\frac{n}{2}} \rceil + 1$ for $3 \le n \le 32$.

Note that for $9 \le n \le 32$, $\frac{n}{2}$ is a square if and only if $n = 18$ or 32. When $n = 18$, $\sqrt{\frac{n}{2}} + 2 = 5$. If $P(18,2)$ has a burning sequence of length 4, then $\sum_{i=1}^{4} |N_{4-i}[x_i]| \le 33$, but $|V(P(18,2))| = 36$. When $n = 32$, $\sqrt{\frac{n}{2}} + 2 = 6$. If $P(32,2)$ has a burning sequence of length 5, then $\sum_{i=1}^{5} |N_{5-i}[x_i]| \le 1 + 4 + 10 + 18 + 26 = 59$, but $|V(P(32,2))| = 64$. Thus, if $\frac{n}{2}$ is a square and $9 \le n \le 32$, then $b(P(n,2)) \ge \sqrt{\frac{n}{2}} + 2$.

Suppose $n \ge 33$. If $P(n,2)$ has a burning sequence of length l, say $(x_1, x_2, \ldots, x_l)$, then by equation (5.1),

$$
\begin{aligned}
2n \le \sum_{i=1}^{l} |N_{l-i}[x_i]| &\le |N_0[x_l]| + |N_1[x_{l-1}]| + |N_2[x_{l-2}]| + \sum_{i=1}^{l-3} |N_{l-i}[x_i]| \\
&\le 1 + 4 + 10 + \sum_{r=3}^{l-1} (8r - 6) \\
&= 4l^2 - 10l + 9.
\end{aligned}
$$

Since $l \geq 1$, by completing the square, we conclude that

$$l \geq \frac{10 + \sqrt{100 - 16(9 - 2n)}}{8} = \frac{5}{4} + \sqrt{\frac{n}{2} - \frac{11}{16}} > \sqrt{\frac{n}{2}} + 1.$$

Hence $b(P(n,2)) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 1$, and if $\frac{n}{2}$ is a square, then $b(P(n,2)) \geq \sqrt{\frac{n}{2}} + 2$. This completes the proof of the lemma. □

Lemma 5.4.2. *For $n \geq 3$,*

$$b(P(n,2)) \leq \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 2.$$

Proof. Let $l = \left\lceil \sqrt{\frac{n}{2}} \right\rceil$.

It is sufficient to show that there is a burning sequence $(x_1, x_2, \ldots, x_l, x_{l+1}, x_{l+2})$ in $H(n)$. Note that for $2 \leq j \leq l$, the term $(2j-1)l - (j-1)^2$ is increasing. Let m_0 be the largest positive integer such that $(2m_0 - 1)l - (m_0 - 1)^2 \leq \left\lfloor \frac{n}{2} \right\rfloor$. Since

$$(2l-1)l - (l-1)^2 = l^2 + l - 1 \geq \left(\sqrt{\frac{n}{2}}\right)^2 + \left(\sqrt{\frac{n}{2}} - 1\right) > \frac{n}{2},$$

we must have $m_0 \leq l - 1$.

Now, we construct the first part of a burning sequence for $H(n)$, say $x_1, x_2, \ldots, x_l$, as follows:

(a) Let $x_1 = t_l$;

(b) For each $2 \leq j \leq m_0$, set $x_j = t_{(2j-1)l-(j-1)^2}$ if j is odd, or $x_j = t'_{(2j-1)l-(j-1)^2}$ if j is even;

(c) For $j \geq m_0 + 1$:

 (i) Suppose $m_0 \leq l - 2$. If $x_{m_0} = t_{(2m_0-1)l-(m_0-1)^2}$, then set $x_{m_0+1} = t'_{\lfloor \frac{n}{2} \rfloor}$, whereas if $x_{m_0} = t'_{(2m_0-1)l-(m_0-1)^2}$, then set $x_{m_0+1} = t_{\lfloor \frac{n}{2} \rfloor}$. For $m_0 + 2 \leq w \leq l$, choose x_w to be any unburned vertex (if possible).

(ii) Suppose $m_0 = l - 1$. If $x_{l-1} = t_{(2l-3)l-(l-2)^2}$, then set $x_l = t'_{\lfloor \frac{n}{2} \rfloor}$, whereas if $x_{l-1} = t'_{(2l-3)l-(l-2)^2}$, then set $x_l = t_{\lfloor \frac{n}{2} \rfloor}$.

In Figure 5.7, the filled vertices are $N_{l-i}[x_i]$ and the shaded vertices are $N_{l+2-i}[x_i] \setminus N_{l-i}[x_i]$. In particular, $L_4 \cup L_5 \cup \cdots \cup L_l \subseteq N_{l-1}[x_1]$. So $(L_1 \cup L_2 \cup \cdots \cup L_l) \setminus \{t'_1\} \subseteq N_{l+1}[x_1]$ (see Figure 5.7 (a)).

