SOME PROPERTIES OF GRAPHS ASSOCIATED WITH CERTAIN FINITE RINGS

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ABSTRACT

In this thesis, we investigate the properties of some graphs associated with finite rings, namely the total graphs, the unit graphs, and the directed graphs (also called digraphs). We start by describing the structure of the total graphs and unit graphs of some rings, particularly those of order p, p^2 and pq where p and q are distinct primes. This involves computing the degrees of vertices and the number of connected components in those graphs. We also obtain explicit formulae for determining the number of sources or non-sources in the digraphs of certain rings. Of special importance is the result related to the digraph associated with the ring of integers modulo p^n, which is then used to extend the result on the number of sources to include the digraph associated with the more general ring $\mathbb{Z}_n$. Some graph parameters concerning the digraph associated with the ring $I = \langle a, b : pa = pb = 0, a^2 = b, ab = 0\rangle$ of order p^2 are also obtained, including the number of non-sources, the number of connected components, and the number of vertices in each connected component.

Keywords: finite ring, total graph, unit graph, directed graph, zero divisor, unit.

BEBERAPA SIFAT BAGI GRAF YANG TERSEKUTU DENGAN GELANGGANG TERHINGGA TERTENTU

ABSTRAK

Dalam tesis ini, kami mengkaji sifat bagi beberapa graf yang tersekutu dengan gelanggang terhingga, khususnya graf jumlah, graf unit, dan graf berarah (juga dipanggil digraf). Kami mula dengan memperincikan struktur graf jumlah dan graf unit bagi beberapa gelanggang, terutamanya gelanggang yang peringkatnya ialah p, p^2 dan pq di mana p dan q ialah nombor perdana berbeza. Ini melibatkan pengiraan darjah bucu dan bilangan komponen terkait dalam graf tersebut. Kami juga memperolehi formula tak tersirat bagi menentukan bilangan punca atau bukan punca dalam digraf beberapa gelanggang. Antara hasil penemuan yang penting ialah yang berkenaan digraf yang tersekutu dengan gelanggang integer modulo p^n; hasil penemuan ini kemudiannya digunakan untuk melanjutkan pencarian bilangan punca untuk merangkumi digraf yang tersekutu dengan gelanggang $\mathbb{Z}_n$ yang lebih umum. Beberapa parameter graf berkenaan dengan digraf yang tersekutu dengan gelanggang $I = \langle a, b : pa = pb = 0, a^2 = b, ab = 0\rangle$ yang berperingkat p^2 juga diperolehi, termasuklah bilangan bukan punca, bilangan komponen terkait, dan bilangan bucu dalam setiap komponen terkait.

Kata kunci: gelanggang terhingga, graf jumlah, graf unit, graf berarah, pembahagi sifar, unit.

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LIST OF SYMBOLS

p: a prime number

$\gcd(m, n)$: the greatest common divisor of the integers m and n

$\phi(n)$: the Euler phi function of the positive integer n

char R: the characteristic of the ring R

$Nil(R)$: the set of nilpotent elements of the ring R

$Reg(R)$: the set of regular elements of the ring R

$U(R)$: the set of units of the ring R

$ZD(R)$: the set of zero divisors of the ring R

K_n: the complete graph on n vertices

$V(G)$: the vertex set of the graph G

$\tau(R)$: the total graph of the ring R

$\Gamma(R)$: the unit graph of the ring R

$\Psi(R)$: the directed graph of the ring R

$S(R)$: the number of sources in the directed graph of the ring R

$NS(R)$: the number of non-sources in the directed graph of the ring R

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Background of study

This dissertation is mainly concerned with relations between finite rings and various graphs associated with them. Research on this topic aims at exposing relations between rings and graphs, and at advancing knowledge of one to the other. The origin of these relationships can be traced to a paper of Beck (1988), where he introduced the idea of a zero divisor graph of a commutative ring with identity. Associating a graph with an algebraic structure has grown to be an area of research that has attracted attention from algebraists and graph theorists.

The main objectives of this research are as follows:

(i) To determine structures of various types of graphs associated with some finite rings;

(ii) To determine properties of rings which can be identified from the graphs associated with them.

The scope of this research is limited to finite rings and three types of graphs associated with them, namely, the total graph, the unit graph, and the directed graph (also called the digraph).

1.2 Significance of the study

Associating a graph to a finite ring opens up an avenue for studying rings via graphs and vice versa. The main contribution of this work are some findings on properties of graphs associated with finite rings and recognition of some types of rings from the associated graphs. Other than knowledge on graphs and rings, the process of obtaining some of these results also involves number-theoretic methods. These combinations of methods play a role in advancing knowledge on graphs and rings.

1.3 Organisation of dissertation

In Chapter 2, we present some background on rings and graphs as well as a literature review on graphs associated with rings. A description of the methodology and some preliminary results used in the dissertation will also be given. Chapter 3 will cover the structures of total graphs associated with some finite rings. In particular, a complete description of the total graphs associated with the rings of order p, p^2 and pq where p and q are distinct prime numbers will be given. In Chapter 4, we discuss how the unit graphs are related to the total graphs of finite rings with identity and present the unit graph structures of some finite rings. We also obtain some general results relating the unit and total graphs of finite rings. Chapter 5 will focus on the directed graphs of some finite rings. In particular, some properties of the graphs such as the number of connected components and the number of sources (or non-sources) will be considered. Finally, in Chapter 6, we give a summary of the work done in this dissertation and discuss possible future work on this topic.

Throughout this dissertation, for any element $a \in R$, the phrase 'the vertex a' will be used to mean 'the vertex associated with the element a in the ring R'.

CHAPTER 2

LITERATURE REVIEW AND PRELIMINARIES

In this chapter, we present some background on rings and graphs as well as a literature review on graphs associated with rings.

2.1 Rings

Definition 2.1.1. Let R be a non-empty set on which two binary operations are defined, called addition and multiplication, and denoted by + and ·, respectively. Then R is called a ring with respect to these operations if the following properties hold.

(i) Associative laws: For all $a, b, c \in R$

$$a + (b + c) = (a + b) + c \quad \text{and} \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

(ii) Zero element: The set R contains an element 0 called the zero element or the additive identity element, such that for all $a \in R$,

$$a + 0 = a \quad \text{and} \quad 0 + a = a.$$

(iii) Additive inverses: For each $a \in R$, the equations

$$a + x = 0 \quad \text{and} \quad x + a = 0$$

has a solution x in R, called the additive inverse of a, and denoted by $-a$.

(iv) Commutative law (for addition): For all $a, b \in R$,

$$a + b = b + a.$$

(v) Distributive laws: For all $a, b, c \in R$,

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

For convenience, the notation xy is used instead of $x \cdot y$ to indicate multiplication. Note also that if there exists an element $e \in R$ such that $x \cdot e = e \cdot x = x$ for all x in R, then e is called a multiplicative identity, and R is then called a ring with identity. In this dissertation, the multiplicative and additive identities in a ring are assumed to be different. If multiplication in R is commutative, then R is called a commutative ring. There is no restriction on the number of elements in a ring, but all rings under consideration throughout this dissertation are assumed to be finite. The notation $|R|$ is used to denote the number of elements in the finite ring R.

Recall that an element α is a zero divisor in the ring R if there exists a non-zero element $\beta \in R$ such that $\alpha\beta = 0$ or $\beta\alpha = 0$. Note that this definition implies that 0 is always a zero divisor in any ring. Now let R be a ring with identity $1 \neq 0$. An element $\gamma \in R$ is a unit if there exists an element $\delta \in R$ such that $\gamma\delta = \delta\gamma = 1$. In a finite ring with identity, there is an interesting relationship between units and zero divisors, as shown in the following proposition.

Proposition 2.1.1. *Let R be a finite ring with identity $1 \neq 0$. Every element in R is either a unit or a zero divisor.*

Proof. Let $R = \{a_1 = 0, a_2, \ldots, a_n\}$. Take $a_i \in R, a_i \neq 0$ and suppose that a_i is not a zero divisor. Then $a_i a_k \neq a_i a_l$ for any $k, l \in \{1, \ldots, n\}, k \neq l$. Indeed, if $a_i a_k = a_i a_l$ for some $k, l \in \{1, \ldots, n\}$, then $a_i(a_k - a_l) = 0$. By the assumption that a_i is not a zero divisor, we have that $a_k - a_l = 0$, that is, $a_k = a_l$. We thus have that $a_i R = \{a_i a_j | j = 1, \ldots, n\} = R$. Therefore, $a_i a_r = 1$ for some $r \in \{2, \ldots, n\}$. Similarly, $Ra_i = R$ and we have $a_s a_i = 1$ for some $s \in \{2, \ldots, n\}$. It follows that $a_r = 1a_r = (a_s a_i)a_r = a_s(a_i a_r) = a_s 1 = a_s$ which implies that a_i is a unit in R. □

For other terminologies on rings, the reader may refer to Beachy (1999).

2.1.1 Presentations of finite rings

In discussing finite rings, it is important that we are able to write and describe them in a concise manner. To this end we make use of ring presentations. A presentation for a finite ring R consists of a set of generators $a_1, \ldots, a_k$ of R together with relations satisfied by the generators. The relations include the additive order(s) of the generator(s). For a ring R, we write its presentation as follows:

$$R = \langle a_1, \ldots, a_k : m_i a_i = 0 \text{ for } i = 1, \ldots, k, \mathcal{R} \rangle$$

where $\mathcal{R}$ are relations between the generators $a_1, \ldots, a_k$. Note that if a relation can be derived from other relations in the presentation, then we omit them. For example, the ring $\langle a, b : pa = pb = 0, a^2 = b, ab = 0 \rangle$ has the relations $b^2 = 0$ and $ba = 0$ omitted as these relations can be derived from the relations $a^2 = b$ and $ab = 0$ as follows:

$$b^2 = a^2b = a(ab) = a \cdot 0 = 0,\ ba = a^2a = aa^2 = ab = 0.$$

Fine (1993) proved that if p is a prime, then there are, up to isomorphism, exactly two rings of order p, given by the following presentations:

$$\mathbb{Z}_p = \langle a : pa = 0, a^2 = a\rangle$$

and

$$C_p(0) = \langle a : pa = 0, a^2 = 0\rangle.$$

In the same paper, Fine also obtained the complete list of non-isomorphic rings of order p^2 and pq for distinct primes p and q. For convenience, we provide both the complete lists below.

Theorem 2.1.1 (Fine, 1993)**.** *Let p be a prime. The non-isomorphic rings of order p^2 are given below:*

$$A = \langle a : p^2a = 0, a^2 = a\rangle \cong \mathbb{Z}_{p^2}$$

$$B = \langle a : p^2a = 0, a^2 = pa\rangle$$

$$C = \langle a : p^2a = 0, a^2 = 0\rangle \cong C_{p^2}(0)$$

$$D = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = ba = 0\rangle \cong \mathbb{Z}_p + \mathbb{Z}_p$$

$$E = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$$

$$F = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a\rangle$$

$$G = \langle a, b : pa = pb = 0, a^2 = 0, b^2 = b, ab = a, ba = a\rangle$$

$$H = \langle a, b : pa = pb = 0, a^2 = 0, b^2 = b, ab = ba = 0 \rangle \cong \mathbb{Z}_p + C_p(0)$$

$$I = \langle a, b : pa = pb = 0, a^2 = b, ab = 0 \rangle$$

$$J = \langle a, b : pa = pb = 0, a^2 = b^2 = 0 \rangle \cong C_p(0) \times C_p(0)$$

$$K = GF(p^2) = \textit{finite field of order } p^2$$

$$= \begin{cases} \langle a, b : pa = pb = 0, a^2 = a, b^2 = ja, ab = b, ba = b \rangle \\ \qquad \textit{where } j \textit{ is not a square, if } p \neq 2 \\ \langle a, b : 2a = 2b = 0, a^2 = a, b^2 = a + b, ab = b, ba = b \rangle, \textit{ if } p = 2 \end{cases}$$

Note that except for the rings E and F, all the rings of order p^2 are commutative. Throughout this dissertation, the rings of order p^2 shall be identified by their respective letters A through K as stated in Theorem 2.1.1. These letters denoting the rings may also be affixed with a subscript denoting the value of p; for example, $D_{(3)}$ denotes the ring D of order 3^2.

Theorem 2.1.2 (Fine, 1993)**.** *Let p and q be distinct prime numbers. The non-isomorphic rings of order pq are as follows:*

$$\mathbb{Z}_{pq} = \langle a : pqa = 0, a^2 = a \rangle$$

$$C_{pq}(0) = \langle a : pqa = 0, a^2 = 0 \rangle$$

$$C_p(0) + \mathbb{Z}_q = \langle a, b : pa = qb = 0, a^2 = 0, b^2 = b, ab = ba = 0 \rangle$$

$$\mathbb{Z}_p + C_q(0) = \langle a, b : pa = qb = 0, a^2 = a, b^2 = 0, ab = ba = 0 \rangle$$

2.2 Graphs

A graph consists of a set of points and lines joining some pairs of these points. A graph may be undirected or directed. For the purpose of this dissertation, we shall formally define (undirected) graphs and directed graphs as follows:

Definition 2.2.1. A graph G is an ordered pair (V, E) where $V = V(G)$ is a finite set called the vertex set of G and $E = E(G)$ is a collection of unordered pairs of elements from V called the edge set of G.

Naturally, the elements in $V(G)$ are called the vertices of G, and the elements in $E(G)$ are called the edges of G. For an edge $e = \{a, b\}$, we say that e connects the vertices a and b. If an edge connects the distinct vertices a and b, then we say that a and b are adjacent. The number of edges connecting a vertex is known as the degree of that vertex. Two graphs G and H are said to be isomorphic if there is a bijection φ from $V(G)$ to $V(H)$ such that the vertices a and b are adjacent in G if and only if $\varphi(a)$ and $\varphi(b)$ are adjacent in H. In this dissertation, all the (undirected) graphs considered are simple graphs, meaning graphs without cycles of length 1 (often called loops or 1-cycles) and multiple edges.

