PROPERTIES OF CLOSE-TO-CONVEX FUNCTIONS AND SPECIAL FUNCTIONS

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KUALA LUMPUR

2019

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DISSERTATION SUBMITTED IN FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

INSTITUTE OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
UNIVERSITY OF MALAYA
KUALA LUMPUR

2019

UNIVERSITI MALAYA

ORIGINAL LITERARY WORK DECLARATION

Name of Candidate: **JONATHAN AARON AZLAN MOSIUN**

Registration/Matric No.: **SGP150004**

Name of Degree: **MASTER OF SCIENCE**

Title of Project Paper/Research Report/Dissertation/Thesis ("this Work"): **"PROPERTIES OF CLOSE-TO-CONVEX FUNCTIONS AND SPECIAL FUNCTIONS"**

Field of Study: **PURE MATHEMATICS**

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PROPERTIES OF CLOSE-TO-CONVEX FUNCTIONS AND SPECIAL FUNCTIONS

ABSTRACT

Let $\mathcal{S}$ be the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are univalent and analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Study on functions derived via geometric properties such as $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$, which are subclasses of $\mathcal{S}$, has been ongoing for many decades and has been done extensively and exhaustively. Among the many subclasses of $\mathcal{K}$, Sakaguchi introduced the class of starlike functions with respect to symmetric point, denoted by $\mathcal{S}_s^*$. Since its introduction in 1959, many authors have introduced generalizations of $\mathcal{S}_s^*$ or classes resembling it. Inspired by this, Gao & Zhou introduced another subclass of $\mathcal{K}$ which was denoted as $\mathcal{K}_s$ which was further generalized by Wang, Gao, & Yuan. Following their inspirations, this dissertation introduces a subclass of close-to-convex functions, denoted by $\mathcal{K}_s^{k,N}$, where $k, N \in \mathbb{N}$, that combines the concepts of $\mathcal{S}_s^*$ and $\mathcal{K}_s$ and investigates them for their properties which include, but not limited to, coefficient estimates, distortion and growth theorems, and radius of convexity. Moreover, we also introduce the class of p-valent functions, denoted by $\mathcal{K}_{s,p}^{k,N}$, in this dissertation which further generalizes the class $\mathcal{K}_s^{k,N}$ and investigate it for its properties. In addition to investigating properties of geometric functions, many other mathematicians have also expressed interest in finding sufficient conditions such that certain special functions has certain geometric properties, such as univalency, starlikeness or convexity. Examples of special functions that have undergone this investigation include Bessel and Struve functions. Motivated by this, this dissertation also investigates sufficient conditions for the function $T_{p,b,c}(z) = (f * g_{p,b,c})(z)$, a convolution between $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and

$g_{p,b,c}(z) = z + \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}} z^n$ with $p, c, b \in \mathbb{C},\ k = p + \frac{b+2}{2} \neq 0, -1, -2, \ldots$, to be univalent, starlike or convex in both the unit disc and $|z| < \frac{1}{2}$.

Keywords: Univalent, Close-to-Convex, Generalized Struve function, Functions with Respect to N-ply Symmetry Point

SIFAT-SIFAT FUNGSI HAMPIR-CEMBUNG DAN FUNGSI ISTIMEWA

ABSTRAK

Biarkan $\mathcal{S}$ adalah kelas fungsi $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ yang bersifat univalen dan analisis dalam cakera unit $U = \{z \in \mathbb{C} : |z| < 1\}$. Kajian mengenai fungsi yang tertakrif melalui sifat geometri seperti $\mathcal{S}^*$, $\mathcal{C}$ dan $\mathcal{K}$, yang merupakan subkelas $\mathcal{S}$ telah dijalankan selama beberapa dekad dan secara mendalam dan menyeluruh. Antara banyak subkelas $\mathcal{K}$, Sakaguchi memperkenalkan kelas fungsi bak-bintang terhadap titik simetri, dilambangkan oleh $\mathcal{S}_s^*$. Sejak diperkenalkan pada 1959, ramai penulis telah perkenalkan generalisasi kelas $\mathcal{S}_s^*$ atau kelas yang mirip dengannya. Diilhamkan oleh ini, Gao dan Zhou memperkenalkan subkelas baharu $\mathcal{K}$ yang dilambangkan sebagai $\mathcal{K}_s$ yang kemudiannya diperluaskan lagi oleh Wang, Gao dan Yuan. Berikutan ini, idea mereka telah mendorong disertasi ini untuk perkenalkan kelas fungsi hampir-cembung yang baharu, dilambangkan oleh $\mathcal{K}_s^{k,N}$, yang menggabungkan konsep $\mathcal{S}_s^*$ dan $\mathcal{K}_s$ serta mengkaji sifat fungsi kelas ini termasuk, tetapi tidak terhad kepada, anggaran koefisien, teorem herotan dan pertumbuhan, dan jejari kecembungan. Selain itu, kami juga perkenalkan kelas fungsi p-valen, dilambangkan oleh $\mathcal{K}_{s,p}^{k,N}$, dalam disertasi ini yang seterusnya mengitlakkan kelas $\mathcal{K}_s^{k,N}$, serta disiasat sifat fungsi dalam kelas ini. Selain dari mengkaji sifat fungsi geometri, ramai ahli matematik lain menunjukkan minat untuk melihat pada syarat yang mencukupi supaya fungsi istimewa tertentu mempunyai ciri geometri tertentu, seperti keunivalenan, kebak-bintangan atau kecembungan. Contoh fungsi istimewa yang telah dikaji termasuk fungsi Bessel dan Struve. Ini seterusnya menjadi motivasi untuk disertasi ini turut menyiasat syarat yang mencukupi untuk fungsi $T_{p,b,c}(z) = (f * g_{p,b,c})(z)$, konvolusi antara $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ dan $g_{p,b,c}(z) = z + \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}} z^n$ dengan $p, c, b \in \mathbb{C},\ k = p + \frac{b+2}{2} \neq 0, -1, -2, \dots$, bersifat univalen, bak-bintang atau cembung dalam kedua-dua cakera unit dan $|z| < \frac{1}{2}$.

Kata Kunci: Univalen, Hampir-Cembung, Umum Funsgi Struve, Fungsi terhadap Titik Simetri N-ply

ACKNOWLEDGEMENTS

I would like to use this opportunity to express my deepest appreciation to my supervisor, Professor Suzeini binti Abdul Halim, for teaching me the ropes of mathematical research and also for providing me encouragement during the course of my study. I would also like to thank Institute of Mathematical Sciences (ISM), Faculty of Science for providing me the necesssary facilities in order to complete my studies.

I also wish to express my gratitude to my family for their never ending support, whether it be financial or familial, to drive me onward and motivated me to never give up on achieving my goals.

I would also like to send my appreciation to the MyBrainSc 2015 Scholarship Program for providing me financial support to pursue my study.

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LIST OF SYMBOLS AND ABBREVIATIONS

arg	:	Argument.
$\mathcal{H}$	:	Class of all analytic functions in U.
$\mathcal{A}$	:	Class of all analytic functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in U$.
$\mathcal{A}_p$	:	Class of all p-valent analytic functions of the form $f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, z \in U$.
$\mathcal{K}$	:	Class of close-to-convex functions in $\mathcal{A}$.
$\mathcal{K}(\alpha)$	:	Class of close-to-convex functions of order α in $\mathcal{A}$.
$\mathcal{K}_s^n(\alpha)$	:	Class of close-to-convex functions with respect to n-ply symmetric point of order α in $\mathcal{A}$.
$\mathcal{K}_s$	:	Class of close-to-convex functions with respect to symmetric point in $\mathcal{A}$.
$\mathcal{K}_s(\alpha)$	:	Class of close-to-convex functions with respect to symmetric point of order α in $\mathcal{A}$.
$\mathcal{K}_p$	:	Class of close-to-convex p-valent functions in $\mathcal{A}_p$.
$\mathcal{K}_p(\alpha)$	:	Class of close-to-convex p-valent functions of order α in $\mathcal{A}_p$.
$\mathcal{K}_{s,p}^n(\alpha)$	:	Class of close-to-convex p-valent functions with respect to n-ply symmetric point of order α in $\mathcal{A}_p$.
$\mathcal{K}_{s,p}$	:	Class of close-to-convex p-valent functions with respect to symmetric point in $\mathcal{A}_p$.
$\mathcal{C}$	:	Class of convex functions in $\mathcal{A}$.
$\mathcal{C}(\alpha)$	:	Class of convex functions of order α in $\mathcal{A}$.
$\mathcal{C}_s^n(\alpha)$	:	Class of convex functions with respect to n-ply symmetric point of order α in $\mathcal{A}$.

$\mathcal{C}_s$	:	Class of convex functions with respect to symmetric point in $\mathcal{A}$.
$\mathcal{C}_s(\alpha)$	:	Class of convex functions with respect to symmetric point of order α in $\mathcal{A}$.
$\mathcal{C}_p$	:	Class of convex p-valent functions in $\mathcal{A}_p$.
$\mathcal{C}_p(\alpha)$	:	Class of convex p-valent functions of order α in $\mathcal{A}_p$.
$\mathcal{C}_{s,p}^n(\alpha)$	:	Class of convex p-valent functions with respect to n-ply symmetric point of order α in $\mathcal{A}_p$.
$\mathcal{C}_{s,p}$	:	Class of convex p-valent functions with respect to symmetric point in $\mathcal{A}_p$.
$\mathcal{P}$	:	Class of functions with positive real part in $\mathcal{A}$.
$\mathcal{S}$	:	Class of normalized univalent functions in $\mathcal{A}$.
$\mathcal{S}^{(k)}$	:	Class of normalized univalent k-fold symmetric functions in $\mathcal{A}$.
$\mathcal{S}^*$	:	Class of starlike functions in $\mathcal{A}$.
$\mathcal{S}^*(\alpha)$	:	Class of starlike functions of order α in $\mathcal{A}$.
$\mathcal{S}_c^*$	:	Class of starlike functions with respect to conjugate points in $\mathcal{A}$.
$\mathcal{S}_s^{*,n}(\alpha)$	:	Class of starlike functions with respect to n-ply symmetric point of order α in $\mathcal{A}$.
$\mathcal{S}_{sc}^*$	:	Class of starlike functions with respect to symmetric conjugate points in $\mathcal{A}$.
$\mathcal{S}_s^*$	:	Class of starlike functions with respect to symmetric point in $\mathcal{A}$.
$\mathcal{S}_s^*(\alpha)$	:	Class of starlike functions with respect to symmetric point of order α in $\mathcal{A}$.
$\mathcal{S}_p^*$	:	Class of starlike p-valent functions in $\mathcal{A}_p$.
$\mathcal{S}_p^*(\alpha)$	:	Class of starlike p-valent functions of order α in $\mathcal{A}_p$.
$\mathcal{S}_{s,p}^{*,n}(\alpha)$	:	Class of starlike p-valent functions with respect to n-ply symmetric point of order α in $\mathcal{A}_p$.
$\mathcal{S}_{s,p}^*$	:	Class of starlike p-valent functions with respect to symmetric point in $\mathcal{A}_p$.
$\in$	:	Element of.

Im	:	Imaginary part of complex numbers.
$\int$	:	Integral.
$\prod$	:	Product.
R_c	:	Radius of convexity.
$\Re$	:	Real part of complex numbers.
$\mathbb{C}$	:	Set of complex numbers.
$\mathbb{N}$	:	Set of natural numbers.
$\mathbb{R}$	:	Set of real numbers.
$\subset$	:	Subset of.
$\sum$	:	Summation.
U	:	Unit disc, $\|z\| < 1$.

CHAPTER 1: PRELIMINARIES

1.1 Introduction

Complex numbers are commonly expressed in the form $z = x + iy$, where $x, y \in \mathbb{R}$ with $\mathbb{R}$ being the set of all real numbers. The expression i is known as an imaginary number as it is a solution to the equation $x^2 = -1$ which no real number can be utilised to solve it. In essence, $i = \sqrt{-1}$. Furthermore, x is called the real part, i.e. $x = \Re(z)$, while y is called the imaginary part, i.e. $y = Im(z)$, of a complex number.

Complex analysis is one of the many branches of mathematical analysis known to the world. The formal theory of complex analysis was first established in the middle of 19th century by Augustin-Louis Cauchy. It is also known as the theory of functions of a complex variable. Since Cauchy, many mathematicians, notably Riemann, Weierstrass and many others, have rigorously worked with complex functions and the study of it. This branch investigates functions of complex variables together with their derivatives, manipulation and other properties. Complex analysis has many uses and it has been used by many branches of mathematics, including but not limited to number theory, algebraic geometry, geometric topology, differential geometry and applied mathematics. Other than its usage in the mathematics field, it is also commonly used in the physics field as complex analysis provides many practical applications and tools to the solution of physical problems. Some examples are the quantum field theory, string theory and two-dimensional Conformal Field Theory.

Let $\mathbb{C}$ be the set of complex numbers. A complex domain D, or domain for short, is any connected open subset of $\mathbb{C}$, i.e. $D \subset \mathbb{C}$. For example, the open upper half of the complex plane, $z = \{x+iy|y > 0; x, y \in \mathbb{R}\}$, and the open unit disc, $|z| = \{x^2+y^2 < 1; x, y \in \mathbb{R}\}$. A domain is simply connected if its complement is connected as well. Geometrically speaking, a simply connected domain is described as one without any hollow space in it. A complex function f is differentiable at $z_0 \in D$ if its derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists at z_0. A function f is said to be analytic at a point $z_0 \in D$ if every point of some neighbourhood of z_0 has a derivative. Furthermore, f is analytic in D if every point in D is complex differentiable. Analytic functions can sometimes be used interchangeably with holomorphic functions as holomorphic functions share a similar definition to that of analytic functions and also because all holomorphic functions are complex analytic functions and vice versa. The theory of holomorphic functions was developed mainly by Cauchy, Weierstrass and Riemann in the 19th century. If an analytic function provides a one-to-one mapping from a domain D onto its image $f(D)$, then that function is said to be univalent.

Definition 1.1.1. *(Goodman, 1983) A function f is said to be univalent in a domain D if the conditions*

$$f(z_1) = f(z_2),\ z_1, z_2 \in D,$$

implies that $z_1 = z_2$.

The study on the theory of univalent functions was first intiated by Koebe (1907). Following this study, many mathematicians and interested researchers began to study this theory extensively even till now and it has developed considerably. The theory of univalent functions is vast, however there are results that can simplify that scope. For one, the arbitary domain D can be replaced with the unit disc $U = \{|z| < 1; z \in \mathbb{C}\}$ by using the Riemann Mapping Theorem. This theorem was first suggested by Riemann in 1851 which states the following.

Theorem 1.1.1. *Let z_0 be any given point in a domain D where D is simply connected and is a proper subset of $\mathbb{C}$. Then there is a unique function f which maps D conformally (angle preserving) onto the unit disc U and has the properties $f(z_0) = 0$ and $f'(z_0) > 0$.*

Any function g that is analytic in the unit disc U has a Maclaurin expansion

$$g(z) = b_0 + b_1 z + b_2 z^2 + ... = \sum_{n=0}^{\infty} b_n z^n,$$

that is convergent in U. If g is univalent in U, then adding an arbitrary constant to the function, which merely translate the image domain, will still result the new function to be univalent in U. With this, we can confidently say that $g(z) - b_0$ is univalent in U. Moreover, if g is a univalent function in U, then $b_1 = g'(0) \neq 0$. Thus, we can divide $g(z)$ by b_1 and consider the function $f(z) = \frac{g(z)-b_0}{b_1}$. Setting $\frac{b_n}{b_1} = a_n$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, we obtained the following form

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1.1}$$

Note that if we let $z = 0$, then $f(0) = 0 = f'(0) - 1$. A function of the form (1.1.1) is said to be normalized. Thus, $f(0) = 0 = f'(0) - 1$ is normally used as conditions for a normalized analytic function.

Denote $\mathcal{H}$ to be the class of all analytic functions in the unit disc U. Now, we denote $\mathcal{A}$ to be the class of analytic functions of the form (1.1.1). Take note that the class $\mathcal{A}$ is a sublcass of class $\mathcal{H}$. The subclass of $\mathcal{A}$ consisting of functions that are univalent in U is denoted by $\mathcal{S}$.

The Koebe Function, given by the following,

$$k(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \frac{(1+z)^2-(1-z)^2}{4(1-z)^2} = \frac{z}{(1-z)^2}$$
$$= \sum_{n=1}^{\infty} nz^n, \tag{1.1.2}$$

was first introduced by Koebe. As this function is in class $\mathcal{S}$, it means that this function is analytic, normalized and univalent in U which is simple to be proven. Firstly, the Koebe function is analytic because it is complex differentiable at every $z \in U$. Secondly, the Koebe function is normalized in U as it satisfies the condition $k'(0) = 1$ and $k(0) = 0$ where $k'(z) = 1 + \sum_{n=2}^{\infty} n^2 z^{n-1}$. Finally, suppose $k(z_1) = k(z_2)$. Then

$$\frac{z_1}{(1-z_1)^2} = \frac{z_2}{(1-z_2)^2}, \ (z_1, z_2 \in U).$$

After some manipulation, the following is obtained

$$(z_1 - z_2)(1 - z_1 z_2) = 0.$$

As $z_1, z_2 \in U$, this implies that $|z_1| < 1$ and $|z_2| < 1$ which then implies that $|z_1 z_2| = |z_1||z_2| < 1$. Therefore, $1 - z_1 z_2 \neq 0$ in U which implies $z_1 - z_2 = 0$ which further implies $z_1 = z_2$. Therefore, the Koebe function, $k(z)$, is univalent in U. In geometrical sense, the Koebe function maps the unit disc U univalently onto the complex plane $\mathbb{C}$ excluding the slit along the negative real axis from $-\infty$ to $-\frac{1}{4}$. The Koebe function is a vital function in the study of class $\mathcal{S}$ especially in the study of determining coefficient bounds. It is usually used as the extremal function in most problems involving class $\mathcal{S}$.

In 1916, Bieberbach stated in his paper that the second coefficient a_2 of the functions f in class $\mathcal{S}$ is bounded by 2. He further expanded this in his paper by conjecturing that all coefficients a_n of functions in class $\mathcal{S}$ are no bigger than n according to their positions.

Conjecture 1.1.1. *(Bieberbach conjecture) The coefficients of all functions $f \in \mathcal{S}$ satisfy the inequality $|a_n| \leq n$ for each $n \geq 2$. The inequality is strict for every n unless f is a rotation of the Koebe function.*

An inequality is considered strict if substituting the signs of any “less than (<)" or “greater than (>)" with their equal counterparts, i.e. $\leq$ and $\geq$, never gives a true expression. To give an example, $a < b$ is a strict inequality while $a \leq b$ is not.

This conjecture inspired several developments in the field of complex function theory by imparting many new methods and generating a considerable number of related problems, some having been solved completely while others are still open. One such inspiration is the conjecture introduced by Robertson (1936b) in which he showed that it implies the Bieberbach conjecture.

Conjecture 1.1.2. *(Robertson conjecture) Let $h(z) = \sum_{n=1}^{\infty} b_{2n-1} z^{2n-1} \in \mathcal{S}$ with $b_1 = 1$. Then $\sum_{k=1}^{n} |b_{2k-1}|^2 \leq n, n = 2, 3, 4, \dots$ with $b_1 = 1$.*

Note that h is an odd function in $\mathcal{S}$ and that every odd function in $\mathcal{S}$ can be represented as a square root transfom, i.e.

$$f(z) = \sqrt{g(z^2)} = z + a_3 z^3 + a_5 z^5 + \dots ,$$

where $g \in \mathcal{S}$. The class of odd functions univalent in U is denoted by $\mathcal{S}^{(2)}$. Another conjecture was introduced by Milin (1977) where the inequality implies the correctness of the Robertson conjecture.

Conjecture 1.1.3. *(Milin conjecture) If $f \in \mathcal{S}$ and*

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} c_n z^n,$$

then

$$\sum_{k=1}^{n}(n-k+1)k|c_k|^2 \leq \sum_{k=1}^{n}\frac{n-k+1}{k},\ n = 1, 2, 3, \dots .$$

Since its introduction, the cases $n = 2, 3, 4, 5, 6$ of the Bieberbach conjecture has been proven by the following authors: Bieberbach (1916) for case $n = 2$, Loewner (1923) for case $n = 3$, Garabedian & Schiffer (1955) for case $n = 4$, Pederson (1968) and Ozawa (1969) for case $n = 6$, and Pederson & Schiffer (1972) for case $n = 5$. The general case, as stated in the Bieberbach conjecture $|a_n| \leq n$, was finally proven by de Branges (1985) with the usage of hypergeometric functions by solving the Milin conjecture which implies the Robertson conjecture which further implies the Bieberbach conjecture as stated before. In current times, the Bieberbach conjecture is synonymous with the de Branges Theorem.

Bieberbach's inequality, that is $|a_2| \leq 2$ for any $f \in \mathcal{S}$, can be used to prove other theorems about the class. One of the many theorems proven using Bieberbach's inequality is the Koebe quarter theorem (conjectured by Koebe in 1907).

