CHAPTER 4

ANALYSES AND RESULTS

4.0 Introduction

In this study, the analysis of the result was based on the students' written responses in the paper and pencil tests, and oral interviews on selected students. The 48 items in the written test were scored, and the incorrect responses were analyzed for patterns of errors. Only systematic errors were analyzed for error types. The same errors that occurred consistently for at least three times were classified as systematic errors. Careless errors were not considered in this study because careless errors would be corrected if the students check their answers. If the student's responses showed no discernible pattern it would be classified under random response.

Oral interviews were conducted on eight selected students to obtain further clarification on the procedures used and the thinking associated with the procedures. The researcher interviewed each student individually in a classroom. During the interview, the students were asked to explain the procedures used in their computations for the written test. Some students were also asked to redo the items and explain their procedures. The researcher used probing questions in order to get to the detail of the thinking involved in the computation. Each interview took about 40 minutes. The interviews were audio recorded.

The systematic errors identified were grouped under four classifications of error types as follows:

(a) Basic fact errors

(b) Errors resulting from zero difficulties

(c) Errors occurring in the use of remainders

(d) Errors due to faulty procedure

The discussion follows the format of error type classifications and the frequency distributions of the error types among the five different achieving groups (UPSR grades: A, B, C, D, and E). For this study, the frequency of errors referred to the number of students who consistently made the particular type of error. Analysis was also made based on the frequency distribution of each type of error among the different groups of achievers.

4.1 Basic Fact Errors

(a) Error Patterns

Division computation entails three basic number facts: multiplication facts, division facts, and subtraction facts. Basic fact errors in division computational errors occur when the student fails to recall these three basic number facts. Table 3 shows the error patterns in basic facts that was observed among the students.

Table 3

Basic Fact Errors

	Examples of Error Patterns		Description of Error
4.1.1.	Division facts (Item 6-c)	$9\overline{)80}$ quotient 9	The wrong division fact, 80÷9=9, was used, the multiplication product was not indicated.
4.1.2.	Multiplication facts (Item 9-c)	$8\overline{)697}$ quotient 82 R 1 68 17 16 1	The incorrect multiplication fact, 8x8=68, was used which led to incorrect remainder.

In this study, subtraction fact error was not observed in the students' computations. Only multiplication fact and division fact errors were identified. Example 4.1.1 shows the use of incorrect division fact; the multiplication product was not indicated. Multiplication facts are used to obtain the quotient digits if multiplication products are written down in the division algorithm as shown in Example 4.1.2. In this example, even though the first quotient digit was correct, the multiplication product was wrong indicating multiplication fact error.

(a) Frequency Distribution of Errors

Table 4 shows the frequency distribution of basic fact errors among the different groups of achievers.

Table 4

Frequency Distribution of Basic Fact Errors

Error Patterns	Students' Grades					Total (54)
	A (10)	B (10)	C (12)	D (10)	E (12)	
1. Multiplication facts	0	6	4	5	2	17
2. Division facts	0	2	3	7	2	14
3. Subtraction facts	0	0	0	0	0	0
Total	0	8	7	12	4	31

Note. The number within the parenthesis indicates the number of students.

The table indicates that all grade A students in the study had no difficulty with basic facts. Grade B students made most errors in multiplication facts while grade D students made most errors in division facts. It was noted that only four grade E students made errors in multiplication facts, whereas, the other grade E students did not indicate the use of any basic fact in their working. Most of them displayed random response.

The frequency for multiplication fact error was higher than the division fact error for most of the division computation. This is because the multiplication facts were used to obtain the multiplication product in the division algorithm.

The table also shows that no subtraction fact error was observed among the students. This suggests that majority of the students had acquired the basic facts in subtraction after completing six years of primary education. This is consistent with the earlier findings made by Grossnickle (1936), which indicated that by grade eight, most of the students had a good mastery of the subtraction facts. No student from the grade A group made error in basic facts. This is in line with Engelhardt's (1977) findings, which indicated that high ability students made less basic fact error. In fact, Stefanich and Rokusek (1992) did not find basic fact errors in their fourth grade pupils.

The written responses of the sample also showed that some students wrote down the multiplication tables to help them in recalling the multiplication facts. Table 5 shows the frequency distribution of students who wrote down multiplication tables.

Table 5

Frequency Distribution of Students Who Wrote Down Multiplication Tables

Students' Grades	A (10)	B (10)	C (12)	D (10)	E (12)	Total (54)
Frequency	1	4	4	3	0	12

Note. The number within the parenthesis indicates the number of students.