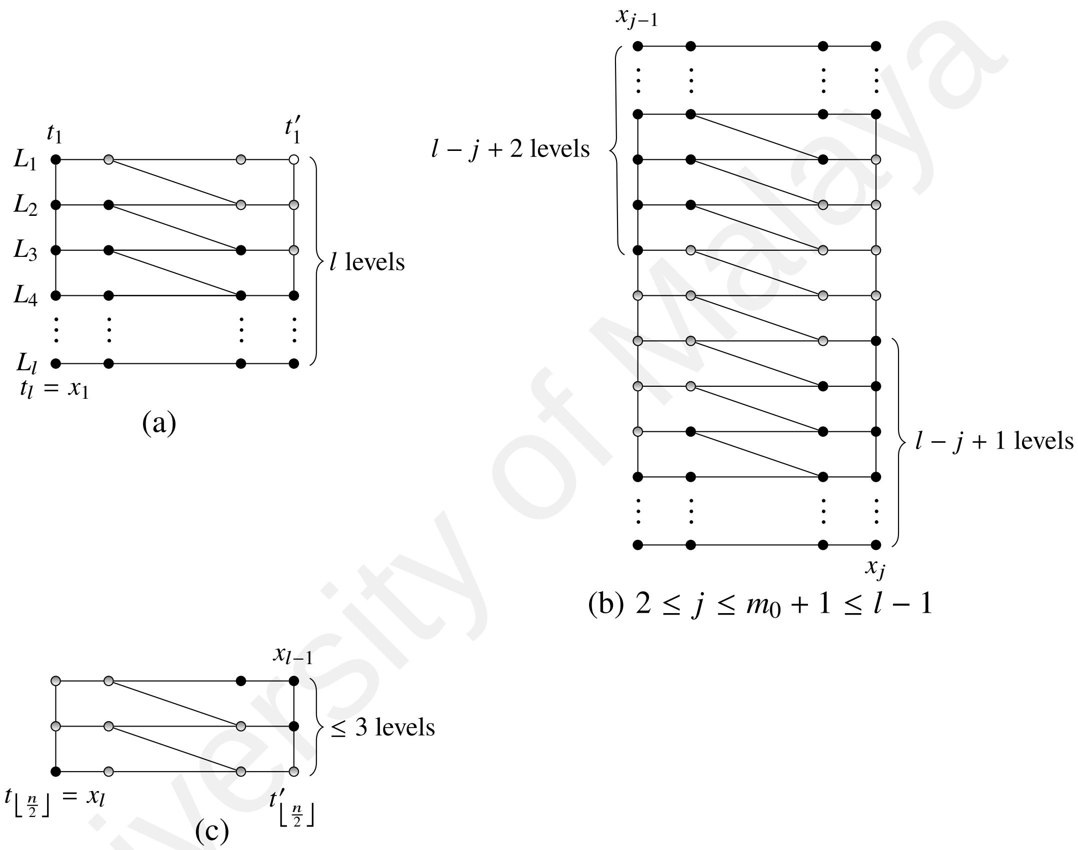


Figure 5.7: Construction

Suppose $2 \leq j \leq m_0$. Note that x_j is contained in level $L_{(2j-1)l-(j-1)^2}$ and x_{j-1} is contained in level $L_{(2j-3)l-(j-2)^2}$. There are exactly $2l - 2j + 4 = ((2j-1)l - (j-1)^2) - ((2j-3)l - (j-2)^2) + 1$ levels between $L_{(2j-1)l-(j-1)^2}$ and $L_{(2j-3)l-(j-2)^2}$ (inclusive). All these levels are contained in $N_{l-j+3}[x_{j-1}] \cup N_{l-j+2}[x_j]$ (see Figure 5.7 (b)).

Suppose $m_0 \le l-2$. By the choice of m_0, $(2m_0+1)l - m_0^2 > \lfloor \frac{n}{2} \rfloor$. So the number of levels between $L_{\lfloor \frac{n}{2} \rfloor}$ and $L_{(2m_0-1)l-(m_0-1)^2}$ (inclusive) is at most

$$\begin{aligned}\left\lfloor \frac{n}{2} \right\rfloor - ((2m_0-1)l - (m_0-1)^2) + 1 &< (2m_0+1)l - m_0^2 - ((2m_0-1)l - (m_0-1)^2) + 1 \\ &= 2l - 2m_0 + 2.\end{aligned}$$

All these levels are contained in $N_{l-m_0+2}[x_{m_0}] \cup N_{l-m_0+1}[x_{m_0+1}]$ (see Figure 5.7 (b)).

Suppose $m_0 = l-1$. Then x_{l-1} is in level $L_{(2l-3)l-(l-2)^2}$ and x_l is in level $L_{\lfloor \frac{n}{2} \rfloor}$. Note that

$$(2l-3)l - (l-2)^2 + 2 = l^2 + l - 2 > \frac{n}{2} - 1 \ge \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Hence we have

$$(2l-3)l - (l-2)^2 + 2 \ge \left\lfloor \frac{n}{2} \right\rfloor.$$

Therefore,

$$L_{(2l-3)l-(l-2)^2} \cup L_{(2l-3)l-(l-2)^2+1} \cup \cdots \cup L_{\lfloor \frac{n}{2} \rfloor} \subseteq N_3[x_{l-1}] \cup N_2[x_l],$$

(see Figure 5.7 (c)).

If we set $x_{l+1} = t_1'$ and x_{l+2} to be any unburned vertex at time step $l+1$ (if possible), then $(x_1, x_2, \ldots, x_l, x_{l+1}, x_{l+2})$ is a burning sequence of $H(n)$ when n is even. If n is odd, it is also a burning sequence by noticing that $\{s_0, t_0\} \in N_{l+1}[x_1]$ (see Figures 5.4 and 5.7 (a)). This completes the proof of the lemma. □

The first part of Theorem 5.1.3 follows from Lemmas 5.4.1 and 5.4.2. Furthermore, if $\frac{n}{2}$ is a square, then $b(P(n,2)) = \sqrt{\frac{n}{2}} + 2$. Finally, by Proposition 5.3.1, $b(P(3,2)) = 3 = \left\lceil \sqrt{\frac{3}{2}} \right\rceil + 1$. So the bounds are tight. This completes the proof of Theorem 5.1.3. □

5.4.3 Proof of Theorem 5.1.4

Proof. The upper bound follows from Theorem 5.1.1.