Definition 2.2.2. A directed graph D is an ordered pair (V, E) where $V = V(D)$ is a finite set called the vertex set of D and $E = E(D)$ is a collection of ordered pairs of elements from V called the directed edge set.

Again, the elements in $V(D)$ are called the vertices of D, and the elements in $E(D)$ are called the directed edges of D. A directed edge (a, b) in a directed graph is often represented by an arrow pointing from a to b: $a \to b$. A vertex in a directed graph has incoming degree n if there are n directed edges pointing to it. A source in a directed graph is a vertex with incoming degree 0. A vertex with incoming degree larger than zero is said to be a non-source. It is important to remember that directed graphs may contain loops (also known as 1-cycles or self-loops). Note that the definition of isomorphic as previously defined for undirected graphs does not take into account the direction of each directed edge. In order to account for the direction of directed edges in a directed graph, we introduce the concept of directly isomorphic:

Definition 2.2.3. Let C and D be directed graphs, then C and D are said to be isomorphic if there exists a bijection φ from $V(C)$ to $V(D)$ such that the vertices a and b are adjacent in C if and only if $\varphi(a)$ and $\varphi(b)$ are adjacent in D. The directed graphs C and D are said to be directly isomorphic if C and D are isomorphic, and $a \to b$ in C if and only if $\varphi(a) \to \varphi(b)$ in D.

The complement of a graph $G = (V, E)$ has vertex set V and two vertices a and b are adjacent in the complement graph if and only if a and b are not adjacent in G. The following result will also prove useful later on.

Proposition 2.2.1. *The complement of a disconnected graph is connected.*

Proof. Let G be a disconnected graph and G' its complement. Take any two vertices $g, h \in G$.

Case 1: The vertices g and h are not adjacent in G. Clearly, g and h are adjacent in G'.

Case 2: The vertices g and h are adjacent in G. Since G is a disconnected graph, there exists a vertex $i \in G$ such that neither g and i nor h and i are adjacent. Hence, g and i are adjacent in G', and h and i are adjacent in G'. Therefore, there exists a path connecting g and h in G'. □

For other terminologies on graphs, the reader may refer to Bollobás (1999).

2.3 Graphs associated with rings

Work on associating a graph with an algebraic structure can be traced back to a paper by Beck (1988), where the notion of a zero divisor graph of a commutative ring with identity was introduced. A zero divisor graph of a commutative ring R is a simple graph with the vertex set consisting of all non-zero zero divisors of R, and two vertices x and y are adjacent if and only if $xy = 0$. Further investigations on zero divisor graphs, specifically on their diameter and girth, have been done by Anderson and Mulay (2007).

Since then, various associations between graphs and algebraic structures have been defined and investigated. Anderson and Badawi (2008) introduced the concept of the total graph of a commutative ring as follows:

Definition 2.3.1 (Anderson & Badawi, 2008)**.** Given a finite commutative ring R, the total graph $\tau(R)$ associated with R is an undirected graph with the vertex set consisting of elements of R, and for distinct $a, b \in R$, the vertices a and b are adjacent if and only if $a + b$ is a zero divisor of the ring R.

Anderson and Badawi (2008) also studied three induced subgraphs of the total graph $\tau(R)$, namely graphs with vertices $Nil(R)$, $ZD(R)$, and $Reg(R)$. Here, $Nil(R)$, $ZD(R)$, and $Reg(R)$ denote the set of nilpotent elements, the set of zero divisors and the set of regular elements of R, respectively.

The definition of total graphs by Anderson and Badawi (2008) was extended by Dolžan and Oblak (2015) to also include non-commutative finite rings as follows:

Definition 2.3.2 (Dolžan & Oblak, 2015)**.** The total graph $\tau(R)$ of a ring R is the graph where the vertex set $V(\tau(R))$ is the set of all elements in R, and two distinct vertices x and y are adjacent if and only if $x + y$ is a (left or right) zero divisor in R.

Dolžan and Oblak (2015) stated that in finite rings, every left zero divisor is also a right zero divisor and vice versa, hence the definition of a total graph they provided above coincides with the commutative version first defined by Anderson and Badawi (2008). They also proved in that same paper that $\tau(R)$ is Hamiltonian if and only if R is not local. Additionally, they also provided an upper bound for the domination number of $\tau(R)$ for all finite rings R.

Chelvam and Asir (2011a) expanded the study on total graphs by obtaining certain fundamental properties of the total graph $\tau(\mathbb{Z}_n)$, including the independent number and clique number of $\tau(\mathbb{Z}_n)$. In another paper, Chelvam and Asir (2011b) studied the domination in the total graph of a commutative ring R; in particular, they characterised all the γ-sets in $\tau(\mathbb{Z}_n)$ where γ is the domination number. Other domination parameters were also obtained by Chelvam and Asir (2011b). The study of domination was continued in Chelvam and Asir (2013a) where they provided lower and upper bounds for the domination number of the total graph for an Artinian ring. The authors also identified certain classes of rings for which the upper bound is sharp.

Dhorajia (2015) studied some fundamental properties of the total graph of $\mathbb{Z}_n \times \mathbb{Z}_m$, where n and m are positive integers. They also determined the independent number and clique number of the total graph of $\mathbb{Z}_n \times \mathbb{Z}_m$. More recently, an excellent survey on various results and properties related to total graphs has been written by Badawi (2014).

Another type of graph associated with rings is the unit graph. This graph was first considered in a paper by Ashrafi, Maimani, Pournaki, and Yassemi (2010) and is defined as follows:

Definition 2.3.3 (Ashrafi et al., 2010)**.** Let R be a ring with identity $1 \neq 0$ and $U(R)$ be the set of unit elements of R. The unit graph of R, denoted by $\Gamma(R)$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices a and b to be adjacent if and only if $a + b \in U(R)$.

Compared to total graphs, unit graphs have received less attention. Basic properties of unit graphs of a finite ring were investigated in the paper by Ashrafi et al. (2010) and some characterisation results regarding connectedness, chromatic number, diameter, girth, and planarity of unit graphs were given. Heydari and Nikmehr (2013) studied the unit graph of a left Artinian ring. They proved some equivalence conditions for the existence of a Hamiltonian cycle in a unit graph. Moreover, they obtained an algorithm to find a Hamiltonian cycle whenever such a cycle exists.

Su and Zhou (2014) proved certain results pertaining to the girth of the unit graph of a ring. Akbari, Estaji, and Khorsandi (2015) obtained necessary and sufficient conditions for the unit graph of a ring (not necessarily commutative) to be a complete r-partite graph. Additionally, they showed that if R is a left Artinian ring, $2 \in U(R)$ and the clique number of $\Gamma(R)$ is finite, then R is a finite ring.

Graphs associated to rings are not just limited to undirected graphs. Lipkovski (2012) introduced the notion of a directed graph $\Psi(R)$ or a digraph of the commutative ring R. Directed graphs encapsulate both the element-wise additive and multiplicative structure of the ring R and provides an alternative to the Cayley Table as a visual representation of a ring. It is defined as follows:

Definition 2.3.4. A directed graph (or digraph) of the ring R, denoted by $\Psi(R)$, is the graph with $V(\Psi(R)) = R \times R$, and for $(a, b), (c, d) \in R \times R$, there is a directed edge, denoted by $(a, b) \to (c, d)$, connecting (a, b) to (c, d) if and only if $a + b = c$ and $a \cdot b = d$.

The following result of Hausken and Skinner (2013) gives us the number of cycles in a connected component of the digraph of a finite ring.

Proposition 2.3.1 (Hausken and Skinner, 2013)**.** *Let R be a finite ring. If $\mathcal{C}$ is a connected component of the digraph $\Psi(R)$, then $\mathcal{C}$ contains exactly one cycle.*

Hausken and Skinner (2013) also proved an equivalence condition for the digraph of an integral domain, namely, that R is an integral domain if and only if there is exactly one connected component consisting of exactly one vertex (namely, the vertex 0) and that all other one-cycles have an incoming degree of two. This is an example of a strong relation between the properties of a digraph and the properties of its associated ring. It was also shown that if two rings R and S are isomorphic, i.e. $R \cong S$, then its digraphs are directly isomorphic. The converse is false, however. Hausken and Skinner (2013) provided a counter-example by showing that the digraphs $\Psi(\mathbb{Z}_4)$ and $\Psi(\mathbb{Z}_2[i])$ are directly isomorphic but the rings $\mathbb{Z}_4$ and $\mathbb{Z}_2[i]$ are not isomorphic.

Ang and Schulte (2013) further expanded the study of digraphs. They proved that in a finite field F, there are $\frac{q^2-q}{2}$ sources in $\Psi(F)$ where $q = |F|$. Properties of digraphs of factor rings and reduced rings have also been investigated by Ang and Schulte (2013). Among others, they proved that a ring R is reduced if and only if its digraph $\Psi(R)$ has a connected component consisting of a looped vertex (also known as a one-cycle).

2.4 Methodology

Preliminary work on this research involved reading and understanding of various advanced concepts on abstract algebra and graph theory. This is followed by reading of various related research articles and identifying as well as understanding the techniques used by other researchers. By analysing known results and investigating some concrete examples of graphs associated with rings for possible patterns, some conjectures are formulated. Work to confirm or refute the conjectures are made by using various tools from number theory, graph theory and ring theory, as well as by building on known results of others.

CHAPTER 3

THE TOTAL GRAPHS OF SOME FINITE RINGS

3.1 A brief overview

In this chapter, we describe the structure of the total graphs associated with some finite rings. Although there have been quite a lot of work done in the literature on total graphs, not many explicit examples of such graphs are known. In this chapter, we attempt to fill this gap by finding the total graphs associated with some finite rings including the rings of order p, p^2 and pq where p, q are distinct prime numbers. More general results on total graphs, including their relations with unit graphs will be discussed in Chapter 4.

3.2 Total graphs of rings without non-zero zero divisors

This section will focus on finite rings with only one zero divisor, namely the zero element 0. A finite field is an example of such a ring. We start with the following proposition, the proof of which is straightforward by the definition of total graphs.

Proposition 3.2.1. *Let R be a finite ring with no non-zero zero divisors. There is an edge connecting the distinct vertices u and v in the total graph $\tau(R)$ if and only if $u + v = 0$ in R.*

As a consequence of Proposition 3.2.1, we have the following structure of total graphs of finite fields.

Theorem 3.2.1. *Let $\mathbb{F}$ be a finite field of order p^n where p is a prime and $n \geq 1$.*

(i) If $p = 2$, then $\tau(\mathbb{F})$ is an empty graph.

(ii) If p is odd, then $\tau(\mathbb{F})$ consists of an isolated vertex and $\frac{p^n-1}{2}$ independent edges. Moreover, each independent edge $\{u, v\}$ is such that $u + v = 0$ in $\mathbb{F}$.

Proof. Note that $\mathbb{F}$ has no zero divisors other than 0, hence two vertices are adjacent in $\tau(\mathbb{F})$ if and only if their sum is 0 in $\mathbb{F}$. We consider the case $p = 2$ and p is odd separately.

Case 1: $p = 2$. In this case, the characteristic of $\mathbb{F}$ is 2. Hence, every element in $\mathbb{F}$ is its own additive inverse. Therefore, there is no edge at all between any two distinct vertices of $\mathbb{F}$. It follows that $\tau(\mathbb{F})$ is an empty graph.

Case 2: p is odd. Since the zero element 0 in $\mathbb{F}$ is its own additive inverse, there is no edge connecting 0 with any other vertex in $\tau(\mathbb{F})$; hence, giving us an isolated vertex in $\tau(\mathbb{F})$. Now let $u \in \mathbb{F}$, $u \neq 0$. Then u is adjacent to its additive inverse $p^n - u$ in $\tau(\mathbb{F})$. Since the additive inverse of any element in $\mathbb{F}$ is unique, u is adjacent to exactly one vertex in $\tau(\mathbb{F})$. We thus have $\frac{p^n-1}{2}$ independent edges connecting the non-zero vertices in $\tau(\mathbb{F})$ such that $\{u, v\}$ is an independent edge in $\tau(\mathbb{F})$ if and only if $u + v = 0$ in $\mathbb{F}$. $\square$

Example 3.2.1. The total graph $\tau(\mathbb{Z}_5)$ is as follows:

$$\begin{array}{ccc} & 1 & 3 \\ 0 & | & | \\ & 4 & 2 \end{array}$$

Example 3.2.2. The total graph of the field of order 9 is as follows:

$$\begin{array}{ccccc} 0 & a & b & a+b & a+2b \\ & | & | & | & | \\ & 2a & 2b & 2a+2b & 2a+b \end{array}$$

3.3 Total graphs of some rings with non-zero zero divisors

In this section, we determine the total graphs of some finite rings with non-zero zero divisors. We begin with a general result as follows:

Theorem 3.3.1. *Let R be a finite ring such that all the elements in R are zero divisors. Then $\tau(R)$ is the complete graph $K_{|R|}$.*

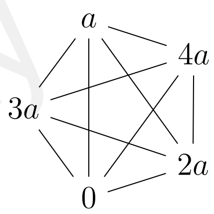
Proof. Since the sum of any two elements in R is also a zero divisor, it follows that every vertex in $\tau(R)$ is adjacent to all the other vertices in $\tau(R)$. Hence, $\tau(R)$ is a complete graph. □

As a consequence of Theorem 3.3.1, we have the following:

Corollary 3.3.1. *The total graph of the ring $C_n(0) = \langle a \mid na = 0, a^2 = 0\rangle$, $n \geq 2$ is the complete graph K_n.*

Proof. Every element in $C_n(0)$ is a zero divisor, hence the result follows readily by Theorem 3.3.1. □

Example 3.3.1. The total graph $\tau(C_5(0))$ is as follows:



By Theorem 3.2.1 and Corollary 3.3.1, we know the total graphs of the rings of order p, as well as those of the rings C and K of order p^2 where C and K are as described in Theorem 2.1.1. We now determine the total graphs of the other rings of order p^2 and pq.

3.3.1 Other rings of order p^2

We begin by identifying the rings with total graphs which are complete.