Theorem 1.1.2. *The image of an injective function f from the unit disc U onto a subset of the complex plance $\mathbb{C}$ contains the disc centered at $f(0)$ and whose radius is $\frac{|f'(0)|}{4}$.*

Suppose that $f \in \mathcal{S}$. Then according to the theorem above the image of U under $f(z)$ must cover an open disc centered at the origin with radius $\frac{1}{4}$. Thus the theorem above can be refered to as a distortion theorem, as it tells us that in the class $\mathcal{S}$ there is a limit to the distortion of the boundary and it cannot be “pushed toward the origin" beyond the limit described in the theorem. As a direct consequence to both the Koebe quarter theorem and Bieberbach’s inequality, the Koebe distortion theorem was found which provides the sharp lower and upper bound of $|f'(z)|$ when $f \in \mathcal{S}$ (Pommerenke, 1975). The following theorem gives a basic estimate which leads to the distortion theorem which is applied to find the growth theorem, i.e. the upper and lower bounds of $|f(z)|$.

Theorem 1.1.3. *(Duren, 1983) For each $f \in \mathcal{S}$,*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}, \quad (|z| = r < 1).$$

Theorem 1.1.4. *(Duren, 1983) (Distortion Theorem) For each $f \in \mathcal{S}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad (|z| = r < 1).$$

Theorem 1.1.5. *(Duren, 1983) (Growth Theorem) For each $f \in \mathcal{S}$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad (|z| = r < 1).$$

All the theorems above share the common phrase “the inequality (or estimation) is sharp if and only if f is a suitable rotation of the Koebe function”. Stating that the inequality is sharp means that the inequality obtained has reached the theoratical maximum and there is no better result for that inequality. For example, the inequality $f(z) \leq g(z)$ is sharp for all $z \in U$ if there exists $z_0 \in U$ such that $f(z_0) = g(z_0)$. This is often followed by showing the extremal function that validates the inequality is sharp or equality holds or some variant of it.

1.2 Functions with Positive Real Part

Let $\mathcal{P}$ denote the class of functions of positive real part of the form $p(z) = 1 + p_1 z + p_2 z^2 + ... = 1 + \sum_{n=1}^{\infty} p_n z^n$ such that $\Re\{p(z)\} > 0$. Geometrically, this means that functions of class $\mathcal{P}$ maps the unit disc U onto the half-plane $H^* : \Re\{p(z)\} > 0$, which is the right half of the complex plane. It should be noted that functions in this class need not be univalent. Similar to how the Koebe function is an important function to class $\mathcal{S}$, there exists an important function in the class $\mathcal{P}$ known as the Möbius function as shown:

$$L_0(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n. \tag{1.2.1}$$

The Möbius function (first introduced by Möbius (1832)) is a function of class $\mathcal{P}$ that is univalent and analytic in U as well as maps U onto the half-plane H^*. An integral formula for functions in class $\mathcal{P}$ was found by Herglotz (1911).

The following lemma, known as the Carathéodory's theorem (Carathéodory, 1907), gives the coefficient bounds of functions in $\mathcal{P}$.

Lemma 1.2.1. *If* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ *with* $z \in U$, *then* $|p_n| \leq 2, n = 1, 2, 3, ...$. *This inequality is sharp for each* n.

With the sharp coefficient bounds provided by Carathéodory's theorem, the following theorem is easily proven.

Theorem 1.2.2. *(Goodman, 1983) If* $p \in \mathcal{P}$ *and* $z = re^{i\theta}$, *then*

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r} \tag{1.2.2}$$

and

$$|p'(z)| \leq \frac{2}{(1-r)^2}, \tag{1.2.3}$$

and these inequalities are sharp. Equality occurs for suitable z if and only if $f(z) = L_0(e^{i\alpha}z)$.

1.3 Subclasses of Univalent Functions

Problems of interest to researchers in this area include the study of functions with geometric properties of the image domain. Similar to how $\mathcal{S}$ is a subclass of $\mathcal{A}$, there are many subclasses of $\mathcal{S}$ which has been introduced and studied throughout the years. The classes of starlike, convex and close-to-convex functions are popular examples of subclasses of $\mathcal{S}$.

The geometric definition of a starlike function is given as follows:

Definition 1.3.1. *(Goodman, 1983) A domain $D \in \mathbb{C}$ is said to be starlike with respect to $w_0 \in D$ if each ray with inital point w_0 intersects the interior of D in a set that is either a ray or a line segment. We say that a function f is starlike with respect to w_0 if it maps the unit disc onto a domain that is starlike with respect to w_0. Furthermore, the function f is called a starlike function when $w_0 = 0$.*

A domain D is called a starlike domain when the domain is starlike with respect to $w_0 = 0$. We denote $\mathcal{S}^*$ to be the class of functions that are starlike in U. The following theorem gives an analytical definition of starlike functions which satisfy both a necessary and a sufficient condition of $f \in \mathcal{S}^*$.

Theorem 1.3.1. *(Duren, 1983) Let f be analytic in the unit disc U, with $f'(0) - 1 =$*

$f(0) = 0$. *Then* $f \in \mathcal{S}^*$ *if and only if*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0,\ (z \in U).$$

The Koebe function (1.1.2) is an example of a starlike function as

$$\begin{aligned}\Re\left\{\frac{zk'(z)}{k(z)}\right\} &= \Re\left\{\frac{z(1+z)}{(1-z)^3}\frac{(1-z)^2}{z}\right\} \\ &= \Re\left\{\frac{1+z}{1-z}\right\} \\ &> 0.\end{aligned}$$

The geometric definition of convex functions is given as follows.

Definition 1.3.2. *(Goodman, 1983) If for every pair w_1 and w_2 along with the line segment joining them are all within the interior of a domain $D \in \mathbb{C}$, then D is called a convex domain. A function f is called a convex function if f maps the unit disc U onto a convex domain.*

We denote $\mathcal{C}$ to be the class of convex functions in U. Similarly to the starlike function, the following theorem gives an analytical definition of convex functions which satisfy both a necessary and a sufficient condition of $f \in \mathcal{C}$.

Theorem 1.3.2. *(Duren, 1983) Let f be analytic in U, with $f'(0) - 1 = f(0) = 0$. Then $f \in \mathcal{C}$ if and only if*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,\ (z \in U).$$

The Möbius function (1.2.1) is a perfect example of a convex function as it maps U onto the half-plane H^* which can be categorized as a convex domain. The following is using the Möbious function to prove the inequality in Theorem 1.3.2.

$$\begin{aligned}\Re\left\{1+\frac{zL_0''(z)}{L_0'(z)}\right\} &= \Re\left\{1+\frac{2z}{1-z}\right\}\\ &= \Re\left\{\frac{1+z}{1-z}\right\}\\ &> 0.\end{aligned}$$

Theorem 1.3.1 and Theorem 1.3.2 showed that there is a connection between the starlike functions and the convex functions which was first observed by Alexander (1915). He showed his findings in the following theorem.

Theorem 1.3.3. *(Alexander's Theorem) Let* $f \in \mathcal{A}$ *in* U. *Then* $f \in \mathcal{C}$ *if and only if* $zf' \in \mathcal{S}^*$ *in the unit disc* U.

The Bieberbach conjecture holds for both the starlike function and the convex function. They were both proven by Nevanlinna (1921) and Loewner (1917), and the growth and distortion bounds for both starlike and convex functions can be found in Goodman (1983). The bounds for the distortion and growth theorem for convex functions were obtained independently by Loewner (1917) and Gronwall (1916).

Robertson (1936a) generalized the classes $\mathcal{S}^*$ and $\mathcal{C}$ by introducing the classes of functions that are starlike and convex of order α for $0 \leq \alpha < 1$. In doing so, he introduced the concept of functions of order α that is still used by many authors today. The classes of functions starlike and convex of order α are denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ respectively.

Definition 1.3.3. *(Goodman, 1983) A function* $f \in \mathcal{A}$ *is in* $\mathcal{S}^*(\alpha)$ *if*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \ (0 \leq \alpha < 1, z \in U).$$

Definition 1.3.4. *(Goodman, 1983) A function* $f \in \mathcal{A}$ *is in* $\mathcal{C}(\alpha)$ *if*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ (0 \leq \alpha < 1, z \in U).$$

Take note that the function in either class $\mathcal{S}^*(\alpha)$ or class $\mathcal{C}(\alpha)$ may not be univalent if $\alpha < 0$ where as if $\alpha \geq 1$ then the classes are empty sets as the inequalities in the definition above will be not be satisfied at $z = 0$. Evidently when $\alpha = 0$, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ are obtained. Furthermore, using a result analogous to Alexander's theorem, $f \in \mathcal{C}(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$ for $\alpha \in [0, 1)$.

As α increases, both classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ become smaller. The geometrical interpretation of the notion of convexity of order α, for $\alpha \in [0, 1)$, is that "the ratio of the angle between two adjacent tangents to the unit circle between the two corresponding tangents of the image of the unit circle is less than $\frac{1}{\alpha}$ and comes arbitarily close to $\frac{1}{\alpha}$ for some point of the unit circle" according to Robertson (1936a). The closer α is to 1, the image of the unit circle becomes more circular. Unfortunately, there is no clear geometric interpretation for the class $\mathcal{S}^*(\alpha)$ for $\alpha \in [0, 1)$.

The coefficient estimate along with the distortion and growth bounds for these two classes have been proven by Robertson (1936a).

The class of close-to-convex functions, introduced by Kaplan (1952), is a key subclass of univalent functions analytic in the unit disc with the following definition.

Definition 1.3.5. *A function* $f \in \mathcal{A}$ *in the unit disc is said to be close-to-convex if there* *exists a convex function g in U such that*

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0, \ (z \in U). \tag{1.3.1}$$

The class of functions f that are close-to-convex and are normalized by the usual conditions $f'(0) - 1 = f(0) = 0$ is denoted by $\mathcal{K}$. Upon observing 1.3.1, it is simple to state that all convex functions are close-to-convex function in U. To show this, suppose f is convex in U. Then by selecting $g = f$ in (1.3.1), we have

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} = \Re\left\{\frac{f'(z)}{f'(z)}\right\} = 1 > 0.$$

Similarly, it can easily be deduced that every starlike function is close-to-convex. Suppose $h(z) = zg'(z)$ is a starlike function in U where $g \in \mathcal{C}$ (this is possible because of Alexander's theorem). Then (1.3.1) can be rewritten into the form

$$\Re\left\{\frac{zf'(z)}{h(z)}\right\} > 0, \ (z \in \ U).$$

If $f \in \mathcal{S}^*$, then we let $h(z) = f(z)$ in the above formula and we have the following:

$$\Re\left\{\frac{zf'(z)}{h(z)}\right\} = \Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

Hence, every starlike function is close-to-convex in U.

With these remarks, we can summarized the subclasses in the following chain of inclusions:

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K}.$$

Just by looking at the chain above, it is simple to state that the Koebe function is an example of a close-to-convex function. However, since we know that $\mathcal{S}^* \subset \mathcal{K}$ from the inclusion above and $\mathcal{S}^* \subset \mathcal{S}$ from the beginning, naturally the next question to ask is whether the functions in class $\mathcal{K}$ are univalent, or in other words, whether the class $\mathcal{K}$ is univalent as a whole. Noshiro (1934) and Warschawski (1935) found a simple criterion for analytic functions to be univalent which is shown in the following theorem.

Theorem 1.3.4. *(Noshiro-Warschawski Theorem) A function f is univalent in D if f is analytic in a convex domain D and $\Re\{f'(z)\} > 0$.*

With the Noshiro-Warschawski theorem, Kaplan showed that all close-to-convex functions are indeed univalent.

Theorem 1.3.5. *(Kaplan, 1952) Every close-to-convex function is univalent.*

Therefore with Theorem 1.3.5 and the previous inclusion, the following inclusion is obtained.

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}.$$

Reade (1955) proved that the Bieberbach conjecture holds true for functions in class $\mathcal{K}$.

Theorem 1.3.6. *If $f \in \mathcal{K}$ of the form (1.1.1), then $|a_n| \leq n$ for all n. The inequality is sharp with Koebe function as the extremal function.*

The sharp distortion and growth bounds for the class $\mathcal{K}$ were proven by Libera (1964).

Similar to the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, the class $\mathcal{K}$ can be generalized to the class $\mathcal{K}(\alpha)$ consisting of functions that are close-to-convex of order α, $0 \leq \alpha < 1$. The following theorem gives an analytical definition for $f \in \mathcal{K}(\alpha)$.

Theorem 1.3.7. *(Libera, 1964) Let function $f \in \mathcal{A}$. Then $f \in \mathcal{K}(\alpha)$ for $0 \leq \alpha < 1$ if there exists a function $g \in \mathcal{S}^*$ such that*

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha, \ (z \in U).$$

Earlier in the chapter, we briefly mentioned the class $\mathcal{S}^{(2)}$. This class can be generalized to the class $\mathcal{S}^{(k)}$ where k is a positive integer number which consists of functions f that are the k-th root transforms of functions g that are univalent in U, i.e.

$$f(z) = \big(g(z^k)\big)^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} a_{nk+1} z^{nk+1}. \tag{1.3.2}$$

The above power series presentation was shown in Gronwall (1916) if f is analytic and k-fold symmetric in U. The converse is also true. The transformation $u = z^k$ maps U onto k copies of U, and $g(u)$ carries this surface onto k copies of $g(U)$ joined by a suitable branch point at $w = 0$. Note that, the k-th root merely "unwinds" the surface to form a plane domain with k-fold symmetry. With this, we can say that all odd univalent functions are 2-fold symmetric functions. The geometric definition of a k-fold symmetric function is given as follows.

Definition 1.3.6. *(Goodman, 1983) Let $k \in \mathbb{N}$. If a domain D rotates onto itself about the origin through an angle $\frac{2\pi}{k}$, then D is said to be k-fold symmetric. If a function f satisifes the equality*

$$f\left(e^{\frac{2\pi i}{k}} z\right) = e^{\frac{2\pi i}{k}} f(z), \tag{1.3.3}$$

for every z in U, then f is said to be k-fold symmetric in U.

A class of normalized univalent k-fold symmetric functions is denoted by $\mathcal{S}^{(k)}$. Noshiro (1934) provided the coefficient estimates and sharp bounds for starlike functions that are k-fold symmetric and by utilizing the Alexander's theorem, Noshiro also found the coefficient estimate for convex k-fold symmetric functions.

Golusin (1929), Robertson (1936a) and Noshiro (1934) pioneered the study of k-fold symmetric functions starlike in the unit disc in the early 1930's with each of them establishing coefficient bounds for the functions. For more information on these functions, refer to Padmanabhan (1968), Janowski (1970), McCarty (1974) and Anh (1985).

1.4 Subclasses of Multivalent Functions

Upon first look at the term "multivalent" of a multivalent function, it is obvious to see that it is a natural generalization of a univalent function as the prefix "multi" means many while the the prefix "uni" means one and as with univalent functions, multivalent functions also play an important role in complex analysis. The following definition is taken from Goodman (1979) to describe p-valent functions (multivalent functions of order p where $p \in \mathbb{N}$).

Definition 1.4.1. *A function f is said to be p-valent in a domain D if it assumes any value in D do not apper more than p times and there is some w_0 such that $f(z) = w_0$ has exactly p solutions in D, when roots are counted in accordance with their multiplicities.*

In terms of geometry, this means at most p points lie above each point of the w_0-plane of the Riemann surface into which $w_0 = f(z)$ maps D.

Let $\mathcal{A}_p$ denote the class of all analytic functions f that are p-valent in the unit disc U and are of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}. \tag{1.4.1}$$

For $p = 1$, $\mathcal{A}_1 = \mathcal{A}$.

Similar to univalent functions, multivalent functions have been studied by various authors in order to determine its properties and also to see how it behaves under certain geometrical limitations. Goodman (1950) introduced the classes of p-valent functions starlike and convex in the unit disc U, denoted by $\mathcal{S}_p^*$ and $\mathcal{C}_p$ respectively, whereas Livingston (1965) introduced the class of p-valent function close-to-convex in U and is denoted by $\mathcal{K}_p$.

Definition 1.4.2. *Let $f \in \mathcal{A}_p$ where $p \in \mathbb{N}$. Then $f \in \mathcal{S}_p^*$ if $f(0) = 0$ and satisfies the inequality*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \ (z \in U).$$

Definition 1.4.3. *Let $f \in \mathcal{A}_p$ where $p \in \mathbb{N}$. Then $f \in \mathcal{C}_p$ if $f(0) = 0$ and it satisfies the inequality*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \ (z \in U).$$

Definition 1.4.4. *Let $f \in \mathcal{A}_p$ where $p \in \mathbb{N}$. Then $f \in \mathcal{K}_p$ if there exists $g \in \mathcal{S}_p^*$ such that*

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \ (z \in U).$$

These classes were expanded upon by Patil & Thakare (1983), Owa (1985) and Aouf (1988) where they explored the classes of functions that are p-valent of order α starlike, convex and close-to-convex, denoted by $\mathcal{S}_p^*(\alpha)$, $\mathcal{C}_p(\alpha)$ and $\mathcal{K}_p(\alpha)$ respectively, in U.

Definition 1.4.5. *Let $f \in \mathcal{A}_p$. Then $f \in \mathcal{S}_p^*(\alpha)$ for $0 \leq \alpha < p$ if it satsifies*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \ (z \in U).$$

Definition 1.4.6. *Let $f \in \mathcal{A}_p$. Then $f \in \mathcal{C}_p(\alpha)$ for $0 \leq \alpha < p$ if it satsifies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ (z \in U).$$

Definition 1.4.7. *Let $f \in \mathcal{A}_p$. Then $f \in \mathcal{K}_p^*(\alpha)$ for $0 \leq \alpha < p$ if there exists $g \in \mathcal{S}_p^*(\alpha)$ such that it satisfies*

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha, \ (z \in U).$$

The sharp bounds for classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ were proven by Goluzina (1974) and Aouf (1988) respectively and the coefficient estimates for classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ were detailed by Aouf in his 1987 and 1988 papers respectively.

Analagous to the Alexander theorem, Ali, Ravichandran, & Lee (2009) found that $f \in \mathcal{C}_p(\alpha)$ if and only if $\frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha)$.

1.5 Scope of Dissertation

In this section, the scope of dissertation is outlined. This dissertation is divided into four chapters ending with a list of references used for the purpose of this dissertation.

The dissertation begins with an introductory chapter as its first chapter. It gives a brief introduction of the general theory of univalent functions and multivalent functions that provides the basic understanding of the geometric function theory of a complex variable in preparation of the work and results presented in the following chapters.

Chapter 2 starts with a brief background on the class of starlike functions with respect to symmetry point introduced by Sakaguchi and the classes that closely resemble to it or the subclass of it. This chapter introduces the class $\mathcal{K}_s^{k,N}$ where $k, N \in \mathbb{N}$ which is a subclass of $\mathcal{K}$ and shows the properties found for the class $\mathcal{K}_s^{k,N}$. This chapter will also briefly look into the subclass of close-to-convex p-valent functions, denoted as $\mathcal{K}_{s,p}^{k,N}$, and its properties.

Chapter 3 is devoted to studying the sufficient conditions for a special function $T_{p,b,c}$, which is a convolution of the function f of the form (1.1.1) and g_p studied by Orhan & Yağmur (2014), to be under certain geometric properties. The chapter begins with briefly discussing the background of Struve function, a special function, followed by presenting the conditions for the function $T_{p,b,c}$ to be under certain geometric properties, such as univalency, starlikeness and convexity.

Finally, chapter 4 will discussed the results obtained in Chapter 2 and Chpater 3, each with their own section, as well as give some ideas for future problems to be looked into.

CHAPTER 2: PROPERTIES OF SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

2.1 Introduction

In the classes mentioned in Chapter 1, the denominator function in the analytical definitions consists only of single functions, such as f for the class $\mathcal{S}^*$, f' for the class $\mathcal{C}$ and g' or g, depending if g is convex or starlike respectively, for the class $\mathcal{K}$. However, the question arises as to what sort of geometric properties we will receive if we replace these single functions with function F that are composed of two functions that are univalent in U, for example $F(z) = f(z) - f(-z)$. This concept was first explored by Sakaguchi (1959).

Sakaguchi introduced starlike functions with respect to symmetric points with the following definition.