We define $P(n,3)$ to be the graph on $2n$ vertices with vertex set

$$V(P(n,3)) = \{u_i, v_i \ : \ i = 0,1,2,\dots,n-1\}$$

and edge set

$$E(P(n,3)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+3} \ : \ i = 0,1,2,\dots,n-1\},$$

where subscripts are taken modulo 3. The set of vertices $D_1 = \{u_i : i = 0,1,2,\dots,n-1\}$ induces the outer rim whereas $D_2 = \{v_i : i = 0,1,2,\dots,n-1\}$ induces the inner rim. Figures 5.8 and 5.9 show part of $P(n,3)$. For ease of understanding, in Figures 5.8 and 5.9, we label the vertices of $P(n,3)$ as follows. For vertex v_i, we write i for $i = 0,1,2,\dots,\lfloor\frac{n-1}{2}\rfloor$ and $i-n$ for $i = \lfloor\frac{n-1}{2}\rfloor+1, \lfloor\frac{n-1}{2}\rfloor+2,\dots,n-1$. Similarly, for vertex u_i, we write i' for $i = 0,1,2,\dots,\lfloor\frac{n-1}{2}\rfloor$ and $(i-n)'$ for $i = \lfloor\frac{n-1}{2}\rfloor+1, \lfloor\frac{n-1}{2}\rfloor+2,\dots,n-1$.

If x is a vertex from the inner rim (as in Figure 5.8), then $|N_0[x]| = 1$, $|N_1[x]| \le 4$, $|N_2[x]| \le 10$, $|N_3[x]| \le 20$, $|N_4[x]| \le 32$ and $|N_r[x]| \le 20 + 12(r-3) = 12r - 16$ for $r \ge 4$. After 4 steps, a maximum of 12 vertices are newly burned in each following step. If x is a vertex from the outer rim (as in Figure 5.9), then $|N_0[x]| = 1$, $|N_1[x]| \le 4$, $|N_2[x]| \le 10$, $|N_3[x]| \le 18$, $|N_4[x]| \le 28$ and $|N_r[x]| \le 28 + 12(r-4) = 12r - 20$ for $r \ge 5$.

In both cases, either x is a vertex in the inner rim or outer rim of $P(n,3)$, $|N_0[x]| = 1$, $|N_1[x]| \le 4$, $|N_2[x]| \le 10$, $|N_3[x]| \le 20$, $|N_4[x]| \le 32$ and $|N_r[x]| \le 20 + 12(r-3) = 12r - 16$ for $r \ge 4$. After 4 steps, at most 12 vertices are newly burned in each following step.

Suppose $4 \le n \le 12$, then $\left\lceil\sqrt{\frac{n}{3}}\right\rceil = 2$. If $P(n,3)$ has burning sequence of length 2, say (x_1, x_2), then by equation (5.1), $8 \le 2n \le \sum_{i=1}^{2} |N_{2-i}[x_i]| \le 1 + 4 = 5$, a contradiction.

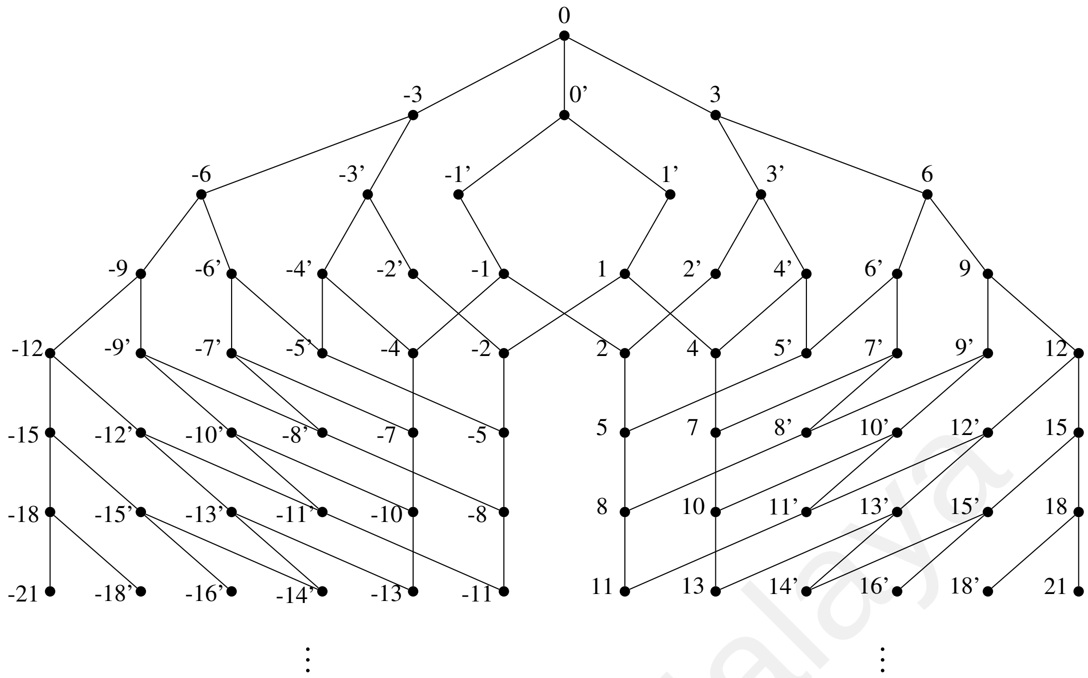


Figure 5.8: Spreading of fire from a burning source x where x is a vertex in the inner rim of $P(n,3)$.

Suppose $13 \leq n \leq 27$, then $\left\lceil \sqrt{\frac{n}{3}} \right\rceil = 3$. If $P(n,3)$ has burning sequence of length 3, say (x_1, x_2, x_3), then $26 \leq 2n \leq \sum_{i=1}^{3} |N_{3-i}[x_i]| \leq 1 + 4 + 10 = 15$, a contradiction. Similarly, suppose $28 \leq n \leq 48$, then $\left\lceil \sqrt{\frac{n}{3}} \right\rceil = 4$. If $P(n,3)$ has burning sequence of length 4, say (x_1, x_2, x_3, x_4), then $56 \leq 2n \leq \sum_{i=1}^{4} |N_{4-i}[x_i]| \leq 1 + 4 + 10 + 20 = 35$, a contradiction.