Theorem 3.3.2. *The total graph of each of the rings B, E, F, H, I and J as defined in Theorem 2.1.1 is the complete graph K_{p^2}.*

Proof. We prove this by showing that every element in the rings B, E, F, H, I and J is a zero divisor or a one-sided zero divisor.

$B = \langle a : p^2a = 0, a^2 = pa \rangle$: Take any non-zero $\alpha a \in B$. Note that $\alpha a \cdot pa = \alpha pa^2 = \alpha p^2 a = 0$. Hence, αa is a zero divisor.

$E = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle$: Take any $\alpha a + \beta b \in E$ where $(\alpha, \beta) \neq (0, 0)$. Note that $(\alpha a + \beta b)(a - b) = \alpha a^2 - \alpha ab + \beta ba - \beta b^2 = 0$. Hence, $\alpha a + \beta b$ is a left zero divisor.

$F = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a \rangle$: Similar to E above, for any $\alpha a + \beta b \in F$ where $(\alpha, \beta) \neq (0, 0)$, note that $(b - a)(\alpha a + \beta b) = \alpha ba - \alpha a^2 + \beta b^2 - \beta ab = 0$. Hence $\alpha a + \beta b$ is a right zero divisor.

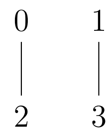
$H = \langle a, b : pa = pb = 0, a^2 = 0, b^2 = b, ab = ba = 0 \rangle = C_p(0) \times \mathbb{Z}_p$: For any $\alpha a + \beta b \in H$ where $(\alpha, \beta) \neq (0, 0)$, note that $(\alpha a + \beta b)a = \alpha a^2 + \beta ba = 0$. Hence $\alpha a + \beta b$ is a zero divisor.

$I = \langle a, b : pa = pb = 0, a^2 = b, ab = 0 \rangle$: Similar to H above, by taking any $\alpha a + \beta b \in I$ where $(\alpha, \beta) \neq (0, 0)$, we have that $(\alpha a + \beta b)b = \alpha ab + \beta b^2 = 0$. Hence $\alpha a + \beta b$ is a zero divisor.

$J = \langle a, b : pa = pb = 0, a^2 = b^2 = 0 \rangle = C_p(0) \times C_p(0)$: Take any $\alpha a + \beta b \in J$ where $(\alpha, \beta) \neq (0, 0)$, we have that $(\alpha a + \beta b)a = \alpha a^2 + \beta ba = 0$. Hence $\alpha a + \beta b$ is a zero divisor.

By Theorem 3.3.1, it follows that the total graph of each of the rings B, E, F, H, I and J is the complete graph K_{p^2}. $\square$

We now consider the other rings of order p^2, namely, $A = \mathbb{Z}_{p^2}$, D and G as defined in Theorem 2.1.1. For $p = 2$, the total graph of A is as follows:



For the total graph of A when $p > 2$, the following lemma is useful in determining the zero divisors of the ring $A = \mathbb{Z}_{p^2}$.

Lemma 3.3.1. *For any prime p, the non-zero zero divisors in $\mathbb{Z}_{p^2}$ are k where $\gcd(k, p^2) = p$.*

Proof. Let $k \neq 0$ be a zero divisor in $\mathbb{Z}_{p^2}$. Then there exists $l \in \mathbb{Z}_{p^2}, l \neq 0$ such that $kl = 0$. Let $k = k_1 k_2 \dots k_m, l = l_1 l_2 \dots l_n$ be the prime factorisations of k and l. Note that $kl = 0$ implies that the prime p appears at least twice in the multiplication $kl = k_1 \dots k_m \cdot l_1 \dots l_n$. Since $k, l \not\equiv 0 \pmod{p^2}$, only one of the primes k_i and one of the primes l_j may be p. Hence the only possibility is that $\gcd(k, p^2) = p$. □

We are now equipped to find the structure of the total graph of the ring $A = \mathbb{Z}_{p^2}$.

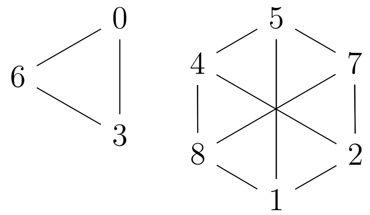
Theorem 3.3.3. *For an odd prime p, the total graph $\tau(\mathbb{Z}_{p^2})$ is the disjoint union of two subgraphs X and Y where X is isomorphic to the complete graph K_p and Y is a p-regular graph on $p(p-1)$ vertices.*

Proof. By Lemma 3.3.1, the zero divisors of the ring $\mathbb{Z}_{p^2}$ are $0, p, \dots, (p-1)p$. Note that for any $ip, jp \in ZD(\mathbb{Z}_{p^2})$, we have that $ip + jp = (i+j)p \in ZD(\mathbb{Z}_{p^2})$; that is, the sum of any two zero divisors is also a zero divisor. Hence, the induced subgraph X associated with the vertices in $ZD(\mathbb{Z}_{p^2})$ is the complete graph K_p.

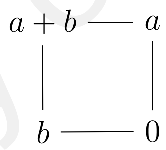
Let $U(\mathbb{Z}_{p^2}) = \mathbb{Z}_{p^2} \setminus ZD(\mathbb{Z}_{p^2})$, the set of units in $\mathbb{Z}_{p^2}$. Note that any $k \in U(\mathbb{Z}_{p^2})$ cannot be adjacent to any $ip \in ZD(\mathbb{Z}_{p^2})$ because that would imply $k + ip \equiv 0 \pmod{p}$ which is not possible because k is not a multiple of p. It is straightforward to check that $k \in U(\mathbb{Z}_{p^2})$ is only adjacent to the vertices $p - k, 2p - k, \dots, (p-1)p - k, p^2 - k$

which are all distinct and also belong to $U(\mathbb{Z}_{p^2})$. Hence, k has degree p. It follows that the induced subgraph Y with vertices in $U(\mathbb{Z}_{p^2})$ consists of $p(p-1)$ vertices, each of degree p. Clearly, $\tau(\mathbb{Z}_{p^2}) = X \cup Y$ and the assertion follows. □

Example 3.3.2. The total graph of the ring $\mathbb{Z}_9$ is as follows:



We now determine the total graph of the ring $D = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = ba = 0\rangle \cong \mathbb{Z}_p + \mathbb{Z}_p$. Note first that for any prime p, the set of zero divisors in D is $ZD(D) = \{ia, jb : i, j \in \mathbb{Z}_p\} = \{0, a, \ldots, (p-1)a, b, \ldots, (p-1)b\}$. When $p = 2$, the total graph of D is as follows:



The case when $p > 2$ is given in the following theorem:

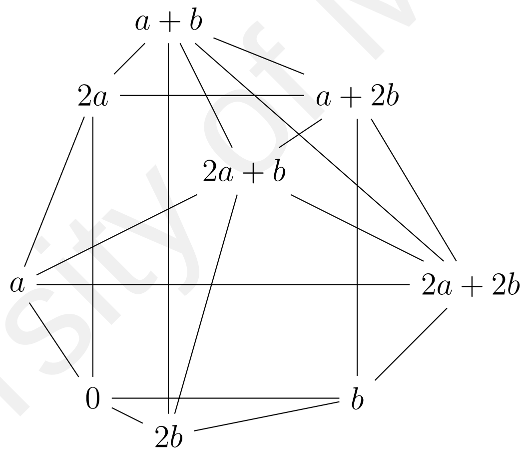
Theorem 3.3.4. *For an odd prime p, the total graph $\tau(D)$ consists of p^2 vertices, of which $2p-1$ are associated with the zero divisors of D and have degree $2p-2$ each. The remaining vertices are associated with the units of D and have degree $2p-1$ each.*

Proof. Note first that there are $2(p-1)$ elements in the set of non-zero zero divisors $ZD(D) \setminus \{0\} = \{a, \ldots, (p-1)a, b, \ldots, (p-1)b\}$. The element 0 is adjacent to these $2(p-1)$ non-zero zero divisors; hence, it has degree $2p-2$. Now fix a vertex $ia \in ZD(D), i \neq 0$. Note that ia is adjacent to 0, every other $ka \in D$ where $k \neq i$, and $(p-i)a + jb$ where $j \in \{1, \ldots, p-1\}$. Adding up gives us $1 + (p-2) + (p-1) = 2p-2$

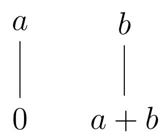
as the degree of ia. A similar line of argument will give us $2p-2$ as the degree of the vertex lb where $l \in \{1, \ldots, p-1\}$.

The remaining $(p-1)^2$ vertices are associated with the units of D. Fix a vertex $\alpha a + \beta b \in U(D)$ where $\alpha, \beta \in \{1, \ldots, p-1\}$. This vertex is adjacent to every vertex of the form $(p-\alpha)a+\gamma b$ where $\gamma \in \{0, 1, \ldots, p-1\}$, and every vertex of the form $\delta a+(p-\beta)b$ where $\delta \in \{0, 1, \ldots, p-1\}$. Summing these up gives us $p+p=2p$. However, there is one vertex that is shared between the sets $\{(p-\alpha)a+\gamma b; \gamma = 1, \ldots, p-1\}$ and $\{\delta a+(p-\beta)b; \delta = 1, \ldots, p-1\}$. We now subtract one from $2p$ to obtain $2p-1$ as the degree of $\alpha a + \beta b \in U(D)$. □

Example 3.3.3. The total graph of the ring D where $p = 3$ is as follows :



We now consider the ring $G = \langle a, b : pa = pb = 0, a^2 = 0, b^2 = b, ab = a, ba = a\rangle$. The set of zero divisors in G is $ZD(G) = \{ia : i \in \mathbb{Z}_p\} = \{0, a, \ldots, (p-1)a\}$. When $p = 2$, the total graph of G is as follows:



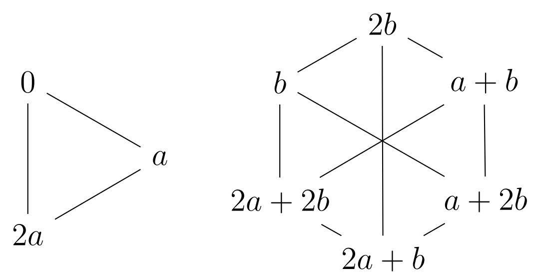
The case when $p > 2$ is given in the following theorem:

Theorem 3.3.5. *For an odd prime p, the total graph $\tau(G)$ is the disjoint union of two subgraphs X and Y where X is the complete graph K_p and Y is a p-regular graph on $p(p-1)$ vertices.*

Proof. Let us first consider all vertices of the form ia where $i \in \{0, 1, \ldots, p-1\}$. Since the set of zero divisors in G is $ZD(G) = \{0, a, \ldots, (p-1)a\}$, it is clear that the vertex ia is adjacent to every vertex ja where $j \in \{0, 1, \ldots, p-1\} \setminus \{i\}$ (contributing $p-1$ to the degree of ia). Moreover, ia cannot be adjacent to any $ka + lb \in U(G)$ where $k \in \{0, 1, \ldots, p-1\}$ and $l \in \{1, \ldots, p-1\}$. Hence, every vertex of the form ia has degree $p-1$. This tells us that the induced subgraph X with vertices in $ZD(G)$ is the complete graph K_p.

The remaining vertices that need to be considered are the $p^2 - p$ units in G of the form $ka + lb$ where $k \in \{0, 1, \ldots, p-1\}$ and $l \in \{1, \ldots, p-1\}$. We see that the vertex $ka + lb$ is adjacent to the vertices $\alpha a + (p-l)b$ where $\alpha \in \{0, 1, \ldots, p-1\}$ and not adjacent to any other vertices. Hence, the vertex $ka + lb$ has degree p. It follows that the induced subgraph Y with vertices in $U(G)$ is a p-regular graph on $p(p-1)$ vertices. □

Example 3.3.4. The total graph of the ring G where $p = 3$ is as follows :



3.3.2 Rings of order pq

In this section we discuss the total graphs associated with the rings of order pq, where p and q are distinct prime numbers. The structure of the total graphs associated with three of them are as follows:

Theorem 3.3.6. *The total graph of each of the rings $C_{pq}(0)$, $C_p(0)+\mathbb{Z}_q$ and $\mathbb{Z}_p+C_q(0)$ is the complete graph K_{pq}.*

Proof. It suffices to show that each of the elements in the rings $C_{pq}(0)$, $C_p(0)+\mathbb{Z}_q$, and $\mathbb{Z}_p+C_q(0)$ is a zero divisor.

(i) For any $\alpha a \in C_{pq}(0)$ where $\alpha \in \mathbb{Z}_{pq}$, note that $a \cdot \alpha a = \alpha a^2 = 0$.

(ii) For any $\alpha a+\beta b \in C_p(0)+\mathbb{Z}_q$ where $\alpha \in \mathbb{Z}_p$ and $\beta \in \mathbb{Z}_q$, note that $a\cdot(\alpha a+\beta b) = \alpha a^2+\beta ab = \alpha 0+\beta 0 = 0$.

(iii) For any $\alpha a+\beta b \in \mathbb{Z}_p+C_q(0)$ where $\alpha \in \mathbb{Z}_p$ and $\beta \in \mathbb{Z}_q$, note that $b\cdot(\alpha a+\beta b) = \alpha ba+\beta b^2 = \alpha 0+\beta 0 = 0$.

□

For the ring $\mathbb{Z}_{pq}$, we first note the following:

Lemma 3.3.2. *In the ring $\mathbb{Z}_{pq}$, there are $p+q-1$ zero divisors.*

Proof. Note that $\alpha \in \mathbb{Z}_{pq}\setminus\{0\}$ is a zero divisor if and only if $\gcd(\alpha, pq) \neq 1$. Therefore, the number of zero divisors in $\mathbb{Z}_{pq}$ is $pq-\phi(pq) = pq-(p-1)(q-1) = p+q-1$. □

Theorem 3.3.7. *For any odd prime p, the total graph $\tau(\mathbb{Z}_{2p})$ is a p-regular graph on $2p$ vertices.*

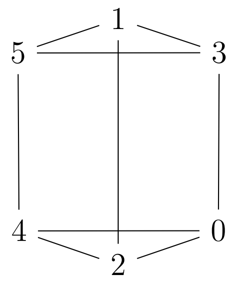
Proof. By Lemma 3.3.2, the set $ZD(\mathbb{Z}_{2p})$ of zero divisors of $\mathbb{Z}_{2p}$ has $p+1$ elements. These are $0, 2, 4, \ldots, 2(p-1), p$. Take any $x \in \mathbb{Z}_{2p}$.

