A STUDY ON SOME GRAPHS ASSOCIATED WITH FINITE GROUPS

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A STUDY ON SOME GRAPHS ASSOCIATED WITH FINITE GROUPS

ABSTRACT

In this dissertation, we study properties of various graphs associated with finite groups and investigate the extent to which these graphs determine the groups. Among the graphs associated with finite groups that we consider here are conjugate graphs, order graphs and generalised order graphs. For conjugate graphs and order graphs associated with certain groups, we determine the number of complete components in the graphs and their clique numbers. The main focus of this research is on generalised order graphs of finite groups. By studying relationships between power graphs and generalised order graphs, we prove that for a finite group, k-connectedness of its power graph implies k-connectedness of its generalised order graph. We also prove that the generalised order graph and the power graph associated with a finite cyclic group are isomorphic. In addition, we classify certain classes of finite groups according to various graph properties of the associated generalised order graphs. We also prove that the generalised order graph of a finite abelian group is 3-connected and Hamiltonian. Along the way we also prove some number-theoretic inequalities.

Keywords: finite groups, abelian groups, connected graphs, Hamiltonian graphs, power graphs

SUATU KAJIAN ATAS BEBERAPA GRAF TERSEKUTU DENGAN KUMPULAN TERHINGGA

ABSTRAK

Dalam disertasi ini, kami mengkaji sifat beberapa jenis graf yang tersekutu dengan kumpulan terhingga dan menyiasat tahap di mana graf tersebut dapat menentukan kumpulan berkaitan. Antara graf yang tersekutu dengan kumpulan terhingga yang dipertimbangkan adalah graf konjugat, graf peringkat dan graf peringkat teritlak. Bagi graf konjugat dan graf peringkat yang tersekutu dengan kumpulan tertentu, kami menentukan bilangan komponen lengkap dan nombor klik graf tersebut. Tumpuan utama dalam penyelidikan ini adalah pada graf peringkat teritlak bagi kumpulan terhingga. Dengan mengkaji hubungan di antara graf kuasa dengan graf peringkat teritlak, kami membuktikan bahawa bagi suatu kumpulan terhingga, *k*-keterkaitan graf kuasa mengimplikasikan *k*-keterkaitan graf peringkat teritlak. Kami membuktikan juga bahawa graf peringkat teritlak dan graf kuasa yang tersekutu dengan kumpulan kitaran terhingga adalah berisomorfik. Tambahan lagi, kami mengelaskan sesetengah kelas kumpulan terhingga mengikut beberapa sifat graf peringkat teritlak yang tersekutu. Kami membuktikan juga bahawa graf peringkat teritlak bagi suatu kumpulan abelan terhingga adalah 3-terkait dan Hamitonan. Dalam perjalanan ini, kami juga membuktikan beberapa ketaksamaan berunsur teori nombor.

Kata kunci: kumpulan terhingga, kumpulan abelan, graf terkait, graf Hamiltonan, graf kuasa

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LIST OF SYMBOLS

$\chi(\mathscr{G})$: the chromatic number of the graph $\mathscr{G}$

$\omega(\mathscr{G})$: the clique number of the graph $\mathscr{G}$

$d(x,y)$: the distance between the vertices x and y in a graph

$D(\mathscr{G})$: the diameter of the graph $\mathscr{G}$

$\gamma(\mathscr{G})$: the domination number of the graph $\mathscr{G}$

$g(\mathscr{G})$: the girth of the graph $\mathscr{G}$

$\kappa(\mathscr{G})$: the connectivity of the graph $\mathscr{G}$

$\kappa'(\mathscr{G})$: the edge-connectivity of the graph $\mathscr{G}$

$\delta(\mathscr{G})$: the minimum degree of the graph $\mathscr{G}$

$H \leq \mathscr{G}$: H is a subgraph of the graph $\mathscr{G}$

$V(\mathscr{G})$: the vertex set of the graph $\mathscr{G}$

$E(\mathscr{G})$: the edge set of the graph $\mathscr{G}$

K_n : the complete graph on n vertices

$\nabla(G)$: the conjugate graph of the group G

$\Delta(G)$: the order graph of the group G

$\Gamma(G)$: the generalised order graph of the group G

$P(G)$: the power graph of the group G

$\Gamma^*(G)$: the punctured generalised order graph of the group G

$P^*(G)$: the punctured power graph of the group G

C_n : the cyclic group of order n

$Z(G)$: the centre of the group G

$C_G(x)$: the centraliser of the element x in the group G

$[x, y]$: the commutator of the elements x and y in a group

$H \leq G$: H is a subgroup of the group G

$o(x)$: the order of x where x is a group element

$|G|$: the order of the group G

$\phi(n)$: the Euler totient function of n

$m|n$: m divides n, where m and n are integers

$\tau(n)$: the number of positive divisors of n

p : a prime number

CHAPTER 1 INTRODUCTION

1.1 Background of study

Associating a group with a graph and finding relationships between them is an area of research that has generated much interest. Research on this subject aims at investigating relationships between groups and graphs and at exploring applications of one to the other.

One of the earliest graphs associated with a group is the power graph. Let G be a finite group. The power graph of G is the graph with the elements of G as its vertices and there is an edge joining two distinct vertices in the power graph if one of the vertices is a power of another. Other examples of graphs associated with groups are the conjugate graphs, identity graphs and order graphs. Information on a group which can be gleaned from the graphs associated with it provides a combinatorial way to study groups, thus making more tools available for group theorists. Moreover, studying groups which share the same corresponding graph-theoretic properties can lead to a classification of certain types of groups.

The main objectives of this research are as follows:

(a) To determine characteristics of some graphs associated with groups,

(b) To investigate the extent to which the graphs associated with a group determine the group.

Among the questions that are considered in this research are the following:

(a) What are properties of the conjugate graphs, order graphs and generalised order graphs associated with finite groups?

(b) What are the information on a finite group that can be retrieved from the graphs associated with it?

1.2 Significance of the study

Associating a graph to a finite group has provided an avenue for studying groups via graphs and vice versa. The main contributions of this work are some new findings on properties of graphs associated with finite groups and classifications of finite groups according to some associated graph properties. The process of obtaining these results could possibly involve or lead to new methods of proving in finite groups as well as in graph theory. All these play a role in the contribution of knowledge on groups and graphs.

1.3 Organisation of dissertation

In Chapter 2, we begin with some preliminaries on groups and graphs that will be useful in this dissertation. This will be followed by a literature review on graphs associated with groups and a brief description of the methodology employed in this research.

In Chapter 3, we consider two types of graphs associated with finite groups, namely, the conjugate graphs and the order graphs. We present some properties of these graphs and also give some explicit examples of the graphs associated with certain groups such as the dihedral groups and the generalised quaternion groups.

In Chapter 4, we introduce the generalised order graphs of finite groups. We determine some properties of these graphs and devote a section on the generalised order graphs of finite p-groups. In the last section, we show how the power graph, conjugate graph, order graph and generalised order graph are related to one another. We also present our findings on relationships between power graphs and generalised order graphs of finite groups.

Chapter 5 contains the main results of this dissertation. We first discuss results on k-connectedness of generalised order graphs of finite groups. We then prove the Hamiltonicity of the generalised order graphs of finite abelian groups. To prove the main result on Hamiltonicity, we also prove a number-theoretic inequality involving the Euler totient function and the number of positive divisors of an integer. We also present some results on connectivity, edge-connectivity and minimum degree of the generalised order graphs of finite groups. We end the chapter with some results on dominating sets and domination numbers of the generalised order graphs of finite groups.

Finally, in Chapter 6, we provide a summary of our findings and give a list of open problems for possible future work.

Throughout this dissertation, for any element x in the group G, the phrase “the vertex x” will be used to mean “the vertex associated with the element x in the group G”.

CHAPTER 2 LITERATURE REVIEW AND SOME PRELIMINARIES

2.1 A brief overview

In this chapter, we first give some preliminaries on groups and graphs that will be useful in this dissertation. This will be followed by a literature review on graphs associated with finite groups. The review will focus on power graphs and conjugate graphs of finite groups as these graphs and some related ones are studied in this dissertation. In the final section we give a brief description of the methodology employed in this dissertation.

2.2 Some preliminaries

The work in this dissertation involves both group theory and graph theory. In this section, we discuss some preliminaries that will be useful later on.

2.2.1 Groups

We first state, without proof, the following fundamental result on the order of the product of two elements in a finite abelian group.

Proposition 2.2.1. *Let G be a finite abelian group with elements x and y such that* $\gcd(o(x), o(y)) = 1$. *Then* $o(xy) = o(x)o(y) = lcm(o(x), o(y))$.

We now prove the following:

Proposition 2.2.2. *Let G be a finite abelian group of order $p_1^{k_1} \dots p_n^{k_n}$, where the p_i's are distinct prime numbers and the k_i's are positive integers, and let $x, y \in G$. Then there exist some positive integers a and b such that $x^a y^b$ has order $lcm(o(x), o(y))$.*

Proof. By factoring $o(x)$ and $o(y)$ into primes, we have $o(x) = p_1^{e_1} \dots p_n^{e_n}$ and $o(y) = p_1^{f_1} \dots p_n^{f_n}$, where $p_1, \dots, p_n$ are distinct primes and e_i, f_i are non-negative integers ($i = 1, \dots, n$). We then have that

$$\operatorname{lcm}(o(x), o(y)) = p_1^{\max(e_1, f_1)} \dots p_n^{\max(e_n, f_n)}. \tag{2.1}$$

Let $r = \prod_{e_i \geq f_i} p_i^{e_i}$ and $s = \prod_{e_i < f_i} p_i^{f_i}$. Then $\operatorname{lcm}(o(x), o(y)) = rs$ and $\gcd(r, s) = 1$. By construction, $r | o(x)$ and $s | o(y)$. Then $x^{\frac{o(x)}{r}}$ has order r and $y^{\frac{o(y)}{s}}$ has order s. It follows by Proposition 2.2.1 that $x^{\frac{o(x)}{r}} y^{\frac{o(y)}{s}}$ has order $rs = \operatorname{lcm}(o(x), o(y))$. $\square$

The following result is known as the Fundamental Theorem of Finitely Generated Abelian Groups. It tells us that every finite abelian group may be written uniquely, up to isomorphism, as a direct product of cyclic groups. The reader may refer to Humphreys (1996, Corollary 14.11) for a proof of this result.

Theorem 2.2.1. *Every finite abelian group G has a unique decomposition in the form*

$$C_{n_1} \times C_{n_2} \times \cdots \times C_{n_r},$$

where n_i is divisible by n_j for $j > i$, $n_r \geq 2$, and $n_1 n_2 \dots n_r = |G|$.

Let G be a group and let $x, y \in G$. We say that x and y are conjugate in G if there exists $g \in G$ such that $y = g^{-1}xg$. The notation (x) is used to denote the set of all elements in G which are conjugate to x, that is, $(x) = \{g^{-1}xg | g \in G\}$. It is known that when G is a finite group, $|(x)| = [G : C_G(x)]$, where $C_G(x)$ is the centraliser of x in G.

2.2.2 Graphs

In this research, we only consider finite simple graphs, which contain neither loops nor multiple edges. By convention, given a graph $\mathscr{G}$, the vertex set and the edge set of $\mathscr{G}$ are denoted by $V(\mathscr{G})$ and $E(\mathscr{G})$, respectively.

A connected graph $\mathscr{G}$ is said to be k-connected if $|V(\mathscr{G})| > k$ and $\mathscr{G} - X$ is connected for every subset $X \subseteq V(\mathscr{G})$ with $|X| < k$. The greatest integer k such that $\mathscr{G}$ is k-connected is called the connectivity $\kappa(\mathscr{G})$ of $\mathscr{G}$. Hence, $\mathscr{G}$ is disconnected or trivial if and only if $\kappa(\mathscr{G}) = 0$. Similarly, if $|E(\mathscr{G})| > l$ and $\mathscr{G} - F$ is connected for every set $F \subseteq E(\mathscr{G})$ with $|F| < l$, then $\mathscr{G}$ is called l-edge-connected and the edge-connectivity, $\kappa'(\mathscr{G})$, of $\mathscr{G}$ is the greatest integer l such that $\mathscr{G}$ is l-edge-connected. The paths in a graph $\mathscr{G}$ are said to be independent if none of them contain an inner vertex of another.

The following theorem gives a necessary and sufficient condition for a graph to be k-connected.

Theorem 2.2.2. *A graph $\mathscr{G}$ is k-connected if and only if $\mathscr{G}$ contains k independent paths between any two vertices.*

Proof. See for example Theorem 3.3.6 in Diestel (2017). □

A graph $\mathscr{G}$ is called a regular graph if every vertex of $\mathscr{G}$ has the same degree. If every pair of distinct vertices in $\mathscr{G}$ is connected by a unique edge, then $\mathscr{G}$ is said to be a complete graph. The notation K_n is used to denote the complete graph on n vertices, where $n = |V(\mathscr{G})| \geq 2$. For convenience, we use K_1 to denote the graph consisting of a single isolated vertex. A component of $\mathscr{G}$ is a maximal connected subgraph of $\mathscr{G}$, whereas a clique of $\mathscr{G}$ is a complete subgraph of $\mathscr{G}$. The clique number $\omega(\mathscr{G})$ of $\mathscr{G}$ is the number of vertices of the largest clique in $\mathscr{G}$. If there is a vertex u of $\mathscr{G}$ with $\deg(u) \leq \deg(v)$ for every $v \in V(\mathscr{G})$, then $\deg(u)$ is called the minimum degree of $\mathscr{G}$, denoted by $\delta(\mathscr{G})$.