The above implies that the lower bound of $b(P(n,3) \geq \left\lceil \sqrt{\frac{n}{3}} \right\rceil + 1$is true for $4 \leq n \leq 48$. So, assume $n \geq 49$. Then $\left\lceil \sqrt{\frac{n}{3}} \right\rceil \geq 5$. If $P(n,3)$ has burning sequence of length l, say $(x_1, x_2, \ldots, x_l)$ where $l \geq 5$, then by equation (5.1),

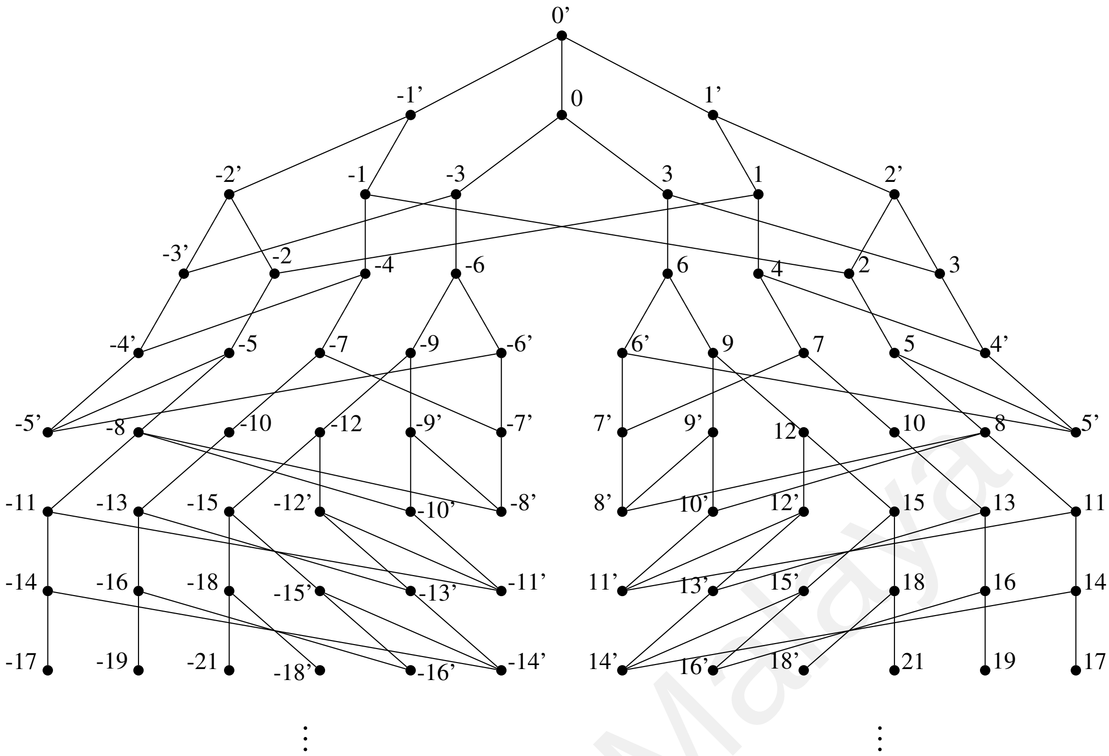


Figure 5.9: Spreading of fire from a burning source x where x is a vertex in the outer rim of $P(n, 3)$.

$$\begin{aligned}
2n \le \sum_{i=1}^{l} |N_{l-i}[x_i]| &\le |N_0[x_l]| + |N_1[x_{l-1}]| + |N_2[x_{l-2}]| + |N_3[x_{l-3}]| + \sum_{i=1}^{l-4} |N_{l-i}[x_i]| \\
&\le 1 + 4 + 10 + 20 + \sum_{r=4}^{l-1} (12r - 16) \\
&= 6l^2 - 22l + 27.
\end{aligned}$$

Since $l \ge 1$, by completing the square, we conclude that

$$l \ge \frac{22 + \sqrt{484 - 24(27 - 2n)}}{12} = \frac{11}{6} + \sqrt{\frac{n}{3} - \frac{41}{36}} > \sqrt{\frac{n}{3}} + 1.$$

Hence $b(P(n, 3)) \ge \left\lceil \sqrt{\frac{n}{3}} \right\rceil + 1$. This completes the proof of Theorem 5.1.4. □

CHAPTER 6: OTHER RESULTS AND CONCLUSIONS

In this chapter, we show some miscellaneous results which stands on its own or maybe useful in future.

6.1 On the shortest path in some k-connected graphs

Let G be a graph and u, v be two distinct vertices of G and $P[u, v]$ be a path with endpoints u and v. A path $P[u, v]$ is *non-separating* if $G - V(P)$ is connected. Two or more paths are said to be *independent* if no internal vertex of one path occurs in the other. This means that two $P[u, v]$ paths are independent if and only if u and v are their only common vertices.

A well-known theorem of Menger (see, e.g. (Diestel, 2010)) states that a graph is k-connected if and only if it contains k independent paths between any two vertices.

The following result was conjectured by Lovász (1975) and later proved by Thomassen (1981).

Theorem 6.1.1. *(Thomassen, 1981) Let $k \geq 1$. If G is a $(k + 3)$-connected graph, then G contains a cycle C such that $G - V(C)$ is k-connected.*

Another well-known conjecture by Lovász (1975) is the following.