The table indicates that 12 students (22%), mostly from the grade B, C and D groups, wrote down the multiplication tables. Interviews with the students revealed that some wrote down the multiplication table from memory; some used skip counting such as 4, 8, 12,16... to arrive at the table; while a number of them used repeated addition to obtain the table. Many average and low achieving students had to refer back to their written tables as they had difficulty in recalling multiplication facts randomly. Some students were able to work out their multiplication facts from known and related facts but they had difficulty in immediate recall of multiplication facts (Kalin,1983).

During the test, it was also observed that a number of the group D and E students used their fingers to help them in direct counting. Steffe and Cobb (1998) found that children used elementary counting schemes in their computation. In this study, counting scheme was also used by some of the 13-year-olds.

An interesting observation was noted on the written work of one grade D student. This student did not write multiplication table, but his working was full of groups of vertical lines and dots. During the interviewed, he explained that to find '6 x 3', he first drew six vertical lines. Then he counted the group of six vertical lines three rounds; each count with his ballpoint pen left a dot at the vertical line. Three rounds of direct counting of the six vertical lines gave the multiplication product of 18. Examination of this student's test paper showed that he often got the multiplication facts wrong when he missed a count; or he simply gave up when the number became too big.

4.2 Errors Resulting from Zero Difficulties

(a) Error Patterns

Ten types of errors resulting from zero difficulties were identified. These error patterns are shown in Table 6.

Table 6

Error Patterns Resulting from Zero difficulties

	Examples of Error Patterns		Description of Error
4.4.1	Omitted embedded zero in quotient, zero is not found in the dividend (Item 5-c)	9 4 9)8136 81 36 36	When the '3' in the tens digit was not divisible by 9, '36' was brought down, Zero as placeholder for the tens quotient digit was omitted.
4.4.2	Omitted embedded zero in quotient, dividend contained embedded zero (Item 3-a)	3 2 3)9*06 9 6 6	The embedded zero as placeholder in the dividend was omitted when the embedded zero in the quotient was ignored.
4.4.3	Wrote embedded zero as final zero (Item 5-c)	940 9)8136 81 *36 36	When the '3' in the tens digit was not divisible by 9, '36' was brought down, and '4' was incorrectly placed in the tens column. Zero was used as placeholder for the ones instead of the tens.
4.4.4	Added extra zero to the last dividend digit (Item 6-a)	a) 672 4)27 24 *30 28 20 20	a) The final remainder '3' was taken as partial dividend and extra zero added to it. The procedure was repeated until there was no remainder; or
	(Item 6-d)	b) 871 R 3 7)61 56 50 49 10 7 3	b) One or more '0' were added to the additional dividend digit before the remainder was taken as final remainder.

	Examples of Error Patterns		Description of Error
4.4.5	Used '0' in place of '1' as identity element in quotient (Item 2-b)	200 4)844	There was confusion between '0' and '1' as identity element. Hence, '4 ÷ 4 = 0' was considered.
4.4.6	Wrote the divisor as quotient digit when the dividend digit is zero (Item 3-b)	423 2)806	When the tens digit in the dividend was '0', instead of '0' being written as the quotient digit, the divisor '2' was written down as the quotient digit; '2 x 0 = 2' was considered.
4.4.7	Omitted the final zero as placeholder when zero is final in dividend (Item 3-d)	23 3)690	Final zero in the dividend was ignored and zero was omitted as placeholder in the quotient.
4.4.8	Omitted the final zero as placeholder when zero was not final in dividend (Item 12-b)	32 R 2 3)962 9 6 6 2	The final digit in the dividend was less than the divisor. The final digit was taken as final remainder, but zero as the placeholder for the ones digit in the quotient was omitted.
4.4.9	Wrote '1' instead of '0' in the final quotient digit (Item 12-b)	321 3)962 9 6 6 2 3	When the final dividend digit was less than the divisor, '1' was written as the quotient digit instead of '0'.

	Examples of Error Patterns		Description of Error
4.4.10	Omitted final '1' in quotient digit (Item 8-a)	(a) 3 * R 1 5) 156 15 6 5 1	The '1' in the final quotient digit was omitted, but the correct partial product for 1 was written down (5x1=5, for (a)).
	(Item 8-b)	(b) 4 * R 3 4) 167 16 7 4 3	

Error patterns of Example 4.4.1 and 4.4.2 involve embedded zero in quotients. They are: (i) embedded zero not found in the dividend (level 5), and (ii) embedded zero found in the dividend (levels 3 a & b, and level 4).

In Example 4.4.1, where embedded zero is not found in the dividend, a number of students explained their procedure as:

> 8 cannot be divided by 9, then take 81; 81 divided by 9 is 9, write down 9; 3 cannot be divided by 9, then take 36; 36 divided by 9 is 4, write down 4; that gives the answer as 94.