A Hamiltonian graph is a graph $\mathscr{G}$ which has a cycle traversing each vertex of $\mathscr{G}$ exactly once. A necessary condition for a graph to be Hamiltonian is stated below.

Proposition 2.2.3. *If a graph $\mathscr{G}$ is Hamiltonian, then $\mathscr{G}$ is 2-connected.*

Proof. See for example Corollary 3.7 in Wallis (2007). □

If a graph $\mathscr{G}$ has a circuit which traverses every edge of $\mathscr{G}$ exactly once, then $\mathscr{G}$ is said to be Eulerian. To determine whether $\mathscr{G}$ is Eulerian or not, we have the following result:

Theorem 2.2.3. *Let $\mathscr{G}$ be a connected graph. Then $\mathscr{G}$ is Eulerian if and only if every vertex of $\mathscr{G}$ has even degree.*

Proof. See for example Diestel (2017). □

The chromatic number of $\mathscr{G}$, denoted by $\chi(\mathscr{G})$, is the smallest k such that $\mathscr{G}$ has k colourings. If $\chi(\mathscr{H}) = \omega(\mathscr{H})$ for every induced subgraph $\mathscr{H}$ of $\mathscr{G}$, then $\mathscr{G}$ is called a perfect graph. The following result is known as the strong perfect graph theorem and has been proven by Chudnovsky et al. (2006).

Theorem 2.2.4. *A graph $\mathscr{G}$ is perfect if and only if neither $\mathscr{G}$ nor its complement $\overline{\mathscr{G}}$ contains an odd cycle of length at least 5 as an induced subgraph.*

If a graph $\mathscr{G}$ does not have overlapping edges, then $\mathscr{G}$ is called a planar graph. For other notations and terminologies in graph theory, the reader may refer to Diestel (2017) or Wallis (2007).

2.3 Literature review

Different types of graphs may be associated to a group. One of the earliest such graphs is the power graph. The directed power graph was first introduced by Kelarev and Quinn (2000). Given a finite group G, the directed power graph $\vec{P}(G)$ is a directed graph with the elements of G as its vertices and with all edges (u,v) such that $u \neq v$ and v is a power of u. The undirected power graph $P(G)$ is the underlying undirected graph of the directed power graph. The undirected power graphs for semigroups were the main objects of study by Chakrabarty et al. (2009). For groups, the associated undirected power graphs were the main focus in the papers by Cameron (2010), Cameron and Ghosh (2011), Curtin et al. (2015) and Pourgholi et al. (2015). In the paper by Cameron (2010), it was shown that two finite groups which have isomorphic undirected power graphs have isomorphic directed power graphs. A consequence of this is that two finite groups which have isomorphic undirected power graphs have the same number of elements of each order. It was shown by Pourgholi et al. (2015) that the undirected power graph of a finite group is Eulerian if and only if the group has odd order. It was also shown in the same paper that the undirected power graph of a finite p-group G has a Hamiltonian cycle if and only if $|G| \neq 2$ and G is cyclic. Let H be a finite group that is a simple group, a cyclic group, a symmetric group, a dihedral group, or a generalised quaternion group. If K is a finite group such that the undirected power graphs of H and K are isomorphic, then H and K are isomorphic (see Mirzargar et al. (2012)).

Chakrabarty et al. (2009) proved that the power graph $P(G)$ is always connected for any finite group G. In the paper by Pourgholi et al. (2015), it was shown that the power graph $P(G)$ is 2-connected if and only if G is a cyclic group or a generalised quaternion 2-group. It was proved by Bubboloni et al. (2017) that the power graph of the alternating group A_n is 2-connected if and only if either $n = 3$ or none of $n, n-1, n-2, \frac{n}{2}$ and $\frac{n-1}{2}$ is a prime.

The punctured power graph $P^*(G)$ of the finite group G is the graph with vertices $G \setminus \{1\}$ where 1 is the identity element of G and with all edges (u,v) such that $u \neq v$ and v is a power of u or u is a power of v. These graphs first appeared in the work by Curtin and Pourgholi (2014). It was shown by Pourgholi et al. (2015) that the diameter of $P^*(G)$ is at most 2 if and only if G is nilpotent and every Sylow subgroup of G is either a cyclic group or a generalised quaternion 2-group. It was also shown that if G is a finite group and $P^*(G)$ has diameter 3, then G is not simple. Another result proven by Pourgholi et al. (2015) is that $P^*(G)$ is Eulerian if and only if G is a cyclic 2-group or a generalised quaternion 2-group.

Another type of graph associated with a finite group is the conjugate graph as defined by Vasantha Kandasamy and Smarandache (2009). Given a finite group G, the conjugate graph of G, denoted by $\nabla(G)$, is the graph with vertex set G and two distinct elements $x, y \in G$ are adjacent in $\nabla(G)$ if x and y are conjugate in G. It is clear that if G is a finite abelian group, then every element in G is self-conjugate and hence, $\nabla(G)$ is an empty graph.

There are various other types of graphs associated with finite groups in the literature. Some examples are the identity graphs (Vasantha Kandasamy and Smarandache (2009)), the non-cyclic graphs (Abdollahi and Hassanabadi (2007)), the cyclic graphs (Ma et al. (2013)), the centre graphs (Balakrishnan et al. (2011)), the coprime graphs (Ma et al. (2014)) and of course, the Cayley graphs which have a long history.

2.4 Methodology

There are three main components in the methodology employed in this research. The first component involves preliminary background work where various advanced concepts on abstract algebra and graph theory are studied together with relevant articles on graphs associated with groups. Techniques used by other researchers are studied and noted for possible applications or extensions to other cases. The second component involves identifying suitable graphs for further investigation. By analysing existing results and investigating some concrete examples of graphs associated with groups, some observations are noted and conjectures are formulated. The third component is the most crucial one and involves proving or disproving the conjectures. Various tools from group theory, graph theory and number theory are used during this stage, as well as results obtained by other researchers.

CHAPTER 3 CONJUGATE GRAPHS AND ORDER GRAPHS OF FINITE GROUPS

3.1 A brief overview

In this chapter, we study two types of graphs associated with finite groups, namely, conjugate graphs and order graphs. We begin in Section 3.2 by presenting some properties of the conjugate graphs of finite groups. We also give some examples of the conjugate graphs of some non-abelian groups such as the dihedral groups and the generalised quaternion groups. In Section 3.3, we define the order graphs of finite groups and present some of their properties. It is also noted that the conjugate graph of a finite group is a subgraph of its order graph. We end the section with some explicit examples of order graphs of finite groups.

3.2 Conjugate graphs of finite groups

As defined in Section 2.3, the conjugate graph of the finite group G, denoted by $\nabla(G)$, is the graph with vertex set G such that for any distinct elements $x, y \in G$, x and y are adjacent in $\nabla(G)$ if x and y are conjugate in G. From the definition, it is clear that elements in G which are conjugate with one another will be adjacent to one another in $\nabla(G)$ and hence, form a complete component in $\nabla(G)$. In other words, the conjugate graph $\nabla(G)$ is a disjoint union of complete components, where each of these components represents a conjugacy class of G. The graph $\nabla(G)$ is therefore disconnected. Note that the identity element 1 of G forms an isolated vertex itself in $\nabla(G)$. The following result gives the number of isolated vertices in $\nabla(G)$.

Proposition 3.2.1. *Let G be a finite group. Then the number of isolated vertices in $\nabla(G)$ is equal to the order of the centre $Z(G)$ of G.*

Proof. Note that a vertex z in $\nabla(G)$ is isolated if and only if the conjugacy class containing z consists of z only. Then since

$$\begin{aligned}(z) = \{z\} &\Leftrightarrow g^{-1}zg = z \; \forall g \in G \\ &\Leftrightarrow gz = zg \; \forall g \in G \\ &\Leftrightarrow z \in Z(G),\end{aligned}$$

we have that the number of isolated vertices in $\nabla(G)$ is the same as the order of $Z(G)$. □

By Proposition 3.2.1, it is clear that if G is a finite abelian group, then its conjugate graph $\nabla(G)$ only consists of isolated vertices, that is, $\nabla(G)$ is an empty graph.

Proposition 3.2.2. *Let G_1 and G_2 be finite groups. If G_1 and G_2 are isomorphic, then $\nabla(G_1)$ and $\nabla(G_2)$ are isomorphic.*

Proof. Let ϕ be an isomorphism from G_1 onto G_2 and let $x, y \in G_1$. It may be shown that x and y are conjugate in G_1 if and only if $\phi(x)$ and $\phi(y)$ are conjugate in G_2. It follows that the conjugacy class structure of G_1 (that is, the number of conjugacy classes and the number of elements in each conjugacy class) is the same as that of G_2. Thus $\nabla(G_1) \cong \nabla(G_2)$. □

The following result is straightforward from the definition of conjugate graphs.

Proposition 3.2.3. *Let G be a finite group and let H be a subgroup of G. Then $\nabla(H)$ is a subgraph of $\nabla(G)$.*

Example 3.2.1. Let $n \geq 3$. Consider the alternating group A_n which is a normal subgroup of the symmetric group S_n. By Proposition 3.2.3, $\nabla(A_n)$ is a subgraph of $\nabla(S_n)$. However, the subgraph $\nabla(A_n)$ is not an induced one. This is because the permutations $(1\ 2\ 3)$ and

(1 3 2) are not conjugate in A_n although they are conjugate in S_n. Therefore, (1 2 3) and (1 3 2) are not adjacent in $\nabla(A_n)$, but adjacent in $\nabla(S_n)$.

Theorem 3.2.1. *Let G be a finite group. Then $\nabla(G)$ is perfect.*

Proof. This is straightforward by Theorem 2.2.4 and the fact that "conjugate" is a transitive relation. □

In the remainder of this section, we determine the structures of the conjugate graphs of some finite non-abelian groups. We first state a result of Vasantha Kandasamy and Smarandache (2009) as follows.

Proposition 3.2.4. *Let G be the dihedral group,*

$$D_n = \langle x, y : x^n = 1 = y^2, yxy^{-1} = x^{-1} \rangle,$$

where n is even. Then $\nabla(G)$ is a disjoint union of complete graphs comprising of two copies of K_1, $\frac{n-2}{2}$ copies of K_2 and two copies of $K_{\frac{n}{2}}$. Furthermore, $\chi(\nabla(G)) = \omega(\nabla(G)) = \frac{n}{2}$.

Proof. This can be found in the proof of Theorem 2.17 in the book by Vasantha Kandasamy and Smarandache (2009). □

For *n* odd, we prove the following:

Proposition 3.2.5. *Let G be the dihedral group,*

$$D_n = \langle x, y : x^n = 1 = y^2, yxy^{-1} = x^{-1} \rangle,$$

where n is odd. Then $\nabla(G)$ is a disjoint union of complete graphs comprising of one copy of K_1, $\frac{n-1}{2}$ copies of K_2 and one copy of K_n. Furthermore, $\chi(\nabla(G) = \omega(\nabla(G))) = n$.

Proof. By the relations in the presentation of G, we have that

$$G = \{1, x, x^2, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\}.$$

Clearly $(1) = \{1\}$ is a conjugacy class in G and this gives us a component K_1 in $\nabla(G)$. Note that for each $i \in \{1, \ldots, n-1\}$, we have $C_G(x^i) = \langle x \rangle = \{1, x, \ldots, x^{n-1}\}$. Therefore $|(x^i)| = [G : C_G(x^i)] = 2$ for $i \in \{1, \ldots, n-1\}$. Since $yx^iy^{-1} = (yxy^{-1})^i = (x^{-1})^i = x^{-i} = x^{n-i}$ for $i \in \{1, \ldots, \frac{n-1}{2}\}$, it follows that $\{x, x^{n-1}\}$, $\{x^2, x^{n-2}\}$, $\ldots$, $\{x^{\frac{n-1}{2}}, x^{\frac{n+1}{2}}\}$ are conjugacy classes in G; thus giving us $\frac{n-1}{2}$ copies of K_2 in $\nabla(G)$. Next, note that $C_G(y) = \{1, y\}$. Therefore $|(y)| = [G : C_G(y)] = n$ and hence, we must have $(y) = \{y, xy, \ldots, x^{n-1}y\}$. The conjugate graph of G is therefore a disjoint union of one copy of K_1, $\frac{n-1}{2}$ copies of K_2 and one copy of K_n. Since the largest clique of $\nabla(G)$ is K_n, we have that $\chi(\nabla(G)) = \omega(\nabla(G)) = n$. □

We next determine the conjugate graphs of generalised quaternion groups.

Proposition 3.2.6. *Let G be the generalised quaternion group*

$$Q_{2^n} = \langle x, y : x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, yxy^{-1} = x^{-1} \rangle$$

where $n \geq 3$. Then $\nabla(G)$ is the union of two copies of K_1, $2^{n-2} - 1$ copies of K_2 and 2 copies of $K_{2^{n-2}}$. Furthermore, $\chi(\nabla(G)) = \omega(\nabla(G)) = 2^{n-2}$.