Definition 2.1.1. *A function f, regular in the unit disc U, is said to be starlike with respect to symmetric points if for every ζ on $|z| = r$ and every r is not more than 1, i.e. $r < 1$, the angular velocity of f is positive about the point $f(-\zeta)$ at $z = \zeta$ as z travels along the circle $|z| = r$ in the positive direction.*

The angular velocity (rate of change of angle) of $f(z)$ about $f(-\zeta)$ is

$$\begin{aligned}\frac{d}{dt}\arg\big(f(z)-f(-\zeta)\big) &= \frac{d}{dt}Im\Big\{\ln\big(f(z)-f(-\zeta)\big)\Big\}\\ &= Im\Big\{\frac{d}{dt}\ln\big(f(z)-f(-\zeta)\big)\Big\}\\ &= Im\Big\{\frac{d}{dz}\ln\big(f(z)-f(-\zeta)\big)\frac{dz}{dt}\Big\}\\ &= Im\Big\{\frac{f'(z)}{f(z)-f(-\zeta)}\frac{dz}{dt}\Big\}.\end{aligned}$$

As z is on the circle $|z| = r$, then $z = x + iy = r(\cos t + i\sin t) = re^{it}$ for $0 \le t < 2\pi$, which upon differentiating gives $\frac{dz}{dt} = ire^{it} = iz$. Substituting this into the equation above gives

$$Im\left\{\frac{f'(z)}{f(z)-f(-\zeta)}\frac{dz}{dt}\right\} = Im\left\{iz\frac{f'(z)}{f(z)-f(-\zeta)}\right\} = \Re\left\{\frac{zf'(z)}{f(z)-f(-\zeta)}\right\}.$$

According to Definition 2.1.1, the angular velocity of f about the point $f(-\zeta)$ is positive when z traverses the circle $|z| = r$ in the positive direction, thus by letting $z = \zeta$,

$$\Re\left\{\frac{\zeta f'(\zeta)}{f(\zeta)-f(-\zeta)}\right\} > 0,$$

and since ζ is arbitary, we let $\zeta = z$,

$$\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0.$$

The class of functions that are starlike with respect to symmetric points is denoted by $\mathcal{S}_s^*$. The following theorem gives an analytical definition for functions to be in $\mathcal{S}_s^*$ which satisfy both the necessary and sufficient conditions.

Theorem 2.1.1. *(Sakaguchi, 1959) Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}_s^*$ if and only if*

$$\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0, \ (z \in U). \tag{2.1.1}$$

In the sufficient part of the proof of Theorem 2.1.1, Sakaguchi showed that $f(z) - f(-z) \in \mathcal{S}^*$. He does this by first substituting $-z$ into z in (2.1.1),

$$\Re\left\{\frac{(-z)f'(-z)}{f(-z)-f(z)}\right\} > 0, \ (z \in U),$$

then adding this to (2.1.1) gives

$$\begin{aligned}&\Re\left\{\frac{(-z)f'(-z)}{f(-z)-f(z)}\right\}+\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}\\&=\Re\left\{\frac{z\big(f'(z)+f'(-z)\big)}{f(z)-f(-z)}\right\}\\&=\Re\left\{\frac{z\big(f(z)-f(-z)\big)'}{f(z)-f(-z)}\right\}>0,\ (z\in U).\end{aligned}$$

As a consequence of this, functions f in the class $\mathcal{S}_s^*$ are close-to-convex in U.

The class $\mathcal{S}_s^*$ includes the classes of odd functions starlike with respect to the origin and convex functions. As an example, suppose $f \in \mathcal{A}$ is an odd starlike function, then letting $k = 2$ in (1.3.3), $f(-z) = -f(z)$ and $\Re\{\frac{zf'(z)}{f(z)}\} > 0$. Thus,

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}=2\Re\left\{\frac{zf'(z)}{2f(z)}\right\}=2\Re\left\{\frac{zf'(z)}{f(z)+f(z)}\right\}=2\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0.$$

Therefore, the class of odd functions starlike with respect to the origin is in the class $\mathcal{S}_s^*$. With all these remarks, the following inclusion is made

$$\mathcal{C}\subset\mathcal{S}_s^*\subset\mathcal{K}\subset\mathcal{S}.$$

The coefficient estimate for this class is shown in the following theorem.

Theorem 2.1.2. *Let* $f(z) = z + \sum_{n=2}^{\infty} \in \mathcal{S}_s^*$ *in* $|z| < 1$. *Then*

$$|a_n|\le 1,\ n\ge 2,$$

with equality being attained with the function $\frac{z}{1+\varepsilon z},\ |\varepsilon| = 1$.

Taking a step further, Sakaguchi generalized the class $\mathcal{S}_s^*$ in the following theorem.

Theorem 2.1.3. *Let $f \in \mathcal{A}$ and suppose that for $k \in \mathbb{N}$ there holds for the inequality*

$$\Re\left\{\frac{zf'(z)}{\sum_{v=0}^{k-1}\varepsilon^{-v}f(\varepsilon^{v}z)}\right\},\ (z \in U),$$

where $\varepsilon = e^{\frac{2\pi i}{k}}$. Then f is close-to-convex and univalent in U.

Robertson (1961), Stankiewicz (1965), Tun (1987), and Owa, Wu, & Ren (1988) has considered the class $\mathcal{S}_s^*$ in their papers. Following this introduction, several authors have also introduced classes that have either generalize the class $\mathcal{S}_s^*$ or are analogous to it.

Das & Singh (1977) introduced the class of convex functions with respect to symmetric points and the class of close-to-convex functions with respect to symmetric points, denoted by $\mathcal{C}_s$ and $\mathcal{K}_s$ respectively, with the following definitions.

Definition 2.1.2. *Let $f \in \mathcal{A}$. Then $f \in \mathcal{C}_s$ if and only if*

$$\Re\left\{\frac{\big(zf'(z)\big)'}{\big(f(z)-f(-z)\big)'}\right\} > 0,\ (z \in U).$$

Definition 2.1.3. *Let $f \in \mathcal{A}$. Then $f \in \mathcal{K}_s$ if there exists $g \in \mathcal{C}_s$ and univalent such that*

$$\Re\left\{\frac{f'(z)}{g'(z)+g'(-z)}\right\} > 0,\ (z \in U).$$

They proceeded to state that functions f in class $\mathcal{K}_s$ are univalent since as $g \in \mathcal{C}_s$, then $G(z) = \frac{1}{2}\big[g(z)-g(-z)\big] \in \mathcal{C}$ and $zG' \in \mathcal{S}^*$, which implies that $f \in \mathcal{K} \subset \mathcal{S}$. Das & Singh remarked that

$$f \in \mathcal{C}_s \iff zf' \in \mathcal{S}_s^*,$$

and hence, an alternative representation for the inequality in $\mathcal{K}_s$ is

$$\Re\left\{\frac{zf'(z)}{h(z)-h(-z)}\right\} > 0, \ (z \in U),$$

where $h \in \mathcal{S}_s^*$. Das & Singh also detailed the coefficient estimates of the classes $\mathcal{C}_s$ and $\mathcal{K}_s$ in their paper

Singh (1977) generalized the classes $\mathcal{S}_s^*$, $\mathcal{C}_s$ and $\mathcal{K}_s$ by introducing the classes of starlike, convex and close-to-convex functions with respect to symmety point of order α, denoted by $\mathcal{S}_s^*(\alpha)$, $\mathcal{C}_s(\alpha)$ and $\mathcal{K}_s(\alpha)$ respectively, for $0 \leq \alpha < 1$ with the following defintions.

Definition 2.1.4. *A function $f \in \mathcal{A}$ is in class $\mathcal{S}_s^*(\alpha)$ for $0 \leq \alpha < 1$ if it satisfies*

$$\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > \alpha, \ (z \in U).$$

Definition 2.1.5. *A function $f \in \mathcal{A}$ is in class $\mathcal{C}_s(\alpha)$ for $0 \leq \alpha < 1$ if it satisfies*

$$\Re\left\{\frac{\big(zf'(z)\big)'}{\big(f(z)-f(-z)\big)'}\right\} > \alpha, \ (z \in U).$$

Definition 2.1.6. *A function $f \in \mathcal{A}$ is in class $\mathcal{K}_s(\alpha)$ for $0 \leq \alpha < 1$ if it satisfies*

$$\Re\left\{\frac{f'(z)}{g'(z)+g'(-z)}\right\} > \alpha, \ (z \in U),$$

where $g \in \mathcal{C}_s(0)$.

Similar to before, Singh remarked that

$$f \in \mathcal{C}_s(\alpha) \iff zf' \in \mathcal{S}_s^*(\alpha),$$

which gives an alternate formulation to the inequality in $\mathcal{K}_s(\alpha)$, i.e.

$$\Re\left\{\frac{zf'(z)}{h(z)-h(-z)}\right\} > \alpha,\ (z \in U),$$

where $h \in \mathcal{S}_s^*(0)$. Following this, Singh proceeded to prove that functions in $\mathcal{C}_s(\alpha)$ and $\mathcal{K}_s(\alpha)$ are univalent in U. Proving that the functions in $\mathcal{S}_s^*(\alpha)$ are univalent in U was not needed as the necessary and sufficient conditions to be so was already derived by Sakaguchi (see Sakaguchi, 1959, p. 73). Singh has also detailed the coefficient estimates, distortion and growth theorems for functions in these classes.

Chand & Singh (1979) further generalize the classes $\mathcal{S}_s^*(\alpha)$, $\mathcal{C}_s(\alpha)$ and $\mathcal{K}_s(\alpha)$ by using the concept of functions with respect to n-ply symmetric points for $n \in \mathbb{N}$, thus introducing the classes of functions that are starlike, convex and close-to-convex with respect to n-ply symmetric points of order $\alpha \in [0,1)$ with $n \in \mathbb{N}$, denoted by $\mathcal{S}_s^{*,n}(\alpha)$, $\mathcal{C}_s^n(\alpha)$ and $\mathcal{K}_s^n(\alpha)$ respectively. The following are the definitons for the aforementioned classes.

Definition 2.1.7. *A function $f \in \mathcal{A}$ is in $\mathcal{S}_s^{*,n}(\alpha)$ if it satisifes the condition*

$$\Re\left\{\frac{zf'(z)}{f_n(z)}\right\} > \alpha,\ (z \in U),$$

where for $w = e^{\frac{2\pi i}{n}}$

$$f_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} w^{-k} f(w^k z).$$

Definition 2.1.8. *A function $f \in \mathcal{A}$ is in $\mathcal{C}_s^n(\alpha)$ if it satisifes*

$$\Re\left\{\frac{(zf'(z))'}{f_n'(z)}\right\} > \alpha,\ (z \in U).$$

where for $w = e^{\frac{2\pi i}{n}}$

$$f_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} w^{-k} f(w^k z).$$

Definition 2.1.9. *A function* $f \in \mathcal{A}$ *is in* $\mathcal{K}_s^n(\alpha)$ *corresponding to the function* $g \in \mathcal{C}_s^n(0)$ *if it satisifes*

$$\Re\left\{\frac{f'(z)}{g_n'(z)}\right\} > \alpha,\ (z \in U),$$

where for $w = e^{\frac{2\pi i}{n}}$

$$g_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} w^{-k} g(w^k z).$$

As it can be observed, when $\alpha = 0$ and $n = 1$, $\mathcal{S}_s^{*,1}(0) = \mathcal{S}_s^*$ was studied by Sakaguchi while $\mathcal{C}_s^1(0) = \mathcal{C}_s$ and $\mathcal{K}_s^1(0) = \mathcal{K}_s$ were both studied by Das & Singh. This concept was first used by Sakaguchi as it can be seen in Theorem 2.1.3 yet in Chand & Singh they multiplied the denominator in Theorem 2.1.3 with $\frac{1}{n}$ for normalization purposes.

It must be noted that $f_n \in \mathcal{S}^*(\alpha)$ when $f \in \mathcal{S}_s^{*,n}(\alpha)$. To show this, suppose that $f \in \mathcal{S}_s^{*,n}(\alpha)$, then $f_n(z) = \frac{1}{n}\sum\limits_{k=0}^{n-1} w^{-k} f(w^k z)$ for $w^n = 1$. Differentiating $f_n(z)$ gives us

$$f_n'(z) = \frac{1}{n}\sum_{k=0}^{n-1} f'(w^k z).$$

Thus,

$$\begin{aligned}
\frac{zf_n'(z)}{f_n(z)} &= \frac{z\left[\frac{1}{n}\sum\limits_{k=0}^{n-1} f'(w^k z)\right]}{f_n(z)} \\
&= \frac{1}{n}\left[\frac{zf'(z)}{f_n(z)} + \frac{zf'(wz)}{f_n(z)} + ... + \frac{zf'(w^{n-1}z)}{f_n(z)}\right] \\
&= \frac{1}{n}\left[\frac{zf'(z)}{f_n(z)} + \frac{(wz)f'(wz)}{f_n(wz)} + ... + \frac{(w^{n-1}z)f'(w^{n-1}z)}{f_n(w^{n-1}z)}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Re\left\{\frac{zf_n'(z)}{f_n(z)}\right\} &= \Re\left\{\frac{1}{n}\left[\frac{zf'(z)}{f_n(z)} + \frac{(wz)f'(wz)}{f_n(wz)} + ... + \frac{(w^{n-1}z)f'(w^{n-1}z)}{f_n(w^{n-1}z)}\right]\right\} \\
&= \frac{1}{n}\left[\Re\left\{\frac{zf'(z)}{f_n(z)}\right\} + \Re\left\{\frac{(wz)f'(wz)}{f_n(wz)}\right\} + ... + \Re\left\{\frac{(w^{n-1}z)f'(w^{n-1}z)}{f_n(w^{n-1}z)}\right\}\right] \\
&> \frac{1}{n}(n\alpha) \\
&= \alpha,
\end{aligned}$$

which implies that $f_n \in \mathcal{S}^*(\alpha)$.

Chand & Singh found the distortion theorems as well as the coefficient estimates for functions in these classes.

The class of starlike functions with respect to conjugate points, denoted by $\mathcal{S}_c^*$, and the class of starlike functions with respect to symmetric conjugate points, denoted by $\mathcal{S}_{sc}^*$, were introduced by El-Ashwah & Thomas (1987) with the following definitions.

Definition 2.1.10. *A function $f \in \mathcal{A}$ is in $\mathcal{S}_c^*$ if and only if*

$$\Re\left\{\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right\} > 0, \ (z \in U).$$

Definition 2.1.11. *A function $f \in \mathcal{A}$ is in $\mathcal{S}_{sc}^*$ if and only if*

$$\Re\left\{\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right\} > 0, \ (z \in U).$$

As it can be seen from the definitions above, these classes are analogous to the class $\mathcal{S}_s^*$. In the recent and current years, these classes have been gaining the attention of many authors and have been continuously investigated. Example authors that have investigated these classes are Abdul Halim (1991), Ravichandran (2004), Kahirnar & Rajas (2010), Karthikeyan (2013), and Selvaraj, Thirupathi, & Stelin (2014).

In 2005, Gao & Zhou introduced another concept of a class of functions that is similar to the class $\mathcal{S}_s^*$ which they denoted as $\mathcal{K}_s$ (not to be confused with the class $\mathcal{K}_s$ by Das & Singh) with the following definition.

Definition 2.1.12. *Let $f \in \mathcal{A}$. Then $f \in \mathcal{K}_s$ if there exists $g \in \mathcal{S}^*(\frac{1}{2})$ such that*

$$\Re\left\{\frac{-z^2 f'(z)}{g(z)g(-z)}\right\} > 0, \; (z \in U).$$

Gou & Zhou's idea was to replace the denominator in (2.1.1) with $\frac{-g(z)g(-z)}{z}$ with $\frac{1}{z}$ being the normalization factor. To show that f is univalent, assume that g is starlike of order $\frac{1}{2}$, i.e. $g \in \mathcal{S}^*(\frac{1}{2})$. Then differentiating $G(z) = \frac{-g(z)g(-z)}{z}$ gives

$$G'(z) = \frac{g(z)g(-z) - z\big[g'(z)g(-z) - g(z)g(-z)\big]}{z^2}.$$

Thus,

$$\begin{aligned}
\Re\left\{\frac{zG'(z)}{G(z)}\right\} &= \Re\left\{\frac{z[g'(z)g(-z) - g(z)g'(-z)] - g(z)g(-z)}{g(z)g(-z)}\right\} \\
&= \Re\left\{\frac{zg'(z)}{g(z)}\right\} + \Re\left\{\frac{(-z)g'(-z)}{g(-z)}\right\} - 1 \\
&> \frac{1}{2} + \frac{1}{2} - 1 \\
&= 0.
\end{aligned}$$

Therefore, function $G(z) = \frac{-g(z)g(-z)}{z}$ is starlike. This shows that f is close-to-convex which implies that f is univalent. They have also detailed the coefficient estimate, distortion and growth theorems, and radius of convexity as their findings for the class $\mathcal{K}_s$.

Following a similar concept, Wang, Gao, & Yuan (2006b) introduced the class $\mathcal{K}_s(\alpha, \beta)$ as a generalization of $\mathcal{K}_s$ which defines as follows.

Definition 2.1.13. *Let $\mathcal{K}_s(\alpha, \beta)$, for $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, denote the class of univalent and analytic functions that satisfy the inequality*

$$\left|\frac{-z^2 f'(z)}{g(z)g(-z)} - 1\right| < \beta\left|\frac{\alpha - z^2 f'(z)}{g(z)g(-z)} + 1\right|, \ (z \in U),$$

where $g \in \mathcal{S}^*(\frac{1}{2})$.

The class $\mathcal{K}_s(\alpha, \beta)$ was further generalized by Wang et al. (2006a) as they introduced the class $\mathcal{K}_s^{(k)}(\alpha, \beta)$ which defines as follows.

Definition 2.1.14. *Let $\mathcal{K}_s^{(k)}(\alpha, \beta)$, for $k \in \mathbb{N}$, $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, denote the class of univalent and analytic functions that satisfy the inequality*

$$\left|\frac{z^k f'(z)}{g_k(z)} - 1\right| < \beta\left|\frac{\alpha z^k f'(z)}{g_k(z)} + 1\right|, \ (z \in U),$$

where $g \in \mathcal{S}^*(\frac{k-1}{k})$ and $g_k(z)$ is defined by the following equality for $\varepsilon^k = 1$

$$g_k(z) = \prod_{v=0}^{k-1} \varepsilon^{-v} g(\varepsilon^v z).$$

Similar to before, $f \in \mathcal{K}_s^{(k)}(\alpha, \beta)$ is univalent. To show this, like before, we assume that $g \in \mathcal{S}^*(\frac{k-1}{k})$ where $k \in \mathbb{N}$. Then differentiating $G_k(z) = \frac{g_k}{z^{k-1}} = \frac{1}{z^{k-1}} \prod\limits_{v=0}^{k-1} \varepsilon^{-v} g(\varepsilon^v z)$

gives

$$\begin{aligned}
G_k'(z) &= \frac{-(k-1)}{z^k}\prod_{v=0}^{k-1}\varepsilon^{-v}g(\varepsilon^v z) + \frac{g'(z)}{z^{k-1}g(z)}\prod_{v=0}^{k-1}\varepsilon^{-v}g(\varepsilon^v z) \\
&\quad + \frac{g'(\varepsilon z)}{z^{k-1}\varepsilon^{-1}g(\varepsilon z)}\prod_{v=0}^{k-1}\varepsilon^{-v}g(\varepsilon^v z) + ... + \frac{g'(\varepsilon^{k-1}z)}{z^{k-1}\varepsilon^{-(k-1)}g(\varepsilon^{k-1}z)}\prod_{v=0}^{k-1}\varepsilon^{-v}g(\varepsilon^v z) \\
&= \frac{1}{z^k}\left[\frac{zg'(z)}{g(z)} + \frac{zg'(\varepsilon z)}{\varepsilon^{-1}g(\varepsilon z)} + ... + \frac{zg'(\varepsilon^{k-1}z)}{\varepsilon^{-(k-1)}g(\varepsilon^{k-1}z)} - (k-1)\right]\prod_{v=0}^{k-1}\varepsilon^{-v}g(\varepsilon^v z) \\
&= \frac{1}{z^k}\left[\frac{zg'(z)}{g(z)} + \frac{(\varepsilon z)g'(\varepsilon z)}{g(\varepsilon z)} + ... + \frac{(\varepsilon^{k-1}z)g'(\varepsilon^{k-1}z)}{g(\varepsilon^{k-1}z)} - (k-1)\right]\prod_{v=0}^{k-1}\varepsilon^{-v}g(\varepsilon^v z).
\end{aligned}$$

Thus,

$$\begin{aligned}
\Re\left\{\frac{zG_k'(z)}{G_k(z)}\right\} &= \Re\left\{\frac{zg'(z)}{g(z)} + \frac{(\varepsilon z)g'(\varepsilon z)}{g(\varepsilon z)} + ... + \frac{(\varepsilon^{k-1}z)g'(\varepsilon^{k-1}z)}{g(\varepsilon^{k-1}z)} - (k-1)\right\} \\
&= \Re\left\{\frac{zg'(z)}{g(z)}\right\} + \Re\left\{\frac{(\varepsilon z)g'(\varepsilon z)}{g(\varepsilon z)}\right\} + ... + \Re\left\{\frac{(\varepsilon^{k-1}z)g'(\varepsilon^{k-1}z)}{g(\varepsilon^{k-1}z)}\right\} \\
&\quad - (k-1) \\
&> k\left(\frac{k-1}{k}\right) - (k-1) \\
&= 0.
\end{aligned}$$

Therefore, $G_k(z) = \frac{g_k(z)}{z^{k-1}}$ is starlike in the unit disc which implies that $f \in \mathcal{K} \subset \mathcal{S}$. Naturally, when $k = 1$, $f \in \mathcal{K}_s^{(1)}(\alpha, \beta) = \mathcal{K}_s(\alpha, \beta)$ is univalent.