This indicates that when the first digit is not divisible by the divisor, the next digit was brought down. When 3 is not divisible by 9, 36 was brought down as the partial dividend; and 4, the quotient digit obtained, was written in the ones place. Consequently, the quotient digit for the tens was left vacant. This error might also be caused by failure to align the place value, resulting in placing the quotient digit in the incorrect place value. The students had neglected to write down zero as placeholder in the vacant tens digit.

Example 4.4.2 shows the zero error in omitting embedded zero in the quotient when dividend contains embedded zero. For this example, every digit is divisible by the divisor; hence, no regrouping is required. This rules out place value difficulty involving regrouping. Thus, the error was due solely to ignoring the embedded zero in the tens place. The students also did not align the place value of the quotient digit obtained to the ones place, consequently the need to place a zero at the tens place was ignored.

Place value difficulty in embedded zero problems was also observed in Example 4.4.3. Failure to align the quotient digit often resulted in incorrect placement of the quotient digit.

In Example 4.4.4, the students added zero to the final dividend figure when the final remainder was less than the divisor, and then repeated the division processes. Consequently, they obtained quotient figure that was bigger than the dividend. One possible explanation for the occurrence of this error is confusion between the algorithms in "division of whole numbers with remainder" and that of the algorithm in "division of decimal numbers". The former requires just writing down the final remainder whereas the latter requires the addition of zero to the remainder and repeating the division process. It is noted that this error occurred only in the older students because division involving decimal numbers are only introduced in higher grades (Grossnickle, 1936).

Confusion between zero and one as identity element in multiplication was observed in Examples 4.4.5, 4.4.6, and 4.4.10. In Example 4.4.5, the student considered "4 ÷ 4 = 0". Zero was taken as the identity element in multiplication. In Example 4.4.6, the student wrote down the divisor as the quotient digit when the dividend digit is zero. The student explained her working as: "2 multiplied by 4 gives 8; 2 multiplied by 0 gives 2 (she reversed divisor with dividend); and 2 multiplied by 3 gives 6." The student considered "2 x 0 = 2". This is another case of confusion between '0' and '1' as identity element in multiplication. Besides, she also made the error of reversing the divisor with the dividend. The error pattern observed was actually caused by a combination of two types of errors.

Similarly, for Example 4.4.10, the student omitted the final '1' in the quotient. During the interview, she was asked to redo four similar items. For four of the items given, she wrote '0' as the final quotient digit in three of them, and '1' in one of the items. She was not very certain whether she should write down or omit the final zero. Counterchecking with her other written responses showed that she also made the error of omitting final zero as placeholder when zero was not final in the dividend (see Example 4.4.8). Hence, the error in Example 4.4.10 was caused by "confusion between zero and one as identity element" and "omitting final zero in quotient".

To examine the error of omitting the final zero in the quotients, two types of items were constructed. The first type contained items with final zero in the dividend (levels 3 c & d, level 4) while the second type did not contain final zero in the dividend (level 12). The purpose was to distinguish between the error in "omitting final zero in quotients" when "zero is also final in the dividend" and when "zero is not final in the dividend" as in Examples 4.4.7 and 4.4.8.

In Example 4.4.7, the students ignored the final zero in the dividend digit, and thus omitted the zero as placeholder in the ones digit for the quotient. In Example 4.4.8, after writing down the final remainder, the students neglected to write down zero as placeholder for the final quotient digit. This error may be caused by place value difficulties, as the students might have considered each digit independently without referring the digit to its place value. In doing so, they neglected to put zero as placeholder for the ones digit.

In Example 4.4.9, when the remainder is less than the divisor, the student approximated the final quotient digit to '1' and ignored the difference between the partial dividend and the partial product.

(b) Frequency Distribution of Errors

A summary of the frequencies of errors resulting from zero is shown in Table 7. Among the ten types of zero errors identified, "omitting the final zero as placeholder, when zero is not final in the dividend" had the highest frequency. Twenty-two of the students (41%) across all achieving groups, including three from the grade A group made this error.

In contrast, "omitting final zero as placeholder when zero is final in the dividend" was made by only two students.

The next most frequent zero error among the students is "omitting embedded zero in the quotient when there is no embedded zero in the dividend". Fourteen students (26%) made such error. However, when the dividend also contained embedded zero, only two students (all from the grade D group) made the error of omitting embedded zero in the quotient. This indicates that when zero is not visible in the dividend, the students are more likely to overlook the need for the place-holding zero in the quotient. The percentage of embedded zero error in this study, however, was considerably lower than those reported by Barr (1983), who found that 42% of the third year secondary school students could not give the correct response to the embedded zero problems.