Case 1: $x \in ZD(\mathbb{Z}_{2p})$. If $x \in \{0, 2, \ldots, 2(p-1)\}$, then x is adjacent to every element in $\{0, 2, \ldots, 2(p-1), p-x\}\setminus\{x\}$. Therefore x has degree p. If $x = p$, then x is adjacent to every element in $\{0, 1, 3, \ldots, 2p-1\}\setminus\{p\}$. Thus $x = p$ also has degree p.

Case 2: $x \notin ZD(\mathbb{Z}_{2p})$. In this case, $x \in \{1, 3, \ldots, 2p-1\}\setminus\{p\}$ is a unit and is adjacent to every element in $\{1, 3, \ldots, 2p-1, p-x\}\setminus\{x\}$. Hence, x has degree p.

Since every element in $\mathbb{Z}_{2p}$ has degree p, it follows that $\tau(\mathbb{Z}_{2p})$ is a p-regular graph on $2p$ vertices. $\square$

Example 3.3.5. The total graph of $\mathbb{Z}_6$ is as follows:



Theorem 3.3.8. *For distinct odd primes p, q, the total graph $\tau(\mathbb{Z}_{pq})$ has $p+q-1$ vertices associated with the zero divisors of $\mathbb{Z}_{pq}$ and each of these vertices has degree $p+q-2$. The remaining vertices of $\tau(\mathbb{Z}_{pq})$ are associated with its units and have degree $p+q-1$ each.*

Proof. By Lemma 3.3.2, there are $p+q-1$ zero divisors in the ring $\mathbb{Z}_{pq}$. These are $0, ip, jq$ where $i \in \{1, 2, \ldots, q-1\}$ and $j \in \{1, 2, \ldots, p-1\}$. The vertex 0 is adjacent to the zero divisors ip and jq; subsequently it has degree $(p-1)+(q-1) = p+q-2$.

For each $i \in \{1, \ldots, q-1\}$, the vertex ip is adjacent to all other vertices of the form $i'p$ where $i' \in \mathbb{Z}_q$. The vertex ip is also adjacent to the vertices $q-ip, 2q-ip, \ldots, (p-1)q-ip \in U(\mathbb{Z}_{pq})$. Thus the degree of ip is $p+q-2$. Similarly, jq has degree $p+q-2$ for each $j \in \{1, \ldots, p-1\}$.

We now consider the remaining vertices which form the unit set $U(\mathbb{Z}_{pq})$. Let $u \in U(\mathbb{Z}_{pq})$. Note that the vertex u is adjacent to the vertices $p-u, 2p-u, \ldots, (q-1)p-u$; $q-u, 2q-u, \ldots, (p-1)q-u$, and to the vertex $pq-u$. Adding these up gives us $(q-1)+(p-1)+1 = p+q-1$, the degree of the vertex u. $\square$

CHAPTER 4

THE UNIT GRAPHS OF SOME FINITE RINGS

4.1 Constructing unit graphs from total graphs

In this chapter, we investigate the unit graphs of some finite rings. These graphs have a close relation with the total graphs in the previous chapter, as explained in the following proposition.

Proposition 4.1.1. *The unit graph of a finite ring with identity* $1 \neq 0$ *is the complement of its total graph.*

Proof. Let R be a finite ring with identity $1 \neq 0$ and let u, v be distinct elements of R. There is an edge connecting u and v in the total graph $\tau(R)$ if and only if $u + v$ is a zero divisor in R, which, by Proposition 2.1.1, is true if and only if $u + v$ is not a unit in R, that is, if and only if there is no edge connecting u and v in the unit graph $\Gamma(R)$. Since the vertex sets of $\tau(R)$ and $\Gamma(R)$ are the same as R, it follows that $\Gamma(R)$ is the complement of $\tau(R)$. □

It is worthwhile to mention here that the finite ring in Proposition 4.1.1 need not be commutative (see Asir and Chelvam (2013b) for a discussion of the commutative case). As a consequence of Propositions 2.2.1 and 4.1.1, we have the following:

Corollary 4.1.1. *Let* R *be a finite ring with identity* $1 \neq 0$*. If the total graph* $\tau(R)$ *is disconnected, then the unit graph* $\Gamma(R)$ *is connected.*

We now set out to describe the structure of unit graphs associated with certain finite rings.

4.2 Unit graphs of finite rings with no units

Let R be a finite ring with no units. Then the sum of any two elements of R is also not a unit and hence, the unit graph $\Gamma(R)$ is an empty graph. Examples of rings with empty unit graphs are $C_n(0) = \langle a \mid na = 0, a^2 = 0 \rangle$ when $n \geq 2$, as well as the rings B, E, F, H, I and J of order p^2 as listed in Theorem 2.1.1 . Among the rings of order pq, we know by Theorem 3.3.6 that none of the elements in the rings $C_{pq}(0)$, $C_p(0) + \mathbb{Z}_q$ and $\mathbb{Z}_p + C_q(0)$ is a unit; hence the unit graphs of each of these rings is an empty graph.

4.3 Unit graphs of finite rings with units

We first consider the extreme case where every non-zero element is a unit. Fields are examples of such rings.

Theorem 4.3.1. *Let $\mathbb{F}$ be a finite field of order p^n where $n \geq 1$ and p is a prime.*

(i) If $p = 2$, then $\Gamma(\mathbb{F})$ is the complete graph K_{2^n}.

(ii) If $p > 2$, then the zero vertex in $\Gamma(\mathbb{F})$ has degree $p^n - 1$ whereas all the other vertices in $\Gamma(\mathbb{F})$ have degree $p^n - 2$ each.

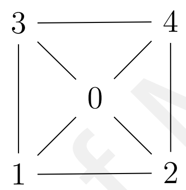
Proof. (i) If $p = 2$, then every element in $\mathbb{F}$ is its own additive inverse. Therefore $u+v \neq 0$ for any $u, v \in \mathbb{F}$, $u \neq v$. It follows that $u + v$ is a unit in $\mathbb{F}$ for any $u, v \in \mathbb{F}$, $u \neq v$. Consequently, any two distinct vertices are adjacent to one another in $\Gamma(\mathbb{F})$. Hence, $\Gamma(\mathbb{F})$ is the complete graph on 2^n vertices.

(ii) If $p > 2$, then any non-zero element $u \in \mathbb{F}$ has a unique additive inverse $-u \in \mathbb{F}$ with $-u \neq u$. It follows that u cannot be adjacent to $-u$ only in $\Gamma(\mathbb{F})$. Therefore every vertex which represents a non-zero element of $\mathbb{F}$ will have degree $p^n - 2$ in $\Gamma(\mathbb{F})$. Since every non-zero element in $\mathbb{F}$ is a unit, the zero vertex in $\Gamma(\mathbb{F})$ is adjacent to all the other vertices in $\Gamma(\mathbb{F})$ and hence, has degree $p^n - 1$. □

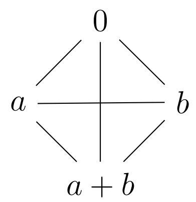
Remark 4.3.1. The unit graph structure of finite fields obtained in Theorem 4.3.1 is the complement of that of the total graph obtained in Theorem 3.2.1, as expected by Proposition 4.1.1.

By Theorem 4.3.1, we have the unit graph structures of the fields $\mathbb{Z}_p$ and K.

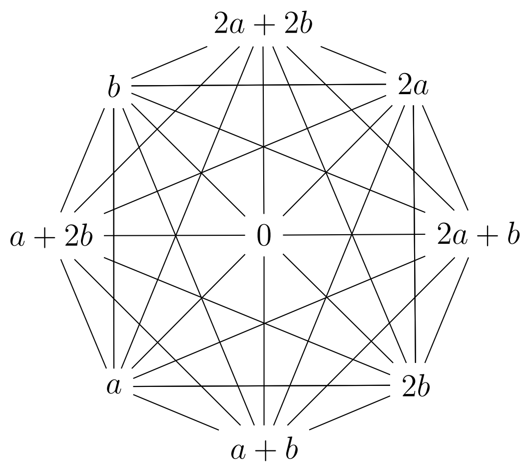
Example 4.3.1. The unit graph $\Gamma(\mathbb{Z}_5)$ is as follows (cf Example 3.2.1):



Example 4.3.2. When $p = 2$, the unit graph $\Gamma(K)$ is isomorphic to the complete graph K_4, as shown below:



Example 4.3.3. When $p = 3$, the unit graph $\Gamma(K)$ is as shown below:



4.4 Unit graphs of other rings of order p^2 and pq

In the previous sections, we have determined the unit graph structures of the rings of order p, the rings B, E, F, H, I, J and K of order p^2, and the rings $C_{pq}(0)$, $C_p(0) + \mathbb{Z}_q$ and $\mathbb{Z}_p + C_q(0)$ of order pq. In this section, we determine the unit graphs of the remaining rings of order p^2 and pq by using Proposition 4.1.1 and the total graph structures obtained in Chapter 3.

4.4.1 Other rings of order p^2

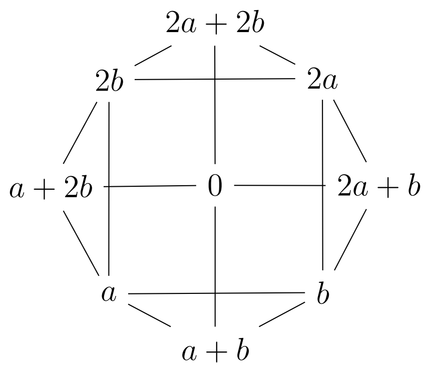
We begin by determining the unit graph structure of the ring $A = \langle a : p^2 a = 0, a^2 = a \rangle = \mathbb{Z}_{p^2}$.

Proposition 4.4.1. *The unit graph of $A = \mathbb{Z}_{p^2}$ is a 4-cycle when $p = 2$. When p is an odd prime, the unit graph $\Gamma(A)$ consists of p^2 vertices, p of which are associated with the zero divisors of A and have degree $p^2 - p$ each whereas the remaining $p^2 - p$ vertices are associated with the units of A and have degree $p^2 - p - 1$ each.*

We now consider the unit graph of the ring $D = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = ba = 0 \rangle = \mathbb{Z}_p + \mathbb{Z}_p$ which has identity $a + b$. The units of D are the elements $ia + jb$ where $i, j \neq 0$.

Proposition 4.4.2. *For $p = 2$, the unit graph $\Gamma(D)$ is a disjoint union of two subgraphs, each isomorphic to the complete graph K_2. For p an odd prime, the unit graph $\Gamma(D)$ consists of p^2 vertices, $2p - 1$ of which are associated with the zero divisors of D and have degree $(p-1)^2$ each whereas the remaining $(p-1)^2$ vertices are associated with the units and have degree $p^2 - 2p$ each.*

Example 4.4.1. The unit graph $\Gamma(D)$ when $p = 3$ is as follows (cf Example 3.3.3):



Now consider $G = \langle a, b : pa = pb = 0, a^2 = 0, b^2 = b, ab = a, ba = a \rangle$. The identity element of G is b and the units in G are of the form $ma + nb$ where $n \neq 0$.

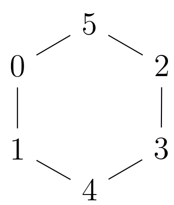
Proposition 4.4.3. *For $p = 2$, the unit graph $\Gamma(G)$ is a 4-cycle. For p an odd prime, the unit graph $\Gamma(G)$ consists of p^2 vertices, p of which are associated with the zero divisors of G and have degree $p^2 - p$ each. The remaining $p^2 - p$ vertices are associated with the units of G and have degree $p^2 - p - 1$ each.*

4.4.2 Ring of order pq

The remaining ring of order pq to be considered is $\mathbb{Z}_{pq}$.

Proposition 4.4.4. *For any odd prime p, the unit graph $\Gamma(\mathbb{Z}_{2p})$ is a $(p-1)$-regular graph on $2p$ vertices.*

Example 4.4.2. The unit graph $\Gamma(\mathbb{Z}_6)$ is as follows:



Proposition 4.4.5. *For distinct odd primes p and q, the unit graph $\Gamma(\mathbb{Z}_{pq})$ has $p + q - 1$ vertices associated with the zero divisors of $\mathbb{Z}_{pq}$ and having degree $(p-1)(q-1)$ each.*

The remaining vertices are associated with the units of $\mathbb{Z}_{pq}$ and have degree $pq - p - q$ each.

4.5 Some properties of unit graphs and their relations with total graphs

Let R be a finite ring with identity $1 \neq 0$. By Proposition 2.1.1, $U(R) = R \setminus ZD(R)$. Let $Reg(\Gamma(R))$ be the induced subgraph of $\Gamma(R)$ with vertices $U(R)$. In this section, we present some properties of unit graphs and their relations with total graphs.