Conjecture 6.1.2. *(Lovász, 1975) Let k be a positive integer. There exists a smallest integer $f(k)$ such that for every $f(k)$-connected graph G and two vertices u and v in G, there exists a $P[u, v]$ such that $G - V(P)$ is k-connected.*

This conjecture is true for $k = 1, 2$. Indeed, a famous theorem of Tutte (1963) states that any 3-connected graph contains a non-separating path connecting any two vertices, implying that $f(1) \leq 3$. The case $k = 2$ was proven by Chen et al. (2003) and independently, by Kriesell (2001), showing that $f(2) = 5$. In fact, they proved that the deleted path is an induced path. Later, Kawarabayashi et al. (2005) showed that if G is not a *double wheel* and is 4-connected, then G contains a $P[u, v]$ such that $G - V(P)$ is 2-connected. Here, a double wheel is a graph obtained from the union of a cycle C with two vertices s, t by adding all possible edges from $\{s, t\}$ to $V(C)$. The set $\{s, t\}$ is called the *center* of the double wheel and C is called the *ring* of the double wheel. Figure 6.1(a) shows an example of a double wheel. Conjecture 6.1.2 still remains open for the cases $k \geq 3$.

Also, Chen et al. (2003) showed that in any $(22k + 2)$-connected graph, there exist k internally vertex disjoint paths between any two vertices such that the deletion of any one of these paths does not disconnect the graph. This motivated Kawarabayashi and Ozeki (2011) to propose the following conjecture, generalizing Conjecture 6.1.2.

Conjecture 6.1.3. *(Kawarabayashi & Ozeki, 2011) Let k, l be positive integers. There exists a function $f = f(k, l)$ such that the following holds. For every $f(k, l)$-connected graph G and two distinct vertices u and v in G, there are k internally disjoint paths $P_1, \ldots, P_k$ with endpoints u and v such that $G - \cup_{i=1}^{k} V(P_i)$ is l-connected.*

Note that when $k = 1$, Conjecture 6.1.3 is exactly Lovász's conjecture.

Kawarabayashi and Ozeki (2011) showed that for any $(2l + 1)$-connected graph G and for any two vertices $u, v \in V(G)$, there exists l internally vertex disjoint $P[u, v]$ paths $P_1, P_2, \ldots, P_l$ such that $G - \cup_{i=1}^{l} V(P_i)$ is 1-connected. Also, they pointed out that if G is $(3l + 2)$-connected, then one can find l internally vertex disjoint paths $P_1, P_2, \ldots, P_l$ between any two given vertices such that $G - \cup_{i=1}^{l} V(P_i)$ is 2-connected.

A weaker version of Lovász conjecture was proposed by Kriesell (n.d.): There exists a function $h(k)$ such that for any $h(k)$-connected graph G and for any two vertices $u, v \in V(G)$, there exists an induced $P[u, v]$ in G such that $G - E(P)$ is k-connected. This weaker version was then proven by Kawarabayashi et al. (2008).

This motivates us to ask: To what extend is the length of non-separating $P[u, v]$ path in non-separating graph $G - V(P)$?

If G is 3-connected, it is not true in general that a non-separating path P is the shortest path in G. For example, the *wheel* graph W_n with n vertices, where $n \geq 7$ and $uv \notin E(W_n)$, is disconnected when the shortest path $P[u, v]$ which includes the center vertex is deleted. A wheel W_n is a graph formed by connecting a single vertex to all vertices of a cycle C of order $n - 1$. Figure 6.1(b) shows a graph W_7. Here, we investigate the maximum length of the shortest non-separating $P[u, v]$ in G for any two distinct vertices u and v.

(a) (b)

Figure 6.1: (a) A double wheel with center $\{s, t\}$**, (b)** W_7

Definition 6.1.4. *Suppose G is a connected graph and u and v are two distinct vertices in G. Let $P[u, v]$ be the shortest path in G with endpoints u and v and $t(G) = max\ \{|V(P[u, v])| : u, v \in V(G)\}$.*

Theorem 6.1.5. *Let $k \geq 1$ and G be a k-connected graph with n vertices where $n \geq k + 1$. Then, $t(G) \leq \lfloor \frac{n-2}{k} \rfloor + 2$.*

Proof. If $n = k + 1$ or G is a complete graph, then $t(G) = 2$. We have nothing to prove. Now, we assume $n \geq k + 2$ and G is not a complete graph.

Let u and v be two distinct vertices in G such that $|V(P[u, v])| = t(G) = t$. Let $A_1 = \{u\}$, $A_2 = N(u)$ and for $3 \leq r \leq t$, $A_r = N(A_{r-1}) \backslash (A_{r-1} \cup A_{r-2})$. Observe that $v \in A_t$ and $|A_2|, |A_3|, |A_4|, ..., |A_{t-1}|$ is at least k and $|A_t| \geq 1$. Otherwise, by deleting the set with less than k vertices will disconnect the graph, contradicting that G is k-connected. Now, since $n \geq k(t-2)+2$, we have $t(G) \leq \lfloor \frac{n-2}{k} \rfloor + 2$. □

Alternative proof of Theorem 6.1.5: Let u, v be any two vertices of G. By Menger's theorem, G has k independent paths between u and v. Then, one of the k independent $u-v$ paths contains at most $\lfloor \frac{n-2}{k} \rfloor$ vertices. By adding u and v, we have $|V(P[u,v])| \leq \lfloor \frac{n-2}{k} \rfloor + 2$, implying $t(G) \leq \lfloor \frac{n-2}{k} \rfloor + 2$. □

One natural question to ask is: When does equality hold? We now present some graphs where the equality holds in Theorem 6.1.5.

Suppose X and Y are two graphs. By the *join* of X and Y, denoted by $X+Y$, we mean the graph with $V(X+Y) = V(X) \cup V(Y)$ and $E(X+Y) = E(X) \cup E(Y) \cup \{ab \mid a \in V(X), b \in V(Y)\}$.

Theorem 6.1.6. *Let* $t(G) = max\ \{|V(P[u,v])| : u, v \in V(G)\}$. *Then*

1. $t(K_m) = 2$;
2. $t(P_r) = r$;
3. $t(C_{2r}) = r+1$;
4. $t(C_{2r+1}) = r+1$;
5. $t(K_{p_1,p_2,\ldots,p_r}) = 3$;
6. $t(K_m \Box P_r) = r+1$ for $m, r \geq 2$;
7. $t(K_m + \overline{K_r}) = 3$ for $m, r \geq 2$.