Table 7

Frequency Distribution of Errors Resulting from Zero Difficulties

	Error Patterns	Students' Grades A (10)	B (10)	C (12)	D (10)	E (12)	Total (54)
1.	Omitted final zero as placeholder when zero was not final in dividend.	3	5	7	6	1	22
2.	Omitted embedded zero in quotient when dividend contained no zero.	1	6	3	3	1	14
3.	Added extra zero to the last dividend figure.	0	0	0	3	0	3
4.	Omitted embedded zero in quotient, dividend contained embedded zero.	0	0	0	2	0	2
5.	Used '0' in place of '1' as identity element in quotient.	0	0	0	1	1	2
6.	Wrote divisor as quotient figure when dividend digit is zero.	0	0	0	2	0	2
7.	Omitted final zero as placeholder when zero was final in dividend.	0	0	0	1	1	2
8.	Wrote '1' instead of '0' in final quotient digit.	1	0	0	0	1	2
9.	Omitted '1' in quotient digit	0	0	0	1	1	2
10.	Wrote embedded zero as final zero.	0	0	0	0	1	1
	Total	5	11	10	19	7	52

Note. The number within the parenthesis indicates the number of students.

4.3 Errors Occurring in the Use of Remainder

(a) Error Patterns

Eight types of error patterns due to remainder difficulties were observed as shown in Table 8.

Table 8

Errors Patterns Occurring in the Use of Remainder

Examples of Error Patterns			Description of Error
4.6.1	Remainder greater than the divisor (Item 7-c)	a) 72 R 8 9)89 63 26* 18 8	a) The estimated first quotient digit 7 was too small, 9 x7=63. This resulted in a remainder within the computation of 26, which was bigger than the divisor.
		b) 8 R 17 9)89 72 17*	b) The estimated first quotient digit 8 was too small. This gave the final remainder of 17, which was bigger than the divisor.
4.6.2	Used remainder as partial dividend without bringing down new dividend digit (Item 7-c)	72 R 8 9)89 63 26 * 18 8	The remainder 26, which was bigger that the divisor, was used as the new partial dividend without bringing down new dividend digit. This resulted in an extra quotient digit.

	Examples of Error Patterns		Description of Error
4.6.3	Divided each digit independently, no regrouping (Item 10-b)	111 6)745	Each digit was divided independently, the differences between the partial dividend 6, and multiplication product 7, was ignored. When the dividend was smaller than the divisor, the quotient digit was approximated to 1.
4.6.4	Wrote decimal values as final remainder (Item 6-a)	6.75 = 6 R 75 4)27 24 30 28 20 20	The student considered decimal point as remainder. The decimal value '.75' was taken to mean "a remainder of 75".
4.6.5	Final dividend digit was taken as remainder (Item 8-b)	4 R 7 4)167 16	When the quotient digit was contained in the first two digit of the dividend, only one quotient digit was written down, the remaining dividend digit was then written as remainders.
4.6.6	Added up the remainders (Item 9-c)	111 R 4 8)697 8 9 8 7 8	The remainders were obtained by subtracting the smaller number from the bigger, regardless of whether it was minuend or subtrahend: 8 – 6 = 2; 9 – 8 = 1, and 8 – 7 = 1. The remainders were then added up, 2+1+1= 4, to give a remainder of 4.

	Examples of Error Patterns		Description of Error
4.6.7	Estimated quotient digit was too big (Item 6-c)	a) 9 9) 80 81	a) The student chose the quotient digit that gave the closest partial product but bigger than the dividend. The difference between the multiplication product and the dividend was ignored.
	(Item 6-b)	b) 7 R 3 8) 53 56 3	b) Same as a) but the remainder was obtained by reversing minuend and subtrahend.
4.6.8	Ignored remainder in the tens digit (Item 10-d)	653 R 2 7) 5573 42 *37 35 23 21 2	The first quotient digit '6' was too small resulting in the remainder '13', which was bigger than the divisor. Only the remainder in the ones digit '3' was written down. The remainder in the tens digit was ignored.

Examples 4.6.1.a and 4.6.1.b show that students made error in using remainder bigger than the divisor both in the remainder within the computation and in the final remainder. This error may be due to carelessness but as it occurred consistently in some students, it is considered as a systematic error in this study.

When the students made the error of using remainder greater than the divisor, the error may lead to other errors such as: (i) using the remainder as the new partial dividend (see Example 4.6.2); (ii) writing two-digit quotient figure (see Example 4.8.3); or (iii) ignoring the tens digit in the remainder (see Example 4.6.8).