Proof. By the relations in the presentation of G, we have that

$$G = \{1, x, \ldots, x^{2^{n-1}-1}, y, xy, \ldots, x^{2^{n-1}-1}y\}.$$

The centre $Z(G)$ of G consists of the elements 1 and $x^{2^{n-2}}$. Hence, the identity element 1 and $x^{2^{n-2}}$ form two conjugacy classes $(1) = \{1\}$ and $(x^{2^{n-2}}) = \{x^{2^{n-2}}\}$ in G, respectively. This gives us two copies of K_1 in $\nabla(G)$. Note that for each $i \in \{1, \ldots, 2^{n-1}-1\}$, we have $C_G(x^i) = \langle x \rangle = \{1, x, \ldots, x^{2^{n-1}-1}\}$. Therefore, $|(x^i)| = [G : C_G(x^i)] = 2$ for $i \in \{1, \ldots, 2^{n-1}-1\}$. Since $yx^iy^{-1} = (yxy^{-1})^i = (x^{-1})^i = x^{-i} = x^{2^{n-1}-i}$ for $i \in \{1, \ldots, 2^{n-2}-1\}$, it follows that $\{x, x^{2^{n-1}-1}\}$, $\{x^2, x^{2^{n-1}-2}\}$, $\ldots$, $\{x^{2^{n-2}-1}, x^{2^{n-2}+1}\}$ are conjugacy classes in G; thus giving us $2^{n-2}-1$ copies of K_2 in $\nabla(G)$. We also note that for each $i \in \{1, \ldots, 2^{n-1}-1\}$, $C_G(x^iy) = \langle x^iy \rangle$ and each $x^iy \in G$ has order 4. Therefore $|(x^iy)| = [G : C_G(x^iy)] = 2^{n-2}$. Let k be the number of conjugacy classes containing 2^{n-2} elements each in G. By the class equation of G, we have that $2^n = |G| = |Z(G)| + \sum_{g \in G, g \notin Z(G)} |(g)| = 2 + (2^{n-2}-1)(2) + k(2^{n-2})$. This give us $k = 2$. Therefore, the conjugate graph of G is a disjoint union of two copies of K_1, $2^{n-2}-1$ copies of K_2 and 2 copies of $K_{2^{n-2}}$. The second assertion follows from the fact that the largest clique of $\nabla(G)$ is $K_{2^{n-2}}$. □

We now consider the conjugate graphs of symmetric groups. First, we state a result on the size of a conjugacy class in a symmetric group. The reader may refer to Herstein (1975) for a proof of the following lemma.

Lemma 3.2.1. *Let G be the symmetric group S_n where $n \geq 3$. The size of a conjugacy class corresponding to a cycle type with α_j parts of length j ($1 \leq j \leq n$) is*

$$\frac{n!}{\prod_j j^{\alpha_j}(\alpha_j!)}. \tag{3.1}$$

Proposition 3.2.7. *Let G be the symmetric group S_n, where $n \geq 3$. Then the chromatic number of $\nabla(G)$ is $\frac{n!}{n-1}$, the number of $(n-1)$-cycles in G.*

Proof. By Lemma 3.2.1, the number N of $(n-1)$-cycles in G is $\frac{n!}{n-1}$. By computation, for $1 \leq j \leq n$, if α_j is the number of parts of size j in a partition of n, the inequality

$n-1 \leq \prod_j j^{\alpha_j}(\alpha_j!)$ holds. Therefore,

$$N = \frac{n!}{n-1} \geq \frac{n!}{\prod_j j^{\alpha_j}(\alpha_j)!}. \tag{3.2}$$

It follows by Lemma 3.2.1 that the maximum size of a conjugacy class of S_n is $N = \frac{n!}{n-1}$. That is, the component in $\nabla(G)$ corresponding to the conjugacy class comprising of all the $(n-1)$-cycles is the largest complete component in $\nabla(G)$. Hence, at least N colours are required for the colouring of the vertices of this component. For other components in the graph, the colouring requires less than N colours. Thus $\chi(\nabla(G)) = N$. □

Proposition 3.2.8. *Let G be the symmetric group S_n, where $n \geq 3$. Then $\nabla(G)$ contains an even number of complete components K_v, where v is odd.*

Proof. By the class equation of G, we have that

$$|G| = \sum_{v \text{ is odd}} |V(K_v)| + \sum_{w \text{ is even}} |V(K_w)|. \tag{3.3}$$

Since $|G| = n!$ is even, it follows by (3.3) that the number of complete components K_v, where v is odd, must be even. This completes the proof. □

In graph theory, it is well-known that the complete graph K_n on n vertices is non-planar for $n \geq 5$ (see for example Corollary 4.2.11 in Diestel (2017)). Hence, we have the following result.

Proposition 3.2.9. *Let G be the symmetric group S_n, where $n \geq 4$. Then $\nabla(G)$ is non-planar.*

Proof. If $n = 4$, then one of the conjugacy classes in G consists of all the 4-cycles and has $3! = 6$ elements. It follows that the component associated to this class in $\nabla(G)$ is non-planar and so, $\nabla(G)$ is non-planar. Now suppose that $n > 4$. Note that S_4 is a subgroup of G. By Proposition 3.2.3, $\nabla(S_4)$ is a subgraph of $\nabla(G)$. Therefore $\nabla(G)$ is non-planar. □

3.3 Order graphs of finite groups

It is known that elements in the same conjugacy class of a finite group have the same order. With this in mind, our study of the conjugate graphs of finite non-abelian groups led us to consider order graphs which we define as follows:

Definition 3.3.1. Let G be a finite group. The order graph of G, denoted by $\Delta(G)$, is the graph with vertex set G and for any distinct elements $x, y \in G$, x and y are adjacent in $\Delta(G)$ if x and y have the same order in G.

By Definition 3.3.1, any set comprising of elements of the same order in G forms a complete component in $\Delta(G)$. Since the identity element of a group is its only element of order 1, it is clear that the order graph of a finite group is disconnected. Moreover, since conjugate elements in a finite group G have the same order, we have that $\nabla(G)$ is a subgraph of $\Delta(G)$.

Remark 3.3.1. There have been other definitions of order graphs of finite groups in the literature. For example, in the paper by Pasebani and Payrovi (2014), the order graph $\mathfrak{T}(G)$ of a finite group G is defined as the graph with non-trivial subgroups of G as vertices such that two distinct vertices H and K are adjacent in $\mathfrak{T}(G)$ if $|H|$ divides $|K|$ or $|K|$ divides $|H|$.

It is known that the order of an element x in a subgroup H of the group G is the same as the order of x in G. We thus have the following:

Proposition 3.3.1. *Let G be a finite group and let H be a subgroup of G. Then $\Delta(H)$ is an induced subgraph of $\Delta(G)$.*

Example 3.3.1. Let $n \geq 3$. The order graph of the alternating group A_n of degree n is an induced subgraph of the order graph of the symmetric group S_n of degree n.

We now consider the perfectness of order graphs.

Theorem 3.3.1. *Let G be a finite group. Then $\Delta(G)$ is perfect.*

Proof. This follows by Theorem 2.2.4 and the fact that if $\phi(a) = \phi(b)$ and $\phi(b) = \phi(c)$, then $\phi(a) = \phi(c)$ where ϕ is the Euler totient function. □

The following result gives a relationship between the order of a finite group and the number of complete components K_v, where v is odd, in the order graph associated with it.

Proposition 3.3.2. *Let G be a finite group of even order. Then there are at least two complete components K_v in $\Delta(G)$, where v is odd.*

Proof. Clearly, the identity element 1 forms a component K_1 on its own. Now we need to show that there exists another component with an odd number of vertices in $\Delta(G)$. Let $|G| = n$, where n is even. Then $2|n$ and it follows by Cauchy's theorem that there is an element $g \in G$ of order 2. Hence, there is at least one involution in G. Now we claim that the number of involutions in G is an odd number. We prove this claim by contradiction.

Let v be the number of involutions in G and suppose that v is even. Then there are $n - v - 1$ non-identity elements of G which are not involutions. Each $x \in G$ which is not an involution together with its inverse x^{-1} form a pair (x, x^{-1}) of distinct elements. It follows that the number of non-identity elements of G which are not involutions is even. But this is a contradiction since $n - v - 1$ is odd. Therefore, v must be odd and we have another component K_v with an odd number of vertices. □

Corollary 3.3.1. *Let G be a finite group. Then $|G|$ is even if and only if the number of involutions in G is an odd number.*

Proof. The necessity part of the proof follows from the proof of Proposition 3.3.2. For the sufficiency, G has at least one involution, say x. Then since $o(x)$ $(=2)$ divides $|G|$, it follows that $|G|$ is even. □

For groups of odd order, we have the following:

Proposition 3.3.3. *Let G be a finite group of odd order. Then every non-trivial complete component of* $\Delta(G)$ *has an even number of vertices.*

Proof. Note that G does not have any involution as $|G|$ is odd. It follows that every element $g \in G \setminus \{1\}$ has an inverse $g^{-1} \neq g$ and both of them have the same order. Hence the number of elements of the same order in G must be even. The assertion thus follows. □

Proposition 3.3.2 implies that the order graph of a group of even order has at least two complete components. For some specific groups, we are able to determine the exact number of complete components as shown in the following results.

Proposition 3.3.4. *If the group G is isomorphic to an elementary abelian p-group or a non-abelian group of exponent p, then* $\Delta(G)$ *has only two complete components, namely* K_1 *and* $K_{|G|-1}$.

Proof. Since all non-trivial elements of G have order p, the corresponding vertices are adjacent to each other in $\Delta(G)$ and hence, form a complete component $K_{|G|-1}$. This completes the proof. □

Proposition 3.3.5. *Let G be a finite cyclic group of order n. Then there is one complete component* $K_{\phi(d)}$ *in* $\Delta(G)$ *for each divisor d of n.*

Proof. We first claim that a cyclic group H of order d has $\phi(d)$ generators. Suppose that $H = \langle h \rangle$. Then, h^k generates H if and only if $h^{km} = h$ for some positive integer m. This is equivalent to the congruence $km \equiv 1 (\text{mod } d)$which occurs if and only if k is a unit in $\mathbb{Z}_n$. Thus there are $\phi(d)$ values of k in order for h^k to be a generator of H. This proves our claim.

Now, let g be a generator of G and let d be a divisor of n. Then $g^{\frac{n}{d}}$ generates a subgroup of G of order d. If $g^t \in G$ has order d, then $g^{td} = 1$ and so, $n|td$. It follows that $\frac{n}{d}|t$. Hence, $g^t \in \langle g^{\frac{n}{d}} \rangle$ and we have $\langle g^t \rangle = \langle g^{\frac{n}{d}} \rangle$. Thus there is only one subgroup of G of order d, say H_d. By our earlier claim, H_d has $\phi(d)$ elements of order d. Thus G has $\phi(d)$ elements of order d for every divisor d of n. If follows by definition that $\Delta(G)$ has one complete component $K_{\phi(d)}$ for each d. □

As an application of Proposition 3.3.5, we have the following:

Corollary 3.3.2. *Let $n \geq 2$ be an integer. The clique number of $\Delta(C_n)$ is $\phi(n)$.*

Proof. Let $n = p_1^{k_1} \dots p_r^{k_r}$, where $p_1, \dots, p_r$ are distinct prime numbers and $k_1, \dots, k_r$ are positive integers. Let $d \neq 1$ be a divisor of n. Then $d = p_1^{j_1} \dots p_r^{j_r}$ for some integer j_i where $0 \leq j_i \leq k_i$ $(i = 1, \dots, r)$. Clearly, $|V(K_{\phi(d)})| = \phi(d) \leq \prod_{i=1}^{r} p_i^{k_i - 1}(p_i - 1) = \phi(n) = |V(K_{\phi(n)})|$. It follows by Proposition 3.3.5 that $K_{\phi(n)}$ is a maximum clique in $\Delta(C_n)$. Therefore, the clique number of $\Delta(C_n)$ is $\phi(n)$. □

For finite non-cyclic p-groups, we have the following:

Proposition 3.3.6. *Let G be a non-cyclic group of order p^n, $n > 1$. Then $\Delta(G)$ is the disjoint union of K_1 and complete components $K_{(1)}, \dots, K_{(n-1)}$, where $K_{(i)}$ consists of the vertices corresponding to the elements of order p^i in G $(i = 1, \dots, n-1)$.*

Proof. Since G has order p^n, there exist elements of order p^i for each $i = 0, 1, \dots, n-1$. Elements of the same order form a complete component in $\Delta(G)$; hence, the result follows. □

The following result tells us that the order graphs associated with two finite groups G_1 and G_2 are isomorphic if G_1 and G_2 are isomorphic.

Proposition 3.3.7. *Let G_1 and G_2 be finite groups. If G_1 and G_2 are isomorphic, then $\Delta(G_1)$ and $\Delta(G_2)$ are isomorphic.*

Proof. Let G_1 and G_2 be isomorphic. Then there exists an isomorphism $\phi : G_1 \to G_2$ and $|G_1| = |G_2|$. We wish to show that G_1 has n elements of order d if and only if G_2 has n elements of order d. Since ϕ is one-to-one and onto, for any $h \in G_2$, there exists a unique $g \in G_1$ such that $\phi(g) = h$. Thus it suffices to show that $g \in G_1$ has order d if and only if $\phi(g) \in G_2$ has order d.