Proposition 4.5.1. *Let R be a finite commutative ring with identity $1 \neq 0$ such that $ZD(R)$ is an ideal of R. If ${}^{R}\!/_{ZD(R)} \ncong \mathbb{Z}_2$ and ${}^{R}\!/_{ZD(R)} \ncong \mathbb{Z}_3$, then $Reg(\Gamma(R))$ is connected.*

Proof. This follows by Theorem 2.4 in Anderson and Badawi (2008) as well as Propositions 2.2.1 and 4.1.1. □

Theorem 4.5.1. *Let R be a finite commutative ring with identity $1 \neq 0$ such that $ZD(R)$ is an ideal of R. Then the following statements are equivalent:*

(i) $Reg(\Gamma(R))$ *is disconnected.*

(ii) *Either $x + y \in ZD(R)$ or $x - y \in ZD(R)$ for all $x, y \in U(R)$.*

(iii) *Either $x + y \in ZD(R)$ or $x - 2y \in ZD(R)$ for all $x, y \in U(R)$. In particular, either $2x \in ZD(R)$ or $3x \in ZD(R)$ (but not both) for all $x \in U(R)$.*

(iv) *Either ${}^{R}\!/_{ZD(R)} \cong \mathbb{Z}_2$ or ${}^{R}\!/_{ZD(R)} \cong \mathbb{Z}_3$.*

Proof. This follows by Theorem 2.9 in Anderson and Badawi (2008) as well as Propositions 2.2.1 and 4.1.1 . □

We next consider the planarity property. It is known that if a simple graph G has 11 or more vertices, then either G or its complement is non-planar. To prove this, we need the following inequality:

Lemma 4.5.1. *In a planar graph with v vertices and e edges, the inequality $e \leq 3v - 6$ holds.*

Proof. Note first that in a planar graph, each face is bounded by at least three edges and each edge borders two faces. Hence, in any planar graph, the number of faces f satisfies $3f \leq 2e$. Combining this with Euler's formula $v - e + f = 2$, we have that $v - e + \frac{2e}{3} \geq 2$, that is, $3v - e \geq 6$. Hence, $3v - 6 \geq e$. □

Theorem 4.5.2. *If a simple graph G has 11 or more vertices, then either G or its complement is non-planar.*

Proof. Let G be a graph on n vertices and assume that both G and G' are planar. Let e and e' be the number of edges in G and G', respectively. The union of the two graphs is the complete graph on n vertices. Thus,

$$e + e' = \binom{n}{2} = \frac{n(n-1)}{2}.$$

By Lemma 4.5.1, $e, e' \leq 3n - 6$. Therefore, $e + e' \leq 6n - 12$. We then have $\frac{n(n-1)}{2} = e + e' \leq 6n - 12$. This implies that $n^2 - 13n + 24 \leq 0$. Solving this inequality, we have that $n < 11$. □

The following result is an immediate consequence of Theorem 4.5.2 and Proposition 4.1.1.

Proposition 4.5.2. *Let R be a finite commutative ring with identity $1 \neq 0$ such that $|R| \geq 11$. If the unit graph $\Gamma(R)$ is planar, then the total graph $\tau(R)$ is non-planar.*

Proposition 4.5.3. *Let R be a finite commutative ring with identity $1 \neq 0$ such that $|R| \geq 11$. If the total graph of R is planar, then R is not isomorphic to any of the following rings:*

(i) $\mathbb{Z}_5$;

(ii) $\mathbb{Z}_3 \times \mathbb{Z}_3$*;*

(iii) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \text{ times}} \times S$ *where* $l \geq 0$ *and* $S \cong \mathbb{Z}_2$*,* $S \cong \mathbb{Z}_3$*,* $S \cong \mathbb{Z}_4$*,* $S \cong \mathbb{F}_4$ *(the field with four elements), or* $S \cong \{\left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] | a, b \in \mathbb{Z}_2\}$.

Proof. This follows by Proposition 4.5.2 and Theorem 5.14 in Ashrafi et al. (2010). □

The degree of vertices in unit graphs associated with finite rings are known and have been obtained by Ashrafi et al. (2010). For convenience, we state their result on the degree of vertices here.

Proposition 4.5.4 (Ashrafi et al., 2010)**.** *Let* R *be a finite ring with identity* $1 \neq 0$*. Then the following statements hold for the unit graph of* R*:*

(i) If $2 \notin U(R)$*, then the unit graph* $\Gamma(R)$ *is a* $|U(R)|$*-regular graph;*

(ii) If $2 \in U(R)$*, then for every* $x \in U(R)$ *we have* $\deg(x) = |U(R)| - 1$ *and for every* $x \in R \setminus U(R)$ *we have* $\deg(x) = |U(R)|$*.*

As a consequence of Propositions 4.1.1 and 4.5.4, we obtain the following:

Proposition 4.5.5. *Let* R *be a finite ring with identity* $1 \neq 0$*. Then the following statements hold for the total graph of* R*.*

(i) If $2 \in ZD(R)$*, then the total graph* $\tau(R)$ *is a* $(|ZD(R)| - 1)$*-regular graph.*

(ii) If $2 \notin ZD(R)$*, then for every* $x \in ZD(R)$ *we have* $\deg x = |ZD(R)| - 1$ *and for every* $x \in U(R)$ *we have* $\deg x = |ZD(R)|$*.*

The following result of Ashrafi et al. (2010) gives a necessary and sufficient condition for the unit graph of a finite ring with identity $1 \neq 0$ to be complete.

Theorem 4.5.3 (Ashrafi et al., 2010)**.** *Let* R *be a finite ring with identity* $1 \neq 0$*. Then the unit graph* $\Gamma(R)$ *is a complete graph if and only if* R *is a division ring with char* $R = 2$*.*

An immediate consequence of Theorem 4.5.3 and Proposition 4.1.1 is the following:

Corollary 4.5.1. *Let R be a finite ring with identity $1 \neq 0$. Then the total graph $\tau(R)$ is an empty graph if and only if R is a division ring with char $R = 2$.*

CHAPTER 5

THE DIRECTED GRAPHS OF SOME FINITE RINGS

5.1 A brief overview

In this chapter, we will discuss the directed graph structures of certain rings. Unlike the total and unit graphs which are simple and undirected, the graphs in this chapter are directed and have loops. The discussion in this chapter will include properties such as the number of sources or non-sources and the number of connected components in a directed graph. By a connected component of a directed graph here, we mean a maximal weakly connected (ignoring the direction of all the edges) subgraph of the directed graph. Recall that a vertex in a directed graph is a source if it has incoming degree 0. Conversely, a non-source is a vertex in a directed graph with incoming degree greater than 0.

The directed graph structures of finite rings appear to be more complex than their total or unit graphs. This will be discussed in Section 5.2 where we will see that some properties of the directed graph of the ring $\mathbb{Z}_n$ do not follow any obvious pattern. In searching for examples of directed graphs of finite rings which share similar characteristics, we found two types of rings whose directed graphs have some "symmetric" properties in its number of connected components and the number of vertices in each component. One of these types is the ring $C_n(0) = \langle a : na = 0, a^2 = 0\rangle$ where $n \geq 2$. The other type is the ring I of order p^2 as described in Theorem 2.1.1. These results will be presented in Sections 5.2.1 and 5.2.2. For the number of non-sources in a directed graph, we found an explicit formula for this number in the directed graphs of the rings I and $\mathbb{Z}_n$ ($n \geq 2$). These results will be presented in Sections 5.2.2 and 5.2.3.

In the remainder of this chapter, we shall refer to directed graphs as digraphs.

5.2 Structures of some digraphs

Unlike the total graphs and unit graphs of finite rings, not much seems to be known about digraphs of finite rings in general. The complexity of the structures of digraphs of finite rings do not necessarily increase as the order of the ring increases. An example would be the digraph of the ring $\mathbb{Z}_n$. Lipkovski (2012) made use of a computer program to obtain the number c_n of connected components, the length p_n of the longest path (including the loop closing the path) and the longest cycle l_n for the digraph of the ring $\mathbb{Z}_n$ where $n \leq 50$. We include his results for $n \leq 15$ below:

Table 5.1: Values of c_n, p_n, and l_n for the digraph $\Psi(\mathbb{Z}_n)$(Lipkovski, 2012)

n	c_n	p_n	l_n
1	1	1	1
2	2	3	1
3	3	5	1
4	5	4	2
5	6	6	4
6	6	5	1
7	7	9	1
8	12	6	4
9	14	6	3
10	12	6	4
11	12	14	6
12	15	6	2
13	14	22	4
14	14	9	1
15	18	8	4

As is evident from the table above, the number of connected components, the length of the longest path, and the length of the longest cycle do not always increase as n increases. More specifically, Lipkovski (2012) made the observation that local peaks of p_n and l_n

appear for (some, but not all) primes n and the peaks of c_n appear also for $n = 2^k$ for some positive integer k.

Nevertheless, there are still some properties of digraphs of certain finite rings which exhibit some uniformity. These will be presented in the following subsections.

5.2.1 Connected components in the digraph of the ring $C_n(0)$, $n \geq 2$

We obtain a complete description of the digraph of the ring $C_n(0) = \langle a : na = 0, a^2 = 0 \rangle$ ($n \geq 2$) as follows:

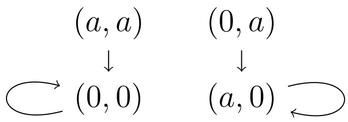
Theorem 5.2.1. *The digraph of the ring* $C_n(0) = \langle a \mid na = 0, a^2 = 0 \rangle$*, where* $n \geq 2$ *is an integer, consists of* n *connected components, each of which has* n *vertices with a loop at exactly one of the vertices. Moreover, every vertex in a connected component points to the vertex with the loop in the component.*

Proof. Since $C_n(0) = \langle a : na = 0, a^2 = 0 \rangle$ is finite, it is possible to list down all the vertices of its digraph as an array as follows. For convenience, we will enumerate the rows and columns starting from 0.

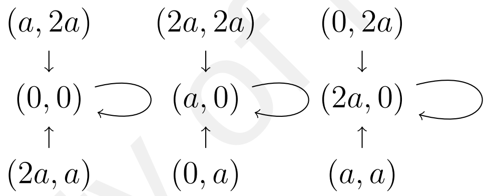
$$\begin{array}{cccccc}
(0,0) & (0,a) & (0,2a) & \cdots & (0,(n-2)a) & (0,(n-1)a) \\
(a,0) & (a,a) & (a,2a) & \cdots & (a,(n-2)a) & (a,(n-1)a) \\
(2a,0) & (2a,a) & (2a,2a) & \cdots & (2a,(n-2)a) & (2a,(n-1)a) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
((n-2)a,0) & ((n-2)a,a) & ((n-2)a,2a) & \cdots & ((n-2)a,(n-2)a) & ((n-2)a,(n-1)a) \\
((n-1)a,0) & ((n-1)a,a) & ((n-1)a,2a) & \cdots & ((n-1)a,(n-2)a) & ((n-1)a,(n-1)a)
\end{array}$$

Note that there is a loop at each vertex in the zeroth column. These vertices must therefore be in different connected components in the digraph because no connected component may have more than one cycle (by Proposition 2.3.1). Now consider the kth row. Its vertices are of the form $(ka, l_i a)$ where $l_i \in \mathbb{Z}_n$ and all point to vertices of the form $(c, 0)$, that is, vertices in the zeroth column. It is obvious that no two distinct vertices in the same row (but not in the zeroth column) point to the same vertex in the zeroth column; otherwise, that would imply that $k + l_i \equiv k + l_j \pmod{n}$ for some $i \neq j$ which contradicts the fact that $l_i \not\equiv l_j \pmod{n}$ when $i \neq j$. Hence, each row contributes exactly 1 degree to each vertex in the zeroth column. Each connected component therefore consists of exactly one loop at a vertex, say v, and $n - 1$ other vertices pointing to v. $\square$

Example 5.2.1. Below are the digraphs for $C_2(0)$ and $C_3(0)$.



The digraph $\Psi(C_2(0))$



The digraph $\Psi(C_3(0))$

By Theorem 5.2.1 we have the following:

Corollary 5.2.1. *The digraph of the ring* $C_n(0) = \langle a : na = 0, a^2 = 0 \rangle$*, where* $n \geq 2$ *is an integer, has exactly* $n(n-1)$ *sources with each of the* n *connected components contributing* $n - 1$ *sources.*

Proof. By Theorem 5.2.1, each of the n connected components in $\Psi(C_n(0))$ has exactly one non-source, namely, the only vertex with a loop in the component. The remaining $n - 1$ vertices in each component all point to the vertex with a loop in the component. Therefore, each component of the digraph contributes $n - 1$ sources to the digraph. It follows that the number of sources in $\Psi(C_n(0))$ is $n(n-1)$. $\square$

Theorem 5.2.2. *The digraph of the ring* $R = C_p(0) + \ldots + C_p(0) = \langle a_1, \ldots, a_k \mid pa_1 = \ldots = pa_k = 0, a_1^2 = \ldots = a_k^2 = 0, a_i a_j = 0 \, \forall i, j = 1, \ldots, k \rangle$ *where* p *is a prime and* k *is a positive integer has exactly* p^k *non-sources.*

Proof. Given any vertex $(x, y) \in \Psi(R)$, note that (x, y) points to the vertex $(x + y, 0)$ in $\Psi(R)$. This implies that the non-sources in $\Psi(R)$ are of the form $(z, 0)$ for some $z \in R$. Since there is a loop (hence, a non-source) at each vertex of the form $(z, 0)$, it follows that the non-sources in $\Psi(R)$ are made up of vertices of the form $(z, 0)$ where $z \in R$. The number of vertices of the form $(z, 0)$ in $\Psi(R)$ is clearly $|R| = p^k$. □

5.2.2 Non-sources and connected components in the digraph of the ring I

In this section, we determine the structure of the digraph $\Psi(I)$ where I is the ring of order p^2 with presentation given by $I = \langle a, b \mid pa = pb = 0, a^2 = b, ab = 0 \rangle$, p is a prime. We obtain an explicit formula for determining the number of non-sources in $\Psi(I)$. We also prove that $\Psi(I)$ has a nice "symmetric" property in that it has exactly p^2 connected components with p^2 vertices in each connected component. Note that the relations in the presentation of I imply that $a^3 = a \cdot a^2 = ab = 0$ and $ba = a^2 \cdot a = a^3 = 0$. To count the number of non-sources in the digraph $\Psi(I)$, we first obtain the following results.