The error in "using remainder bigger than the divisor" could probably be caused by inadequate procedural understanding of the distributive algorithm. This error may be averted if subtractive algorithm is employed in the computation. In the subtractive algorithm, one just needs to repeatedly subtract the divisor from the dividend, until the remainder is smaller than the divisor. In contrast, the distributive algorithm requires the bringing down of the dividend digit before dividing for new quotient digit. For those students who encounter place value difficulty, they may find this method confusing. Besides, place value difficulty also leads to the error of "dividing each digit independently, regardless of its place value" as shown in Example 4.6.3. However, dividing each digit independently works fine when there is no carrying and no remainder (as for items in level 1 to level 4). But when there is regrouping and remainder, this method leads to errors in computation.

In Example 4.6.4, the student wrote the decimal digits as the remainder. The interview revealed that she considered decimal and remainder were equivalent. This is another example of confusion between the newly learned decimal concepts with the remainder concept learned in the earlier stage (see Example 4.4.4).

The student could be confused between division involving two-digit dividend and division involving three-digit dividend in Example 4.6.5. In the former, where the quotient is contained in the first two digits, the computation was completed after a single digit quotient was obtained and the final remainder was written down. In the latter, the remainder had to be regrouped with the next dividend digit to obtain the quotient for the ones digit.

Example 4.6.6 was a unique case, the student's actual written response (refer to case study) did not indicate the procedure used. The researcher added in the procedure as shown in Example 4.6.6 based on the interview. Her written response did not indicate how she obtained such a large remainder. During interview, the explanation given was that she added up each remainder of each subdivision and then wrote their sum as the final remainder. Buswell and John (1926) had identified remainder error in "writing remainder within example" and "writing all remainders at the end of example"; but to the knowledge of the researcher, adding up all the remainders within the computation as final remainder was not observed in previous studies.

When the estimated quotient figure used is too small, it gives rise to error in "using remainder bigger than the divisor". However, when the estimated quotient figure is too big (see Example 4.6.7), it gives a partial product that is bigger than the partial dividend. In the latter, the student then ignored the difference between the multiplication product and the dividend (see Example 4.6.7.a), or obtained the remainder by "reversing the minuend and subtrahend" (see Error due to faulty procedure) as in Example 4.6.7.b.

(b) Frequency Distribution of Errors

Table 9 shows the frequency distribution of remainder errors among the different groups of achievers.

Table 9

Frequency Distribution of Errors Occurring in the Use of Remainder

	Error Patterns	Students' Grades					Total
		A (10)	B (10)	C (12)	D (10)	E (12)	(54)
1.	Remainder greater than the divisor	1	3	4	2	2	12
2.	Used remainder as partial dividend without bringing down new digit	0	3	1	0	0	4
3.	Divided each digit independently, no regrouping	0	0	1	1	1	3
4.	Wrote decimal values as final remainder	0	1	0	0	0	2
5.	Estimated quotient digit too big	0	0	0	2	0	2
6.	Final dividend digit was taken as the remainder	0	0	0	1	0	1
7.	Added up the remainders	0	0	0	1	0	1
8.	Ignored remainder in the tens digit	0	0	1	0	0	1
	Total	1	7	7	7	3	25

Note. The number within the parenthesis indicates the number of students.

The most frequent error made was "Using remainder greater than the divisor ". Twelve of the students (22%) across all achieving groups made this error. The study by Lim (1980) also indicated that this error type occurred most frequently in the primary four pupils. The next most frequent error is "using remainder as new partial dividend without bringing down new digit". Four students from the grade B and C groups made this error. Students in the grade B, C, and D groups made 21 out of the 25 (84%) errors in remainder difficulties.

4.4 Errors due to Faulty Procedure

(a) Error Patterns

Table 10 shows the nine error patterns due to faulty procedure or defective algorithm that were identified.

Table 10

Error Patterns Due to Faulty Procedure

	Examples of Error Patterns		Description of Errors
4.8.1	Missed out digit in dividend (Item 10-d)	79 R 4 7)5573 49 67 63 4	The student missed out '3' in the ones digit when the last partial dividend 63 also ends in 3.
4.8.2	Used 2-digit dividend unnecessarily (Item 12-c)	930 R 3 4)8523 36 *12 12 3 0 3	The first quotient '8' is divisible by the divisor but '85' was used as the partial dividend. The remainder in the tens digit '3' was ignored. (The remainder in the ones digit was obtained by reversion of minuend and subtrahend.)