Suppose that $g \in G_1$ has order d. Then $\phi(g)^d = \phi(g^d) = \phi(1) = 1$, that is, m divides d where $m = o(\phi(g))$. We also note that $\phi(g^m) = \phi(g)^m = 1$. Since $\phi(1) = 1$ and ϕ is one-to-one, it follows that $g^m = 1$ and hence, d divides m. The equality $d = m$ thus follows. Now suppose that $\phi(g) \in G_2$ has order d and let $m = o(g)$. Then $1 = \phi(1) = \phi(g^m) = \phi(g)^m$ and hence, $d|m$. We also note that $\phi(g^d) = \phi(g)^d = 1$. Since $\phi(1) = 1$ and ϕ is one-to-one, it follows that $g^d = 1$. Therefore $m|d$. Thus $d = m$ as required. □

Remark 3.3.2. The converse of Proposition 3.3.7 does not hold in general. As an example, let G_1 be the elementary abelian p-group and let G_2 be the p-group of order p^3 with exponent p given by the presentation

$$\langle x, y, z : x^p = y^p = z^p = [x, z] = [y, z] = 1, [x, y] = z \rangle.$$

Then, by Proposition 3.3.4, $\Delta(G_1)$ and $\Delta(G_2)$ are isomorphic. However, since G_2 is non-abelian, these two groups cannot be isomorphic. This shows that two non-isomorphic groups may have the same order graphs.

We end this section by giving some explicit examples of order graphs of finite groups.

Example 3.3.2. Let $G = C_p$, the cyclic group of order p. The $p-1$ non-trivial elements of G have order p. Thus, the order graph $\Delta(G)$ of G is the disjoint union of K_1 and K_{p-1}.

Example 3.3.3. Let $G = C_p \times C_q$ where p and q are distinct prime numbers. Then G has one element of order 1, $p-1$ elements of order p, $q-1$ elements of order q and $(p-1)(q-1)$ elements of order pq. Thus, the order graph $\Delta(G)$ of G is the disjoint union $K_1 \cup K_{p-1} \cup K_{q-1} \cup K_{(p-1)(q-1)}$.

Example 3.3.4. Let G be the dihedral group $\langle a,b : a^{2^n} = b^2 = 1, ba = a^{-1}b\rangle$ where $n \geq 2$. It is clear that the identity element and b have orders 1 and 2, respectively. The element a generates the subgroup $\langle a\rangle$ of order 2^n. By Proposition 3.3.5, $\langle a\rangle$ has 2^{k-1} elements of order 2^k for $k = 1,\ldots,n$. By induction, it may be shown that $ba^k = a^{2^n-k}b$ for $k = 0,1,\ldots,2^n-1$. Then for any $k \in \{0,1,\ldots,2^n-1\}$, we have $(a^kb)^2 = a^k(ba^k)b = a^ka^{2^n-k}b^2 = 1$. It follows that G has 2^n+1 elements of order 2 and 2^{k-1} elements of order 2^k for $k = 2,\ldots,n$. Thus, the order graph $\Delta(G)$ of G is the disjoint union $K_1 \cup K_{2^n+1} \cup K_2 \cup K_{2^2} \cup \cdots \cup K_{2^{n-1}}$.

Example 3.3.5. Let G be the generalised quaternion group $\langle a,b : a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, ba = a^{2^{n-1}-1}b\rangle$ where $n \geq 3$. Clearly, the identity element and $a^{2^{n-2}}$ have orders 1 and 2, respectively. Note that the element a generates the subgroup $\langle a\rangle$ of order 2^{n-1}. By Proposition 3.3.5, $\langle a\rangle$ has $\phi(2^i) = 2^{i-1}$ elements of order 2^i for $i = 1,\ldots,n-1$. We now determine the orders of b and ba^k for $k = 1,\ldots,2^{n-1}-1$. By induction, it may be shown that $ba^k = a^{2^{n-1}-k}b$ for $k = 1,\ldots,2^{n-1}-1$. Then for any $k \in \{1,\ldots,2^{n-1}-1\}$, we have $(ba^k)^2 = a^{2^{n-1}-k}b^2a^k = a^{2^{n-1}-k}a^{2^{n-2}}a^k = a^{2^{n-1}}a^{2^{n-2}} = a^{2^{n-2}}$. It follows that $o(ba^k) = 4$ for $k = 1,\ldots,2^{n-1}-1$. We also have $o(b) = 4$. Therefore, there are $2 + (2^{n-1}-1) + 1 = 2^{n-1}+2$ elements of order 4 in G. It follows that the order graph $\Delta(G)$ of G is the disjoint union $K_1 \cup K_1 \cup K_{2^{n-1}+2} \cup K_{2^2} \cup K_{2^3} \cup \cdots \cup K_{2^{n-2}}$.

CHAPTER 4 GENERALISED ORDER GRAPHS OF FINITE GROUPS

4.1 Some background

In the previous chapter, we have seen that the conjugate graphs and order graphs of finite groups are disconnected. The disconnectedness of these graphs tells us that they are neither Eulerian nor Hamiltonian. This also somewhat limits the graph properties that can be studied as many graph properties require that the graph be connected. This led us to investigate another type of graph associated with a finite group which we call the generalised order graph. This graph is defined as follows:

Definition 4.1.1. *Let G be a finite group. The generalised order graph of G, denoted by $\Gamma(G)$, is the graph with vertex set G such that for any distinct elements $x, y \in G$, x and y are adjacent in $\Gamma(G)$ if $o(x) \mid o(y)$ or $o(y) \mid o(x)$.*

For convenience, we shall refer to the generalised order graph of a finite group G as the GO-graph of G. It is clear that $\nabla(G) \leq \Delta(G) \leq \Gamma(G)$.

In Section 4.2, we present some properties of the GO-graphs of finite groups. We then focus on the GO-graphs of finite p-groups in Section 4.3. In the final section of this chapter, we discuss relationships between the GO-graphs and the undirected power graphs of finite groups.

In the remainder of this dissertation, all power graphs are assumed to be undirected power graphs.

4.2 Some general properties of GO-graphs

The following two results are straightforward by the definition of GO-graphs.

Proposition 4.2.1. *Let G be a finite group and let H be a subgroup of G. Then $\Gamma(H)$ is a subgraph of $\Gamma(G)$.*

Proposition 4.2.2. *Let G_1 and G_2 be finite groups. If G_1 and G_2 are isomorphic, then $\Gamma(G_1)$ and $\Gamma(G_2)$ are isomorphic.*

Remark 4.2.1. The converse of Proposition 4.2.2 does not hold in general. As an example, let G_1 be the cyclic group C_{27} and G_2 be the group of order 27 with exponent 3 given by the presentation

$$\langle x,y,z : x^3 = y^3 = z^3 = [x,z] = [y,z] = 1, [x,y] = z\rangle.$$

Both $\Gamma(G_1)$ and $\Gamma(G_2)$ are complete graphs K_{27} and so, $\Gamma(G_1) \cong \Gamma(G_2)$. However, since G_1 is abelian but G_2 is not abelian, G_1 and G_2 cannot be isomorphic with one another. We will show later that the converse of Proposition 4.2.2 holds for finite abelian groups.

Proposition 4.2.3. *Let G be a finite group with $|G| \geq 3$. Then $\Gamma(G)$ is a connected graph of diameter at most 2. Furthermore, the girth of $\Gamma(G)$ is 3.*

Proof. The identity element 1 of G has order 1 and this order divides the order of every other element in G. Hence, the vertex 1 is adjacent to all other vertices of $\Gamma(G)$. The graph $\Gamma(G)$ is therefore connected. If G is a p-group, then every non-identity element of G has as its order some power of p and so, it follows by definition that all the vertices of $\Gamma(G)$ are adjacent to each other. Hence, $\Gamma(G)$ is complete. Therefore, $\operatorname{diam}(\Gamma(G)) = 1$.

Suppose that G is not a p-group. Let $x,y \in G$. If $o(x)|o(y)$ or $o(y)|o(x)$, then x,y is the shortest path connecting x and y in $\Gamma(G)$ and so, $d(x,y) = 1$. Suppose that $o(x) \nmid o(y)$

and $o(y) \nmid o(x)$. Then the shortest path from x to y in $\Gamma(G)$ is $x, 1, y$ and hence, $d(x,y) = 2$. Thus $\text{diam}(\Gamma(G)) = 2$.

Now we prove the last assertion. If $|G| = 3, 4$ or 5, then G is a p-group and so, $\Gamma(G)$ is a complete graph. Hence, $\Gamma(G)$ contains a K_3. Assume that $|G| \geq 6$. Then there must exist $x \in G$ such that $x^2 \neq 1$. It follows that x has an inverse x^{-1} and so, $1, x$ and x^{-1} form K_3 in $\Gamma(G)$. Thus $g(\Gamma(G)) = 3$. □

Proposition 4.2.3 tells us that $\Gamma(G)$ contains a cycle of length 3 and hence, we have the following corollary:

Corollary 4.2.1. *Let G be a finite group with $|G| \geq 3$. Then the GO-graph of G is not a tree.*

In the case when G is finite abelian, its GO-graph $\Gamma(G)$ is in fact 3-connected and this result will be proven in the next chapter. In graph theory, it is known that a connected graph is Eulerian if and only if evey vertex in the graph has even degree. By using this result, we are able to characterise the Eulerian GO-graphs of finite groups. Before that, we need the following preliminary results.

Lemma 4.2.1. *Let G be a finite group of even order. Then $\Gamma(G)$ is not Eulerian.*

Proof. Note that the identity element 1 of G is adjacent to every other vertex of $\Gamma(G)$. Since $|G| = n$ is even, it follows that the degree of the vertex 1 is $n-1$, which is odd. Thus $\Gamma(G)$ is not Eulerian. □

Lemma 4.2.2. *Let G be a finite group of odd order. Then $\Gamma(G)$ is Eulerian.*

Proof. Suppose $|G| = n$, where n is odd. Then, the degree of 1 in $\Gamma(G)$ is $n-1$, which is even. Let $x \in G$ such that $x \neq 1$. Since n is odd, $x^2 \neq 1$ and hence, $x \neq x^{-1}$. Then since $o(x) = o(x^{-1})$, it follows that x^{-1} is adjacent to x. Thus x is adjacent to 1 and x^{-1} in $\Gamma(G)$.

Now if $y \in G \setminus \{1, x, x^{-1}\}$ is adjacent to x in $\Gamma(G)$, so is y^{-1} as $o(y) = o(y^{-1})$. If follows that the degree of x in $\Gamma(G)$ is even. Since every vertex in $\Gamma(G)$ has even degree, we have that $\Gamma(G)$ is Eulerian. □

By combining Lemmas 4.2.1 and 4.2.2, we have the following result:

Theorem 4.2.1. *Let G be a finite group. Then $\Gamma(G)$ is Eulerian if and only if $|G|$ is odd.*

4.3 Properties of GO-graphs of finite p-groups

The following result gives a necessary and sufficient condition for the GO-graph of a finite group to be complete.

Theorem 4.3.1. *Let G be a finite group. Then $\Gamma(G)$ is complete if and only if G is a p-group.*

Proof. Assume that $\Gamma(G)$ is complete and suppose that G is not a p-group. Then there exists another prime q which divides $|G|$. By Sylow's Theorem, G has a Sylow p-subgroup P and a Sylow q-subgroup Q. Let $x \in P$ and $y \in Q$. Since $\gcd(p,q) = 1$, x and y do not have order dividing each other. Hence, x and y are not adjacent in $\Gamma(G)$. This contradicts the assumption that $\Gamma(G)$ is complete. Thus G must be a p-group.

Conversely, suppose that G is a p-group so that every element of G has some power of p as its order. By definition, all the vertices in $\Gamma(G)$ are adjacent to each other. Thus $\Gamma(G)$ is a complete graph. □

It is known that the graph $\mathscr{G}$ is the complete graph K_n if and only if $\mathscr{G}$ is simple and $(n-1)$-regular. By using this fact and Theorem 4.3.1, we have the following:

Corollary 4.3.1. *Let G be a finite group of order n. Then $\Gamma(G)$ is $(n-1)$-regular if and only if G is a p-group.*

As a consequence of Theorem 4.3.1, we also have other properties of GO-graphs of finite p-groups as follows:

Corollary 4.3.2. *Let G be a finite p-group, where $p \geq 3$. Then $\Gamma(G)$ is Hamiltonian.*

Proof. By Theorem 4.3.1, the graph $\Gamma(G)$ is complete. Since a complete graph is Hamiltonian, it follows that $\Gamma(G)$ is Hamiltonian. □

Corollary 4.3.3. *Let G be a finite p-group, where $p \geq 5$. Then $\Gamma(G)$ is non-planar.*

Proof. This follows by Theorem 4.3.1 and the fact that any graph which contains K_5 as a subgraph is non-planar (by Corollary 4.2.11 in Diestel (2017)). □

Corollary 4.3.4. *Let G be a finite group of order p^n, where $n \geq 1$. Then $\chi(\Gamma(G)) = p^n - 1$.*

Proof. This follows by Theorem 4.3.1 and the fact that $\chi(K_m) = m - 1$. □

Chakrabarty et al. (2009) showed that the power graph $P(G)$ of G is complete if and only if G is trivial or a cyclic p-group. Hence, we have the following corollary:

Corollary 4.3.5. *Let G be a finite cyclic p-group. Then $P(G)$ and $\Gamma(G)$ coincide, and are both complete.*

4.4 Relationships between power graphs and GO-graphs of finite groups

We have shown in the previous section that when G is a finite cyclic p-group, its power graph and its GO-graph are isomorphic. This result led us to ask whether the isomorphism $P(G) \cong \Gamma(G)$ also holds for other classes of groups. In this section, we will consider this question and look at how the power graphs and the GO-graphs of finite groups are related. The following result is straightforward from the definition of a power graph and a GO-graph.

Proposition 4.4.1. *Let G be a finite group. Then $P(G)$ is a subgraph of $\Gamma(G)$.*

The following diagram illustrates how the conjugate graphs, order graphs, GO-graphs and power graphs of finite groups are related to one another.