Backtracking slightly, Singh (1977) also introduced the classes of p-valent functions starlike, convex and close-to-convex with respect to symmetry point, denoted by $\mathcal{S}_{s,p}^*$, $\mathcal{C}_{s,p}$ and $\mathcal{K}_{s,p}$ respectively.

Definition 2.1.15. *Let $f \in \mathcal{A}_p$. Then $f \in \mathcal{S}_{s,p}^*$ if it satisfies*

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \ (z \in U).$$

Definition 2.1.16. *Let $f \in \mathcal{A}_p$. Then $f \in \mathcal{C}_{s,p}$ if it satisfies*

$$\Re\left\{\frac{(zf'(z))'}{f'(z)+f'(-z)}\right\} > 0,\ (z \in U).$$

Definition 2.1.17. *Let $f \in \mathcal{A}_p$. Then $f \in \mathcal{K}_{s,p}$ if there exists $g \in \mathcal{S}^*_{s,p}$ such that*

$$\Re\left\{\frac{zf'(z)}{g(z)-g(-z)}\right\} > 0,\ (z \in U).$$

The definitions for the classes of p-valent functions starlike, convex and close-to-convex with respect to n-ply points of order α with $0 \leq \alpha < p$ were established by Ali, Badghaish, & Ravichandran (2010).

Definition 2.1.18. *Let $f \in \mathcal{A}_p$ where $p \in \mathbb{N}$. Then $f \in \mathcal{S}^{*,n}_{s,p}(\alpha)$ for $0 \leq \alpha < p$ if*

$$\Re\left\{\frac{zf'(z)}{f_{n,p}(z)}\right\} > \alpha,\ (z \in U),$$

where for $w = e^{\frac{2\pi i}{n}}$

$$f_{n,p}(z) = \frac{1}{n}\sum_{k=0}^{n-1} w^{-pk} f(w^k z). \tag{2.1.2}$$

Definition 2.1.19. *Let $f \in \mathcal{A}_p$ where $p \in \mathbb{N}$. Then $f \in \mathcal{C}^{n}_{s,p}(\alpha)$ for $0 \leq \alpha < p$ if*

$$\Re\left\{\frac{\left(zf'(z)\right)'}{f'_{n,p}(z)}\right\} > \alpha,\ (z \in U),$$

where for $w = e^{\frac{2\pi i}{n}}$

$$f_{n,p}(z) = \frac{1}{n}\sum_{k=0}^{n-1} w^{-pk} f(w^k z).$$

Definition 2.1.20. *Let $f \in \mathcal{A}_p$ where $p \in \mathbb{N}$. Then $f \in \mathcal{K}^{n}_{s,p}(\alpha)$ for $0 \leq \alpha < p$ if there* *exists* $g \in \mathcal{S}^{*,n}_{s,p}(0)$ *such that*

$$\Re\left\{\frac{zf'(z)}{g_{n,p}(z)}\right\} > \alpha,\ (z \in U),$$

where for $w = e^{\frac{2\pi i}{n}}$

$$g_{n,p}(z) = \frac{1}{n}\sum_{k=0}^{n-1} w^{-pk} g(w^k z).$$

Using the same method to show that $f_n \in \mathcal{S}^*(\alpha)$, from (2.1.2) we have

$$\begin{aligned}\frac{zf'_{n,p}(z)}{f_{n,p}(z)} &= \frac{z[\frac{1}{n}\sum_{k=0}^{n-1} w^{k(1-p)} f'(w^k z)]}{f_{n,p}(z)} \\ &= \frac{1}{n}\left[\frac{zf'(z)}{f_{n,p}(z)} + \frac{(wz)f'(wz)}{w^p f_{n,p}(z)} + \ldots + \frac{(w^{n-1}z)f'(w^{n-1}z)}{w^{p(n-1)} f_{n,p}(z)}\right].\end{aligned}$$

Thus with $w^{pm} f_{n,p}(z) = f_{n,p}(w^m z)$,

$$\begin{aligned}\Re\left\{\frac{zf'_{n,p}(z)}{f_{n,p}(z)}\right\} &= \frac{1}{n}\left[\Re\left\{\frac{zf'(z)}{f_{n,p}(z)} + \frac{(wz)f'(wz)}{f_{n,p}(wz)} + \ldots + \frac{(w^{n-1}z)f'(w^{n-1}z)}{f_{n,p}(w^{n-1}z)}\right\}\right] \\ &= \frac{1}{n}\left[\Re\left\{\frac{zf'(z)}{f_{n,p}(z)}\right\} + \Re\left\{\frac{(wz)f'(wz)}{f_{n,p}(wz)}\right\} + \ldots \right. \\ &\qquad \left. + \Re\left\{\frac{(w^{n-1}z)f'(w^{n-1}z)}{f_{n,p}(w^{n-1}z)}\right\}\right] \\ &> \frac{1}{n}(n\alpha) \\ &= \alpha.\end{aligned}$$

Therefore, $f_{n,p} \in \mathcal{S}^*_p(\alpha)$.

Just recently, Vyas and Kant (2018) introduced the class of p-valent functions denoted by $\mathcal{K}^{(k)}_p$ with the following definition.

Definition 2.1.21. *Let* $f \in \mathcal{A}_p$. *Then* $f \in \mathcal{K}^{(k)}_p(\alpha, \beta)$ *for* $0 \leq \alpha \leq 1$ *and* $0 < \beta \leq 1$, *if* *there exists* $g \in \mathcal{S}_p^*\big(\frac{p(k-1)}{k}\big)$ *such that*

$$\left|\frac{zf'(z)}{G_{k,p}(z)} - p\right| < \beta\left|\frac{\alpha z^{(p(k-1)+1)}f'(z)}{g_k(z)} + p\right|,$$

where for $\varepsilon = e^{\frac{2\pi i}{k}}$

$$G_{k,p}(z) = \frac{1}{z^{p(k-1)}}\prod_{v=0}^{k-1}\varepsilon^{-vp}g(\varepsilon^v z). \tag{2.1.3}$$

Remark 2.1.1. For $\alpha = \beta = 1$, $f \in \mathcal{K}_p^{(k)}$ if there exists $g \in \mathcal{S}_p^*\big(\frac{p(k-1)}{k}\big)$ such that

$$\Re\left\{\frac{zf'(z)}{G_{k,p}(z)}\right\} > 0,$$

where $G_{k,p}(z)$ is of the form (2.1.3).

Obviously, $\mathcal{K}_p^{(1)} = \mathcal{K}_p$. To show $G_{k,p} \in \mathcal{S}_p^*$ when $g \in \mathcal{S}_p^*\big(\frac{(k-1)p}{k}\big)$, we first differentiate (2.1.3) logarithmically.

$$\frac{G'_{k,p}(z)}{G_{k,p}(z)} = -\frac{(k-1)p}{z} + \frac{g'(z)}{g(z)} + \frac{\varepsilon g'(\varepsilon z)}{g(\varepsilon z)} + ... + \frac{\varepsilon^{k-1}g'(\varepsilon^{k-1}z)}{g(\varepsilon^{k-1}z)}.$$

Thus,

$$\begin{aligned}
\Re\left\{\frac{zG'_{k,p}(z)}{G_{k,p}(z)}\right\} &= \Re\left\{\frac{zg'(z)}{g(z)} + \frac{(\varepsilon z)g'(\varepsilon z)}{g(\varepsilon z)} + ... + \frac{(\varepsilon^{k-1}z)g'(\varepsilon^{k-1}z)}{g(\varepsilon^{k-1}z)}\right\} - (k-1)p \\
&= \Re\left\{\frac{zg'(z)}{g(z)}\right\} + \Re\left\{\frac{(\varepsilon z)g'(\varepsilon z)}{g(\varepsilon z)}\right\} + ... + \Re\left\{\frac{(\varepsilon^{k-1}z)g'(\varepsilon^{k-1}z)}{g(\varepsilon^{k-1}z)}\right\} \\
&\quad - (k-1)p \\
&> k\left(\frac{(k-1)p}{k}\right) - (k-1)p \\
&= 0.
\end{aligned}$$

Therefore, $G_{k,p} \in \mathcal{S}_p^*$.

Motivated by their findings, this dissertation explores the possibility of finding a unified representation combining the concepts suggested by Wang et al. and Chand & Singh. The class given in the next definition is an attempt to formulate a generalization of classes of functions encompassing both concepts.

Definition 2.1.22. *Let* $f \in \mathcal{S}$. *Then* $f \in \mathcal{K}_s^{k,N}$ *if there exists* $g \in \mathcal{S}_s^{*,N}(\frac{k-1}{k})$ *for* $k \geq 1$ *and* $N \geq 1$ *such that*

$$\Re\left\{\frac{zf'(z)}{G_{k,N}(z)}\right\} > 0, \ (z \in U), \tag{2.1.4}$$

where for $\varepsilon = e^{\frac{2\pi i}{n}}$

$$G_{k,N}(z) = \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} g_N(\varepsilon^j z), \tag{2.1.5}$$

and for $\gamma = e^{\frac{2\pi i}{n}}$

$$g_N(z) = \frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} g(\gamma^p z). \tag{2.1.6}$$

Easily, $\mathcal{K}_s^{k,1} \equiv \mathcal{K}_s^{(k)}$ (Wang et al., 2006a) and $\mathcal{K}_s^{1,N} \equiv \mathcal{K}_s^N$ (Chand & Singh, 1979). For other examples to illustrate the genrealization, $\mathcal{K}_s^{2,2}$ with $g(z) \equiv z$ is the Macgregor's class, i.e. $\Re\{f'(z)\} > 0$ (MacGregor, 1962) and $\mathcal{K}_s^{1,1}$ with $g(z) = \frac{z}{(1-z)^2}$ results in the class that satisfies the inequality $\Re\big\{(1-z)^2 f'(z)\big\} > 0$ (Janowksi, 1970).

Moreover, we generalized the class $\mathcal{K}_p^{(k)}$ (Vyas & Kant, 2018) and the class $\mathcal{K}_{s,p}^n$ (Ali et al., 2010) with the following definition.

Definition 2.1.23. *Let* $f \in \mathcal{A}_p$ *where* $p \in \mathbb{N}$. *Then for* $k, N \in \mathbb{N}$, $f \in \mathcal{K}_{s,p}^{k,N}$ *if there exists* $g \in \mathcal{S}_{s,p}^{*,N}\big(\frac{(k-1)p}{k}\big)$ *such that*

$$\Re\left\{\frac{zf'(z)}{G_{k,N,p}(z)}\right\} > 0, \ (z \in U), \tag{2.1.7}$$

where for $\varepsilon = e^{i\frac{2\pi}{k}}$

$$G_{k,N,p}(z) = \frac{1}{z^{(k-1)p}} \prod_{j=0}^{k-1} \varepsilon^{-jp} g_{N,p}(\varepsilon^j z), \tag{2.1.8}$$

and for $\gamma = e^{i\frac{2\pi}{N}}$

$$g_{N,p}(z) = \frac{1}{N} \sum_{m=0}^{N-1} \gamma^{-mp} g(\gamma^m z). \tag{2.1.9}$$

When $p = 1$, $\mathcal{K}_{s,1}^{k,N} = \mathcal{K}_s^{k,N}$. Sections 2.3 and 2.4 give results obtained for the class $\mathcal{K}_s^{k,N}$ and the class $\mathcal{K}_{s,p}^{k,N}$ respectively. However, to establish these results, preliminary results, obtained by other mathematicians, are required.

2.2 Preliminary Results

As stated before, the following preliminary results are required.

Lemma 2.2.1. *(Modified from (Wang et al., 2006a)) Let* $\phi_i \in \mathcal{S}_s^{*,N}(\alpha_i)$ *where* $0 \leq \alpha_i < 1$ $(i = 0, 1, 2, ..., k-1; N \in \mathbb{N})$. *Then for* $k - 1 \leq \beta = \sum\limits_{i=0}^{k-1} \alpha_i < k$,

$$\frac{\prod\limits_{i=0}^{k-1} \phi_{i,N}(z)}{z^{k-1}} \in \mathcal{S}^*\big(\beta - (k-1)\big),$$

where

$$\phi_{i,N}(z) = \frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} \phi_i(\gamma^p z), \ (\gamma^N = 1).$$

Proof. If $\phi_i \in \mathcal{S}_s^{*,N}(\alpha_i)$, $(i = 0, 1, 2, ..., k-1; N \in \mathbb{N})$, then $\phi_{i,N} \in \mathcal{S}_s^{*,N}(\alpha_i)$ which makes it easy to deduce that

$$\Re\left\{\frac{z\phi_{0,N}'(z)}{\phi_{0,N}(z)}\right\} > \alpha_0, \ \Re\left\{\frac{z\phi_{1,N}'(z)}{\phi_{1,N}(z)}\right\} > \alpha_1, \ \ldots, \ \Re\left\{\frac{z\phi_{k-1,N}'(z)}{\phi_{k-1,N}(z)}\right\} > \alpha_{k-1}.$$

Next, let

$$F(z) = \frac{\phi_{0,N}(z)\phi_{1,N}(z)\ ...\ \phi_{k-1,N}(z)}{z^{k-1}} = \frac{\prod_{i=0}^{k-1}\phi_{i,N}(z)}{z^{k-1}}. \tag{2.2.1}$$

Logarithmic differentiating (2.2.1) gives

$$\frac{zF'(z)}{F(z)} = \frac{z\phi'_{0,N}(z)}{\phi_{0,N}(z)} + \frac{z\phi'_{1,N}(z)}{\phi_{1,N}(z)} + ... + \frac{z\phi'_{k-1,N}(z)}{\phi_{k-1,N}(z)} - (k-1).$$

Thus,

$$\begin{aligned}\Re\left\{\frac{zF'(z)}{F(z)}\right\} &= \Re\left\{\frac{z\phi'_{0,N}(z)}{\phi_{0,N}(z)} + \frac{z\phi'_{1,N}(z)}{\phi_{1,N}(z)} + ... + \frac{z\phi'_{k-1,N}(z)}{\phi_{k-1,N}(z)}\right\} - (k-1)\\ &= \Re\left\{\frac{z\phi'_{0,N}(z)}{\phi_{0,N}(z)}\right\} + \Re\left\{\frac{z\phi'_{1,N}(z)}{\phi_{1,N}(z)}\right\} + ... + \Re\left\{\frac{z\phi'_{k-1,N}(z)}{\phi_{k-1,N}(z)}\right\} - (k-1)\\ &> \sum_{i=0}^{k-1}\alpha_i - (k-1)\\ &= \beta - (k-1).\end{aligned}$$

Therefore, since $k-1 \leq \beta < k$,

$$F(z) = \frac{\prod\limits_{i=0}^{k-1}\phi_{i,N}(z)}{z^{k-1}} \in \mathcal{S}^*\big(\beta - (k-1)\big).$$

□

Lemma 2.2.2. *(Silverman, 1976) Suppose that $p \in \mathcal{P}$, $\gamma \in [0,1)$, $\delta = \frac{\gamma}{1-\gamma}$. Moreover, suppose that $|z| = r$ and $a = \frac{1+r^2}{1-r^2}$. Then*

$$\Re\left\{\frac{zp'(z)}{p(z)+\delta}\right\} \geq \begin{cases}\frac{-2r}{(1+r)(1+\delta-(1-\delta)r)} & , \text{ for } 0 \leq r \leq r_\gamma,\\ 2\sqrt{\delta^2 + a\delta} - a - 2\delta & , \text{ for } r_\gamma < r < 1,\end{cases} \tag{2.2.2}$$

where r_γ is the unique root of the equations

$$(1-2\gamma)r^3 - 3(1-2\gamma)r^2 + 3r - 1 = 0, \tag{2.2.3}$$

in the interval $(0,1]$. This result is sharp where equality in (2.2.2) is attained at the point $z = -r$ for

$$\tilde{p}(z) = \begin{cases} \frac{1+z}{1-z} & , \text{ for } 0 \le r \le r_\gamma, \\ \frac{1-z^2}{1-2z\cos\theta_0 + z^2} & , \text{ for } r_\gamma < r < 1, \end{cases} \tag{2.2.4}$$

where $\cos\theta_0$ is defined by the equations

$$\cos\theta_0 = \frac{1 - r_0^2 - (1+r_0^2)(\sqrt{\delta^2 + a\delta})}{-2r_0(\sqrt{\delta^2 + a\delta})},$$

with

$$r_0 = 2\sqrt{\delta^2 + a\delta} - (a + 2\delta).$$

Lemma 2.2.3. *(Ma & Minda, 1992) If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, then*

$$|p_2 - vp_1^2| \le \begin{cases} -4v + 2 & , v \le 0, \\ 2 & , 0 \le v \le 1, \\ 4v - 2 & , v \ge 1. \end{cases}$$

Equality holds for $v < 0$ or $v > 1$ if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations. Equality holds for $0 < v < 1$ if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. At $v = 0$, equality *holds if and only if*

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z}, \ (0 \leq \gamma \leq 1),$$

or one of its rotation. At $v = 1$, equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that equality holds in the case of $v = 0$.

Lemma 2.2.4. *(Koepf, 1987) Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$. Then for any $\gamma \in \mathbb{C}$,*

$$|b_3 - \gamma b_2^2| \leq \max\left\{1, |3 - 4\gamma|\right\}.$$

The inequality is sharp for $k(z) = \frac{z}{(1-z)^2}$, or its rotations if $|3 - 4\gamma| \geq 1$, and for $\sqrt{k(z^2)}$, or its rotations if $|3 - 4\gamma| < 1$.

Lemma 2.2.5. *Let $\phi_{i,p} \in \mathcal{S}^{*,N}_{s,p}(\alpha_i)$ where $0 \leq \alpha_i < p \ (i = 0, 1, ..., k-1; N \in \mathbb{N})$. Then for $(k-1)p \leq \beta = \sum_{i=0}^{k-1} \alpha_i < kp$,*

$$\frac{1}{z^{(k-1)p}} \prod_{i=0}^{k-1} \phi_{i,N,p}(z) \in \mathcal{S}^*_p\big(\beta - (k-1)p\big).$$

Proof. Suppose $\phi_{i,p} \in \mathcal{S}^{*,N}_{s,p}(\alpha_i)$ where $0 \leq \alpha_i < p \ (i = 0, 1, ..., k-1; N \in \mathbb{N})$. Then $\phi_{i,N,p} \in \mathcal{S}^*_p(\alpha_i)$. Write

$$F_p(z) = \frac{1}{z^{(k-1)p}} \prod_{i=0}^{k-1} \phi_{i,N,p}(z). \tag{2.2.5}$$

Differentiating (2.2.5) logarithmically,

$$\frac{F_p'(z)}{F_p(z)} = \frac{\phi_{0,N,p}'(z)}{\phi_{0,N,p}(z)} + \frac{\phi_{1,N,p}'(z)}{\phi_{1,N,p}(z)} + ... + \frac{\phi_{k-1,N,p}'(z)}{\phi_{k-1,N,p}(z)} - \frac{(k-1)p}{z}.$$

Thus,

$$\begin{aligned}
\Re\left\{\frac{zF_p'(z)}{F_p(z)}\right\} &= \Re\left\{\frac{z\phi'_{0,N,p}(z)}{\phi_{0,N,p}(z)} + \frac{z\phi'_{1,N,p}(z)}{\phi_{1,N,p}(z)} + ... + \frac{z\phi'_{k-1,N,p}(z)}{\phi_{k-1,N,p}(z)}\right\} - (k-1)p \\
&= \Re\left\{\frac{z\phi'_{0,N,p}(z)}{\phi_{0,N,p}(z)}\right\} + \Re\left\{\frac{z\phi'_{1,N,p}(z)}{\phi_{1,N,p}(z)}\right\} + ... + \Re\left\{\frac{z\phi'_{k-1,N,p}(z)}{\phi_{k-1,N,p}(z)}\right\} \\
&\quad - (k-1)p \\
&> \alpha_0 + \alpha_1 + ...\alpha_{k-1} - (k-1)p \\
&= \beta - (k-1)p,
\end{aligned}$$

where $\beta = \sum_{i=0}^{k-1} \alpha_i$. Therefore,

$$F_p(z) = \frac{1}{z^{(k-1)p}} \prod_{i=0}^{k-1} \phi_{i,N,p}(z) \in \mathcal{S}_p^*\big(\beta - (k-1)p\big).$$

□

2.3 Class $\mathcal{K}_s^{k,N}$

In this section, we present properties of the class $\mathcal{K}_s^{k,N}$. The properties include distortion and growth bounds, various coefficient estimates, and its radius of convexity. The following theorem shows that $G_{k,N}$ of the form (2.1.5) is starlike.