	Examples of Error Patterns		Description of Errors
4.8.3	Used 2-digit number as quotient figure (Item 10-c)	6812 R 2 5) 3462 30 46 40 *62 60 2	This error followed from using remainder bigger than the divisor. The remainder 6 is bigger than the divisor. Bringing down the next digit 2 gave the new partial dividend of 62. The quotient digit was written as '12', because 5 x 12 = 60.
4.8.4	Reversed minuend and subtrahend (Item 6-d)	9 R 2 7) 61 63 2	When the minuend is smaller than the subtrahend, then the smaller number was subtracted from the bigger number.
4.8.5	Reversed divisor and dividend (Item 8-d)	632 6) 128	When the divisor is bigger than the dividend figure, the divisor and the dividend were interchanged: 1÷6 = 6, 2÷6 = 3
4.8.6	Incorrect placement of quotient digit (Item 1-c)	32 3) 96 9 6 6	The quotient digits were not written at the correct place value. They were written one place to the right.
4.8.7	Incorrect placement of partial product (Item 9-c)	10 R 617 8) 697 80 617	The partial product was placed below the dividend digits that could minus it, instead of the correct place.

	Examples of Error Patterns		Description of Errors
4.8.8	Stopped dividing when dividend digit is less than divisor (Item 5-a) (Item 5-c)	a) 3 5)1515 15 *1 b) 9 9)8136 81 3	When the dividend digit '1' (example a) brought down is smaller than the divisor, the division process was stopped resulting in an incomplete computation.
4.8.9	Divided only first dividend digit, wrote down first two dividend digits as the remaining quotient digits, and last dividend digit as remainder (Item 2-c) (Item 3-c) (Item 4-b)	a) 269 R 6 3)696 0 96 b) 284 R 0 4)840 0 40 c) 390 R 0 3)900 0 00	The first dividend digit '6' was divided by 3 to get 2. This was written down as the first quotient digit. However, the next two quotient digits '69', were taken from the first two dividend digits, while the last dividend digit '6', was written down as the remainder (example a).

In Example 4.8.1, the students made the error of missing out a digit. This error occurred when the student did not align the numbers according to their correct place value. Consequently, they missed some of the digits. This seemingly careless error occurred quite consistently in some students when the dividend consisted of four or more digits.

In Example 4.8.2, using two-digit dividend unnecessarily, when 8 is divisible by 4, has resulted in a remainder bigger than the divisor. The student then made a further error of "ignoring the remainder in the tens digit" (Example 4.6.8).

When a student uses a remainder that is bigger than the divisor, bringing down the next dividend digit will give a partial dividend that is too big and requires a two-digit quotient as in Example 4.8.3. Interviews revealed that the student obtained the two-digit quotient by multiplying the estimated two-digit quotient with the divisor to get the multiplication product closest to the partial dividend. This error resulted in an extra quotient digit. However, the student did not realize that the answer obtained was not reasonable.

Errors due to inversion of order were observed in Examples 4.8.4 and 4.8.5. In Example 4.8.4, reversing the minuend and subtrahend helped him to get a quick answer by avoiding the regrouping process that was necessary to obtain the remainder. The other inversion of order was "reversing divisor and dividend" as in Example 4.8.5. The dividend digit is smaller than the divisor. The students interchanged the dividend and divisor to avoid regrouping. These observations indicate that the concept of reversibility in addition and multiplication was incorrectly applied for subtraction and division.

In Examples 4.8.6 and 4.8.7, the students did not align the digits to their correct place value. In Example 4.8.7, the partial product was written under the incorrect dividend digits. Interview with the student revealed that he did not understand the reason for subtracting the partial product from the partial dividend in the division algorithm. He chose the dividend digit that could subtract the partial product.

In Example 4.8.8, the student did not consider the possibility of bringing down the next dividend digit when the first dividend digit brought down was smaller than the divisor. He stopped the computation when the dividend digit was not divisible by the divisor.

A peculiar algorithm was observed in Example 4.8.9. The student's incorrect response actually followed certain rules, which were more complicated than the correct algorithm (Rudnistky et al., 1981).

(b) Frequency Distribution of Errors

Table 11 shows the frequency distribution of error patterns due to faulty procedure.

Table 11

Frequency Distribution of Errors Due to Faulty Procedure

Error Patterns	Students' Grades A (10)	B (10)	C (12)	D (10)	E (12)	Total (54)
1. Used 2-digit number as quotient digit	1	2	2	1	0	6
2. Reversed minuend and subtrahend	0	0	1	3	0	4
3. Revered divisor and dividend	0	0	0	3	0	3
4. Missed out digit in dividend	1	1	0	0	0	2
5. Incorrect placement of partial product	0	0	0	2	0	2
6. Incorrect placement of quotient figure	0	0	0	1	0	1
7. Used 2-digit dividend unnecessarily	0	0	1	0	0	1
8. Stopped dividing when dividend digit is less than divisor	0	0	1	0	0	1
9. Divided only first dividend digit, wrote first two dividend digits as the remaining quotient digit and last digit as remainder	0	0	0	1	0	1
Total	2	3	5	11	0	21

Note. The number within the parenthesis indicates the number of students.