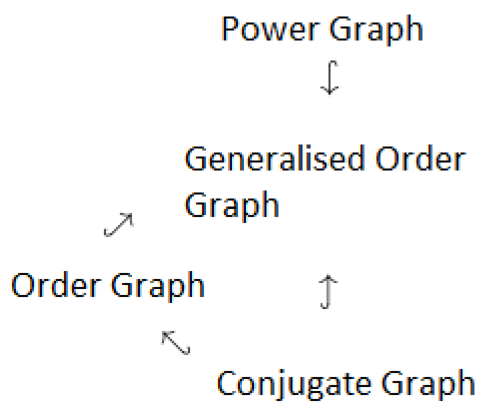

Figure 4.1: Relationship between various types of graphs

We now show how connectedness in power graphs is related to connectedness in GO-graphs of finite groups.

Theorem 4.4.1. *Let G be a finite group and let $k \geq 1$ be an integer. If $P(G)$ is k-connected, then $\Gamma(G)$ is k-connected.*

Proof. Suppose that $\Gamma(G)$ is not k-connected. By Theorem 2.2.2, there exists two distinct vertices $x, y \in V(\Gamma(G))$ such that the number of independent paths connecting x and y in $\Gamma(G)$ is less than k. Now since the power graph $P(G)$ of G is a subgraph of $\Gamma(G)$, it follows that the number of independent paths connecting x and y in $P(G)$ is also less than k. Thus $P(G)$ is not k-connected. □

Example 4.4.1. It has been shown by Bubboloni et al. (2017) that $P(A_n)$ is 2-connected if and only if either $n = 3$ or none of $n, n-1, n-2, \frac{n}{2}$ and $\frac{n-1}{2}$ is a prime. Hence, by Theorem 4.4.1, we have $\Gamma(A_n)$ is 2-connected if either $n = 3$ or none of $n, n-1, n-2, \frac{n}{2}$ and $\frac{n-1}{2}$ is a prime. In particular, the smallest integer value of n which fulfills the second restriction is 16.

For a finite group G, let $\Gamma^*(G)$ denote the punctured GO-graph of G, that is, the subgraph of $\Gamma(G)$ obtained after deleting the vertex 1. The following corollary is straightforward.

Corollary 4.4.1. *Let G be a finite group. If $P^*(G)$ is connected, then $\Gamma^*(G)$ is connected.*

We next show that Corollary 4.3.5 may be extended to finite cyclic groups. First, we prove the following lemma.

Lemma 4.4.1. *Let G be a finite cyclic group and let x,y be elements of G. Then $x \in \langle y \rangle$ if and only if $o(x)|o(y)$.*

Proof. Suppose that $x \in \langle y \rangle$. Then, $o(x)$ divides $|\langle y \rangle| = o(y)$.

Conversely, suppose that $o(x)|o(y)$. If y is a generator of G, then it is clear that $x \in \langle y \rangle$. Assume that y is not a generator of G. Let $o(x) = s$ and $o(y) = t$. Thus s and t are positive integers with $s,t < |G|$. Since $s|t$, the number of elements of order s in $\langle y \rangle$ is $\phi(s)$. Similarly, the number of elements of order s in G is also $\phi(s)$. Since $\langle y \rangle$ is a subgroup of G, the elements of order s in both $\langle y \rangle$ and G must be the same. Thus x is in $\langle y \rangle$. □

By Lemma 4.4.1, we have the following result:

Theorem 4.4.2. *Let G be a finite cyclic group. Then $P(G)$ and $\Gamma(G)$ are isomorphic.*

It was shown by Doostabadi et al. (2015) that the power graph of a finite group is perfect. Since the power graph and the GO-graph of a finite cyclic group are isomorphic, therefore the GO-graphs of finite cyclic groups are also perfect. A natural question to ask is whether this holds for finite groups in general. We answer this in the affirmative in the following theorem, the proof of which is motivated by the work of Doostabadi et al. (2015).

Theorem 4.4.3. *Let G be a finite group. Then $\Gamma(G)$ is perfect.*

Proof. Since a complete graph is perfect, it follows by Theorem 4.3.1 that the GO-graph of a finite p-group is perfect. We thus assume that G is not a p-group; hence, $|G| \geq 6$. We prove that $\Gamma(G)$ is perfect by contradiction. Suppose that $\Gamma(G)$ contains an odd cycle $\mathscr{C}$ of length at least 5 as an induced subgraph. Let x, y be two distinct vertices which are adjacent in the cycle $\mathscr{C}$. If 1 is also in the same cycle, then x, y and 1 form a triangle as an induced subgraph of $\mathscr{C}$, which contradicts our assumption that $\mathscr{C}$ is a cycle. Thus, 1 is not in $\mathscr{C}$. Let $a_1, a_2, \ldots, a_{2m-1}, a_{2m} = a_1$ be the cycle $\mathscr{C}$, where $m \geq 3$. Since all the vertices of $\Gamma(G)$ are adjacent to 1, we may replace one of the vertices of $\mathscr{C}$ with 1 to form another cycle $\mathscr{C}'$ of equal length. The cycle $\mathscr{C}'$ is also an induced subgraph of $\Gamma(G)$. However, $\mathscr{C}'$ contains a triangle as an induced subgraph, which gives a contradiction again.

Now suppose that $\overline{\Gamma(G)}$ contains an odd cycle $\overline{\mathscr{C}}$ of length at least 5 as an induced subgraph. If $\overline{\mathscr{C}}$ has length 5, then $\mathscr{C}$ also has length 5. By a similar argument as above, we may replace one of the vertices in $\mathscr{C}$ with 1 to form another cycle $\mathscr{C}'$ of equal length. But then $\mathscr{C}'$ would contain a triangle as an induced subgraph, which gives a contradiction. If $\overline{\mathscr{C}}$ has length at least 7, then the cycle $\mathscr{C}$ would contain a triangle, which contradicts our assumption again. Thus neither $\Gamma(G)$ nor $\overline{\Gamma(G)}$ contains an odd cycle of length at least 5 as an induced subgraph. It follows by Theorem 2.2.4 that $\Gamma(G)$ is perfect. □

CHAPTER 5 CONNECTEDNESS AND HAMILTONICITY OF GENERALISED ORDER GRAPHS OF FINITE GROUPS

5.1 A brief overview

In the previous chapter, we have shown in Proposition 4.2.3 that if G is a finite group with $|G| \geq 3$, then its GO-graph $\Gamma(G)$ is connected. In Section 5.2 in this chapter, we extend this result partially by showing that $\Gamma(G)$ is 3-connected for the case when G is abelian. As a consequence of this result and by Theorem 4.4.2, we may deduce that the power graph of a finite cyclic group is 3-connected. In the case of non-abelian groups, we will show in Section 5.3 that the GO-graph of a finite non-abelian group is not necessarily 3-connected.

In Section 5.5, we prove that the GO-graph of a finite abelian group contains a Hamiltonian cycle. The proof of this result makes use of a number-theoretic inequality involving the Euler totient function and the number of positive divisors of an integer, which will be proven in Section 5.4. We also prove in Section 5.5 that two finite abelian groups are isomorphic if and only if their GO-graphs are isomorphic to one another.

Next, in Section 5.6, we consider other properties related to connectedness in the GO-graphs of finite groups. In particular, we obtain some relations between the connectivity, edge-connectivity and minimum degree of the GO-graphs of finite groups. We also determine the minimum degree of the GO-graphs of various finite abelian groups. Finally, in Section 5.7, we determine the domination number and some dominating sets of the GO-graphs of finite abelian groups.

5.2 On 3-connectedness of GO-graphs of finite abelian groups

In this section we prove the first main result in this chapter which is as follows:

Theorem 5.2.1. *Let G be a finite abelian group with $|G| \geq 4$. Then $\Gamma(G)$ is 3-connected.*

We first prove the following lemmas:

Lemma 5.2.1. *Let G be a finite p-group with $|G| \geq 4$. Then $\Gamma(G)$ is 3-connected.*

Proof. By Theorem 4.3.1, $\Gamma(G)$ is the complete graph K_n, where $n = |G|$. Since K_n is 3-connected for $n \geq 4$, we have that $\Gamma(G)$ is 3-connected. □

Lemma 5.2.2. *Let G be a finite abelian group with $|G| \geq 6$ such that $|G|$ has at least two distinct prime divisors. Let $x, y \in G$ such that $o(x) = p_1^m$ and $o(y) = p_2^n$ for some prime numbers p_1, p_2 and positive integers m, n. Then there exist at least three independent paths connecting x and y in $\Gamma(G)$.*

Proof. We consider three different cases.

Case 1: $p_1 > p_2 > 2$. It is clear that $x, 1, y$ is a path in $\Gamma(G)$. Since G is abelian and $\gcd(o(x), o(y)) = 1$, we have that $o(xy) = o(x)o(y)$ and hence, x, xy, y is a path connecting x and y in $\Gamma(G)$. Since p_1 is odd, it follows that $o(x^2) = o(x)$ and hence, $o(x^2y) = o(x^2)o(y) = o(x)o(y)$. Thus, x, x^2y, y is another path connecting x and y in $\Gamma(G)$.

Case 2: $p_1 = p_2 \geq 2$. In this case, the paths x, y and $x, 1, y$ are clearly two independent paths between x and y in $\Gamma(G)$. Since $|G|$ is divisible by two distinct primes, there exists a prime q with $q \neq p_1$ such that q divides $|G|$. By Cauchy's Theorem, there exists an element $z \in G$ of order q. Since G is abelian and $\gcd(p_1, q) = 1$, it follows that $o(xz) = o(x)o(z)$ and $o(yz) = o(y)o(z)$. Therefore, x, xz, z, yz, y is a path connecting x and y in $\Gamma(G)$.

Case 3: $p_1 > 2, p_2 = 2$ or $p_1 = 2, p_2 > 2$. It is clear that $x, 1, y$ is a path in $\Gamma(G)$. Since G is abelian and $\gcd(2, p) = 1$ for any odd prime p, we have another path x, xy, y in $\Gamma(G)$.

Without loss of generality, assume that $p_1 > 2$ and $p_2 = 2$. Then, $o(x^2) = o(x)$ and hence, x, x^2y, y is another path connecting x and y in $\Gamma(G)$.

In each of the above cases, we have shown the existence of three independent paths connecting x and y in $\Gamma(G)$. This completes the proof. □

Lemma 5.2.3. *Let G be a finite abelian group with $|G| \geq 6$ such that $|G|$ has at least two distinct prime divisors. Let x,y $\in$ G such that $o(x)$ is a power of a prime and $o(y)$ is divisible by at least two distinct prime numbers. Then there exist at least three independent paths connecting x and y in $\Gamma(G)$.*

Proof. By Proposition 2.2.2, there exist positive integers a and b such that $x^a y^b$ has order $\operatorname{lcm}(o(x), o(y))$. Thus $x, 1, y$ and $x, x^a y^b, y$ are two independent paths in $\Gamma(G)$. Hence, we need to find a third path connecting x and y in $\Gamma(G)$ which is independent from the other two paths. We divide the remaining proof into two cases.

Case 1: Let $o(x) = p^k$ and $o(y) = p^l r$ where p is a prime, k, l, r are positive integers with $r > 1$ and $\gcd(p, r) = 1$. Suppose first that $1 \leq k \leq l$. Then $o(x)$ divides $o(y)$ and we have another path x, y connecting x and y in $\Gamma(G)$. Now suppose that $k > l \geq 1$. By Cauchy's Theorem, there exists $u \in G$ with $o(u) = p$. Thus x, u, y is another path connecting x and y in $\Gamma(G)$.

Case 2: Let $o(x) = p^k$ where p is a prime and k is a positive integer, and let $o(y)$ be co-prime to p. Suppose first that $p = 2$ and $k > 1$ or p is odd. Take c to be a positive integer such that $c \neq a$ and $o(x^c) = p^k$. Then $x^c y \neq x^a y^b$ and by Proposition 2.2.1, $o(x^c y) = \operatorname{lcm}(o(x^c), o(y)) = \operatorname{lcm}(o(x), o(y))$. Thus $x, x^c y, y$ is a third path connecting x and y in $\Gamma(G)$. Now suppose that $p = 2$ and $k = 1$. Then y has odd order. Take d to be a positive integer such that $d \neq b$ and $o(y^d) = o(y)$. Then $xy^d \neq x^a y^b$ and so, $o(xy^d) = o(x)o(y^d) = o(x)o(y)$. Thus x, xy^d, y is a third path connecting x and y in $\Gamma(G)$.

In both cases, we have three independent paths connecting x and y in $\Gamma(G)$. □

Lemma 5.2.4. *Let G be a finite abelian group with $|G| \geq 6$ such that $|G|$ has at least two distinct prime divisors. Let $x, y \in G$ such that $o(x)$ and $o(y)$ are each divisible by at least two distinct prime numbers. Then there exist at least three independent paths connecting x and y in $\Gamma(G)$.*

Proof. It is clear that $x, 1, y$ is a path in $\Gamma(G)$. By Proposition 2.2.2, there exist positive integers a and b such that $x^a y^b$ has order $\operatorname{lcm}(o(x), o(y))$. Thus $x, x^a y^b, y$ is another path connecting x and y in $\Gamma(G)$. By the assumption, $\operatorname{lcm}(o(x), o(y)) \neq 2$. Therefore, $(x^a y^b)^{-1} \neq x^a y^b$. Since $o((x^a y^b)^{-1}) = o(x^a y^b) = \operatorname{lcm}(o(x), o(y))$, we have that $x, (x^a y^b)^{-1}, y$ is a third path connecting x and y in $\Gamma(G)$. This completes the proof. □

Now we give the proof of Theorem 5.2.1.