Theorem 2.3.1. *Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$ for $k, N \in \mathbb{N}$. Then the function $G_{k,N}$ of the form (2.1.5) is a function of the class $\mathcal{S}^*$ and can be written as*

$$G_{k,N} = z + \sum_{n=2}^{\infty} B_{N(n-1)+1} z^{N(n-1)+1}. \tag{2.3.1}$$

The coefficients $B_{N(n-1)+1}$ are given by

$$B_{N(n-1)+1} = \begin{cases} b_{N(n-1)+1} & , \ k=1, \\ \\ \chi_{k-1} & , \ k \geq 2, \end{cases}$$

where

$$\chi_{k-1} = \sum_{p_{k-1}=1}^{n} \Psi \Big\{ \sum_{p_{k-2}=1}^{p_{k-1}} \psi_{k-2} \Big\{ \dots \Big\{ \sum_{p_2=1}^{p_3} \psi_2 \Big\{ \sum_{p_1=1}^{p_2} \psi_1 b_{N(p_1-1)+1} \Big\} \Big\} \dots \Big\} \Big\},$$

with

$$\Psi = \varepsilon^{N(k-1)(n-p_{k-1})} b_{N(n-p_{k-1})+1}$$

and

$$\psi_j = \varepsilon^{Nj(p_{j+1}-p_j)} b_{N(p_{j+1}-p_j)+1},$$

for $1 \leq j \leq k-2$.

Proof. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}_s^{*,N}(\frac{k-1}{k})$ where $k, N \in \mathbb{N}$. From (2.1.5),

$$\begin{aligned} G_{k,N}(z) &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} g_N(\varepsilon^j z) \\ &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} g(\gamma^p \varepsilon^j z) \right] \\ &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} \Big\{ \gamma^p \varepsilon^j z + \sum_{n=2}^{\infty} b_n \gamma^{np} \varepsilon^{nj} z^n \Big\} \right] \\ &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \Big\{ \varepsilon^j z + \sum_{n=2}^{\infty} b_n \gamma^{p(n-1)} \varepsilon^{nj} z^n \Big\} \right] \\ &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \left[\frac{1}{N} \sum_{p=0}^{N-1} \Big\{ z + \sum_{n=2}^{\infty} b_n \gamma^{p(n-1)} \varepsilon^{j(n-1)} z^n \Big\} \right] \\ &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \left[z + \sum_{n=2}^{\infty} \phi_n \varepsilon^{j(n-1)} b_n z^n \right], \end{aligned}$$

where $\phi_n = \frac{1}{N}\Big(1 + \gamma^{n-1} + \gamma^{2(n-1)} + ... + \gamma^{(N-1)(n-1)}\Big)$. Observe that

$$\phi_n = \begin{cases} 1 & , \frac{n-1}{N} \in \mathbb{N}, \\ 0 & , \frac{n-1}{N} \notin \mathbb{N}. \end{cases}$$

It can be easily deduced from above that $\phi_1 = \phi_{N+1} = \phi_{2N+1} = ... = \phi_{Nm+1} = 1$ for any positive integer m, otherwise $\phi_n = 0$. This allows us to rewrite $\sum_{n=2}^{\infty} \phi_n \varepsilon^{j(n-1)} b_n z^n$ to $\sum_{m=1}^{\infty} \varepsilon^{Nmj} b_{Nm+1} z^{Nm+1}$ and upon re-indexing gives

$$G_{k,N}(z) = \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \left[z + \sum_{n=2}^{\infty} \varepsilon^{Nj(n-1)} b_{N(n-1)+1} z^{N(n-1)+1} \right]. \tag{2.3.2}$$

Write $P_j(z) = z + \sum_{n=2}^{\infty} \varepsilon^{Nj(n-1)} b_{N(n-1)+1} z^{N(n-1)+1}$ for $0 \leq j \leq k-1$ and denote $D_j(z)$ as follows

$$\begin{aligned} D_0(z) &= P_0(z), \\ D_1(z) &= \frac{1}{z} P_1(z) P_0(z), \\ D_2(z) &= \frac{1}{z^2} P_2(z) P_1(z) P_0(z), \\ &... \\ D_j(z) &= \frac{1}{z^j} P_j(z) P_{j-1}(z) ... P_1(z) P_0(z). \end{aligned}$$

Under the assumption $D_j(z) = \frac{1}{z} P_j(z) D_{j-1}(z)$ is true, then

$$\begin{aligned} D_{j+1}(z) &= \frac{1}{z^{j+1}} P_{j+1}(z) P_j(z) P_{j-1}(z) ... P_2(z) P_1(z) P_0(z) \\ &= \frac{1}{z} P_{j+1}(z) \frac{1}{z^j} P_j(z) P_{j-1}(z) ... P_1(z) P_0(z) \\ &= \frac{1}{z} P_{j+1}(z) D_j(z). \end{aligned}$$

Therefore, by induction,

$$D_j(z) = \frac{1}{z} P_j(z) D_{j-1}(z). \tag{2.3.3}$$

Expanding (2.3.3) gives

$$\begin{aligned} D_j(z) &= \frac{1}{z} P_j(z) D_{j-1}(z) \\ &= \frac{1}{z^j} P_j(z) P_{j-1}(z) ... P_1(z) P_0(z) \\ &= z + \sum_{n=2}^{\infty} \chi_j z^{N(n-1)+1}, \end{aligned}$$

where

$$\chi_j = \sum_{p_j=1}^{n} \Psi \Bigg\{ \sum_{p_{j-1}=1}^{p_j} \psi_{j-1} \Bigg\{ ... \Bigg\{ \sum_{p_2=1}^{p_3} \psi_2 \Bigg\{ \sum_{p_1=1}^{p_2} \psi_1 b_{N(p_1-1)+1} \Bigg\} \Bigg\} ... \Bigg\} \Bigg\},$$

with $\Psi = \varepsilon^{Nj(n-p_j)} b_{N(n-p_j)+1}$ and $\psi_j = \varepsilon^{Nj(p_{j+1}-p_j)} b_{N(p_{j+1}-p_j)+1}$. This gives

$$\begin{aligned} G_{k,N}(z) &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \left[z + \sum_{n=2}^{\infty} \varepsilon^{Nj(n-1)} b_{N(n-1)+1} z^{N(n-1)+1} \right] \\ &= D_{k-1}(z) \\ &= z + \sum_{n=2}^{\infty} B_{N(n-1)+1} z^{N(n-1)+1}, \end{aligned}$$

where

$$B_{N(n-1)+1} = \begin{cases} b_{N(n-1)+1} & , \ k = 1, \\ \chi_{k-1} & , \ k \geq 2. \end{cases}$$

As $g \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$, then by Lemma 2.2.1, $G_{k,N} \in \mathcal{S}^*$.

□

Remark 2.3.1. From Theorem 2.3.1, observed that if $f \in \mathcal{K}_s^{k,N}$ then $f \in \mathcal{K}$. Therefore, $\mathcal{K}_s^{k,N}$ is a subclass of $\mathcal{K}$.

Theorem 2.3.2. *For $f \in \mathcal{K}_s^{k,N}$, the distortion and growth bounds are given, respectively, by*

$$\frac{1-r}{(1+r)(1+r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} \leq |f'(z)| \leq \frac{1+r}{(1-r)(1-r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}$$

and

$$\int_0^r \frac{1-\rho}{(1+\rho)(1+\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho \leq |f(z)| \leq \int_0^r \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho,$$

where $|z| = r < 1$ and m is the highest common factor of N and k. Equality is attained at the right-hand side for the function

$$f_1(z) = \int_0^z \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho$$

with respect to $g_1(z) = \frac{z}{(1-z^N)^{\frac{2}{Nk}}} \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$.

Proof. Suppose $f \in \mathcal{K}_s^{k,N}$ for $k, N \in \mathbb{N}$, then there exists a function $g \in \mathcal{S}_s^{*,N}(\frac{k-1}{k})$ such that (2.1.4) holds. As the form $g(z) = z + \sum_{n=2}^{\infty} b_{N(n-1)+1} z^{N(n-1)+1} \in \mathcal{S}_s^{*,N}(\frac{k-1}{k})$ is similar to that of (1.3.2), we can infer that g is a N-fold symmetric function of order $\frac{k-1}{k}$. With this inference, the boundaries for g is given as follows.

$$\frac{r}{(1+r^N)^{\frac{2}{Nk}}} \leq |g(z)| \leq \frac{r}{(1-r^N)^{\frac{2}{Nk}}}, (|z| = r < 1).$$

Furthermore, setting $g(z) = \frac{z}{(1-z^N)^{\frac{2}{Nk}}}$, gives

$$\begin{aligned} G_{k,N}(z) &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} g(\gamma^p \varepsilon^j z) \right] \\ &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} \frac{\gamma^p \varepsilon^j z}{\left(1 - (\gamma^p \varepsilon^j z)^N\right)^{\frac{2}{Nk}}} \right] \\ &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \left[\frac{1}{N} \sum_{p=0}^{N-1} \frac{z}{(1 - \varepsilon^{Nj} z^N)^{\frac{2}{Nk}}} \right] \\ &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \frac{z}{(1 - \varepsilon^{Nj} z^N)^{\frac{2}{Nk}}} \\ &= \frac{z}{\prod_{j=0}^{k-1} (1 - \varepsilon^{Nj} z^N)^{\frac{2}{Nk}}}, \end{aligned}$$

which upon expanding and simplifying gives

$$G_{k,N}(z) = \frac{z}{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk}}},$$

where m is the highest common factor of N and k. Thus, for $|z| = r < 1$,

$$\left|G_{k,N}(z)\right| = \left| \frac{z}{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} \right| \leq \frac{r}{(1 - r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}. \tag{2.3.4}$$

Applying the same method with $g(z) = \frac{z}{(1+z^N)^{\frac{2}{Nk}}}$ gives

$$\left|G_{k,N}(z)\right| \geq \frac{r}{(1 + r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}. \tag{2.3.5}$$

Combining (2.3.4) and (2.3.5) results in

$$\frac{r}{(1 + r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} \leq \left|G_{k,N}(z)\right| \leq \frac{r}{(1 - r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}. \tag{2.3.6}$$

Since there exists a $p \in \mathcal{P}$ such that

$$\frac{zf'(z)}{G_{k,N}(z)} = p(z), \tag{2.3.7}$$

the following is easily obtained using (1.2.2), (2.3.6) and (2.3.7),

$$\frac{1-r}{(1+r)(1+r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} \leq |f'(z)| \leq \frac{1+r}{(1-r)(1-r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}. \tag{2.3.8}$$

From (2.3.8), the upper bound for $|f(z)|$ is

$$|f(z)| = \Big|\int_0^z f'(\rho)d\rho\Big| \leq \int_0^r |f'(\rho)|d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}d\rho.$$

To find the lower bound, it is sufficient to show that $|f(z)| \geq \int_0^r \frac{1-\rho}{(1+\rho)(1+\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}d\rho$ holds for the nearest point $f(z_0)$ $(|z_0| = r)$ from zero for some $r \in (0,1)$, otherwise $|f(z)| \geq |f(z_0)|$ with $|z| = r$ $(0 \leq r < 1)$. Since f is univalent in the unit disc U, as f is a close-to-convex function, the original image of the line segment $\big[0, f(z_0)\big]$ is a piece of arc R in $\big\{|z| \leq r\big\}$, then

$$|f(z_0)| = \int_{f(R)} |dw| = \int_R |f'(\rho)|d\rho \geq \int_R \frac{1-\rho}{(1+\rho)(1+\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}d\rho.$$

Therefore,

$$\int_0^r \frac{1-\rho}{(1+\rho)(1+\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}d\rho \leq |f(z)| \leq \int_0^r \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}d\rho.$$

With $g_1(z) = \frac{z}{(1-z^N)^{\frac{2}{Nk}}}$, we obtain

$$\begin{aligned}\Re\left\{\frac{zf_1'(z)}{G_{1,k,N}(z)}\right\} &= \Re\left\{z\left(\frac{1+z}{(1-z)(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}\right)\left(\frac{(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}{z}\right)\right\} \\ &= \Re\left\{\frac{1+z}{1-z}\right\} \\ &> 0\end{aligned}$$

and

$$\Re\left\{\frac{zg_1'(z)}{g_{1,N}(z)}\right\} = \Re\left\{\frac{zg_1'(z)}{g_1(z)}\right\} > \frac{k-1}{k},$$

because $g_{1,N}(z) = \frac{1}{N}\sum_{p=0}^{N-1}\gamma^{-p}g_1(\gamma^p z) = \frac{1}{N}\sum_{p=0}^{N-1}g_1(z) = g_1(z)$ as $\gamma^N = 1$. This proves that $f_1 \in \mathcal{K}_s^{k,N}$ with respect to $g_1 \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$. □

Theorem 2.3.3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_s^{k,N}$ for $k, N \in \mathbb{N}$. Then*

$$|a_n| \leq \begin{cases} \frac{1}{n}\left(2\sum_{d=0}^{d_1-1}\eta_d + \eta_{d_1}\right), & n = Nd_1 + 1 \text{ for } d_1 \geq 1, \\ \frac{2}{n}\sum_{d=0}^{d_2}\eta_d, & Nd_2 + 1 < n < N(d_2+1) + 1 \text{ for } d_2 \geq 0. \end{cases}$$

where $\eta_0 = 1$ and $\eta_d = \frac{(\frac{2m}{Nk})(\frac{2m}{Nk}+1)\dots(\frac{2m}{Nk}+d-1)}{d!}$ for $d \geq 1$ with m being the highest common factor of N and k. The inequality is sharp with the extremal function

$$f_1(z) = \int_0^z \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}d\rho,$$

with respect to $g_1(z) = \frac{z}{(1-z^N)^{\frac{2}{Nk}}} \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$.

Proof. Let $f \in \mathcal{K}_s^{k,N}$. Then there exists $g \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$ such that

$$\Re\left\{\frac{zf'(z)}{G_{k,N}(z)}\right\} > 0, \ (z \in U),$$

where $k, N \in \mathbb{N}$. $G_{k,N}$ of the form (2.3.1) is a starlike function by Theorem 2.3.1 and there exists $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ such that

$$\frac{zf'(z)}{G_{k,N}(z)} = p(z). \tag{2.3.9}$$

Using series representations for p, f and $G_{k,N}$ in (2.3.9) gives

$$\begin{aligned}
&z + 2a_2z^2 + 3a_3z^3 + ... + (Nn)a_{Nn}z^{Nn} + ... \\
= &z(1 + p_1z + p_2z^2 + ... + p_{Nn-1}z^{Nn-1} + p_{Nn}z^{Nn} + ...) + \\
&B_{N+1}z^{N+1}(1 + p_1z + p_2z^2 + ... + p_{Nn-1}z^{Nn-1} + p_{Nn}z^{Nn} + ...) + \\
&B_{2N+1}z^{2N+1}(1 + p_1z + p_2z^2 + ... + p_{Nn-1}z^{Nn-1} + p_{Nn}z^{Nn} + ...) + ... + \\
&B_{N(n-1)+1}z^{N(n-1)+1}(1 + p_1z + p_2z^2 + ... + p_{Nn-1}z^{Nn-1} + p_{Nn}z^{Nn} + ...) + ...
\end{aligned}$$

which upon simplification yields

$$na_n = \begin{cases} \sum\limits_{d=0}^{d_1-1} p_{n-(Nd+1)}B_{Nd+1} + B_n & , \ n = Nd_1 + 1 \text{ for } d_1 \geq 1, \\ \sum\limits_{d=0}^{d_2} p_{n-(Nd+1)}B_{Nd+1} & , \ Nd_2 + 1 < n < N(d_2+1) + 1 \text{ for } d_2 \geq 0. \end{cases}$$

Using Lemma 1.2.1, the following is obtained.

$$|a_n| \leq \begin{cases} \frac{1}{n}\left(2\sum\limits_{d=0}^{d_1-1} |B_{Nd+1}| + |B_n|\right) & , \ n = Nd_1 + 1 \text{ for } d_1 \geq 1, \\ \frac{2}{n}\sum\limits_{d=0}^{d_2} |B_{Nd+1}| & , \ Nd_2 + 1 < n < N(d_2+1) + 1 \text{ for } d_2 \geq 0. \end{cases} \tag{2.3.10}$$

Expanding (2.3.4) using its series representation gives

$$
\begin{aligned}
\left|G_{k,N}(z)\right| &\leq \frac{r}{(1-r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} \\
&= r + \sum_{n=1}^{\infty} \frac{(\frac{2m}{Nk})(\frac{2m}{Nk}+1)...(\frac{2m}{Nk}+n-1)}{n!} r^{\frac{Nk}{m}n+1},
\end{aligned}
$$

where m is the highest comman factor of N and k, shows that $|B_1| = \eta_0 = 1$ and

$$
|B_{Nd+1}| \leq \eta_d = \frac{(\frac{2m}{Nk})(\frac{2m}{Nk}+1)...(\frac{2m}{Nk}+d-1)}{d!}, \tag{2.3.11}
$$

for d positive integers. Substituting (2.3.11) into (2.3.10) gives

$$
|a_n| \leq \begin{cases} \frac{1}{n}\left(2\sum_{d=0}^{d_1-1} \eta_d + \eta_{d_1}\right) & , \ n = Nd_1 + 1 \text{ for } d_1 \geq 1, \\ \frac{2}{n}\sum_{d=0}^{d_2} \eta_d & , \ Nd_2 + 1 < n < N(d_2+1) + 1 \text{ for } d_2 \geq 0. \end{cases}
$$

□

Theorem 2.3.4. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_s^{k,N}$*. Then*

$$
|a_3^2 - a_2^2| \leq \begin{cases} 13 & , \ k = N = 1, \\ 2 & , \ k = 1, N = 2; \ k = 2, N = 1; \ k = N = 2, \\ \frac{13}{9} & , \ \textit{otherwise}. \end{cases}
$$

Proof. Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_s^{k,N}$. Then there exists $g \in \mathcal{S}_s^{*,N}(\frac{k-1}{k})$ such that

$$
\Re\left\{\frac{zf'(z)}{G_{k,N}(z)}\right\} > 0,
$$

where $G_{k,N}(z) = z + \sum_{m=2}^{\infty} B_m z^m$. This implies there exists $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$

such that

$$\frac{zf'(z)}{G_{k,N}(z)} = p(z).$$

Comparing coefficients on both sides of the equation gives

$$\begin{aligned} 2a_2 &= B_2 + p_1 \\ 3a_3 &= B_3 + p_1 B_2 + p_2. \end{aligned} \tag{2.3.12}$$

Thus,

$$\begin{aligned} 36(a_3^2 - a_2^2) &= 4(B_3 + p_1 B_2 + p_2)^2 - 9(B_2 + p_1)^2 \\ &= 4(B_3^2 + p_1^2 B_2^2 + p_2^2 + 2p_1 B_3 B_2 + 2p_2 B_3 + 2p_1 p_2 B_2) \\ &\quad - 9(B_2^2 + 2p_1 B_2 + p_1^2) \\ &= (4B_3^2 - 9B_2^2) + (8p_1 B_3 B_2 + 8p_2 B_3 - 18p_1 B_2) \\ &\quad + (4p_1^2 B_2^2 + 4p_2^2 + 8p_1 p_2 B_2 - 9p_1^2) \\ &= (4B_3^2 - 9B_2^2) + 8p_2 B_3 + (8B_3 - 18)p_1 B_2 \\ &\quad + (4B_2^2 - 9)p_1^2 + 4p_2^2 + 8p_1 p_2 B_2. \end{aligned}$$

Utilizing the triangle inequality and Lemma 1.2.1

$$\begin{aligned} 36|a_3^2 - a_2^2| &\leq |4B_3^2 - 9B_2^2| + 8|p_2||B_3| + |8B_3 - 18||p_1||B_2| + |4B_2^2 - 9||p_1^2| \\ &\quad + 4|p_2|^2 + 8|p_1||p_2||B_2| \\ &\leq |4B_3^2 - 9B_2^2| + 16|B_3| + 2|8B_3 - 18||B_2| + 4|4B_2^2 - 9| + 16 + 32|B_2|. \end{aligned} \tag{2.3.13}$$

Considering case by case, for different values of k and N, the coefficients obtained are as follows:

$$
k=1\begin{cases}N=1: \ B_2=b_2 \text{ and } B_3=b_3,\\ N=2: \ B_2=0 \text{ and } B_3=b_3,\\ N\geq 3: \ B_2=B_3=0,\end{cases}
$$

$$
k=2\begin{cases}N=1: \ B_2=0 \text{ and } B_3=2b_3-b_2^2,\\ N=2: \ B_2=0 \text{ and } B_3=2b_3,\\ N\geq 3: \ B_2=B_3=0,\end{cases}
$$

$$
k\geq 3, \ B_2=B_3=0 \, \forall N\in\mathbb{N},
$$

and since $g\in\mathcal{S}_s^{*,N}(\frac{k-1}{k})$,

$$
\begin{aligned}
&k=1\begin{cases}N=1: \ |B_2|=|b_2|\leq 2 \text{ and } |B_3|=|b_3|\leq 3,\\ N=2: \ |B_2|=0 \text{ and } |B_3|=|b_3|\leq 1, \ (\text{due to Theorem 2.1.2})\\ N\geq 3: \ |B_2|=|B_3|=0,\end{cases}\\
&k=2\begin{cases}N=1: \ |B_2|=0 \text{ and } |B_3|=|2b_3-b_2^2|\leq 1, \ (\text{see Gao \& Zhou, 2005, p.124})\\ N=2: \ |B_2|=0 \text{ and } |B_3|=|2b_3|\leq 1,\\ N\geq 3: \ |B_2|=|B_3|=0,\end{cases}\\
&k\geq 3, \ |B_2|=|B_2|=0 \, \forall N\in\mathbb{N}.
\end{aligned} \tag{2.3.14}
$$

Substituting (2.3.14) into (2.3.13) and going case by case gives:

$k = 1,\ N = 1$:

$$|a_3^2 - a_2^2| \le \frac{1}{36}\big(|4b_3^2 - 9b_2^2| + 16|b_3| + 2|8b_3 - 18||b_2| + 4|4b_2^2 - 9| + 16 + 32|b_2|\big)$$
$$\le 13;$$

$k = 1,\ N = 2$:

$$|a_3^2 - a_2^2| \le \frac{1}{36}\big(|4b_3^2| + 16|b_3| + 4|-9| + 16\big) \le 2;$$

$k = 2,\ N = 1$:

$$|a_3^2 - a_2^2| \le \frac{1}{36}\big(|4(2b_3 - b_2^2)^2| + 16|2b_3 - b_2^2| + 4|-9| + 16\big) \le 2;$$

$k = 2,\ N = 2$:

$$|a_3^2 - a_2^2| \le \frac{1}{36}\big(|4(2b_3)^2| + 16|2b_3| + 4|-9| + 16\big) \le 2;$$

Others:

$$|a_3^2 - a_2^2| \le \frac{1}{36}\big(4|-9| + 16\big) \le \frac{13}{9}.$$

□

Theorem 2.3.5. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_s^{k,N}$*. Then the Fekete-Szegö inequality is*

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{1}{3}\max\{1, 3|1-\gamma|\} + \frac{4}{3} + \frac{2}{3}|2-3\gamma| & , \ k = N = 1, \\ \frac{5}{3} & , \ \begin{array}{l} k=1, N=2; \ k=2, N=1; \\ k = N = 2, \end{array} \\ \frac{4}{3} & , \ \textit{otherwise}, \end{cases}$$

where $\gamma \in [0, 1)$.