Considering the overall frequencies among the groups, using faulty procedure occurred mostly in the low-achieving students. Eleven out of the 21(52%) errors due to faulty procedure were made by the grade D students, and five (24%) were from the grade C students. Higher achievers (grade A and B groups) made errors only in missing out digits in dividend and using two-digit number as quotient figure. It is noted that the frequency of faulty procedure is rather low in grade E students. This is because most of the errors made by grade E students do not show a consistent pattern. As such, they were classified under random response.

The most frequent procedural error was using two-digit number as quotient digit, which occurred six times across the grade A to grade D students. Next was the error in reversing minuend and subtrahend, followed by reversing divisor and dividend. Other errors due to faulty procedures occurred only in one or two students.

4.5 Random response

Table 12 shows the frequency distribution of random responses among the different groups of achievers.

Table 12

Frequency Distribution of Random Responses

Error	Students' Grades					Total (54)
	A (10)	B (10)	C (12)	D (10)	E (12)	
Random Responses	0	0	0	0	9	9

Note. The number within the parenthesis indicates the number of students.

All the nine random responses came from the low achieving grade E group. These nine students made up 75% of the grade E students. Their responses did not show the use of any algorithm in computation, but resembled more of random juxtaposition of numbers. There was no observable relationship between the numbers written. This indicates that these students most probably did not have any conceptual and procedural understanding of the division process. The division algorithm was not used and appeared to have no meaning to them.

Lim (1980) found that random responses in division computation occurred in about 38% of the primary four pupils. In comparison, random responses occurred only in 17% of the Form One students in this study. This suggests that as the students progress to higher grades, they are less prone to giving random responses.

4.6 Frequency Distribution of Errors According to Classification

All the 137 errors identified were summarized according to the five classifications of error patterns. The first four classifications are systematic errors whereas the fifth is the random responses. Table 13 shows the frequency distribution of errors according to these five classifications of errors.

Table 13

Frequency Distribution of Errors by Classification

	Classification	Students' Grades A (10)	B (10)	C (12)	D (10)	E (12)	Total (54)	%
1.	Basic fact errors	0	8	7	12	4	31	22.6
2.	Zero errors	5	11	10	19	7	52	38.0
3.	Remainder errors	1	7	7	7	3	25	18.2
4.	Faulty procedure	2	3	5	10	0	20	14.6
5.	Random responses	0	0	0	0	9	9	6.6
	Total	8	29	29	48	23	137	100
	(%)	5.8	21.1	21.1	35.0	16.8	100	

Notes. The number within the parenthesis indicates the number of students.

Based on the frequency distribution above, the five classifications of errors follow the order as:

(a) Zero errors (38.0%)

(b) Basic fact errors (22.6%)

(c) Remainder errors (18.2%)

(d) Faulty procedure (14.6%)

(e) Random responses (6.6%)

4.7 Common Error Patterns among the Students

Among the 137 errors, 29 patterns of systematic errors in division computation were identified. Table 14 shows the number of error patterns corresponding to each error type classification. The highest number of error patterns is observed in the zero errors classification. The number of error patterns for faulty procedure and remainder errors are almost the same.

Table 14

Number of Error Patterns Identified for Each Classification

	Classification	Number of Error Patterns
1.	Zero errors	10
2.	Faulty procedure	9
3.	Remainder errors	8
4.	Basic fact errors	2
	Total	29

Further examination of the errors indicates that some errors occurred rather frequently among the students, while others occurred only in isolated cases. Table 15 shows the five types of error patterns which occurred in frequencies of more then ten. These five types of error patterns alone accounted for 79 (58%) of the total 137 incorrect responses made by the students.

Table 15

Common Error Patterns among the Students

	Error Patterns	Frequency
1.	Omitted final zero in quotient, zero not final in dividend	22
2.	Multiplication fact errors	17
3.	Division fact errors	14
4.	Omitted embedded zero, zero not found in dividend	14
5.	Using remainder bigger than the divisor	12
	Total	79

In this study, “using wrong operation”, which was the second most common error in Lim's (1980) study on the primary four pupils, was not observed in the Form One students. This observation, however, is consistent with the findings on the division exercise of the NAEP of 1980 which showed that using wrong operations did not occur in the 13-year-olds (cited by McKillip, 1981). This suggests that for those students who have completed six years of primary education, a great majority of them have acquired sufficient understanding of the four basic operations and confusion between the operations would no longer arise.