Proof. Let G be a k-connected graph with n vertices. If G has two non-adjacent vertices, then $t(G) \geq 3$. Hence, $t(K_m) = 2$ and if G is not a complete graph, we have $t(G) \geq 3$. To show that the upper bound of Theorem 6.1.5 is best possible, we need to show that for each G in the theorem, there must exist two vertices u and v such that $|V(P[u,v])| = \lfloor \frac{n-2}{k} \rfloor + 2$.

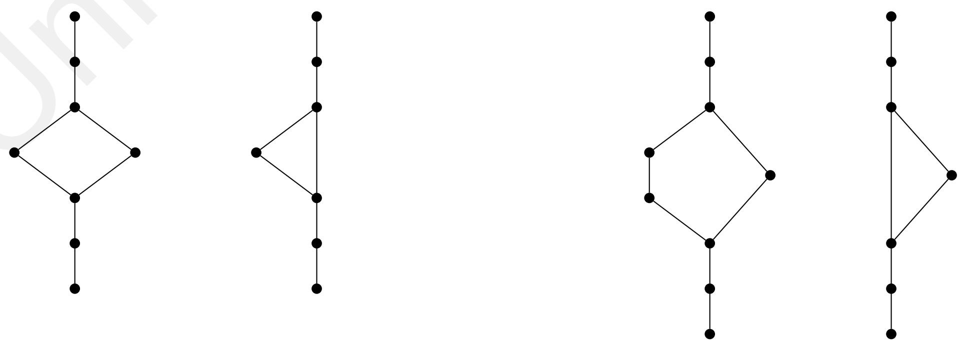
Noting that K_m is $(m-1)$-connected, P_r is 1-connected and C_m is 2-connected, the first four cases are straightforward. For the case of complete r-partite graph $K_{p_1,p_2,\ldots,p_r}$, any two vertices u, v in different partite sets have $|V(P[u,v])| = 2$, whereas any two vertices u, v in the same partite set have $|V(P[u,v])| = 3$, implying that $t(K_{p_1,p_2,\ldots,p_r}) = 3$.

Next, observe that for $r \geq 2$, $K_m \square P_r$ is m-connected an so $t(K_m \square P_r) \leq r+1$ (by Theorem 6.1.5). Let $Z_m = \{0, 1, 2, \ldots, m-1\}$ and $W_r = \{0, 1, 2, \ldots, r-1\}$. Suppose that $V(K_m \square P_r) = Z_m \times W_r$. Then the shortest path between $(x_1, 0)$ and $(x_2, r-1)$, where $x_1 \neq x_2$, has $|V(P[(x_1,0),(x_2,r-1)])| = r+1$, and so $t(K_m \square P_r) = r+1$.

Finally, let $K_m + \overline{K_r}$ has vertex set $Z_m \cup Z_r$. If at least one of the vertices u, v is in Z_m, then they are adjacent, otherwise $|V(P[u,v])| = 3$. So $t(K_m + \overline{K_r}) = 3$. $\square$

6.2 Lazy cop number

In finding the minimum order of graphs with fixed lazy cop number, we are curious whether the lazy cop number of a connected graph G remains the same if we delete some vertices of G and add edges to it. In Figure 6.2, compare G_1 with H_1, and G_2 with H_2. These are the counterexamples where $c_L(G) \neq c_L(H)$ if some vertices of degree 2 is replaced by an edge.



$c_L(G_1) = 2$ $\quad$ $c_L(H_1) = 1$ $\quad$ $c_L(G_2) = 2$ $\quad$ $c_L(H_2) = 1$

Figure 6.2: $c_L(G) \neq c_L(H)$

Theorem 6.2.1. *Suppose G is a graph with n vertices such that there exists a path $P[x, y]$ with every vertex of $V_G(P(x, y))$ is of degree 2. Let H be a graph with $V(H) = V(G - P(x, y))$ and $E(H) = E(G) - E(P[x, y]) + xy$. Then, we have $c_L(H) \leq c_L(G) + 1$.*

Proof. Suppose $c_L(G) = k$ and let S_1 be a cop winning strategy in G. We show that $k + 1$ cops are sufficient to catch the robber R in H. Here, we describe a cop winning strategy in H as follows. Let $C_1, C_2, \ldots, C_{k+1}$ be the cops at our disposal. At the beginning of the game, we place a cop C_{k+1} at x in H and assign the cops $C_i, i = 1, 2, \ldots, k$ as in strategy S_1. If there is a cop C_j for some $j = 1, 2, \ldots, k$ is assigned in $P(x, y)$ as in strategy S_1 initially, then we place C_j in y. If at any round, it is on C_j's turn to move in $P(x, y)$, then C_j just remains stationary on y. Otherwise, the cops play as S_1. After a finite number of rounds, $k + 1$ cops can capture R in H because C_{k+1} in x prevents R from moving to or to be placed in y at the beginning of the game.

□

Offner and Ojakian (2014) gave Lemma 6.2.2. For x and y are integers, x active cops (denoted as $x\mathbf{a}$) and y passive cops (denoted as $y\mathbf{p}$) means that only x cops can move and y cops have to remain stationary in each round of the game. Then $(x\mathbf{a} + y\mathbf{p}) \xrightarrow{*} Q_n$ means that x active cops and y passive cops in each round are sufficient to catch robber in Q_n regardless initial position of the game.