Proof. Suppose $f \in \mathcal{K}_s^{k,N}$. Then there exists $g \in \mathcal{S}_s^{*,N}(\frac{k-1}{k})$ such that

$$\Re\left\{\frac{zf'(z)}{G_{k,N}(z)}\right\} > 0,$$

where $G_{k,N}(z) = z + \sum_{m=2}^{\infty} B_m z^m$. This implies there exists $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ such that

$$\frac{zf'(z)}{G_{k,N}(z)} = p(z).$$

Using (2.3.12),

$$\begin{aligned} 12(a_3 - \gamma a_2^2) &= 4(B_3 + p_1 B_2 + p_2) - 3\gamma(B_2^2 + 2p_1 B_2 + p_1^2) \\ &= 4B_3 + 4p_1 B_2 + 4p_2 - 3\gamma B_2^2 - 6\gamma p_1 B_2 - 3\gamma p_1^2. \end{aligned}$$

Utilizing the triangle inequality and Lemma 1.2.1,

$$\begin{aligned} 12|a_3 - \gamma a_2^2| &\leq |4B_3 - 3\gamma B_2^2| + |4 - 6\gamma||B_2||p_1| + 4|p_2| + |4p_2 - 3\gamma p_1^2| \\ &\leq |4B_3 - 3\gamma B_2^2| + 4|2 - 3\gamma||B_2| + 8 + |4p_2 - 3\gamma p_1^2|. \end{aligned}$$

Note that

$$|4p_2 - 3\gamma p_1^2| = 4|p_2 - vp_1^2|,$$

where $v = \frac{3}{4}\gamma$. Since $v \in [0,1]$ for all $\gamma \in [0,1)$, then by Lemma 2.2.3,

$$|4p_2 - 3\gamma p_1^2| = 4|p_2 - vp_1^2| \leq 4(2) = 8.$$

Thus,

$$\begin{aligned} 12|a_3 - \gamma a_2^2| &\leq |4B_3 - 3\gamma B_2^2| + 4|2-3\gamma||B_2| + 8 + 8 \\ &= |4B_3 - 3\gamma B_2^2| + 4|2-3\gamma||B_2| + 16. \end{aligned} \tag{2.3.15}$$

Substituting (2.3.14) into (2.3.15) and going case by case gives:

$k = 1,\ N = 1$:

$$|a_3 - \gamma a_2^2| \leq \frac{1}{12}\big(|4b_3 - 3\gamma b_2^2| + 4|2-3\gamma||b_2| + 16\big).$$

By Lemma 2.2.4,

$$|4b_3 - 3\gamma b_2^2| = 4|b_3 - \mu b_2^2| \leq 4\max\{1, |3-4\mu|\},$$

where $\mu = \frac{3}{4}\gamma$. Therefore,

$$\begin{aligned} |a_3 - \gamma a_2^2| &\leq \frac{1}{12}\big(4\max\{1,|3-4\mu|\} + 4|2-3\gamma||b_2| + 16\big) \\ &\leq \frac{1}{3}\max\{1,|3-4\mu|\} + \frac{4}{3} + \frac{2}{3}|2-3\gamma|; \end{aligned}$$

$k = 1,\ N = 2$:

$$|a_3 - \gamma a_2^2| \leq \frac{1}{12}\big(|4b_3| + 16\big) \leq \frac{5}{3};$$

$k = 2,\ N = 1$:

$$|a_3 - \gamma a_2^2| \le \frac{1}{12}\big(|4(2b_3 - b_2^2)| + 16\big) \le \frac{5}{3};$$

$k = N = 2$:

$$|a_3 - \gamma a_2^2| \le \frac{1}{12}\big(|4(2b_3)| + 16\big) \le \frac{5}{3};$$

Otherwise:

$$|a_3 - \gamma a_2^2| \le \frac{1}{12}\big(16\big) \le \frac{4}{3}.$$

□

Theorem 2.3.6. *The radius of convexity R_c for the class $\mathcal{K}_s^{k,N}$ is the positive real root of the equation*

$$1 - 2r - r^2 - r^{\frac{2m}{Nk}} - 2r^{\frac{2m}{Nk}+1} + r^{\frac{2m}{Nk}+2} = 0.$$

Proof. Suppose $f \in \mathcal{K}_s^{k,N}$ where $k, N \in \mathbb{N}$. Then there exists a $p \in \mathcal{P}$ such that

$$\frac{zf'(z)}{G_{k,N}(z)} = p(z).$$

Logarithmic differentation gives

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + \frac{zG'_{k,N}(z)}{G_{k,N}(z)}. \tag{2.3.16}$$

When $|z| = r$,

$$\frac{1 - r^{\frac{2m}{Nk}}}{1 + r^{\frac{2m}{Nk}}} \le \Re\left\{\frac{zG'_{k,N}(z)}{G_{k,N}(z)}\right\} \le \left|\frac{zG'_{k,N}(z)}{G_{k,N}(z)}\right| \le \frac{1 + r^{\frac{2m}{Nk}}}{1 - r^{\frac{2m}{Nk}}}. \tag{2.3.17}$$

Using (2.3.16), (2.3.17) and Lemma 2.2.2 when $\delta = 0$,

$$\begin{aligned}\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} &= \Re\left\{\frac{zp'(z)}{p(z)}\right\} + \Re\left\{\frac{zG'_{k,N}(z)}{G_{k,N}(z)}\right\} \\ &\geq \frac{-2r}{1-r^2} + \frac{1 - r^{\frac{2m}{Nk}}}{1 + r^{\frac{2m}{Nk}}} \\ &= \frac{-2r(1 + r^{\frac{2m}{Nk}}) + (1-r^2)(1 - r^{\frac{2m}{Nk}})}{(1-r^2)(1 + r^{\frac{2m}{Nk}})} \\ &= \frac{1 - 2r - r^2 - r^{\frac{2m}{Nk}} - 2r^{\frac{2m}{Nk}+1} + r^{\frac{2m}{Nk}+2}}{(1-r^2)(1 + r^{\frac{2m}{Nk}})}.\end{aligned}$$

The next step is to find under which values of r such that $\Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0$. Evidently, $(1-r^2)(1+r^{\frac{2m}{Nk}}) > 0$ for any $N, k \in \mathbb{N}$ as $1 - r^2 > 0$ and $1 + r^{\frac{2m}{Nk}} > 0$ in $0 \leq r < 1$. Let $F(r) = 1 - 2r - r^2 - r^{\frac{2m}{Nk}} - 2r^{\frac{2m}{Nk}+1} + r^{\frac{2m}{Nk}+2}$, $0 \leq r < 1$. A simple differentiation gives $F'(r) = -2 - 2r - \frac{2m}{Nk}r^{\frac{2m}{Nk}-1} - 2(\frac{2m}{Nk}+1)r^{\frac{2m}{Nk}} + (\frac{2m}{Nk}+2)r^{\frac{2m}{Nk}+1}$ and upon rearranging $F'(r) = -2(1+r) - r^{\frac{2m}{Nk}-1}\big[2r(\frac{2m}{Nk} - r) + 2r + \frac{2m}{Nk}(1-r^2)\big] < 0$ for all $0 \leq r < 1$. This means that F is a monotonically decreasing function within $r \in [0, 1)$. Since $F(0) = 1$ and $F(1) = 4$, this implies that there exists a root R_c wihin $(0, 1)$ such that $F(R_c) = 0$. Therefore, $\Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0$, i.e. f is convex, for $0 \leq |z| = r \leq R_c$. $\square$

2.4 Class $\mathcal{K}_{s,p}^{k,N}$

This section present properties obtained for the class $\mathcal{K}_{s,p}^{k,N}$. Before proceeding further, we present a theorem which shows that $G_{k,N,p}$ is a starlike p-valent function.

Theorem 2.4.1. *Let* $g \in \mathcal{S}_{s,p}^{*,N}\big(\frac{(k-1)p}{k}\big)$ *for* $k, N \in \mathbb{N}$. *Then*

$$G_{k,N,p}(z) = z^p + \sum_{n=1}^{\infty} B_{Nn+p} z^{Nn+p} \in \mathcal{S}_p^*. \qquad (2.4.1)$$

The coefficients B_{Nn+p} can be written as

$$B_{Nn+p} = \begin{cases} b_{Nn+p} & , \ k = 1, \\ \\ \chi_{p,k-1} & , \ k \geq 2, \end{cases}$$

where

$$\chi_{p,k-1} = \sum_{q_{k-1}=1}^{n+1} \Psi_p \Bigg\{ \sum_{q_{k-2}=1}^{q_{k-1}} \varphi_{k-2} \Bigg\{ \cdots \Bigg\{ \sum_{q_2=1}^{q_3} \varphi_2 \Bigg\{ \sum_{q_1=1}^{q_2} \varphi_1 b_{N(q_1-1)+p} \Bigg\} \Bigg\} \cdots \Bigg\} \Bigg\},$$

with

$$\Psi_p = \varepsilon^{N(k-1)(n+1-q_{k-1})} b_{N(n+1-q_{k-1})+p}$$

and

$$\varphi_j = \varepsilon^{Nj(q_{j+1}-q_j)} b_{N(q_{j+1}-q_j)+p},$$

for $1 \leq j \leq k-2$.

Proof. The approach of finding B_{Nn+p} is similar to finding B_{Nn+1} in the proof of Theorem 2.3.1. Let $g \in \mathcal{S}^{*,N}_{s,p}\left(\frac{(k-1)p}{k,N}\right)$ where $k, N, p \in \mathbb{N}$. From (2.1.8),

$$\begin{aligned} G_{k,N,p}(z) &= \frac{1}{z^{(k-1)p}} \prod_{j=0}^{k-1} \varepsilon^{-jp} g_N(\varepsilon^j z) \\ &= \frac{1}{z^{(k-1)p}} \prod_{j=0}^{k-1} \varepsilon^{-jp} \left[\frac{1}{N} \sum_{m=0}^{N-1} \gamma^{-mp} g(\gamma^m \varepsilon^j z) \right] \\ &= \frac{1}{z^{(k-1)p}} \prod_{j=0}^{k-1} \varepsilon^{-jp} \left[\frac{1}{N} \sum_{m=0}^{N-1} \gamma^{-mp} \left\{ \gamma^{mp} \varepsilon^{jp} z^p + \sum_{n=1}^{\infty} b_{n+p} \gamma^{m(n+p)} \varepsilon^{j(n+p)} z^{n+p} \right\} \right] \\ &= \frac{1}{z^{(k-1)p}} \prod_{j=0}^{k-1} \left[\frac{1}{N} \sum_{m=0}^{N-1} \left\{ z^p + \sum_{n=1}^{\infty} b_{n+p} \gamma^{mn} \varepsilon^{jn} z^{n+p} \right\} \right] \\ &= \frac{1}{z^{(k-1)p}} \prod_{j=0}^{k-1} \left[z^p + \sum_{n=1}^{\infty} \phi_n \varepsilon^{jn} b_{n+p} z^{n+p} \right], \end{aligned}$$

where $\phi_n = \frac{1}{N}\left(1+\gamma^n+\gamma^{2n}+...+\gamma^{(N-1)n}\right)$ which we can simplify to

$$\phi_n = \begin{cases} 1 & , \frac{n-p}{N} \in \mathbb{N}, \\ \\ 0 & , \frac{n-p}{N} \notin \mathbb{N}. \end{cases}$$

This shows that $\phi_p = \phi_{N+p} = ... = \phi_{Nm+p} = 1$ for any m positive integer and $\phi_{n+p} = 0$ otherwise. With this, we can rewrite $\sum\limits_{n=1}^{\infty} \phi_n \varepsilon^{jn} b_{n+p} z^{n+p}$ to $\sum\limits_{m=1}^{\infty} \varepsilon^{Nmj} b_{Nm+p} z^{Nm+p}$, which upon re-indexing gives

$$G_{k,N,p}(z) = \frac{1}{z^{(k-1)p}} \prod_{j=0}^{k-1} \left[z^p + \sum_{n=1}^{\infty} \phi_n \varepsilon^{Nnj} b_{Nn+p} z^{Nn+p} \right].$$

Let $Q_j(z) = z^p + \sum\limits_{n=1}^{\infty} \phi_{n+p} \varepsilon^{Nnj} b_{Nn+p} z^{Nn+p}$ for $0 \leq j \leq k-1$ and denote $E_j(z)$ as follows

$$\begin{aligned} E_0(z) &= Q_0(z), \\ E_1(z) &= \frac{1}{z^p} Q_1(z) Q_0(z), \\ E_2(z) &= \frac{1}{z^{2p}} Q_2(z) Q_1(z) Q_0(z), \\ &... \\ E_j(z) &= \frac{1}{z^{jp}} Q_j(z) Q_{j-1}(z) ... Q_1(z) Q_0(z). \end{aligned}$$

By induction, $E_j(z) = \frac{1}{z^p} Q_j(z) E_{j-1}(z)$ which upon expansion gives

$$\begin{aligned} E_j(z) &= \frac{1}{z^p} Q_j(z) E_{j-1}(z) \\ &= \frac{1}{z^{jp}} Q_j(z) Q_{j-1}(z) ... Q_1(z) Q_0(z) \\ &= z^p + \sum_{n=1}^{\infty} \chi_{p,j} z^{Nn+p}, \end{aligned}$$

where

$$\chi_{p,j} = \sum_{q_j=1}^{n+1} \Psi_p \Bigg\{ \sum_{q_{j-1}=1}^{q_j} \varphi_{j-1} \Bigg\{ \ldots \Bigg\{ \sum_{q_2=1}^{q_3} \varphi_2 \Bigg\{ \sum_{q_1=1}^{q_2} \varphi_1 b_{N(q_1-1)+p} \Bigg\} \Bigg\} \ldots \Bigg\} \Bigg\},$$

with $\Psi_p = \varepsilon^{Nj(n+1-q_j)} b_{N(n+1-q_j)+p}$ and $\varphi_j = \varepsilon^{Nj(q_{j+1}-q_j)} b_{N(q_{j+1}-q_j)+p}$. Thus,

$$\begin{aligned} G_{k,N,p}(z) &= \frac{1}{z^{(k-1)p}} \prod_{j=0}^{k-1} \left[z^p + \sum_{n=1}^{\infty} \varepsilon^{jNn} b_{Nn+p} z^{Nn+p} \right] \\ &= E_{k-1}(z) \\ &= z^p + \sum_{n=1}^{\infty} B_{Nn+p} z^{Nn+p}, \end{aligned}$$

where

$$B_{Nn+p} = \begin{cases} b_{Nn+p} & , \ k = 1, \\ \chi_{p,k-1} & , \ k \geq 2. \end{cases}$$

As $g \in \mathcal{S}^{*,N}_{s,p}\left(\frac{(k-1)p}{k}\right)$, then by Lemma (2.2.5), $G_{k,N,p} \in \mathcal{S}^{*}_{p}$. $\square$

Remark 2.4.1. With Theorem 2.4.1, it is simple to state that all $f \in \mathcal{K}^{k,N}_{s,p}$ are also in $\mathcal{K}_p$. Therefore, $\mathcal{K}^{k,N}_{s,p}$ is a subclass of $\mathcal{K}_p$.

Theorem 2.4.2. *For $f \in \mathcal{K}^{k,N}_{s,p}$ where $p \in \mathbb{N}$, the distortion and growth bounds are given, respectively, by*

$$\frac{p(1-r)r^{p-1}}{(1+r)(1+r)^{2p}} \leq |f'(z)| \leq \frac{p(1+r)r^{p-1}}{(1-r)(1-r)^{2p}},$$

and

$$\int_0^r \frac{p(1-\rho)\rho^{p-1}}{(1+\rho)(1+\rho)^{2p}} d\rho \leq |f(z)| \leq \int_0^r \frac{p(1+\rho)\rho^{p-1}}{(1-\rho)(1-\rho)^{2p}} d\rho,$$

where $|z| = r < 1$.

Proof. Suppose $f \in \mathcal{K}_{s,p}^{k,N}$ for $k, N, p \in \mathbb{N}$. Then there exists a function $g \in \mathcal{S}_{s,p}^{*,N}(\frac{p(k-1)}{k})$ such that (2.1.7) holds. As $G_{k,N,p} \in \mathcal{S}_p^*$ by Theorem 2.4.1, then

$$\frac{r^p}{(1+r)^{2p}} \leq |G_{k,N,p}(z)| \leq \frac{r^p}{(1-r)^{2p}}, \tag{2.4.2}$$

and there exists a $p \in \mathcal{P}$ such that

$$\frac{zf'(z)}{pG_{k,N,p}(z)} = p(z). \tag{2.4.3}$$

Using (1.2.2), (2.4.2) and (2.4.3),

$$\frac{p(1-r)r^{p-1}}{(1+r)(1+r)^{2p}} \leq |f'(z)| \leq \frac{p(1+r)r^{p-1}}{(1-r)(1-r)^{2p}}. \tag{2.4.4}$$

From (2.4.4), the upper bound for $|f(z)|$ is

$$|f(z)| = \Big|\int_0^z f'(\rho)d\rho\Big| \leq \int_0^r |f'(\rho)|d\rho \leq \int_0^r \frac{p(1+\rho)\rho^{p-1}}{(1-\rho)(1-\rho)^{2p}}d\rho.$$

To prove the lower bound, it is sufficient to prove that it holds for the point $z_0 \in U$ with $|z_0| = r$ $(0 < r < 1)$ such that $|f(z_0)| = \min\big\{|f(z)| : |z| = r\big\}$, otherwise $|f(z)| \geq |f(z_0)|$ with $|z| = r,$ $(0 \leq r < 1)$. The image of the closed line segment $\big[0, f(z_0)\big]$ is a piece of arc L in the unit disc $\big\{|z| \leq r\big\}$, then

$$|f(z_0)| = \int_{f(L)} |dw| = \int_L |f'(\rho)|d\rho \geq \int_L \frac{p(1-\rho)\rho^{p-1}}{(1-\rho)(1-\rho)^{2p}}d\rho.$$

Therefore,

$$\int_0^r \frac{p(1-\rho)\rho^{p-1}}{(1+\rho)(1+\rho)^{2p}}d\rho \leq |f(z)| \leq \int_0^r \frac{p(1+\rho)\rho^{p-1}}{(1-\rho)(1-\rho)^{2p}}d\rho.$$

$\square$

Remark 2.4.2. The bounds in Theorem 2.4.2 are only sharp for the case $k = N = 1$.

Theorem 2.4.3. *Let $f \in \mathcal{K}_{s,p}^{k,N}$ for $k, N, p \in \mathbb{N}$. Then*

$$|a_n| \leq \begin{cases} \frac{p}{n}\Big[\zeta_{d_1} + 2\Big(1 + \sum\limits_{d=1}^{d_1-1} \zeta_d\Big)\Big] & , \ n = Nd_1 + p \text{ for } d_1 \geq 1, \\ \frac{2p}{n}\left(1 + \sum\limits_{d=1}^{d_2} \zeta_d\right) & , \ Nd_2 + p < n < N(d_2+1) + p \text{ for } d_2 \geq 0, \end{cases}$$

where $\zeta_d = \frac{2p(2p+1)...(2p+d-1)}{d!}$ for positive integers d.