4.8 Case Study

The researcher includes this case study in this report in view of the peculiar nature of the error pattern displayed, in particular, the "ingenious" ways this student arrived at the error pattern. While her written work did not show the procedures employed, interview with the student threw light on how she arrived at her error patterns.

Nina (not her real name) scored grade D for her mathematics achievement in her UPSR examination. A wide range of computational errors was observed in her written response. She could carry out simple division computation that did not require regrouping and without remainders. Her written responses to some items in division with no carrying, and no remainders are shown below:

42	32	341
2) 84	3) 96	2) 682
4	3	3
2	2	4
		1

Her responses indicate that she got the correct quotient figures. However, she did not follow the correct division algorithm which requires writing down the partial product and subtracting for remainders, in her computation. She just copied down the quotient digit below its dividend digit.

This pattern was carried through to items involving "zero not final" and "zero final in the dividend" with no regrouping. Her response indicates that she did not make the error of "omitting embedded zero" and "omitting final zero". She gave the correct quotient figures as shown below:

```
   302            200            230
3)906          4)800          3)690
   3              2              2
    0              0              3
     2              0              0
```

However, for items that involve identity element, her written computations were:

```
   200             203            200
4)844          2)2426          4)840
   2               2              2
    0               0              0
     0               3              0
```

During the interview, she explained her procedure for $4\overline{)844}$ as: "Four divides eight gives two, four divides four gives zero, and four divides four gives zero."

In example $2\overline{)2426}$, her explanation was: "Two divides two gives zero, two divides four gives two, two divides two gives zero, and two divides six gives three". Her explanation showed that she was confused over the identity element involving zero and one. She considered a number divided by itself gave a quotient of zero instead of one.

For division computation that involves regrouping and with remainders, Nina's written response seemed like random response. Closer examination of her work disclosed an underlying consistent pattern. During the interview, she demonstrated the various faulty strategies used in her computation. An example of her computation is:

```
  2011 R 5
8)4856
  2
   0
    1
     1
```

She explained her computation process as:

> Eight divides four gives two; eight divides eight is zero; eight divides five gives one; eight minus five gives a remainder of three. Eight divides six gives one; eight minus six gives a remainder of two. Adding up the remainders 2 and 3, gives a remainder of five.

Nina had used several incorrect strategies in her computation. For her first quotient digit, she reversed the divisor and dividend, thus obtained '$4 \div 8 = 2$'. The next error is related to the identity element. Instead of '1', '0' was used as the identity element as shown in '$8 \div 8 = 0$'. For the next quotient digit, she considered '$5 \div 8 = 1$', she made the error of using an estimate that was too big. She approximated to '1' for the quotient figure when the dividend was smaller than the divisor. This was then followed by the error of reversing the minuend and subtrahend to obtain the remainder of 3. She employed the same procedure again to obtain the final digit, and obtained a remainder of 2. She then added up the two remainders as the final remainder of 5. Consequently, she obtained the quotient as 2011 with remainder 5.

She displayed the same pattern in her other computations as shown:

```
   12 R 5           111 R 4          112 R 4
8)53             8)697            3)248
  1                1                1
   2                1                1
                     1                2
```

In one item, her strategies contained five types of errors. The procedures that Nina employed indicated that she lacked adequate conceptual understanding of division as well as procedural understanding of division algorithm. Consequently, she invented her own rules or "invented algorithm", which were more complicated than the correct procedure as noted by Rudnitsky et al. (1981).

4.9 Sources of Errors

Based on the examination of the errors and interviews with the students, the possible sources of the computational errors are inferred.

Students made computational errors due to lack of conceptual understanding of the division concept and procedural understanding of the algorithm in division. The meaning of division for the *measurement* situation, which requires repeated subtraction was seldom considered.

Consequently, the students made the error by leaving a remainder bigger than the divisor. They also had difficulty in making meaningful connection between the division algorithm and the meaning of division process. They had memorised the steps in the computation without understanding the reasoning behind each step.

Many students encountered difficulty with place value concept. They divided each digit in the dividend independently and ignored its overall place value. They did not align the digits to their correct place values. These students with place value difficulty also had difficulty with regrouping. They tried to avoid regrouping by reversing minuend and subtrahend in subtraction, or reversing divisor and dividend in division. They also ignored the remainders in their computation.

Some students did not understand the role of zero as placeholder. Consequently, they made errors in ignoring the zeros in the dividend. This situation led to the errors in omitting the embedded zero and final zero in the quotient. Another common zero errors arise from the students' confusion between zero and one as identity element in the multiplication.

The findings also indicate that some students still lack mastery of the multiplication facts, which is necessary in division algorithm. They may be able to work out the multiplication facts, but they have difficulty in spontaneous recall of the multiplication facts.