Proof of Theorem 5.2.1. If G is a p-group with $|G| \geq 4$, then the result follows by Lemma 5.2.1. Suppose that G is not a p-group. Then $|G| \geq 6$ and the result follows by Lemmas 5.2.2, 5.2.3 and 5.2.4. □

By Theorem 5.2.1 and the fact that the GO-graph of a finite cyclic group of order 3 is 2-connected, we have the following corollary:

Corollary 5.2.1. *Let G be a finite abelian group with $|G| \geq 3$. Then $\Gamma(G)$ is 2-connected.*

Since power graphs and GO-graphs of finite cyclic groups are isomorphic by Theorem 4.4.2, we have the following corollary which enhances Theorem 2.1 in the work by Pourgholi et al. (2015).

Corollary 5.2.2. *Let G be a finite cyclic group with $|G| \geq 4$. Then $P(G)$ is 3-connected.*

Remark 5.2.1. In general, the GO-graph of a finite abelian group of order at least 6 is not 4-connected. As an example, let G be the cyclic group $C_6 = \langle x \rangle$. Deleting the vertices $1, x, x^5$ disconnects $\Gamma(G)$. Hence, $\Gamma(G)$ is not 4-connected. Thus 3 is the best upper bound k for the k-connectedness of GO-graphs of finite abelian groups. By Proposition 4.4.1, this upper bound also holds for $P(G)$. We will discuss more about the upper bound of k-connectedness of $\Gamma(G)$ in Section 5.6.

5.3 Connectedness of GO-graphs of some finite non-abelian groups

The GO-graph of a finite non-abelian group is not necessarily 3-connected, as shown in the following example:

Example 5.3.1. Let G be the symmetric group $S_3 = \langle x, y : x^3 = y^2 = 1, xy = yx^{-1} \rangle$. Then $\Gamma(G)$ is a union of the complete graphs K_3 (which is formed by $1, x$ and x^2) and K_4 (which is formed by $1, y, yx$ and yx^2). These two subgraphs of $\Gamma(G)$ have one common vertex, namely 1. Hence, deleting the vertex 1 disconnects $\Gamma(G)$. It follows that $\Gamma(G)$ is not 2-connected and therefore, not 3-connected. Let H be the symmetric group S_n of degree n, where $n \geq 3$. Since $G \leq H$, it follows that $\Gamma(G) \leq \Gamma(H)$. Thus the GO-graph of the symmetric group of degree $n \geq 3$ is only 1-connected.

Unlike the symmetric group, there are some finite non-abelian groups which have k-connected GO-graphs ($k > 1$), as shown in the following results.

Proposition 5.3.1. *Let G be the dihedral group*

$$D(n) = \langle a, b : a^n = 1 = b^2, bab^{-1} = a^{-1} \rangle,$$

where $n \geq 4$. Then $\Gamma(G)$ is 3-connected if and only if n is even.

Proof. Suppose that $\Gamma(G)$ is 3-connected. We prove that n is even by using contradiction. Assume that n is odd. Note that $G = \langle a \rangle \cup \{b, ba, \ldots, ba^{n-1}\}$. Since $\langle a \rangle$ has order n, it follows that 1 is the only element in this subgroup which has order dividing 2. Hence, 1 is the cut vertex of $\Gamma(G)$, which implies that $\Gamma(G)$ is only 1-connected; a contradiction. Thus n is even.

Conversely, suppose that n is even. Let H be the subgroup $\langle a \rangle$ of G. Since H is abelian, it forms a subgraph $\Gamma(H)$ of $\Gamma(G)$ which is 3-connected. Note that the elements a and a^{-1} both have order n, which is even. We consider three distinct cases:

Case 1: Let $x, y \in H$. Since the subgraph $\Gamma(H)$ is 3-connected, there are three independent paths connecting them in $\Gamma(G)$.

Case 2: Let $x \in H$ and $y \in \{ba^i : 0 \leq i \leq n-1\}$. It is clear that $x, 1, y$ is a path connecting x and y in $\Gamma(G)$. If $x \neq a$ and $x \neq a^{-1}$, then x, a, y and x, a^{-1}, y are two other independent paths connecting x and y in $\Gamma(G)$. Assume that $x = a$ or $x = a^{-1}$. Without loss of generality, suppose that $x = a$. Then x, y and x, a^{-1}, y are two other independent paths connecting x and y in $\Gamma(G)$.

Case 3: Let $x, y \in \{ba^i : 0 \leq i \leq n-1\}$. It is clear that $x, 1, y$ is a path in $\Gamma(G)$. Since x and y both have order 2 in G, x, y is also a path in $\Gamma(G)$. It follows by the assumption $n \geq 4$ that there is an element $z \in \{ba^i\} \setminus \{x, y\}$. Therefore, x, z, y is a third independent path connecting x and y in $\Gamma(G)$.

By all three cases above, $\Gamma(G)$ is 3-connected. $\square$

Proposition 5.3.2. *Let G be the alternating group A_n. If either $n = 3$ or none of $n, n-1, n-2, \frac{n}{2}$ and $\frac{n-1}{2}$ is a prime, then $\Gamma(G)$ is 2-connected .*

Proof. Refer to Example 4.4.1. $\square$

5.4 A number-theoretic inequality

In this section, we prove the following number-theoretic inequality:

Theorem 5.4.1. *For any integer $n \geq 2$, the inequality $\tau(n) < \phi(n) + 3$ holds.*

The inequality in Theorem 5.4.1 plays a key role in the proof of the Hamiltonicity of the GO-graphs of finite abelian groups, which will be presented in the next section. In order to prove Theorem 5.4.1, we need some preliminary results. The first is a well-known result in number theory that is given in Proposition 5.4.1. The notations $\phi(n)$ and $\tau(n)$ are used to denote the Euler totient function of n and the number of positive divisors of n, respectively. For a proof of the following result, the reader may refer to Burton (2011).

Proposition 5.4.1. *If m and n are relatively prime positive integers, then $\tau(mn) = \tau(m)\tau(n)$ and $\phi(mn) = \phi(m)\phi(n)$. For a prime p and positive integer k, $\tau(p^k) = k+1$ and $\phi(p^k) = p^{k-1}(p-1)$.*

We now consider relations between $\tau(n)$ and $\phi(n)$ in several cases.

Lemma 5.4.1. *Let p be a prime and let k be a positive integer. Then the following hold:*

(a) If $k \in \{1,2\}$, then $\tau(2^k) \leq \phi(2^k)+1$.

(b) If $k \geq 3$, then $\tau(2^k) \leq \phi(2^k)$.

(c) If p is odd, then $\tau(p^k) \leq \phi(p^k)$.

Proof. It can be checked by hand that if $k \in \{1,2\}$, then $\tau(2^k) \leq \phi(2^k)+1$. Now let p be any prime and suppose that $\tau(p^t) = t+1 \leq p^{t-1}(p-1) = \phi(p^t)$ for some positive integer t. Then

$$\begin{aligned}\tau(p^{t+1}) &= (t+1)+1 \\ &\leq p^{t-1}(p-1)+1 \\ &\leq p^{t-1}(p-1)+p^{t-1}(p-1)^2 \\ &= \phi(p^{t+1}).\end{aligned}$$

Thus, for a given prime p, if we can show that $\tau(p^t) \leq \phi(p^t)$, it follows by induction on the exponent that $\tau(p^k) \leq \phi(p^k)$ for all $k > t$. The proof follows since

$$\tau(2^3) = 4 = \phi(2^3)$$

and for any prime $p \geq 3$,

$$\tau(p^1) = 2 = (3-1) \leq (p-1) = \phi(p^1).$$

□

Lemma 5.4.2. *Let p be an odd prime and let t and k be positive integers. Then the following hold:*

(a) If $k \geq 3$, then $\tau(2^k p^t) \leq \phi(2^k p^t)$.

(b) If $k \in \{1,2\}$, then $\tau(2^k \cdot 3) = \phi(2^k \cdot 3) + 2$.

(c) If $t \geq 2$, then $\tau(2^k \cdot 3^t) \leq \phi(2^k \cdot 3^t)$.

(d) If $p \geq 5$, then $\tau(2^k p^t) \leq \phi(2^k p^t)$.

Proof. If $k \geq 3$, then the first assertion follows from Lemma 5.4.1 and Proposition 5.4.1 since $\tau(2^k p^t) = \tau(2^k)\tau(p^t) \leq \phi(2^k)\phi(p^t) = \phi(2^k p^t)$. So suppose that $k \in \{1,2\}$. It can be checked by hand that $\tau(2^k \cdot 3) = \phi(2^k \cdot 3) + 2$. Now let $p \geq 3$ and suppose that $\tau(2^k p^m) = (k+1)(m+1) \leq 2^{k-1} p^{m-1}(p-1) = \phi(2^k p^m)$ for some positive integer m. Then

$$\begin{aligned}
\tau(2^k p^{m+1}) &= (k+1)((m+1)+1) \\
&= (k+1)(m+1)+(k+1) \\
&\leq 2^{k-1}p^{m-1}(p-1)+3 \\
&< 2^{k-1}p^{m-1}(p-1)+2^{k-1}p^{m-1}(p-1)^2 \\
&= 2^{k-1}p^m(p-1) \\
&= \phi(2^k p^{m+1}).
\end{aligned}$$

Thus, for a fixed odd prime p and $m \in \{1,2\}$, if we can show that $\tau(2^k p^m) = (k+1)(m+1) \leq 2^{k-1}p^{m-1}(p-1) = \phi(2^k p^m)$, it follows by induction on exponent of p that $\tau(2^k p^t) = (k+1)(t+1) \leq 2^{k-1}p^{t-1}(p-1) = \phi(2^k p^t)$ for all $t > m$. The proof follows since

$$\tau(2 \cdot 3^2) = 6 = \phi(2 \cdot 3^2) \text{ and } \tau(2^2 \cdot 3^2) = 9 \leq 12 = \phi(2^2 \cdot 3^2)$$

and for any prime $p \geq 5$,

$$\tau(2 \cdot p^1) = 4 = (5-1) \leq (p-1) = \phi(2 \cdot p^1)$$

and

$$\tau(2^2 \cdot p) = 6 \leq 2(5-1) \leq 2(p-1) = \phi(2^2 \cdot p).$$

□

We are now ready for the proof of Theorem 5.4.1.

Proof of Theorem 5.4.1. If n has only one prime divisor, then the theorem follows from Lemma 5.4.1. If n has only odd prime divisors, then the theorem follows from Lemma 5.4.1 and Proposition 5.4.1. Now suppose that n is even and has exactly two distinct prime divisors. Then we can write $n = 2^k p^t$ for some odd prime p and positive integers t and k. In this case, the theorem follows from Lemma 5.4.2. Lastly, suppose that the n is even and has three or more distinct prime divisors. Then we can write $n = 2^k p_1^{k_1} \dots p_r^{k_r}$ where $r \geq 2$ and $p_1 < \cdots < p_r$ are distinct odd primes and k_i is a positive integer for each $1 \leq i \leq r$. Since p_1 and p_2 are odd primes with $p_1 < p_2$, we know that $p_2 \geq 5$. Thus $\tau(2^k p_2^{k_2}) \leq \phi(2^k p_2^{k_2})$ by Lemma 5.4.2. Consequently, $\tau(n) = \tau(2^k p_2^{k_2}) \cdot \tau(p_1^{k_1} p_3^{k_3} \dots p_r^{k_r}) \leq \phi(2^k p_2^{k_2}) \cdot \phi(p_1^{k_1} p_3^{k_3} \dots p_r^{k_r}) = \phi(n)$. □

5.5 Hamiltonicity of GO-graphs of finite abelian groups

In Section 5.2, we proved that the GO-graph of a finite abelian group is 3-connected. It is commonly known in graph theory that a Hamiltonian graph is 2-connected. This led us to ask whether the GO-graph of a finite abelian group is also Hamiltonian. In this section, we show that the answer to this question is in the affirmative. We first prove the following lemma.

Lemma 5.5.1. *Let G be a finite abelian group with $|G| \geq 6$ and let m be the largest element order in G. Then $o(g)|m$ for any $g \in G$.*

Proof. Since G is abelian, it follows by Theorem 2.2.1 that $G \cong C_{n_1} \times \cdots \times C_{n_r}$, where $n_j | n_i$ for $j > i$, $n_r \geq 2$ and $n_1 \dots n_r = |G|$. Hence, n_1 is divisible by n_j for $j \geq 2$. We have $n_1 \geq 6$ since $|G| \geq 6$. We also note that for any $x, y \in G$ with $xy \neq 1$,

$$o(xy) = \frac{o(x)o(y)}{\gcd(o(x), o(y))}.$$

Then since $n_j | n_i$ for $j > i$, it follows that $n_1 = m$ and the order of $g \in G$ divides n_1. □

Recall that a spanning path of a graph $\mathscr{G}$ is a path which passes through every vertex of $\mathscr{G}$. We are now ready for the second main result in this chapter, the proof of which is motivated by a result on the Hamiltonicity of undirected power graphs of finite cyclic groups in Chakrabarty et al. (2009, Theorem 4.13).