Proof. Let $f \in \mathcal{K}_{s,p}^{k,N}$. Then there exists $g \in \mathcal{S}_{s,p}^{*,N}\left(\frac{p(k-1)}{k}\right)$ such that (2.1.7) holds. There exists $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ such that (2.4.3) holds. Using series representations for p, f and $G_{k,N,p}$ in (2.4.3) gives

$$\begin{aligned} & pz^p + (p+1)a_{p+1}z^{p+1} + (p+2)a_{p+2}z^{p+2} + ... \\ = & \, p\Big[z\big(1 + p_1 z + p_2 z^2 + ...\big) + \\ & B_{N+p}z^{N+p}\big(1 + p_1 z + p_2 z^2 + ... + p_n z^n + ...\big) + \\ & B_{2N+p}z^{2N+p}\big(1 + p_1 z + p_2 z^2 + ... + p_n z^n + ...\big) + ... + \\ & B_{Nn+p}z^{Nn+p}\big(1 + p_1 z + p_2 z^2 + ... + p_n z^n + ...\big) + ...\Big], \end{aligned}$$

which upon simplification

$$a_n = \begin{cases} \frac{p}{n}\Big(p_n + \sum\limits_{d=1}^{d_1-1} p_{n-(Nd+p)}B_{Nd+p} + B_n\Big) & , \ n = Nd_1 + p \text{ for } d_1 \geq 1, \\ \frac{p}{n}\Big(p_n + \sum\limits_{d=1}^{d_2} p_{n-(Nd+p)}B_{Nd+p}\Big) & , \ Nd_2 + p < n < N(d_2+1) + p \text{ for } d_2 \geq 0. \end{cases}$$

Using Lemma 1.2.1, the following is obtained.

$$|a_n| \leq \begin{cases} \frac{p}{n}\left[|B_n| + 2\Big(1 + \sum_{d=1}^{d_1-1} |B_{Nd+p}|\Big)\right] & , \ n = Nd_1 + p \text{ for } d_1 \geq 1, \\ \frac{2p}{n}\left(1 + \sum_{d=1}^{d_2} |B_{Nd+p}|\right) & , \ Nd_2 + p < n < N(d_2+1) + p \\ & \quad \text{for } d_2 \geq 0. \end{cases} \tag{2.4.5}$$

From the right-hand side (2.4.2),

$$\begin{aligned} |G_{k,N,p}(z)| &\leq \frac{r^p}{(1-r)^{2p}} \\ &= r^p + \sum_{n=1}^{\infty} \frac{2p(2p+1)...(2p+n-1)}{n!} r^{n+p}, \end{aligned}$$

shows that

$$|B_{Nd+p}| \leq \zeta_d = \frac{2p(2p+1)...(2p+d-1)}{d!}, \tag{2.4.6}$$

for d positive integers. Substituting (2.4.6) into (2.4.5) gives

$$|a_n| \leq \begin{cases} \frac{p}{n}\left[\zeta_{d_1} + 2\Big(1 + \sum_{d=1}^{d_1-1} \zeta_d\Big)\right] & , \ n = Nd_1 + p \text{ for } d_1 \geq 1, \\ \frac{2p}{n}\left(1 + \sum_{d=1}^{d_2} \zeta_d\right) & , \ Nd_2 + p < n < N(d_2+1) + p \text{ for } d_2 \geq 0. \end{cases}$$

□

Remark 2.4.3. The coefficient bounds in Theorem 2.4.3 is only sharp for the case $k = N = 1$.

CHAPTER 3: PROPERTIES OF FUNCTIONS ASSOCIATED WITH THE STRUVE FUNCTION

3.1 Introduction

Special functions are functions that have established their names and notations in history due to their importance in various fields of science, namely mathematical analysis (both real and complex), funnctional analysis and physics, for their various applications and usefulness. Many of these special functions appear as integral of elementary functions or as solutions of differential equations. Some examples of special functions are the generalized hypergeometric function, Bessel functions $\big(J_\alpha(x)\big)$ and Gamma function $\big(\Gamma(n) = (n-1)!\big)$. In response to Louis de Brange proving the Milin conjecture by utilizing the hypergeometric function, the mathematics community became interested in function theory which culminated into a sizable literature that is dedicated to studying the geometric properties of divergent types of special functions, such as Bessel functions, Struve functions and the generalized hypergeometric functions to name a few. Sufficient conditions on the parameters for special functions to belong to a certain subclass of $\mathcal{S}$, for example $\mathcal{S}^*$, $\mathcal{C}$ or $\mathcal{K}$, were also determined by many authors. In this chapter, we will be focusing solely on a special function that has recently been gaining a lot of attention, the Struve function.

The following equation

$$H_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\frac{3}{2})\Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \tag{3.1.1}$$

is defined as the Struve function where $p \in \mathbb{C}$ and is the particular solution to the non-homogeneous differential equation

$$z^2w''(z) + zw'(z) + (z^2 - p^2)w(z) = \frac{4(\frac{z}{2})^{p+1}}{\sqrt{\pi}\Gamma(p + \frac{1}{2})}, \tag{3.1.2}$$

where the homogeneous part (as in the left-hand side of the equation) is Bessel's equation. This function was first introduced by Struve (1882) (see Watson, 1966, p. 328). Struve introduced this function to describe the luminous line's intensity. He also used it to describe solutions of Bessel's differential equation thus widely utilized in both physics and mathematics by many authors. However, Struve only studied the special functions of this type in the orders of zero and unity. Siemon and Walker have studied the properties of the general function at some length (Watson, 1966).

A modified version of the Struve function, henceforth known as the modified Struve function, is defined by

$$\begin{aligned} L_p(z) &= -ie^{-ip\frac{\pi}{2}}H_p(iz) \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n + \frac{3}{2})\Gamma(p + n + \frac{3}{2})}\left(\frac{z}{2}\right)^{2n+p+1}, \end{aligned} \tag{3.1.3}$$

and is the particular solution to the non-homogeneous differential equation

$$z^2w''(z) + zw'(z) - (z^2 + p^2)w(z) = \frac{4(\frac{z}{2})^{p+1}}{\sqrt{\pi}\Gamma(p + \frac{1}{2})}. \tag{3.1.4}$$

This function was first introduced and studied (in the case of $p = 0$) by Nicholson (1911) (see Watson, 1966, p. 329).

As with all special functions, the application of Struve function (both normal and modified) can be found in multiple areas of physics and applied mathematics. For examples, they can be found in unsteady aerodynamics (Shaw, 1985) and optical diffraction (Levine & Schwinger, 1948).

As the Struve function and modified Struve function are very similar in form, this raises a question as to whether generalizing them is possible. Following this, several generalizations of the Struve functions have been introduced and their investigations conducted by many authors. Most notably, Bhowmick in his 1962 and 1963 papers, Joshi (1967) and Kanth (1981). Yağmur & Orhan (2013) introduced a generalization of the Struve function which they defined on the complex plane as

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{\Gamma(n+\frac{3}{2})\Gamma(p+n+\frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1},$$

where $p, c, b \in \mathbb{C}$ and is known as the generalized Struve function of order p. This function is also the particular solution of the non-homogeneous differential equation

$$z^2 w''(z) + bzw'(z) + \Big[cz^2 - p^2 + (1-b)p\Big] w(z) = \frac{4(\frac{z}{2})^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}. \qquad (3.1.5)$$

Evidently, it can be seen that under certain conditions (3.1.5) will reduce to either (3.1.2) (when $b = 1,\ c = 1$) or (3.1.4) (when $b = 1,\ c = -1$). Similary, following the same conditions as (3.1.5), when $b = 1,\ c = 1$, $w_{p,1,1} = H_p(z)$ and $w_{p,1,-1} = L_p(z)$, when $b = 1,\ c = -1$. With this generalization, it allows both Struve functions to be studied together. Though the function $w_{p,b,c}$ is not univalent in U, the series representation of $w_{p,b,c}$ is convergent at every point in $\mathbb{C}$ (Yağmur & Orhan, 2013). Consider the function

$u_{p,b,c}$ shown below

$$u_{p,b,c}(z) = 2^p\sqrt{\pi}\Gamma\Big(p+\frac{b+2}{2}\Big)z^{\frac{-p-1}{2}}w_{p,b,c}(\sqrt{z}).$$

By using the Pochhammer symbol, $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma(\gamma+1)...(\gamma+n-1)$, $u_{p,b,c}$ can be rewritten as

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty}\frac{(-\frac{c}{4})^n}{(\frac{3}{2})_n(k)_n}z^n = b_0 + b_1z + b_2z^2 + ... ,$$

where $k = p + \frac{b+2}{2} \neq 0, -1, -2, ...$ and

$$b_n = \frac{(-1)^n c^n \Gamma(\frac{3}{2})\Gamma(k)}{4^n\Gamma(n+\frac{3}{2})\Gamma(n+k)},\ (n \geq 0).$$

Thus the function $u_{p,b,c}$ satisfies the condition $u_{p,b,c}(0) = 1$, is analytic in $\mathbb{C}$ and satisfies the differential equation

$$4z^2u''(z) + 2(2p+b+3)zu'(z) + (cz+2pb)u(z) = 2p+b.$$

Yağmur & Orhan obtained sufficient conditions for this generalized Struve function to be univalent, starlike, convex and close-to-convex in their 2013 and 2014 papers. They have also obtained conditions for the function $g_{p,b,c}(z) = zu_{p,b,c}(z)$ to belong to the Hardy space of analytic functions (Yağmur & Orhan, 2014). In the year 2015, Raza & Yağmur defined and investigated a new operator using the generalized Struve function.

The function $T_{p,b,c}$ is a Hadamard product (or convolution) of $g_{p,b,c}(z) = zu_{p,b,c}(z)$ and $f \in \mathcal{A}$, i.e.

$$T_{p,b,c}(z) = (g_{p,b,c} * f)(z) = z + \sum_{n=2}^{\infty}\frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}}a_nz^n, \tag{3.1.6}$$

where $p, c, b \in \mathbb{C}$ and $k = p + \frac{b+2}{2}$. Promptly motivated by Orhan & Yağmur seeking conditions or bounds for the function $g_{p,b,c}$ to be under a certain subclass of $\mathcal{S}$, we strove to find properties not dissimilar to that of $g_{p,b,c}$ for the function $T_{p,b,c}$ of which the function can be found in Raza & Yağmur (2015). As such, this chapter seeks sufficient conditions for this function to be, but not limited to, univalent, starlike or convex.

3.2 Preliminary Results

The following preliminary results are required to establish the main properties.

Lemma 3.2.1. *(Xanthopoulos, 1993) The inequality* $\Re\{f'(z)\} > 0$ *is true if an analytic function* f *in* $\mathcal{A}$ *satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \ (z \in U),$$

where M *is the solution of the equation* $\cos\ M = M$.

Lemma 3.2.2. *(MacGregor, 1962) The function* f *is univalent if* $\Re\{f'(z)\} > 0$ *where* $f \in \mathcal{A}$.

Lemma 3.2.3. *(MacGregor, 1963) The function* f *is starlike and univalent for* $|z| < \frac{1}{2}$ *if* $f \in \mathcal{A}$ *satisfies*

$$\left|\frac{f(z)}{z} - 1\right| < 1, \ (z \in U).$$

Lemma 3.2.4. *(MacGregor, 1964) The function* f *is convex for* $|z| < \frac{1}{2}$ *if* $f \in \mathcal{A}$ *satisfies*

$$|f'(z) - 1| < 1, \ (z \in U).$$

Lemma 3.2.5. *(Owa & Srivastava, 1987) The function* $f \in \mathcal{S}^*(\alpha)$ *if* $f \in \mathcal{A}$ *and*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\beta} \left|\frac{zf''(z)}{f'(z)}\right|^{\beta} < (1-\alpha)^{1-2\beta}\left(1 - \frac{3\alpha}{2} + \alpha^2\right)^{\beta},$$

for some $\alpha \in [0, \frac{1}{2}]$ *and* $\beta \geq 0$ *for all* $z \in U$.

Proposition 3.2.6. *(Cardano, 1545) Consider* $x \in \mathbb{R}$ *and the the quartic equation* $Y(x)$ *of the form:*

$$Y(x) = rx^4 + sx^3 + tx^2 + ux + v,$$

where $r, s, t, u, v \in \mathbb{R}$*. Then the solutions of* $Y(x) = 0$ *are given as*

$$x_{1,2} = -\frac{s}{4r} - S \pm \frac{1}{2}\sqrt{-4S^2 - 2a + \frac{d}{S}},$$
$$x_{3,4} = -\frac{s}{4r} + S \pm \frac{1}{2}\sqrt{-4S^2 - 2a - \frac{d}{S}},$$

where a *and* d *are*

$$a = \frac{8rt - 3s^2}{8r^2},$$
$$d = \frac{s^3 - 4rst + 8r^2u}{8r^3},$$

and where

$$S = \frac{1}{2}\sqrt{-\frac{2}{3}a + \frac{1}{3r}\Big(Q + \frac{\Delta_0}{Q}\Big)},$$
$$Q = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}},$$

with

$$\Delta_0 = t^2 - 3su + 12rv,$$

$$\Delta_1 = 2t^3 - 9stu + 27s^2v + 27ru^2 - 72rtv.$$

3.3 Univalency, Starlikenss and Convexity of $T_{p,b,c}$

The following results illustrate sufficient conditions for the function $T_{p,b,c}$ to be univalent, starlike or convex, respectively.

Theorem 3.3.1. *Let $T_{p,b,c}$ be defined by (3.1.6) and $f \in \mathcal{A}$. If $p, c, b \in \mathbb{C}$, $k_1 = p + \frac{b+2}{2}$ and*

$$\begin{aligned} k_1 &> \frac{7M + 2 + \phi + \sqrt{\frac{16M^3+(17\phi+100)M^2+(40\phi+80)M+12\phi+16-\phi^3}{\phi}}}{24M}|c| \\ &\approx 1.098143352|c|, \end{aligned}$$

where M is the solution of the equation $\cos M = M$ and

$$\begin{aligned} \phi &= \sqrt{\frac{1}{3}\left[\frac{49}{\varphi}M^3 + \left(\frac{28}{\varphi} + 17\right)M^2 + \left(\frac{4}{\varphi} + \varphi + 40\right)M + 12\right]} \\ &\approx 3.469588501, \end{aligned} \tag{3.3.1}$$

with

$$\begin{aligned} \varphi^3 &= 24\sqrt{3M}\sqrt{1029M^4 + 5113M^3 + 3138M^2 + 540M + 8} \\ &\quad - 343M^3 - 2886M^2 - 948M - 8 \\ &\approx -28.2795216, \end{aligned} \tag{3.3.2}$$

then $\Re\{T'_{p,b,c}(z)\} > 0$ for all $z \in U$.

Proof. Let $f \in \mathcal{A}$. Using the inequalities $(k)_n \geq k^n$, $(\frac{3}{2})_n \geq \frac{3}{2}n$ $(n \in \mathbb{N})$, and

$|z_1 + z_2| \leq |z_1| + |z_2|$ for $|z| < 1$,

$$\begin{aligned}
\left| T'_{p,b,c}(z) - \frac{T_{p,b,c}(z)}{z} \right| &= \left| \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}} (n-1) a_n z^{n-1} \right| \\
&\leq \frac{2}{3} \sum_{n=2}^{\infty} n \left(\frac{|c|}{4k} \right)^{n-1} \\
&= \frac{2}{3} \left(\frac{|c|}{4k} \right) \left[\sum_{n=1}^{\infty} \left(\frac{|c|}{4k} \right)^{n-1} + \sum_{n=1}^{\infty} n \left(\frac{|c|}{4k} \right)^{n-1} \right] \\
&= \frac{2}{3} \left(\frac{|c|}{4k} \right) \left[\frac{1}{1 - \frac{|c|}{4k}} + \frac{1}{(1 - \frac{|c|}{4k})^2} \right] \\
&= \frac{16k|c| - 2|c|^2}{3(4k - |c|)^2}.
\end{aligned} \tag{3.3.3}$$

To ensure that $16k|c| - 2|c|^2 > 0$, we let $k > \frac{|c|}{8}$. Then, the inequalities $(k)_n \geq k^n$, $(\frac{3}{2})_n \geq (\frac{3}{2})^n$ $(n \in \mathbb{N})$, and $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ for $|z| < 1$ gives

$$\begin{aligned}
\left| \frac{T_{p,b,c}(z)}{z} \right| &= \left| 1 + \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}} a_n z^{n-1} \right| \\
&\geq 1 - \sum_{n=2}^{\infty} (n+1) \left(\frac{|c|}{6k} \right)^{n-1} \\
&= 1 - \frac{|c|}{6k} \left[\sum_{n=1}^{\infty} \left(\frac{|c|}{6k} \right)^{n-1} + \sum_{n=1}^{\infty} n \left(\frac{|c|}{6k} \right)^{n-1} \right] \\
&= 1 - \frac{|c|}{6k} \left[\frac{1}{1 - \frac{|c|}{6k}} + \frac{1}{(1 - \frac{|c|}{6k})^2} \right] \\
&= \frac{2(18k^2 - 12k|c| + |c|^2)}{(6k - |c|)^2}.
\end{aligned} \tag{3.3.4}$$

Applying the same argument as before, we let $k > \frac{2+\sqrt{2}}{6}|c|$ to ensure that $18k^2 - 12k|c| + |c|^2 > 0$. Merging (3.3.3) and (3.3.4) gives

$$\begin{aligned}
\left| \frac{zT'_{p,b,c}(z)}{T_{p,b,c}(z)} - 1 \right| &= \left| T'_{p,b,c}(z) - \frac{T_{p,b,c}(z)}{z} \right| \left| \frac{z}{T_{p,b,c}(z)} \right| \\
&< \left(\frac{16k|c| - 2|c|^2}{3(4k - |c|)^2} \right) \left(\frac{(6k - |c|)^2}{2(18k^2 - 12k|c| + |c|^2)} \right) \\
&= \Phi(k, |c|).
\end{aligned} \tag{3.3.5}$$

We now find the range of k_1 such that the following inequality is satisfied.

$$\Phi(k_1, |c|) < M, \tag{3.3.6}$$

where M is the solution of the equation $\cos\ M = M$. Expanding (3.3.6) implies

$$\begin{aligned}
&(8k_1|c| - |c|^2)(6k_1 - |c|)^2 < 3M(4k_1 - |c|)^2(18k_1^2 - 12k_1|c| + |c|^2) \\
\Rightarrow\ & 288k_1^3|c| - 132k_1^2|c|^2 + 20k_1|c|^3 - |c|^4 \\
& < 3M(288k_1^4 - 336k_1^3|c| + 130k_1^2|c|^2 - 20k_1|c|^3 + |c|^4) \\
\Rightarrow\ & F(k_1) = Mk_1^4 - \frac{7M+2}{6}k_1^3|c| + \frac{65M+22}{144}k_1^2|c|^2 - \frac{15M+5}{216}k_1|c|^3 + \frac{3M+1}{864}|c|^4 \\
& \qquad > 0.
\end{aligned} \tag{3.3.7}$$

Proposition 3.2.6 is utilised to find the zeros of F. First let,

$$Mx^4 - \frac{7M+2}{6}|c|x^3 + \frac{65M+22}{144}|c|^2x^2 - \frac{15M+5}{216}|c|^3x + \frac{3M+1}{864}|c|^4 = 0.$$

Let

$$
\begin{aligned}
a &= -\frac{17M^2+40M+12}{288M^2}|c|^2,\\
d &= -\frac{4M^3+25M^2+20M+4}{864M^3}|c|^3,\\
\Delta_0 &= \frac{49M^2+28M+4}{20736}|c|^4,\\
\Delta_1 &= -\frac{343M^3+2886M^2+948M+8}{1492992}|c|^6,\\
\Delta_1^2-4\Delta_0^3 &= \frac{1728M(1029M^4+5113M^3+3138M^2+540M+8)}{1492992^2}|c|^{12},\\
Q &= \frac{\varphi}{144}|c|^2 \text{ where } \varphi \text{ is (3.3.2)},\\
S &= \frac{\phi}{24M}|c| \text{ where } \phi \text{ is (3.3.1)},
\end{aligned}
$$

and solving $\cos\ M = M$ results in the approximation $M \approx 0.7390851332$. With these parameters, the zeros of F are

$$
\begin{cases}
x_{1,2} \in \mathbb{C} \text{ for } -4S^2-2a+\frac{d}{S} < 0,\\
x_{3,4} \in \mathbb{R} \text{ for } -4S^2-2a-\frac{d}{S} > 0.
\end{cases}
$$

Since the zeros of F are real, only $x_{3,4}$ are considered. The real roots $x_{3,4}$ are given by

$$
\begin{aligned}
x_{3,4} &= -\frac{s}{4r}+S\pm\frac{1}{2}\sqrt{-4S-2a-\frac{d}{S}}\\
&= \frac{7M+2+\beta\pm\sqrt{\frac{16M^3+(17\beta+100)M^2+(40\beta+80)M+12\beta+16-\beta^3}{\beta}}}{24M}|c|,
\end{aligned}
$$

the following approximation, which is achieved by simplication, is

$$
\begin{aligned}
x_3 &= \frac{7M+2+\beta-\sqrt{\frac{16M^3+(17\beta+100)M^2+(40\beta+80)M+12\beta+16-\beta^3}{\beta}}}{24M}|c|\\
&\approx 0.1018970715|c|
\end{aligned}
$$

and

$$x_4 = \frac{7M + 2 + \beta + \sqrt{\frac{16M^3 + (17\beta+100)M^2 + (40\beta+80)M + 12\beta + 16 - \beta^3}{\beta}}}{24M}|c|$$

$$\approx 1.09813352|c|.$$

From the above approximations, it is simple to conclude that $F(k_1) > 0$ for $k_1 < x_3$ or $k_1 > x_4$. Since $k > \frac{2+\sqrt{2}}{6}|c|$, therefore

$$k_1 > x_4 = \frac{7M + 2 + \beta + \sqrt{\frac{16M^3 + (17\beta+100)M^2 + (40\beta+80)M + 12\beta + 16 - \beta^3}{\beta}}}{24M}|c|$$

satisfies (3.3.6). By Lemma 3.2.1, $\Re\{T'_{p,b,c}(z)\} > 0$ for all $z \in U$ as $T_{p,b,c}$ is an analytic function. □

Remark 3.3.1. As $T_{p,b,c} \in \mathcal{A}$ and $\Re\{T'_{p,b,c}(z)\} > 0$ when c and k satisfies the hypothesis in Theorem 3.3.1, then $T_{p,b,c}$ is univalent in U by Lemma 3.2.2

Theorem 3.3.2. *Let $T_{p,b,c}$ be defined by (3.1.6) and $f \in \mathcal{A}$. If $p, c, b \in \mathbb{C}$, $k_2 = p + \frac{b+2}{2}$ and*

$$k_2 > \frac{9 + \theta + \sqrt{\frac{-\theta^3 + 69\theta + 212}{\theta}}}{24}|c| \approx 0.970886809|c|,$$

where

$$\theta = \sqrt{23 + \sqrt[3]{-155 + 16\sqrt{91}} + \frac{9}{\sqrt[3]{-155 + 16\sqrt{91}}}} \approx 3.862149964, \qquad (3.3.8)$$

then $T_{p,b,c}$ is starlike in U.