Lemma 5.2.1. *Let* $i, i', j, j' \in \mathbb{Z}_p \setminus \{0\}$ *such that* $(i, j) \neq (i', j')$*. Then the equations* $i + j = i' + j'$ *and* $ij = i'j'$ *hold if and only if* $i = j'$ *and* $j = i'$*.*

Proof. (⇒): Assume that $i + j = i' + j'$ and $ij = i'j'$. Then

$$\begin{aligned} 0 &= ij - ij' + ij' - i'j' \\ &= i(j - j') + (i - i')j' \\ &= i(j - j') - (j - j')j' \qquad (\because i + j = i' + j') \\ &= (i - j')(j - j'). \end{aligned}$$

Since $\mathbb{Z}_p$ is an integral domain (hence, has no zero divisors), it follows that $i = j'$ or $j = j'$. If $j = j'$, then $i = i'$ and we have $(i, j) = (i', j')$, a contradiction. Thus, $i = j'$ and hence, $j = i'$.

(⇐): This is clear. □

Lemma 5.2.2. *The number of distinct pairs* $(i + j, ij)$ *where* $i, j \in \mathbb{Z}_p \setminus \{0\}$ *is* $p(p - 1)/2$.

Proof. By Lemma 5.2.1, the number of distinct pairs $(i + j, ij)$ where $i, j \in \mathbb{Z}_p \setminus \{0\}$ is the same as the number of pairs (i, j) where $1 \leq i \leq j \leq p - 1$. Thus, the total number of distinct pairs $(i + j, ij)$ is $1 + 2 + \ldots + (p - 1) = p(p - 1)/2$. □

We now obtain a formula for determining the number of non-sources in the digraph $\Psi(I)$.

Theorem 5.2.3. *The digraph* $\Psi(I)$ *of the ring* $I = \langle a, b \mid pa = pb = 0, a^2 = b, ab = 0\rangle$ *has exactly* $p^2(p + 1)/2$ *non-sources.*

Proof. Note that a non-source may occur in $\Psi(I)$ in one of the following ways:

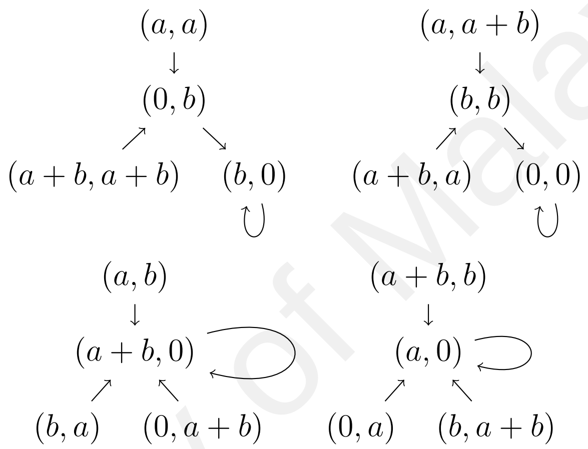
(a) $(i, 0) \to (i, 0)$ where $i \in I$;

(b) $(i, j) \to (i + j, ij)$ where $i, j \in I, ij \neq 0$.

The number of non-sources of the form $(i, 0)$ where $i \in I$ is the same as the number of elements in I, that is, p^2. To count the non-sources which occur via (b), consider the vertex

$(\lambda_1 a + \lambda_2 b, \gamma_1 a + \gamma_2 b)$ which points to $((\lambda_1 + \gamma_1)a + (\lambda_2 + \gamma_2)b, \lambda_1\gamma_1 b)$. By Lemma 5.2.2, there are $p(p-1)/2$ distinct pairs $(\lambda_1 + \gamma_1, \lambda_1\gamma_1)$ where $\lambda_1, \gamma_1 \in \mathbb{Z}_p \setminus \{0\}$. For each of these pairs, there are p possible values for $\lambda_2 + \gamma_2$. Hence, the number of non-sources which occur via (b) is $p \cdot \frac{p(p-1)}{2} = \frac{p^2(p-1)}{2}$. Combining cases (a) and (b), the number of non-sources in the digraph of the ring I is $p^2 + \frac{p^2(p-1)}{2} = \frac{p^2(p+1)}{2}$. □

Example 5.2.2. The digraph of the ring $I_{(2)} = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0\rangle$ is illustrated in the following diagram.



The number of non-sources seen here is 6 which agrees with the result obtained by Theorem 5.2.3.

We now determine the number of connected components in $\Psi(I)$ as well as the number of vertices in each connected component.

Theorem 5.2.4. *The digraph of the ring $I = \langle a, b \mid pa = pb = 0, a^2 = b, ab = 0\rangle$ has p^2 connected components.*

Proof. Note that for any vertex $(\lambda_1 a + \lambda_2 b, \gamma_1 a + \gamma_2 b) \in V(\Psi(I))$,

$$\begin{aligned}(\lambda_1 a + \lambda_2 b, \gamma_1 a + \gamma_2 b) &\to ((\lambda_1 + \gamma_1)a + (\lambda_2 + \gamma_2)b, \lambda_1\gamma_1 b) \\ &\to ((\lambda_1 + \gamma_1)a + (\lambda_2 + \gamma_2 + \lambda_1\gamma_1)b, 0)\end{aligned}$$

is a directed path in $\Psi(I)$ ending with a loop. By Proposition 2.3.1, every connected component of $\Psi(I)$ has exactly one cycle. It follows that the only cycle in each connected component is the loop at $(x, 0)$ for some $x \in I$. Since $|I| = p^2$, we conclude that there are exactly p^2 connected components in $\Psi(I)$. $\square$

To determine the number of connected components in the digraph of the ring I, it will be useful to know about quadratic residues. Let n be a positive integer and q an integer such that $\gcd(q, n) = 1$. Then q is called a quadratic residue modulo n if the congruence $x^2 \equiv q \pmod{n}$ has a solution (that is, if q is a square modulo n). A known result in number theory concerns the number of quadratic residues. One such formulation of this enumeration can be found in a paper of Stangl (1996) and is as follows:

Proposition 5.2.3. *If p is an odd prime, then the number of quadratic residues in the ring $\mathbb{Z}_{p^n}$ is $(p^n - p^{n-1})/2$ for all $n \geq 1$.*

Theorem 5.2.5. *Each connected component in the digraph of the ring $I = \langle a, b \mid pa = pb = 0, a^2 = b, ab = 0\rangle$ consists of p^2 vertices.*

Proof. The case $p = 2$ has been shown to be true in Example 5.2.2. We thus assume that p is odd. By the proof of Theorem 5.2.4, every connected component in $\Psi(I)$ has exactly one loop at $(x, 0)$ for some $x \in I$. We consider the following two cases.

Case 1: $x = \beta b$ for some $\beta \in \{0, 1, \ldots, p-1\}$ where β is fixed. We first determine the number of vertices which point to $(\beta b, 0)$ in $\Psi(I)$. Any such vertex takes the form $(\alpha b, \gamma b)$ where $\alpha + \gamma = \beta$. There are p possible pairs (α, γ) which satisfy $\alpha + \gamma = \beta$; thus p distinct vertices $(\alpha b, \gamma b)$ which point to $((\alpha + \gamma)b, 0) = (\beta b, 0)$.

Next we determine the number of vertices which point to $(\alpha b, \gamma b)$ where $\alpha + \gamma = \beta$ and $\gamma \neq 0$. In other words, we determine the number of vertices which are of distance two from $(\beta b, 0)$. Let $(\lambda_1 a + \lambda_2 b, \gamma_1 a + \gamma_2 b)$ be a vertex which points to $(\alpha b, \gamma b)$ in $\Psi(I)$.

Then

$$\lambda_1 + \gamma_1 = 0, \tag{5.1}$$

$$\lambda_2 + \gamma_2 = \alpha, \tag{5.2}$$

$$\lambda_1 \gamma_1 = \gamma. \tag{5.3}$$

By (5.1) and (5.3), we have that $-\lambda_1^2 = \gamma$. That is, $\lambda_1^2 = (p-1)\gamma$. This equation has a solution if and only if $(p-1)\gamma$ is a quadratic residue modulo p. Since there are $(p-1)/2$ quadratic residues modulo p (by Proposition 5.2.3), it follows that there are $(p-1)/2$ vertices of the form $(\alpha b, \gamma b)$ where $\alpha + \gamma = \beta$ with $\gamma \neq 0$, that are non-sources with each of these vertices pointed at by $(\lambda_1 a + \lambda_2 b, \gamma_1 a + \gamma_2 b)$ where $\lambda_1, \lambda_2, \gamma_1, \gamma_2$ satisfy equations (5.1), (5.2), and (5.3). Now assume that $(p-1)\gamma$ is a quadratic residue modulo p. Note that if λ_1 satisfies the equation $\lambda_1^2 = (p-1)\gamma$, so does $\gamma_1 = p - \lambda_1$. Therefore $(\lambda_1 a + \lambda_2 b, \gamma_1 a + \gamma_2 b)$ points to $(\alpha b, \gamma b)$ in $\Psi(I)$ if and only if $(\gamma_1 a + \lambda_2 b, \lambda_1 a + \gamma_2 b)$ also points to $(\alpha b, \gamma b)$. Clearly, there are p solutions for (λ_2, γ_2) so that $\lambda_2 + \gamma_2 = \alpha$. Let $(\lambda_2^{(i)}, \gamma_2^{(i)}), i = 1, \ldots, p$ be these solutions. For each of the pair $(\lambda_2^{(i)}, \gamma_2^{(i)})$, we have that $(\lambda_1 a + \lambda_2^{(i)} b, \gamma_1 a + \gamma_2^{(i)} b)$ and $(\gamma_1 a + \lambda_2^{(i)} b, \lambda_1 a + \gamma_2^{(i)} b)$, where (λ_1, γ_1) satisfies (5.1) and (5.3), point to $(\alpha b, \gamma b)$ where $\alpha + \gamma = \beta$, $\gamma \neq 0$. This implies that there are altogether $2p$ vertices pointing to the aforementioned $(\alpha b, \gamma b)$. It follows that the number of vertices in the connected component with loop at $(\beta b, 0)$ is $p + 2p(\frac{p-1}{2}) = p^2$.

Case 2: $x = \alpha a + \beta b$ where $\alpha \neq 0$ and α, β are fixed. Note first that any vertex that points to $(\alpha a + \beta b, 0)$ in $\Psi(I)$ is of the form $(\alpha a + \gamma b, \lambda b)$ or $(\gamma b, \alpha a + \lambda b)$ where $\gamma + \lambda = \beta$. There are p pairs of (γ, λ) such that $\gamma + \lambda = \beta$. For each of these pairs we have the directed edges $(\alpha a + \gamma b, \lambda b) \to (\alpha a + \beta b, 0)$ and $(\gamma b, \alpha a + \lambda b) \to (\alpha a + \beta b, 0)$. Therefore, there are altogether $2p$ vertices which point to $(\alpha a + \beta b, 0), \alpha \neq 0$.

Note that any vertex of the form $(\gamma b, \alpha a + \lambda b)$ where $\alpha \neq 0$ is a source, that is, there is no vertex pointing to it. We now determine the number of vertices which point to $(\alpha a + \gamma b, \lambda b)$ where $\gamma + \lambda = \beta$ and $\lambda \neq 0$. Let $(\lambda_1 a + \lambda_2 b, \gamma_1 a + \gamma_2 b)$ be one such vertex. Then

$$\lambda_1 + \gamma_1 = \alpha, \tag{5.4}$$

$$\lambda_2 + \gamma_2 = \gamma, \tag{5.5}$$

$$\lambda_1 \gamma_1 = \lambda. \tag{5.6}$$

By (5.4) and (5.6), $\lambda_1(\alpha - \lambda_1) = \lambda$, that is, $\lambda_1^2 - \alpha\lambda_1 = -\lambda$. Completing the square gives us $(\lambda_1 - 2^{-1}\alpha)^2 = (2^{-1}\alpha)^2 - \lambda$. A solution for this exists if and only if $(2^{-1}\alpha)^2 - \lambda$ is 0 or a quadratic residue modulo p.

Subcase 1: $(2^{-1}\alpha)^2 - \lambda = 0$. In this case, $\lambda_1 = 2^{-1}\alpha = \frac{p+1}{2}\alpha$ and hence, $\gamma_1 = \alpha - \lambda_1 = (1 - 2^{-1})\alpha = \frac{p+1}{2}\alpha$. There are p pairs (λ_2, γ_2) which satisfy equation (5.5). Let $(\lambda_2^{(i)}, \gamma_2^{(i)})$, $i = 1, \ldots, p$ be these solutions. Therefore the vertices in $\Psi(I)$ which point to $(\alpha a + \gamma b, \lambda b)$ where $\lambda = (\frac{p+1}{2}\alpha)^2, \gamma = \beta - \lambda$ are $(\frac{p+1}{2}\alpha a + \lambda_2^{(i)} b, \frac{p+1}{2}\alpha a + \gamma_2^{(i)} b)$ $(i = 1, \ldots, p)$. Hence there are altogether p vertices which point to $(\alpha a + \gamma b, \lambda b)$ where $\lambda = (2^{-1}\alpha)^2$, $\alpha \neq 0$ and $\gamma = \beta - \lambda$.