Theorem 5.5.1. *Let G be a finite abelian group of order $n \geq 3$. Then $\Gamma(G)$ is Hamiltonian.*

Proof. If n is a power of a prime p, then G is a p-group. By Corollay 4.3.2, $\Gamma(G)$ is Hamiltonian. Now suppose that $n = p^i m$, where $m > 1$, $i \geq 1$ and $\gcd(p,m) = 1$. Hence, $n \geq 6$. Let n_1 be the largest element order in G. By Lemma 5.5.1, the elements of order n_1 are adjacent to every other element of G in $\Gamma(G)$. Let U be the set of elements of order n_1 in G and let $\mathscr{C} = U \cup \{1\}$. It is clear that all the elements in $\mathscr{C}$ are adjacent to each other in $\Gamma(G)$.

Let d be a non-trivial divisor of n_1 (that is, $d \neq 1$ and n_1) and let U_d be the set of elements of order d in G. Since the elements of U_d have the same order, they are adjacent to each other in $\Gamma(G)$ and so, form a clique in $\Gamma(G)$. It is clear that all the vertices of this clique are adjacent to every vertex of $\mathscr{C}$. Now we have s such cliques in $\Gamma(G)$, where s is the number of non-trivial divisors of n_1. Let $\mathscr{C}_1, \ldots, \mathscr{C}_s$ be these cliques. As mentioned in the previous paragraph, every element $x \in G \setminus \mathscr{C}$ is adjacent to all elements of $\mathscr{C}$ in $\Gamma(G)$. Hence, x lies in one of the cliques $\mathscr{C}_i$. It is clear that $s = \tau(n_1) - 2$ and $|\mathscr{C}| \geq \phi(n_1) + 1$. By Theorem 5.4.1, we have that $s < |\mathscr{C}|$.

Since $\mathscr{C}_i$ ($i = 1, \ldots, s$) is a clique, it has a spanning path, where the end vertices are adjacent to every vertex of $\mathscr{C}$. Thus a Hamiltonian cycle of $\Gamma(G)$ can be constructed as follows: Start from a vertex of $\mathscr{C}$, then move to one of the cliques $\mathscr{C}_i$. If $|V(\mathscr{C}_i)| > 1$, then traverse its spanning path and return to another vertex c of $\mathscr{C}$. Otherwise, move to the single vertex of $\mathscr{C}_i$ and then return to another vertex c of $\mathscr{C}$. From this vertex

c, move to traverse the spanning path of another clique $\mathscr{C}_j$ (where $j \neq i$) and return to an untraversed vertex of $\mathscr{C}$. This process is repeated until all the $\mathscr{C}_i$ have been traversed. This process of traversing the cliques will terminate at a vertex of $\mathscr{C}$ as $s < |\mathscr{C}|$. Therefore a Hamiltonian cycle is formed by traversing the remaining untraversed vertices (if any) of $\mathscr{C}$ and returning to the starting vertex of $\mathscr{C}$. □

Remark 5.5.1. From the proof of the above result, it can be seen that for a finite abelian group G of order $n \geq 6$, where n is a composite number, the GO-graph $\Gamma(G)$ is $(\phi(n_1)+1)$-connected, where n_1 is the largest element order of G. In particular, $\Gamma(G)$ is 3-connected. If $|G| = p$, where p is a prime, then $\Gamma(G)$ is complete and hence, $(|G|-1)$-connected. In particular, $\Gamma(G)$ is 3-connected when $|G| \geq 5$. This observation reinforces the assertion in Theorem 5.2.1.

Remark 5.5.2. For a finite cyclic group G with $|G| \geq 4$, we have by Theorem 4.4.2 and Remark 5.5.1 that $P(G) \cong \Gamma(G)$ is 3-connected. In particular, when $|G| = n \geq 6$ is composite, then $P(G)$ is $(\phi(n)+1)$-connected.

The following corollary is a straightforward consequence of Theorem 5.5.1.

Corollary 5.5.1. *Let G be a finite cyclic group of order $n \geq 3$. Then $\Gamma(G)$ is Hamiltonian.*

By Corollary 5.5.1 and Theorem 4.4.2, we obtain the following corollary; thus giving an alternative proof of Theorem 4.13 in Chakrabarty et al. (2009).

Corollary 5.5.2. *Let G be a finite cyclic group of order $n \geq 3$. Then $P(G)$ is Hamiltonian.*

For a finite abelian group G, let $n_k(G)$ denote the number of elements of order k in G where k is a divisor of $|G|$. We say that two finite groups G_1 and G_2 have the same sequence of orders if $|G_1| = |G_2|$ and $n_k(G_1) = n_k(G_2)$ for each positive integer k which divides $|G|$. It is known by a result of McHaffey (1965) that if the sequence of orders of two finite abelian groups are the same, then the groups are isomorphic. In other words, a finite abelian group can be determined uniquely by the orders of its elements. We thus have the following:

Theorem 5.5.2. *Let G_1 and G_2 be any two finite abelian groups. Then G_1 and G_2 are isomorphic if and only if $\Gamma(G_1)$ and $\Gamma(G_2)$ are isomorphic.*

In general, the GO-graph of a finite group is not necessarily Hamiltonian. This is illustrated in the example below.

Example 5.5.1. Let G be the dihedral group $D_3 = \langle a, b : a^3 = b^2 = 1, ab = ba^{-1} \rangle$. The elements in the set $\{a, a^2\}$ have order 3, whereas all the elements in the set $\{b, ab, a^2b\}$ have order 2. Thus any cycle which contains all vertices of the graph $\Gamma(G)$ must traverse the vertex 1 at least twice. Therefore $\Gamma(G)$ is not Hamiltonian.

Remark 5.5.3. Theorem 5.5.2 does not hold for finite groups in general. We provide two examples as follows:

1. Let $G_1 = C_2 \times Q_8$ and $G_2 = \langle x, y : x^4 = 1 = y^4, y^{-1}xy = x^3 \rangle$. Then G_1 and G_2 each have 1 identity element, 3 elements of order 2 and 12 elements of order 4. Thus $\Gamma(G_1) \cong \Gamma(G_2)$ although $G_1 \ncong G_2$.

2. Let $G_1 = \langle x, y, z : x^4 = 1 = y^2 = z^2, xy = yx, xz = zx, y^{-1}zy = zx^2 \rangle$ and $G_2 = \langle x, y, z : x^4 = 1 = y^2 = z^2, xz = zx, yz = zy, y^{-1}xy = xz \rangle$. Then G_1 and G_2 each have 1 identity element, 7 elements of order 2 and 8 elements of order 4. Thus $\Gamma(G_1) \cong \Gamma(G_2)$ although $G_1 \ncong G_2$.

5.6 Connectivity, edge-connectivity and minimum degree of GO-graphs of finite groups

In Remark 5.5.1, it was stated that for a finite abelian group G with largest element order n, its GO-graph $\Gamma(G)$ is $(\phi(n)+1)$-connected. Hence, it is of interest to determine an upper bound of k such that $\Gamma(G)$ is k-connected.

In the work by Whitney (1932), it was shown that the connectivity, the edge-connectivity and the minimum degree of a finite simple graph $\mathscr{G}$ satisfy the inequality $\kappa(\mathscr{G}) \leq \kappa'(\mathscr{G}) \leq \delta(\mathscr{G})$. Besides, it was shown by Plesník (1975) that any graph $\mathscr{G}$ with diameter at most 2 satisfies the equality $\kappa'(\mathscr{G}) = \delta(\mathscr{G})$. Hence, by Proposition 4.2.3, we have the following result:

Proposition 5.6.1. *Let G be a finite group. Then $\kappa'(\Gamma(G)) = \delta(\Gamma(G))$.*

By Proposition 5.6.1, we may focus our attention on the minimum degree of $\Gamma(G)$ to determine an upper bound k of the k-connectedness of $\Gamma(G)$. For the following result, the reader may refer to West (1996).

Lemma 5.6.1. *If a graph $\mathscr{G}$ is 3-regular, then $\kappa(\mathscr{G}) = \kappa'(\mathscr{G})$.*

By Theorem 4.3.1, the GO-graph $\Gamma(G)$ of a p-group G is complete and so, $\Gamma(G)$ is 3-regular if $|G| \geq 4$. Therefore, we have the following corollary.

Corollary 5.6.1. *Let G be a finite p-group of order $|G| \geq 4$. Then*

$$\kappa(\Gamma(G)) = \kappa'(\Gamma(G)) = \delta(\Gamma(G)).$$

The following result provides a necessary and sufficient condition for $\delta(\Gamma(G))$ to be $|G|-1$.

Proposition 5.6.2. *Let G be a finite group. Then* $\delta(\Gamma(G)) = |G|-1$ *if and only if G is a p-group.*

Proof. This follows by Theorem 4.3.1. □

We now give some results on the minimum degree of GO-graphs of some finite abelian groups of composite order. In the remainder of this section, let $\mathscr{C}$ be as described in the proof of Theorem 5.5.1. That is, $\mathscr{C} = U \cup \{1\}$, where U is the set consisting of elements with the largest element order in the group.

Theorem 5.6.1. *Let G be a finite abelian group of order* $p^i k$*, where* $k > 1$ *and* $gcd(p,k) = 1$*. Then* $\delta(\Gamma(G)) = |\mathscr{C}| + \delta(\Gamma(G) \setminus \mathscr{C})$.

Proof. By Lemma 5.5.1, it is clear that each vertex u of $\Gamma(G) \setminus \mathscr{C}$ is adjacent to every vertex of $\mathscr{C}$. Moreover, each vertex c of $\mathscr{C}$ is adjacent to every other vertex of $\Gamma(G)$. Hence, $\deg(u) \geq |\mathscr{C}|$ and $\deg(c) = |G|-1$. Clearly, $\delta(\Gamma(G)) \geq |\mathscr{C}|$. Let v be a vertex of $\Gamma(G)$ such that $\deg(v) = \delta(\Gamma(G))$. Then v must be a vertex in the subgraph $\Gamma(G) \setminus \mathscr{C}$. Note that v has degree $\delta(\Gamma(G) \setminus \mathscr{C})$ in the subgraph $\Gamma(G) \setminus \mathscr{C}$. Since v is adjacent to all vertices of $\mathscr{C}$ in $\Gamma(G)$, therefore $\delta(\Gamma(G)) = \deg(v) = \delta(\Gamma(G) \setminus \mathscr{C}) + |\mathscr{C}|$. □

As a consequence of Theorem 5.6.1, we have the following:

Corollary 5.6.2. *Let G be a finite cyclic group of order n, where n is a composite number. Then* $\delta(\Gamma(G)) = \phi(n) + 1 + \delta(\Gamma(G) \setminus \mathscr{C})$.

By Theorem 4.4.2, we know that $P(G) \cong \Gamma(G)$ when G is a finite cyclic group. Therefore we have the following immediate consequence of Corollary 5.6.2, thus giving an alternative proof of Theorem 4.4(i) in the paper by Panda and Krishna (2018).

Corollary 5.6.3. *Let G be a finite cyclic group of order n, where n is composite. Then $\delta(P(G)) = \phi(n) + 1 + \delta(P(G) \setminus \mathscr{C})$.*

Lemma 5.6.2. *Let $G = C_{pq} = \langle x | x^{pq} = 1 \rangle$, where p and q are distinct primes. Then $\Gamma(G) \setminus \mathscr{C}$ is disconnected. Furthermore, $\Gamma(G) \setminus \mathscr{C}$ has two disjoint cliques.*

Proof. Since G has order pq, it follows that $\mathscr{C} = \{1\} \cup \{x \in G : o(x) = pq\}$. By Cauchy's Theorem, G has elements of order p and elements of order q. Hence, $G \setminus \mathscr{C}$ is the union of two disjoint subsets, namely $\{y \in G : o(y) = p\}$ and $\{z \in G : o(z) = q\}$. Then let $y, z \in G$ such that $o(y) = p$ and $o(z) = q$. Since $o(y) \nmid o(z)$ and $o(z) \nmid o(y)$, the vertices y and z are not adjacent in $\Gamma(G)$. Thus the subgraph $\Gamma(G) \setminus \mathscr{C}$ is disconnected. The last assertion is clear. □

Proposition 5.6.3. *Let $G = C_{pq}$, where p and q are distinct primes and $p > q$. Then $\delta(\Gamma(G) \setminus \mathscr{C}) = q - 2 = \phi(q) - 1$.*

Proof. By the proof of Lemma 5.6.2, $\Gamma(G) \setminus \mathscr{C} = A \cup B$, where $A = \{x \in G : o(x) = p\}$ and $B = \{y \in G : o(y) = q\}$ are two disjoint cliques. Since $|A| = p - 1 > q - 1 = |B|$, we have $\delta(\Gamma(G) \setminus \mathscr{C}) = q - 2 = \phi(q) - 1$. □

By combining Corollary 5.6.2 and Proposition 5.6.3, we have

Corollary 5.6.4. *Let $G = C_{pq}$, where p and q are distinct primes and $p > q$. Then $\delta(\Gamma(G)) = \phi(pq) + \phi(q) = p(q-1)$.*

We end this section by providing a result on the minimum degree of the GO-graph of the abelian group $C_p \times C_p \times C_q$, where $p > q$.