Lemma 6.2.2. *(Offner & Ojakian, 2014) Suppose $x, y \geq 1$ are integers and $(x\boldsymbol{a} + y\boldsymbol{p}) \xrightarrow{*} Q_n$. Then $((x + 1)\boldsymbol{a} + y\boldsymbol{p}) \xrightarrow{*} Q_{n+2}$.*

Besides, they proposed Conjecture 6.2.3. Let $C_{1a}(n)$ denote the lazy cop number in hypercube Q_n. Let $C_{ka}(n)$ denote the cop number in a hypercube Q_n such that in each round of the game, only k cops are active. Even if we cannot determine the values of

$C_{ka}(n)$ exactly, it would still be interesting to understand how this quantity behaves as the parameter is changed.

One can easily verify that $C_{(k+1)a}(n) \leq C_{ka}(n) + 1$ as follows. Let T be a cop winning strategy in Q_n with k cops active in each round. Then, by assigning an extra cop C^* in any vertex of Q_n where C^* moves freely in every round of the game, and following T, we have $C_{(k+1)a}(n) \leq C_{ka}(n) + 1$. Actually, Offner and Ojakian (2014) expected that $C_{ka}(n)$ would decrease in k.

Conjecture 6.2.3. *(Offner & Ojakian, 2014) For* $1 \leq k < \lceil n/2 \rceil$,

$$C_{(\lceil n/2 \rceil - 1)a}(n) \leq C_{(\lceil n/2 \rceil - 2)a}(n) \leq \ldots \leq C_{1a}(n).$$

Although we are not able to prove the Conjecture 6.2.3 in general, we show Lemma 6.2.4, which will then lead to Theorem 6.2.5.

Lemma 6.2.4. $C_{1a}(n) > C_{1a}(n-2)$ *for* $n - 2 \geq 3$.

Proof. It suffices to show that $C_{1a}(n-2) = r$ cops is insufficient to capture the robber in Q_n such that only one cop is active in each round of the game. We arrange the vertices of Q_n into the Cartesian product graph $G = Q_n = Q_{n-2} \square C_4$ such that every vertex in each copy of Q_{n-2} is the same in the last two coordinates. Let $H_i, i = 1, 2, 3, 4$ denote the i-th copy of Q_{n-2} in G, i is of reduced modulo 4. Here, we describe a robber winning strategy in Q_n to evade capture from the r cops indefinitely.

Let S be a robber winning strategy in Q_{n-2} with $\leq r - 1$ cops. The robber will move according to strategy S as follows. Initially, the robber identifies a cop C^* in H_i and places himself in H_{i+2}. The robber may assume every cops (except C^*) is at corresponding vertices in H_{i+2}. Then the robber can avoid being caught by the $\leq r - 1$ cops and moves within H_{i+2} following strategy S. If C^* remains stationary, then the robber moves within

H_{i+2} following strategy S. If C^* moves within H_i in any round, the robber just remains stationary. If C^* moves from H_i across to H_{i+1} (or H_{i-1}), then the robber moves to H_{i+3} (or H_{i+1} respectively). In any round, the robber will always be at distance two away from all the cops and in particular, distance two is in the first $n-2$ coordinates for cops not C^*. Hence, $C_{1a}(n) > C_{1a}(n-2)$.

□

Theorem 6.2.5. *For $n \geq 3$, we have $C_{2a}(n) \leq C_{1a}(n)$.*

Proof. Lemma 6.2.2 indicates that $C_{2a}(n) \leq C_{1a}(n-2)+1$. Then, following Lemma 6.2.4, we have $C_{2a}(n) - C_{1a}(n) \leq [C_{1a}(n-2)+1] - [C_{1a}(n-2)+1] = 0$. □

6.3 Open problems and future work

In this thesis, we bound the lazy cop number in generalized hypercubes (as in Chapter 3) and found the minimum order of graphs with lazy cop number ≥ 4 (as in Chapter 4). Besides, we also bound the burning number of generalized Petersen graphs (as in Chapter 5).

We conclude with some reflections on the literature reviews together with our results and hence propose some open problems. We include citations where relevant.

1. (Bonato & Mohar, 2017) Determine a tight bound on the capture time of planar graphs with cop number 2 and 3.
2. Determine the capture time of lazy cops and robbers in hypercube Q_n. Bonato et al. (2013) showed that the capture time of the hypercube capt(Q_n)=$\Theta(n \ln n)$. Their methods include a novel randomized strategy for the players, which involve the coupon collector problem. This motivates us to investigate the capture time of lazy cops and robbers in hypercube Q_n, since we already know that $c_L(Q_n) = \Omega\left(\frac{2^n}{n^{5/2+\epsilon}}\right)$ for every $\epsilon > 0$, see Bal et al. (2015).

3. We shall try to explore the capture time of cops and robbers in general subcubic graphs.
4. Using the results in Chapter 4, we shall continue our work in finding the minimum order of graphs for k-lazy cops-winning for $k \geq 5$. We would like to find a more general solution without investigating the maximum degree of vertices of each possible graph.
5. It seems reasonable to expect that Conjecture 6.2.3 is true. We shall continue the work in Theorem 6.2.5 to prove the general case.
6. Determine the minimum order of k-cop-win (or k-lazy cop-win) graphs in general.
7. Determine the burning number of some other special graphs such as Halin graphs.
8. In finding the burning number of generalized Petersen graphs $P(n, k)$, since n and k are finite, the number of newly burned vertices in each step is believed to be constant after some finite steps. However, when k gets larger, the calculations is more complex and more steps are required for the number of newly burned vertices in each step to be constant. Hence, one might need to develop further techniques in order to reduce the large amount of computations involved. We shall continue the work in (**Sim, K. A.** et al., 2018) by finding the burning number of generalized Petersen graph in general.

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On the Burning Number of Generalized Petersen Graphs

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Abstract The burning number $b(G)$ of a graph G is used for measuring the speed of contagion in a graph. In this paper, we study the burning number of the generalized Petersen graph $P(n, k)$. We show that for any fixed positive integer k, $\lim_{n\to\infty} \frac{b(P(n,k))}{\sqrt{\frac{n}{k}}} = 1$. Furthermore, we give tight bounds for $b(P(n, 1))$ and $b(P(n, 2))$.