Subcase 2: $(2^{-1}\alpha)^2 - \lambda$ is a quadratic residue modulo p. Suppose $(2^{-1}\alpha)^2 - \lambda = r^2$ for $r \in \{1, 2, \ldots, \frac{p-1}{2}\} \setminus \{2^{-1}\alpha\}$. There are $\frac{p-3}{2}$ possibilities for r. There are also p pairs $(\lambda_2^{(i)}, \gamma_2^{(i)})$ such that $\lambda_2^{(i)} + \gamma_2^{(i)} = \gamma$ $(i = 1, \ldots, p)$. Therefore the vertices which point to

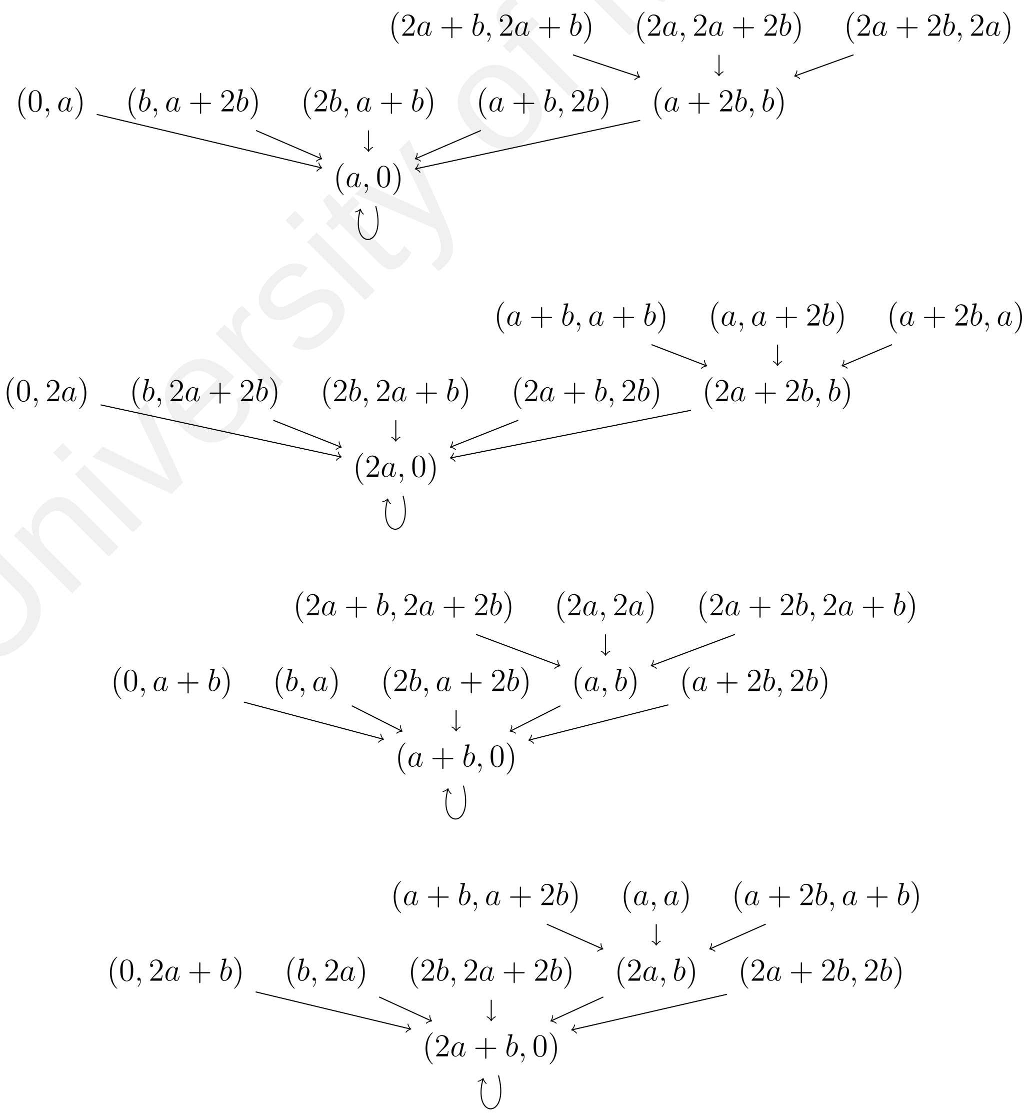
$(\alpha a+\gamma b, \lambda b)$, where $\gamma+\lambda=\beta$ and $\lambda \neq 0$ are

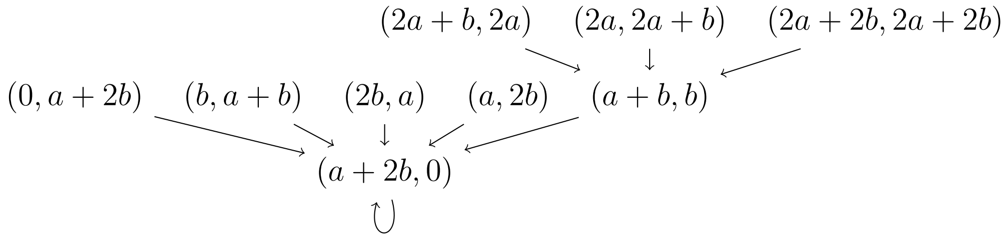
$$(\lambda_1 a+\lambda_2^{(i)} b, (\alpha-\lambda_1)a+(\gamma-\lambda_2^{(i)})b) \tag{5.7}$$

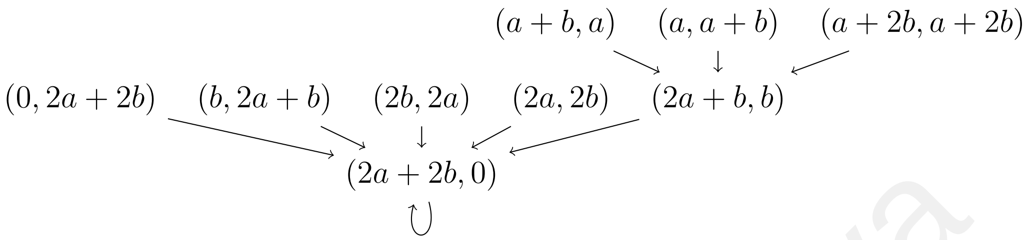
where $\lambda_1=\pm r+2^{-1}\alpha$. The number of vertices of the form (5.7) is $2\cdot(\frac{p-3}{2})\cdot p=p(p-3)$.

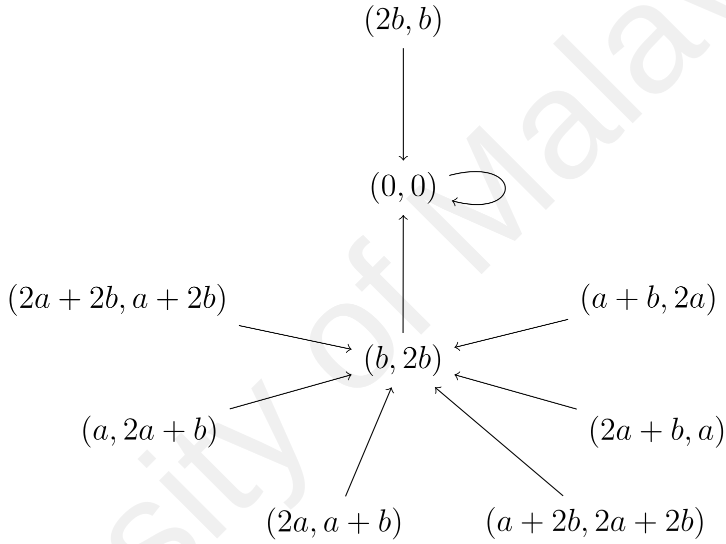
By collecting the above cases, we have that the number of vertices in the connected component with loop at $(\alpha a+\beta b, 0)$ when $\alpha \neq 0$ (α, β are fixed) is $2p+p+p(p-3)=p^2$. This completes the proof. □

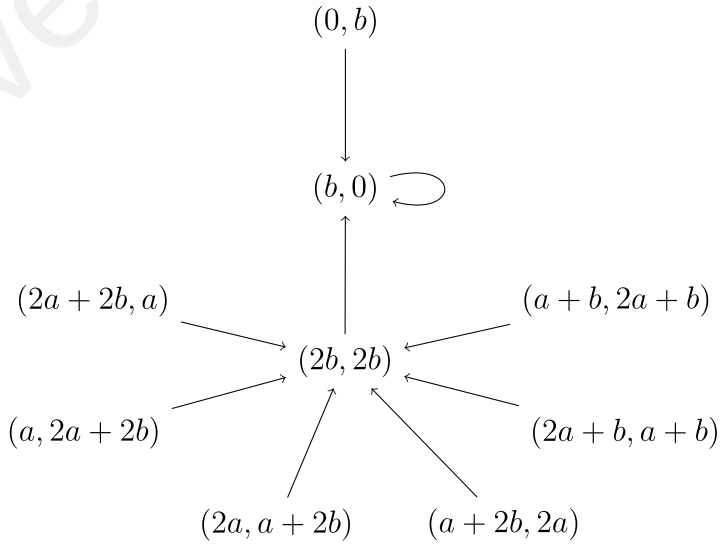
Example 5.2.4. The digraph of the ring $I_{(3)}=\langle a, b \mid 3a=3b=0, a^2=b, ab=0\rangle$ is illustrated below.

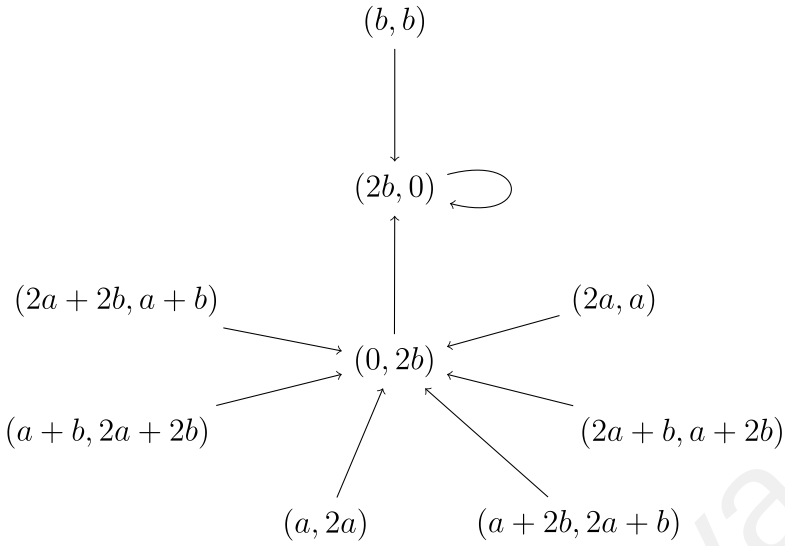



$(2a+b,2a)$
$(2a,2a+b)$
$(2a+2b,2a+2b)$
$(0,a+2b)$
$(b,a+b)$
$(2b,a)$
$(a,2b)$
$(a+b,b)$
$(a+2b,0)$


$(a+b,a)$
$(a,a+b)$
$(a+2b,a+2b)$
$(0,2a+2b)$
$(b,2a+b)$
$(2b,2a)$
$(2a,2b)$
$(2a+b,b)$
$(2a+2b,0)$


$(2b,b)$
$(0,0)$
$(2a+2b,a+2b)$
$(a+b,2a)$
$(b,2b)$
$(a,2a+b)$
$(2a+b,a)$
$(2a,a+b)$
$(a+2b,2a+2b)$


$(0,b)$
$(b,0)$
$(2a+2b,a)$
$(a+b,2a+b)$
$(2b,2b)$
$(a,2a+2b)$
$(2a+b,a+b)$
$(2a,a+2b)$
$(a+2b,2a)$



The number of non-sources seen here is 18 which agrees with the result obtained by Theorem 5.2.3. There are also 9 connected components with 9 vertices in each component, thus agreeing with Theorems 5.2.4 and 5.2.5.

5.2.3 Non-sources in the digraph of the ring $\mathbb{Z}_n$

In this section, we obtain an explicit formula for determining the number of non-sources in the digraph $\Psi(\mathbb{Z}_{p^n})$. This formula may then be used to determine the number of non-sources in the digraph of the ring $\mathbb{Z}_m$ where m is a positive integer. The results in this section are motivated by a question raised by Ang and Schulte (2013) on the number of sources in a connected component of the digraph associated with a ring. Ang and Schulte (2013) proved the following theorem which allows for the enumeration of the number of sources in the digraph of a finite field:

Theorem 5.2.6 (Ang & Schulte, 2013)**.** *Let F be a finite field. Then there are $\frac{q^2-q}{2}$ sources in $\Psi(F)$ where $q = |F|$.*

We now extend Theorem 5.2.6 to the ring $\mathbb{Z}_{p^n}$, where p is a prime. More precisely, we determine the number of non-sources in $\Psi(\mathbb{Z}_{p^n})$. This is equivalent to finding all pairs

$(a, b) \in \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$ such that $a = c + d$ and $b = cd$ for some $c, d \in \mathbb{Z}_{p^n}$. It turns out that the results on the number of non-sources are useful for determining the number of non-sources in $\Psi(\mathbb{Z}_m)$, where m is a positive integer. This will be discussed at the end of this section. It is worthwhile to remark here that the techniques used in this section may be adapted to find the number of non-sources for the digraphs of finite commutative rings of prime power order.

We first prove the following theorem, which gives a formula for the number of non-sources in the digraph of $\mathbb{Z}_{p^n}$, where p is an odd prime and $n \geq 1$.

Theorem 5.2.7. *Let p be an odd prime and n a positive integer. Then there are exactly*

$$NS(\Psi(\mathbb{Z}_{p^n})) = p^n \left[1 + \frac{p-1}{2}\left(p^{n-1} + p^{n-3} + \cdots + p^{n-1-2\lfloor \frac{n-1}{2} \rfloor}\right)\right]$$

non-sources in the digraph $\Psi(\mathbb{Z}_{p^n})$.

Proof. We begin by showing that for any fixed $b \in \mathbb{Z}_{p^n}$, there are exactly

$$1 + \frac{p-1}{2}\left(p^{n-1} + p^{n-3} + \ldots + p^{n-1-2\lfloor \frac{n-1}{2} \rfloor}\right)$$

non-sources of the form (b, a) in $\Psi(\mathbb{Z}_{p^n})$ where $a \in \mathbb{Z}_{p^n}$.

Fix $b \in \mathbb{Z}_{p^n}$. Note that the vertex (b, a) is a non-source in $\Psi(\mathbb{Z}_{p^n})$ if and only if there exists an $x \in \mathbb{Z}_{p^n}$ such that

$$x(b - x) \equiv a \pmod{p^n}. \tag{5.8}$$

Since 2 and p^n are coprime, the inverse of 2 exists in $\mathbb{Z}_{p^n}$. Hence, we can rewrite 5.8 as

$$(x - 2^{-1}b)^2 \equiv (2^{-1}b)^2 - a \pmod{p^n},$$

implying such an x exists if and only if

$$y^2 \equiv A \pmod{p^n} \tag{5.9}$$

is solvable, where $A \equiv (2^{-1}b)^2 - a$.