Proposition 5.6.4. *Let G be the group $C_p \times C_p \times C_q$, where p and q are two distinct odd primes and $p > q$. Then $\delta(\Gamma(G)) = p^2(q-1)$.*

Proof. By assumption, pq is the largest element order of G. Let $\mathscr{C} = \{x \in G : o(x) = pq\} \cup \{1\}$. Note that every vertex of $\mathscr{C}$ is adjacent to all other vertices of $\Gamma(G)$ and hence, has degree $|G|-1$. It follows that any vertex of minimum degree in $\Gamma(G)$ has to be in $\Gamma(G) \setminus \mathscr{C}$. Note that $\Gamma(G) \setminus \mathscr{C} = A \cup B$, where $A = \{y : o(y) = p\}$ and $B = \{z : o(z) = q\}$. By computation, we have $|A| = |C_p \times C_p| - 1 = p^2 - 1$ and $|B| = |C_q| - 1 = q - 1$. Since $p > q$, it is clear that any vertex of minimum degree in $\Gamma(G)$ must be in B. Thus

$$\begin{aligned}
\delta(\Gamma(G)) &= (|B|-1) + |\mathscr{C}| \\
&= (|B|-1) + (|G| - |A| - |B|) \\
&= |G| - |A| - 1 \\
&= p^2 q - p^2 \\
&= p^2(q-1).
\end{aligned}$$

□

5.7 Dominating sets of GO-graphs of finite groups

We first give the definitions of dominating set and domination number of a graph $\mathscr{G}$ as in West (1996).

Definition 5.7.1. *Let $\mathscr{G}$ be a graph. A set $S \subseteq V(\mathscr{G})$ is a dominating set if every vertex $g \in V(\mathscr{G})$ not in S has a neighbour in S. The domination number $\gamma(\mathscr{G})$ is the minimum size of a dominating set in $\mathscr{G}$.*

Example 5.7.1. *The complete graph K_n with n vertices has domination number $\gamma(K_n) = 1$.*

The following result is clear.

Proposition 5.7.1. *Let G be a finite group. Then $\gamma(\Gamma(G)) = 1$.*

Proof. The identity element 1 is adjacent to all other elements of G in $\Gamma(G)$. Hence, the vertex 1 forms a dominating set of $\Gamma(G)$. Thus $\gamma(\Gamma(G)) = 1$. □

Since the GO-graph of a finite group has the set $\{1\}$ as a dominating set, it seems natural to ask about dominating sets of the punctured generalised order graph $\Gamma^*(G)$ of G, that is, the subgraph of $\Gamma(G)$ obtained by deleting the vertex 1.

Theorem 5.7.1. *Let G be a finite abelian group of order $|G| \geq 3$. Then $\gamma(\Gamma^*(G)) = 1$.*

Proof. If G is a p-group, then $\Gamma(G)$ is complete. It follows that $\Gamma^*(G)$ is also complete and so, $\gamma(\Gamma^*(G)) = 1$. Now we assume that G is not a p-group. Then $|G| \geq 6$. Let $x \in G$ such that $o(x)$ is the largest element order of G. By Lemma 5.5.1, for any $g \in G$, $o(g)|o(x)$. It follows that g is adjacent to x in $\Gamma^*(G)$ for any $g \in G$. Thus $\{x\}$ is a dominating set of $\Gamma^*(G)$ and hence, $\gamma(\Gamma^*(G)) = 1$. □

The following corollary is straightforward.

Corollary 5.7.1. *Let G be a finite cyclic group of order $|G| \geq 3$. Then $\gamma(\Gamma^*(G)) = 1$.*

We have the following result which relates domination number and connectedness of a graph.

Theorem 5.7.2. *Let G be a finite group of order $|G| \geq 3$. If $\gamma(\Gamma^*(G)) = 1$, then $\Gamma(G)$ is 2-connected.*

Proof. If $|G| = 3$, then G is a cyclic group and $\Gamma(G)$ is clearly 2-connected. Now suppose that $|G| > 3$. In the graph $\Gamma(G)$, the vertex 1 is adjacent to all other vertices. Since $\gamma(\Gamma^*(G)) = 1$, there exists $g \in G \setminus \{1\}$ such that $\{g\}$ is a dominating set of $\Gamma^*(G)$. Hence, all vertices of $\Gamma^*(G)$ are adjacent to g. Let x, y be two distinct vertices of $\Gamma(G)$. We consider 4 distinct cases:

Case 1: Let $x = 1$ and $y = g$. Then there are two independent paths connecting x and y in the graph $\Gamma(G)$, these are x, y and x, a, y, where $a \in G \setminus \{1, g\}$.

Case 2: Let $x = 1$ and $y \in G \setminus \{1, g\}$. Then x, y and x, g, y are two independent paths in $\Gamma(G)$ which connect x and y.

Case 3: Let $x = g$ and $y \in G \setminus \{1, g\}$. Then x, y and $x, 1, y$ are two independent paths in $\Gamma(G)$ which connect x and y.

Case 4: Let $x, y \in G \setminus \{1, g\}$. Then $x, 1, y$ and x, g, y are two independent paths connecting x and y in $\Gamma(G)$.

By Cases 1 to 4, the assertion follows. □

Example 5.7.2. We have shown in Example 5.3.1 that the GO-graph of S_n ($n \geq 3$) is not 2-connected. Therefore, $\gamma(\Gamma^*(S_n)) > 1$ for $n \geq 3$.

We now give some results on domination numbers of punctured GO-graphs of some finite non-abelian groups.

Proposition 5.7.2. *Let G be the dihedral group $D_n = \langle a, b : a^n = 1 = b^2, ab = ba^{-1} \rangle$. Then*

(a) n is even if and only if $\gamma(\Gamma^*(G)) = 1$;

(b) n is odd if and only if $\gamma(\Gamma^*(G)) = 2$.

Proof. (a) Assume that n is even. Then the element $a \in G$ generates the normal subgroup $\langle a \rangle$ of even order. Let g be a non-identity element of $\langle a \rangle \setminus \{a\}$. Since the order of an element of a finite group divides the order of that group, we have $o(g) | o(a)$. Hence, g is adjacent to a in $\Gamma^*(G)$. All elements of the set $\{ba^i : 0 \leq i \leq n-1\}$ have order 2 and so, ba^i is also adjacent to a in $\Gamma^*(G)$. Thus $\{a\}$ is a dominating set of $\Gamma^*(G)$ and hence, $\gamma(\Gamma^*(G)) = 1$.

Conversely, suppose that $\gamma(\Gamma^*(G)) = 1$. Assume to the contrary that n is odd. Then none of the elements of $\langle a \rangle$ would be adjacent to any of the elements in the set $\{ba^i : 0 \leq i \leq n-1\}$ in $\Gamma^*(G)$. Therefore, any dominating set of $\Gamma^*(G)$ must contain at least one element of the form a^i and at least one element of the form ba^j, which is a contradiction. Thus n is even.

(b) Suppose that n is odd. It is clear then that none of the elements of $\langle a\rangle$ is adjacent to any of the elements in the set $\{ba^i : 0 \leq i \leq n-1\}$ in $\Gamma^*(G)$. Therefore, any dominating set of $\Gamma^*(G)$ must contain at least one element of the form a^i and at least one element of the form ba^j. Thus $\gamma(\Gamma^*(G)) = 2$.

Conversely, assume that $\gamma(\Gamma^*(G)) = 2$. If n is even, then it follows by part (a) that $\gamma(\Gamma^*(G)) = 1$; a contradiction. Thus n must be odd.

□

We have a similar result for dicyclic groups.

Proposition 5.7.3. *Let G be the dicyclic group $Q_n = \langle a,b : a^{2n} = 1, a^n = b^2, ab = ba^{-1}\rangle$, for $n \geq 2$. Then*

(a) n is even if and only if $\gamma(\Gamma^*(G)) = 1$;

(b) n is odd if and only if $\gamma(\Gamma^*(G)) = 2$.

Proof. (a) Assume that $n = 2k$ for some positive integer k. Then the element $a \in G$ has order $2n = 4k$ and so, generates the normal subgroup $\langle a\rangle$ of even order. It is clear that $g \in \langle a\rangle \setminus \{a\}$ is adjacent to a in $\Gamma^*(G)$. All elements of the set $\{ba^i : 0 \leq i \leq n-1\}$ have order 4 and so, ba^i is also adjacent to a in $\Gamma^*(G)$. Thus $\{a\}$ is a dominating set of $\Gamma^*(G)$ and $\gamma(\Gamma^*(G)) = 1$.

Conversely, suppose that $\gamma(\Gamma^*(G)) = 1$. Assume that n is odd. Then none of the elements of $\langle a\rangle$ would be adjacent to any of the elements in the set $\{ba^i : 0 \leq i \leq n-1\}$ in $\Gamma^*(G)$. Therefore, any dominating set of $\Gamma^*(G)$ must contain at least one element of the form a^i and at least one element of the form ba^j, which is a contradiction. Thus n is even.

(b) Suppose that n is odd. Then none of the elements of $\langle a \rangle$ would be adjacent to any of the elements in the set $\{ba^i : 0 \leq i \leq n-1\}$ in $\Gamma^*(G)$. Therefore, any dominating set of $\Gamma^*(G)$ must contain at least one element of the form a^i and at least one element of the form ba^j. Thus $\gamma(\Gamma^*(G)) = 2$.

Conversely, assume that $\gamma(\Gamma^*(G)) = 2$. If n is even, then it follows by part (a) that $\gamma(\Gamma^*(G)) = 1$; a contradiction. Thus n must be odd.

□

In general, it is not an easy task to find a dominating set of the punctured GO-graph of a finite non-abelian group. However, we are able to determine the size of a minimal dominating set as follows:

Theorem 5.7.3. *Let G be a finite group of order $p_1^{k_1} \dots p_r^{k_r}$, where $p_1, \dots, p_r$ are distinct prime numbers and $k_1, \dots, k_r$ are positive integers. Then $\Gamma^*(G)$ has a dominating set of size r which is minimal with respect to set inclusion.*

Proof. By Cauchy's Theorem, for each prime p_i, there is an element $x_i \in G$ such that $o(x_i) = p_i$. Let $D = \{x_1, \dots, x_r\}$ and let $y \in G \setminus \{1\}$ such that $y \notin D$. Since $o(y)$ divides $|G|$, it follows that $o(y)$ is divisible by at least one of the p_i's. Hence, y is adjacent to some $x_i \in D$. Thus D is a dominating set of $\Gamma^*(G)$. We prove the minimality of D by contradiction. Assume that $D' \subsetneq D$ is also a dominating set of $\Gamma^*(G)$. Then there exists $x_j \in D \setminus D'$ which is adjacent to some $x_k \in D'$ in $\Gamma^*(G)$. But this implies that $p_j | p_k$ or $p_k | p_j$; a contradiction. Therefore D is minimal with respect to set inclusion. □

The following corollary is a straightforward consequence of Theorem 5.7.3.

Corollary 5.7.2. *Let G be a finite group of order $p_1^{k_1} \dots p_r^{k_r}$, where $p_1, \dots, p_r$ are distinct prime numbers and $k_1, \dots, k_r$ are positive integers. Then $\gamma(\Gamma^*(G)) \leq r$.*

CHAPTER 6 CONCLUSION

6.1 Summary

In this dissertation, we associate different types of graphs to a finite group with the aim of finding relationships between groups and graphs, and at advancing knowledge of one to the other.

Given a finite group G, there are three main graphs associated with G that are considered in this dissertation, namely, the conjugate graph $\nabla(G)$, the order graph $\Delta(G)$ and the generalised order graph $\Gamma(G)$. These graphs satisfy the subgraph relation $\nabla(G) \leq \Delta(G) \leq \Gamma(G)$.

We begin by investigating properties of the conjugate graphs of finite groups. Among others, we show that $\nabla(G)$ is perfect when G is a finite group. For abelian groups, the conjugate graphs are just empty graphs. In the non-abelian case, we determine the conjugate graphs of the dihedral and generalised quaternion groups, and also obtain some properties of the conjugate graph of the symmetric group S_n. As conjugate elements of a finite group have the same order, our investigation on conjugate graphs led us to study order graphs of finite groups. We show that order graphs of finite groups are also perfect. We also show that if G is a finite group of even order, then there are at least two complete components K_v in $\Delta(G)$, where v is odd. On the other hand, if G has odd order, then every complete component of $\Delta(G)$ other than K_1 has an even number of vertices. We also obtain the structures of order graphs for cyclic groups and various p-groups.

The main focus in this dissertation is on the generalised order graph (GO-graph) of a finite group. Given a finite group G, we prove that the GO-graph $\Gamma(G)$ is Eulerian if and only if $|G|$ is odd, and that $\Gamma(G)$ is complete if and only if G is a p-group. It is also shown that the GO-graph and the power graph of a finite cyclic group are isomorphic. Moreover, just like the conjugate and order graphs, we show that $\Gamma(G)$ is a perfect graph when G is a finite group. Among the main results obtained on properties of GO-graphs is that the GO-graph of a finite abelian group of order at least 4 is 3-connected. Another main result is that the GO-graph of a finite abelian group of order at least 3 is Hamiltonian. In order to prove this result, we first obtained a number-theoretic inequality involving the Euler totient function and the number of positive divisors of an integer.

6.2 Some open problems

We end this chapter with some open problems related to the work in this dissertation.

1. Determine the structure of the conjugate graph of the alternating group A_n, where $n \geq 4$.

2. Let G be a group of order n. Is the clique number of $\Delta(G)$ the largest when G is the cyclic group C_n?

3. Let G be a group of order n. Is the clique number of $\Gamma(G)$ the largest when G is the cyclic group C_n?

4. Let G be a finite group. Determine an upper bound k for the k-connectedness of $\Gamma(G)$.

5. Determine the finite groups which have planar GO-graphs.

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