Keywords Burning number · Generalized Petersen graphs

Mathematics Subject Classification 05C57 · 05C80

1 Introduction

Graph burning is a discrete-time process that can be used to model the spread of social contagion in social networks. It was introduced by Bonato et al. [2,3,8]. This process is defined on the vertex set of a simple finite graph. Throughout the process, each vertex is either *burned* or *unburned*. Initially, at time step $t = 0$, all vertices are unburned. At the beginning of every time step $t \geq 1$, an unburned vertex is chosen to burn (if

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Lazy cops and robbers on generalized hypercubes

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ABSTRACT

The lazy cop number is the minimum number of cops needed for the cops to have a winning strategy in the game of Cops and Robbers where at most one cop may move in any round. This variant of the game of Cops and Robbers, called Lazy Cops and Robbers, was introduced by Offner and Ojakian, who provided bounds for the lazy cop number of the hypercube. In this paper, we are interested in the game of Lazy Cops and Robbers on generalized hypercubes. Generalizing existing methods, we will give asymptotic upper and lower bounds on the lazy cop number of the generalized hypercube.

1. Introduction

The game of Cops and Robbers is a well-known two-player game played on a finite connected undirected graph. It was independently introduced by Quilliot [14] and by Nowakowski and Winkler [12]. The first player occupies some vertices with some number of cops (multiple cops may occupy a single vertex) and the second player occupies a vertex with a single robber. After that they move alternatively along the edges of the graph. On the cops' turn, each of the cops may remain stationary or move to an adjacent vertex. On the robber's turn, he may remain stationary or move to an adjacent vertex. A *round* of the game is a cop move together with the subsequent robber move. The cops win if after a finite number of rounds, one of them can move to *catch* the robber, that is, the cop and the robber occupy the same vertex. The main object of study in the game of Cops and Robbers is the *cop number*, the minimum number of cops required to catch the robber, introduced by Aigner and Fromme [1]. The most famous unsolved question in this context is Meyniel's conjecture [9]: the cop number of a connected graph with n vertices is $O(\sqrt{n})$.

Many variants of Cops and Robbers have been studied. (See [5] for a survey of some of the variants.) We are interested in a variant introduced by Offner and Ojakian [13], where at most one cop moves in any round. It is called the game of *Lazy Cops and Robbers* and the *lazy cop number* is the minimum number of cops required to catch the robber in this setting. We write $c_L(G)$ for the lazy cop number of a graph G. Offner and Ojakian were interested in Lazy Cops and Robbers played on the hypercube Q_n and they proved the following asymptotic bounds:

$$2^{\lfloor\sqrt{n}/20\rfloor} \leq c_L(Q_n) = O\left(\frac{2^n \ln n}{n^{3/2}}\right).$$

The lower bound was later improved by Bal, Bonato, Kinnersley, and Pralat [3], by using the probabilistic method coupled with a potential function argument. They showed that for every $\varepsilon > 0$,

$$c_L(Q_n) = \Omega\left(\frac{2^n}{n^{5/2+\varepsilon}}\right).$$

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On the Shortest Path in some k-connected Graphs

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Abstract. Suppose G is a connected graph and u and v are two distinct vertices of G. Let $P[u,v]$ be the shortest path in G with endpoints u and v. Let $t(G) = max\{|V(P[u,v])| : u,v \in V(G)\}$. A graph G is said to be k-connected if it has more than k vertices and removal of fewer than k vertices does not disconnect the graph G. We show that in any k-connected graph G with n vertices, $t(G) \leq \left\lfloor \frac{n-2}{k} \right\rfloor + 2$. We also present some graphs where the equality holds.

INTRODUCTION

Let G be a graph and u, v be two distinct vertices of G. A $u-v$ path is a path with endpoints u and v. A $u-v$ path P is *non-separating* if $G-V(P)$ is connected. Two or more paths are said to be *independent* if no internal vertex of one path occurs in the other. This means that two $u-v$ paths are independent if and only if u and v are their only common vertices. A graph G is said to be *k-connected* if it has more than k vertices and removal of any set of fewer than k vertices from G does not disconnect G.

A well-known theorem of Menger (see, e.g. [1]) states that a graph is k-connected if and only if it contains k independent paths between any two vertices.

The following result was conjectured by Lovász [2] in 1975 and later proven by Thomassen [3] in 1981.

Theorem 1 *Let $k \geq 1$. If G is a (k + 3)-connected graph, then G contains a cycle C such that $G-V(C)$ is k-connected.*

Another well-known conjecture due to Lovász [2] is the following.

Conjecture 1 *Let k be a positive integer. There exists a smallest integer $f(k)$ such that for every $f(k)$-connected graph G and two vertices u and v in G, there exists a $u-v$ path P such that $G-V(P)$ is k-connected.*

This conjecture is true for $k = 1, 2$. Indeed, a famous theorem of Tutte [4] states that any 3-connected graph contains a non-separating path connecting any two vertices, implying that $f(1) \leq 3$. The case $k = 2$ was proven by Chen, Gould and Yu [5] and independently, by Kriesell [6], showing that $f(2) = 5$. In fact, they proved that the deleted path is an induced path. Later, Kawarabayashi, Lee and Yu [7] showed that if G is not a double wheel and is 4-connected, then G contains a $u-v$ path P such that $G-V(P)$ is 2-connected. Here, a double wheel is a graph obtained from the union of a cycle C with two vertices s, t by adding all possible edges from $\{s,t\}$ to $V(C)$. The set $\{s,t\}$ is called the *center* of the double wheel and C is called the *ring* of the double wheel. Figure 1(a) shows an example of a double wheel. Conjecture 1 still remains open for the cases $k \geq 3$.

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