We now determine the number of $A \in \mathbb{Z}_{p^n}$ such that (5.9) is solvable. For $A \neq 0$, writing $A = p^l q$ where $p \nmid q$, we can deduce that (5.9) is solvable if and only if l is even and q is a quadratic residue modulo p^{n-l}. By Proposition 5.2.3, there are $(p^{n-l} - p^{n-l-1})/2$ such q's for each even l. Summing all possible values of l, together with the case when $A = 0$, we conclude that there are

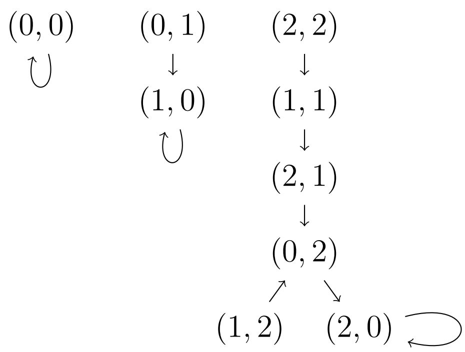
$$1 + \frac{p-1}{2}\left(p^{n-1} + p^{n-3} + \cdots + p^{n-1-2\lfloor \frac{n-1}{2} \rfloor}\right)$$

non-sources of the form (b, a) when $a \in \mathbb{Z}_{p^n}$. Since there are p^n possible values for b, the sum of the expression above p^n times gives us

$$p^n\left[1 + \frac{p-1}{2}\left(p^{n-1} + p^{n-3} + \cdots + p^{n-1-2\lfloor \frac{n-1}{2} \rfloor}\right)\right]$$

as the total number of non-sources in the digraph $\Psi(\mathbb{Z}_{p^n})$. □

Example 5.2.5. By substituting $p = 3$ and $n = 1$ into the formula in Theorem 5.2.7, we have that $NS(\Psi(\mathbb{Z}_3)) = 6$. This is illustrated in the digraph of $\mathbb{Z}_3$ given below.



We now consider the case $p = 2$. To determine the number of non-sources in the digraph $\Psi(\mathbb{Z}_{2^n})$, we will apply the following result of Stangl (1996) on the number $s(2^n)$ of squares in $\mathbb{Z}_{2^n}$:

Proposition 5.2.6.

$$s(2^n) = \begin{cases} \frac{1}{3}(2^{n-1}+4), & \text{if } n \text{ is even} \\ \\ \frac{1}{3}(2^{n-1}+5), & \text{if } n \text{ is odd and } n \geq 3 \end{cases}.$$

Theorem 5.2.8. *Let n be a positive integer. The number $NS(\Psi(\mathbb{Z}_{2^n}))$ of non-sources in the digraph of $\mathbb{Z}_{2^n}$ is*

$$NS(\Psi(\mathbb{Z}_{2^n})) = \begin{cases} 2^{n-1}\left(2^{n-1} + \dfrac{2^{n-1}+4}{3}\right), & \text{if } n \text{ is even} \\ \\ 2^{n-1}\left(2^{n-1} + \dfrac{2^{n-1}+5}{3}\right), & \text{if } n \text{ is odd} \end{cases}.$$

Proof. The result is easily checked to be true for $n = 1$. We thus assume that $n \geq 2$. Let b be fixed in $\mathbb{Z}_{2^n}$. Clearly, the vertex (b, a) is a non-source if and only if there exists $x \in \mathbb{Z}_{2^n}$ such that

$$x(b - x) \equiv a \pmod{2^n}. \tag{5.10}$$

We first show that if b is odd, then there are 2^{n-1} non-sources of the form (b, a) in $\Psi(\mathbb{Z}_{2^n})$. Let $b = 2k + 1$. We want to find the number of distinct expressions of the form $x((2k +$

$1) - x) \pmod{2^n}$. Note that for $x \not\equiv y \pmod{2^n}$,

$$\begin{aligned} & (2k+1)x - x^2 \equiv (2k+1)y - y^2 \pmod{2^n} \\ \Leftrightarrow \quad & (x + y - (2k+1))(x-y) \equiv 0 \pmod{2^n} \\ \Leftrightarrow \quad & x + y \equiv 2k+1 \pmod{2^n}. \end{aligned}$$

The following table gives all the solutions for x and y:

x	y
0	$2k+1$
1	$2k$
2	$2k-1$
$\vdots$	$\vdots$
k	$k+1$
$k+1$	k
$\vdots$	$\vdots$
$2k$	1
$2k+1$	0
$2k+2$	2^n-1
$2k+3$	2^n-2
$\vdots$	$\vdots$
$2^{n-1}+k$	$2^{n-1}+k+1$

Note that $bx - x^2 = (2k+1)x - x^2$ is always even. Since there are 2^{n-1} distinct expressions of the form $(2k+1)x - x^2$ (as can be seen from the table above) and there are 2^{n-1} even values in $\mathbb{Z}_{2^n}$, it follows that the non-sources of the form (b, a) in $\Psi(\mathbb{Z}_{2^n})$, where b is odd, are $(b, 0), (b, 2), (b, 4), \ldots, (b, 2^n - 2)$.

Now suppose that b is even, say $b = 2l$. By (5.10), (b, a) is a non-source in $\Psi(\mathbb{Z}_{2^n})$ if and only if there exists $x \in \mathbb{Z}_{2^n}$ such that

$$x(2l - x) \equiv a \pmod{2^n}.$$

We may rewrite the above as

$$(x - l)^2 \equiv A \pmod{2^n}, \tag{5.11}$$

where $A = l^2 - a$. The number of solutions for x in (5.11) is the number of distinct squares in $\mathbb{Z}_{2^n}$. By Proposition 5.2.6, the number of squares in $\mathbb{Z}_{2^n}$ is

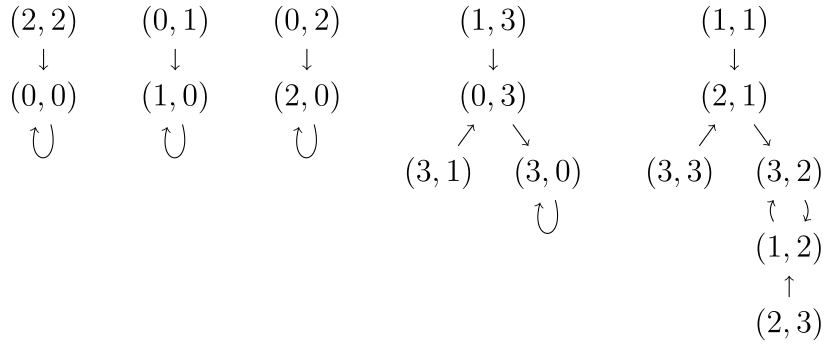
$$s(2^n) = \begin{cases} \frac{1}{3}(2^{n-1} + 4), & \text{if } n \text{ is even} \\ \\ \frac{1}{3}(2^{n-1} + 5), & \text{if } n \text{ is odd and } n \geq 3 \end{cases}. \tag{5.12}$$

There are as many odd values for b as there are even values, namely 2^{n-1}. Thus by multiplying 2^{n-1} to the number of non-sources obtained in the case b is odd and b is even, and then adding them up together we obtain

$$NS(\mathbb{Z}_{2^n}) = \begin{cases} 2^{n-1}\left(2^{n-1} + \frac{2^{n-1}+4}{3}\right), \text{ if } n \text{ is even} \\ 2^{n-1}\left(2^{n-1} + \frac{2^{n-1}+5}{3}\right), \text{ if } n \text{ is odd and } n \geq 3 \end{cases}.$$

□

Example 5.2.7. By substituting $n = 2$ into the formula in Theorem 5.2.8, we have that $NS(\Psi(\mathbb{Z}_4)) = 8$. This is also apparent from the digraph $\Psi(\mathbb{Z}_4)$ given below.



We are now equipped to find the number of non-sources in digraphs of the ring $\mathbb{Z}_n$. Let $n \geq 2$ be a positive integer. By the Fundamental Theorem of Arithmetic, we may write n uniquely in the form $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where $p_1, \dots, p_k$ are distinct prime numbers and $\alpha_1, \dots, \alpha_k$ are positive integers. Since $\gcd(p_i, p_j) = 1$ for $i, j \in \{1, \dots, k\}, i \neq j$, we have the (ring) isomorphism

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}. \tag{5.13}$$

Let $\mathcal{S}(D)$ denote the number of sources in the digraph D. Clearly, $\mathcal{S}(\Psi(R)) = |R|^2 - NS(\Psi(R))$ for any finite ring R. The number of sources in the digraph of a direct product of two finite rings has been obtained by Hausken and Skinner (2013) and is as follows:

Proposition 5.2.8. *If R_1 and R_2 are finite rings, then*

$$\mathcal{S}(\Psi(R_1 \times R_2)) = \mathcal{S}(\Psi(R_1)) \cdot |R_2|^2 + \mathcal{S}(\Psi(R_2)) \cdot |R_1|^2 - \mathcal{S}(\Phi(R_1)) \cdot \mathcal{S}(\Psi(R_2)).$$

By using induction, we may extend Proposition 5.2.8 to obtain the number of sources in the digraph of a direct product of finitely many finite rings.

Theorem 5.2.9. *Let $R_1, \dots, R_n$ be finite rings. For $i = 1, \dots, n$, let $r_i = |R_i|$ and $s_i = S(\Psi(R_i))$. For any $i_1, \dots, i_k \in \{1, \dots, n\}$, let $\alpha_{i_1,\dots,i_k}(r_1, \dots, r_n) = \prod_{i=1; i \neq i_1,\dots,i_k}^{n} r_i^2$.*

Then

$$S(\Psi(R_1 \times \cdots \times R_i)) = \sum_{i=1}^{n} s_i \alpha_i(r_1, \ldots r_n) - \sum_{1 \le i_1 < i_2 \le n} s_{i_1} s_{i_2} \alpha_{i_1,i_2}(r_1, \ldots, r_n)$$
$$+ \sum_{1 \le i_1 < i_2 < i_3 \le n} s_{i_1} s_{i_2} s_{i_3} \alpha_{i_1,i_2,i_3}(r_1, \ldots, r_n) + \ldots$$
$$+ \sum_{1 \le i_1 < \cdots < i_k \le n} (-1)^{k+1} s_{i_1} \ldots s_{i_k} \alpha_{i_1,\ldots,i_k}(r_1, \ldots, r_n)$$
$$+ \cdots + (-1)^{n+1} s_1 \ldots s_n.$$

Proof. This can be proven by using Proposition 5.2.8 and induction. □

Theorem 5.2.9, together with the isomorphism in (5.13) as well as Theorems 5.2.7 and 5.2.8, can be used to determine the number of sources (hence, the number of non-sources) in the digraph of $\mathbb{Z}_n$.

CHAPTER 6

CONCLUSION

6.1 Summary

As per our research objective, we have now obtained various properties of total graphs, unit graphs and digraphs associated with finite rings. In particular, we have completely determined the total graph and unit graph structures of rings of order p, p^2 and pq where p, q are distinct prime numbers. In the case of digraphs, we have investigated the number of connected components as well as the number of vertices and non-sources in each connected component of digraphs associated with various finite rings. Some results on these numbers were obtained for digraphs associated with the ring $C_n(0)$ where $n \geq 2$, the ring I of order p^2 and the ring of integers modulo p^m where $m \geq 1$. Owing to the fact that the total graph and unit graph of a finite ring with identity $1 \neq 0$ are complements of each other, we have been able to extend results on total graphs in the literature to include unit graphs and vice versa. By extending a result on the number of sources in a direct product of two finite rings, we are also able to apply our results on the number of non-sources in the digraphs of $\mathbb{Z}_{p^n}$ to determine the number of sources in the digraph of the ring of integers modulo m ($m \geq 2$).

An observation that can be made regarding the three types of graphs investigated here is that relative to total graphs and unit graphs, the properties of digraphs, such as the number of connected components, do not follow any obvious pattern. This is evident when large graphs, typically having more than 40 vertices, are taken into consideration. It is clear that we do need more examples of digraphs in order to identify a common pattern for a particular graph property that is worth investigating. With that in mind, for future work, it is worthwhile to consider the use of computer programs to expedite calculations for concrete examples.

6.2 Possible future work

Compared to total graphs and unit graphs, the study of the association of digraphs with rings is quite new, hence future work can be done in a variety of different directions. Throughout this dissertation, we have found formulae for the number of sources in several different digraphs. Since the structure of some digraphs do not follow any obvious pattern, an alternative to this is to find bounds on the number of sources instead, for the digraph as a whole and for specific connected components in a digraph.

One of the more fundamental questions regarding digraphs was posed in Hausken and Skinner (2013) as follows:

Conjecture 6.2.1. *If the digraph $\Psi(R)$ is directly isomorphic to $\Psi(S)$, then the additive structure of R and S are the same.*

In the current literature, the above conjecture has not been proven, nor has a counter-example been obtained. Another conjecture pertaining to the direct isomorphism of digraphs was posed by Ang and Schulte (2013), as follows:

Conjecture 6.2.2. *Let R, S be finite reduced rings. Then $\Psi(R)$ is directly isomorphic to $\Psi(S)$ if and only if $R \cong S$.*

Another question posed by Ang and Schulte (2013) is what sort of ring structure do the sources in a digraph retain.

Regarding the structure of the ring I of order p^2 as was discussed in Section 5.2.2, it is worthwhile to mention here that the methods used in that section can be used or extended to study the digraphs associated with other rings of prime power order. A classification of rings whose corresponding digraphs share similar "symmetric" properties as the digraph of the ring I here would bring us closer to understanding digraphs associated with finite rings. Additionally, it is known that a finite commutative ring with identity can be expressed as a direct sum of local rings. In other words, local rings are building blocks of finite commutative rings with identity. The ring $\mathbb{Z}_p$ where p is a prime is an example of a local ring. It would therefore be of interest to investigate similar questions on the digraphs of other finite local rings.

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