Proof. Let $f \in \mathcal{A}$. With the same goal as the previous theorem, the goal is to find the bounds on k_2 such that $T_{p,b,c}$ is starlike in U. To achieve this, we replace k_1 with k_2 and M

with 1 in (3.3.6) gives the following:

$$\begin{aligned}
&\Phi(k_2, |c|) < 1 \\
&\Rightarrow 4|c|^4 - 80k_2|c|^3 + 522k_2^2|c|^2 - 1296k_2^3|c| + 864k_2^4 > 0 \qquad (3.3.9)\\
&\Rightarrow G(k_2) = k_2^4 - \frac{3}{2}k_2^3|c| + \frac{29}{48}k_2^2|c|^2 - \frac{5}{54}k_2|c|^3 + \frac{1}{216}|c|^4 > 0.
\end{aligned}$$

Proposition 3.2.6 is once again used to find the zeros of

$$G(y) = y^4 - \frac{3}{2}y^3|c| + \frac{29}{48}y^2|c|^2 - \frac{5}{54}y|c|^3 + \frac{1}{216}|c|^4.$$

Putting

$$\begin{aligned}
a &= -\frac{23}{96}|c|^2, \\
d &= -\frac{53}{864}|c|^3, \\
\Delta_0 &= \frac{1}{256}|c|^4, \\
\Delta_1 &= -\frac{155}{55296}|c|^6, \\
\Delta_1^2 - 4\Delta_0^3 &= \frac{91}{11943936}|c|^{12}, \\
Q &= \frac{\sqrt[3]{-155 + 16\sqrt{91}}}{48}|c|^2 \\
S &= \frac{\theta}{24}|c| \text{ where } \theta \text{ is (3.3.8),}
\end{aligned}$$

the zeros of G are

$$\begin{cases} y_{1,2} \in \mathbb{C} \text{ as } -4S^2 - 2a + \frac{d}{S} < 0, \\ y_{3,4} \in \mathbb{R} \text{ as } -4S^2 - 2a - \frac{d}{S} > 0. \end{cases}$$

The real roots $y_{3,4}$ are given by

$$y_{3,4} = -\frac{s}{4r} + S \pm \frac{1}{2}\sqrt{-4S - 2a - \frac{d}{S}} = \frac{9+\theta \pm \sqrt{\frac{-\theta^3+69\theta+212}{\theta}}}{24}|c|,$$

results in the following approximation upon simplification:

$$y_3 = \frac{9+\theta - \sqrt{\frac{-\theta^3+69\theta+212}{\theta}}}{24}|c| \approx 0.100958749|c|,$$
$$y_4 = \frac{9+\theta + \sqrt{\frac{-\theta^3+69\theta+212}{\theta}}}{24}|c| \approx 0.970886809|c|.$$

This concludes that $G(k_2) > 0$ for $k_2 < y_3$ or $k_2 > y_4$. Since $k > \frac{2+\sqrt{2}}{6}|c|$,

$$k_2 > y_4 = \frac{9+\theta + \sqrt{\frac{-\theta^3+69\theta+212}{\theta}}}{24}|c|$$

satisfies (3.3.9). Therefore, by Lemma 3.2.5 when $\alpha = \beta = 0$, $T_{p,b,c}$ is starlike in U. □

Theorem 3.3.3. *Let $T_{p,b,c}$ be defined by (3.1.6) and $f \in \mathcal{A}$. If $p, c, b \in \mathbb{C}$, $k_3 = p + \frac{b+2}{2}$ and $k_3 > \frac{13+\sqrt{130}}{12}|c|$, then $T_{p,b,c}$ is convex in U.*

Proof. Let $f \in \mathcal{A}$. Using the inequalities $(k)_n \geq k^n$, $(\frac{3}{2})_n > \frac{n(n+1)}{2}$ $(n \in \mathbb{N})$ and $|z_1 + z_2| \leq |z_1| + |z_2|$ for $|z| < 1$,

$$
\begin{aligned}
|zT''_{p,b,c}(z)| &= \left| \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}} n(n-1)a_n z^{n-1} \right| \\
&\leq 2\sum_{n=2}^{\infty} n\left(\frac{|c|}{4k}\right)^{n-1} \\
&= 2\left(\frac{|c|}{4k}\right)\left[\sum_{n=1}^{\infty}\left(\frac{|c|}{4k}\right)^{n-1} + \sum_{n=1}^{\infty} n\left(\frac{|c|}{4k}\right)^{n-1}\right] \qquad (3.3.10) \\
&= 2\left(\frac{|c|}{4k}\right)\left[\frac{1}{1-\frac{|c|}{4k}} + \frac{1}{(1-\frac{|c|}{4k})^2}\right] \\
&= \frac{16k|c| - 2|c|^2}{(4k-|c|)^2}.
\end{aligned}
$$

Let $k > \frac{|c|}{8}$ to ensure that $16k|c| - 2|c|^2 > 0$. Using the inequalities $(k)_n \geq k^n$, $(\frac{3}{2})_n > \frac{3(n+1)}{4}$ $(n \in \mathbb{N})$ and $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ for $|z| < 1$,

$$
\begin{aligned}
|T'_{p,b,c}(z)| &= \left| 1 + \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}} n a_n z^{n-1} \right| \\
&\geq 1 - \frac{4}{3}\sum_{n=2}^{\infty} n\left(\frac{|c|}{4k}\right)^{n-1} \\
&= 1 - \frac{4}{3}\left(\frac{|c|}{4k}\right)\left[\sum_{n=1}^{\infty}\left(\frac{|c|}{4k}\right)^{n-1} + \sum_{n=1}^{\infty} n\left(\frac{|c|}{4k}\right)^{n-1}\right] \qquad (3.3.11) \\
&= 1 - \frac{4}{3}\left(\frac{|c|}{4k}\right)\left[\frac{1}{1-\frac{|c|}{4k}} + \frac{1}{(1-\frac{|c|}{4k})^2}\right] \\
&= \frac{48k^2 - 56k|c| + 7|c|^2}{3(4k-|c|)^2}.
\end{aligned}
$$

Let $k > \frac{7+2\sqrt{7}}{12}|c|$ to ensure that $48k^2 - 56k|c| + 7|c|^2 > 0$. Combining (3.3.10) and (3.3.11) gives

$$
\left|\frac{zT''_{p,b,c}(z)}{T'_{p,b,c}(z)}\right| = \left|zT'_{p,b,c}(z)\right|\left|\frac{1}{T_{p,b,c}(z)}\right| < \frac{48k|c| - 6|c|^2}{48k^2 - 56k|c| + 7|c|^2}.
$$

We now find the range of k_3 such that the following is satsified.

$$\frac{48k_3|c| - 6|c|^2}{48k_3^2 - 56k_3|c| + 7|c|^2} < 1. \tag{3.3.12}$$

From (3.3.12),

$$\begin{aligned}
&\frac{48k_3|c| - 6|c|^2}{48k_3^2 - 56k_3|c| + 7|c|^2} < 1 \\
&\Rightarrow 48k_3|c| - 6|c|^2 < 48k_3^2 - 56k_3|c| + 7|c|^2 \\
&\Rightarrow 48k_3^2 - 104k_3|c| + 13|c|^2 > 0 \\
&\Rightarrow k_3^2 - \frac{13}{6}k_3|c| + \frac{13}{48}|c|^2 > 0 \\
&\Rightarrow k_3 < \frac{13 - \sqrt{130}}{12}|c| \approx 0.133187|c|,\ k_3 > \frac{13 + \sqrt{130}}{12}|c| \approx 2.03348|c|.
\end{aligned}$$

Since $k > \frac{7+2\sqrt{7}}{12}|c|$, $k_3 > \frac{13+\sqrt{130}}{12}|c|$ satisfies the hypothesis in Theorem 3.3.2, therefore $zT'_{p,b,c}$ is starlike in U which, by the Alexander theorem, implies that $T_{p,b,c}$ is convex in U. □

The next two results present contraints or conditions of the parameters to ensure that $T_{p,b,c}$ is either starlike or convex in $|z| < \frac{1}{2}$.

Theorem 3.3.4. *Let $T_{p,b,c}$ be defined by (3.1.6) and $f \in \mathcal{A}$. If $p, c, b \in \mathbb{C}$, $k_4 = p + \frac{b+2}{2}$ and $k_4 > \frac{2+\sqrt{2}}{6}|c|$, then $T_{p,b,c}$ is starlike for $|z| < \frac{1}{2}$.*

Proof. Let $f \in \mathcal{A}$. Using the inequalities $(k)_n \geq k^n$, $(\frac{3}{2})_n > (\frac{3}{2})^n$ $(n \in \mathbb{N})$ and

$|z_1 + z_2| \leq |z_1| + |z_2|$ for $|z| < 1$,

$$\begin{aligned}\left|\frac{T_{p,b,c}(z)}{z} - 1\right| &= \left|\sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}} a_n z^{n-1}\right| \\ &\leq \sum_{n=2}^{\infty} n\left(\frac{|c|}{6k}\right)^{n-1} \\ &= \frac{|c|}{6k}\left[\sum_{n=1}^{\infty}\left(\frac{|c|}{6k}\right)^{n-1} + \sum_{n=1}^{\infty} n\left(\frac{|c|}{6k}\right)^{n-1}\right] \\ &= \frac{|c|}{6k}\left[\frac{1}{1-\frac{|c|}{6k}} + \frac{1}{(1-\frac{|c|}{6k})^2}\right] \\ &= \frac{12k|c| - |c|^2}{(6k - |c|)^2}.\end{aligned}$$

Let $k > \frac{|c|}{12}$ to ensure that $12k|c| - |c|^2 > 0$. We now find the range of k_4 such that

$$\left|\frac{T_{p,b,c}(z)}{z} - 1\right| \leq \frac{12k_4|c| - |c|^2}{(6k_4 - |c|)^2} < 1. \tag{3.3.13}$$

Then from (3.3.13)

$$12k_4|c| - |c|^2 < 36k_4^2 - 12k_4|c| + |c|^2$$

$$\Rightarrow 36k_4^2 - 24k_4|c| + 2|c|^2 > 0$$

$$\Rightarrow k_4^2 - \frac{2}{3}k_4|c| + \frac{1}{18}|c|^2 > 0$$

$$\Rightarrow k_4 < \frac{2-\sqrt{2}}{6}|c| \approx 0.097631073|c|,\ k_4 > \frac{2+\sqrt{2}}{6}|c| \approx 0.5690356|c|.$$

Since $k > \frac{|c|}{12}$, $k_4 > \frac{2+\sqrt{2}}{6}|c|$ satisfies (3.3.13). Therefore, by Lemma 3.2.4, $T_{p,b,c}$ is starlike for $|z| < \frac{1}{2}$. □

Theorem 3.3.5. *Let $T_{p,b,c}$ be defined by (3.1.6) and $f \in \mathcal{A}$. If $p, c, b \in \mathbb{C}$, $k_5 = p + \frac{b+2}{2}$ and $k_5 > \frac{4+\sqrt{10}}{6}|c|$, then $T_{p,b,c}$ is convex for $|z| < \frac{1}{2}$.*

Proof. Let $f \in \mathcal{A}$. Using the inequalities $(k)_n \geq k^n$ and $(\frac{3}{2})_n > \frac{3(n+1)}{4}$ $(n \in \mathbb{N})$ and

$|z_1 + z_2| \leq |z_1| + |z_2|$ for $|z| < 1$,

$$
\begin{aligned}
|T'_{p,b,c}(z) - 1| &= \left| \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}} n a_n z^{n-1} \right| \\
&\leq \frac{4}{3} \sum_{n=2}^{\infty} n \left(\frac{|c|}{4k} \right)^{n-1} \\
&= \frac{4}{3} \left(\frac{|c|}{4k} \right) \left[\sum_{n=1}^{\infty} \left(\frac{|c|}{4k} \right)^{n-1} + \sum_{n=1}^{\infty} n \left(\frac{|c|}{4k} \right)^{n-1} \right] \\
&= \frac{4}{3} \left(\frac{|c|}{4k} \right) \left[\frac{1}{1 - \frac{|c|}{4k}} + \frac{1}{(1 - \frac{|c|}{4k})^2} \right] \\
&= \frac{32k|c| - 4|c|^2}{3(4k - |c|)^2}.
\end{aligned}
$$

Let $k > \frac{|c|}{8}$ to ensure that $32k|c| - 4|c|^2 > 0$. Next is to find the range of k_5 such that

$$
|T'_{p,b,c}(z) - 1| \leq \frac{32k_5|c| - 4|c|^2}{3(4k_5 - |c|)^2} < 1. \tag{3.3.14}
$$

From (3.3.14),

$$
\begin{aligned}
& 32k_5|c| - 4|c|^2 < 42k_5^2 - 24k_5|c| + 3|c|^2 \\
\Rightarrow\ & 42k_5^2 - 56k_5|c| + 7|c|^2 > 0 \\
\Rightarrow\ & k_5^2 - \frac{4}{3}k_5|c| + \frac{1}{6}|c|^2 > 0 \\
\Rightarrow\ & k_5 < \frac{4 - \sqrt{10}}{6}|c| \approx 0.13962039|c|,\ k_5 > \frac{4 + \sqrt{10}}{6}|c| \approx 1.193712943|c|.
\end{aligned}
$$

Since $k > \frac{|c|}{8}$, $k_5 > \frac{4+\sqrt{10}}{6}|c|$ satisfies (3.3.14). Therefore, $T_{p,b,c}$ is convex for $|z| < \frac{1}{2}$ by Lemma 3.2.4. □

Finally, in concluding this Chapter, conditions on the parameters are determined so that $T_{p,b,c}$ is either starlike or convex of order α for $\alpha \in [0, \frac{1}{2}]$.

Theorem 3.3.6. *Let $T_{p,b,c}$ be defined by (3.1.6) and $f \in \mathcal{A}$. If $p, c, b \in \mathbb{C}$, $k_6 = p + \frac{b+2}{2}$*

and

$$k_6 > \frac{9-7\alpha+\Psi+\sqrt{\frac{-16\alpha^3+(17\Psi+148)\alpha^2-(74\Psi+328)\alpha+69\Psi+212-\Psi^3}{\Psi}}}{24(1-\alpha)}|c|,$$

where $0 \leq \alpha \leq \frac{1}{2}$ *and*

$$\Psi = \sqrt{\frac{1}{3}\left[-\frac{49}{\psi}\alpha^3+\left(\frac{175}{\psi}+17\right)\alpha^2-\left(\frac{207}{\psi}+\psi+74\right)\alpha+69+\psi+\frac{81}{\psi}\right]}, \quad (3.3.15)$$

with

$$\psi^3 = 343\alpha^3-3915\alpha^2+7749\alpha-4185$$
$$+24\sqrt{3}\sqrt{-1029\alpha^5+10258\alpha^4-33880\alpha^3+50992\alpha^2-36099\alpha+9828}, \quad (3.3.16)$$

then $T_{p,b,c}$ *is starlike of order* α *in* U.

Proof. Let $f \in \mathcal{A}$. Using the same method used to proved Theorem 3.3.2, we replace k_1 with k_6 and M with $1-\alpha$ in (3.3.6) to find constraints on k_6, such that $T_{p,b,c}$ is starlike of order α, $0 \leq \alpha \leq \frac{1}{2}$ in U. Thus,

$$\Phi(k_6,|c|) < 1-\alpha$$
$$\Rightarrow (8k_6|c|-|c|^2)(6k_6-|c|)^2 < 3(1-\alpha)(4k_6-|c|)^2(18k_6^2-12k_6|c|+|c|^2)$$
$$\Rightarrow 288k_6^3|c|-132k_6^2|c|^2+20k_6|c|^3-|c|^4$$
$$< 3(1-\alpha)(288k_6^4-336k_6^3|c|+130k_6^2|c|^2-20k_6|c|^3+|c|^4) \quad (3.3.17)$$
$$\Rightarrow H(k_6) = (1-\alpha)k_6^4-\frac{9-7\alpha}{6}k_6^3|c|+\frac{87-65\alpha}{144}k_6^2|c|^2$$
$$-\frac{20-15\alpha}{216}k_6|c|^3+\frac{4-3\alpha}{864}|c|^4$$
$$> 0.$$

Finding the zeros of $H(h)$ by using Proposition 3.2.6,

$$(1-\alpha)h^4 - \frac{9-7\alpha}{6}h^3|c| + \frac{87-65\alpha}{144}h^2|c|^2 - \frac{20-15\alpha}{216}h|c|^3 + \frac{4-3\alpha}{864}|c|^4 = 0.$$

Letting

$$\begin{aligned}
a &= -\frac{17\alpha^2 - 74\alpha + 69}{288(1-\alpha)^2}|c|^2, \\
d &= \frac{4\alpha^3 - 37\alpha^2 + 82\alpha - 53}{864(1-\alpha)^3}|c|^3, \\
\Delta_0 &= \frac{49\alpha^2 - 126\alpha + 81}{20736}|c|^4, \\
\Delta_1 &= \frac{343\alpha^3 - 3915\alpha^2 + 7749\alpha - 4185}{1492992}|c|^6, \\
\Delta_1^2 - 4\Delta_0^3 &= \frac{1728(-1029\alpha^5 + 10258\alpha^4 - 33880\alpha^3 + 50922\alpha^2 - 36099\alpha + 9828)}{1492992^2}|c|^{12}, \\
Q &= \frac{\psi}{144}|c|^2 \ \text{ where } \psi \text{ is (3.3.16)}, \\
S &= \frac{\Psi}{24(1-\alpha)}|c| \text{ where } \Psi \text{ is (3.3.15)},
\end{aligned}$$

the zeros of H are

$$\begin{cases} h_{1,2} \in \mathbb{C} \text{ for } -4S^2 - 2a + \frac{d}{S} < 0, \\ h_{3,4} \in \mathbb{R} \text{ for } -4S^2 - 2a - \frac{d}{S} > 0. \end{cases}$$

The real roots $h_{3,4}$ are given by

$$\begin{aligned}
h_{3,4} &= -\frac{s}{4r} + S \pm \frac{1}{2}\sqrt{-4S - 2a - \frac{d}{S}} \\
&= \frac{9 - 7\alpha + \Psi \pm \sqrt{\frac{-16\alpha^3 + (17\Psi+148)\alpha^2 - (74\Psi+328)\alpha + 69\Psi + 212 - \Psi^3}{\Psi}}}{24(1-\alpha)}|c|,
\end{aligned}$$

results in the following upon simplification

$$h_3 = \frac{9 - 7\alpha + \Psi - \sqrt{\frac{-16\alpha^3+(17\Psi+148)\alpha^2-(74\Psi+328)\alpha+69\Psi+212-\Psi^3}{\Psi}}}{24(1-\alpha)}|c|$$

$$\Rightarrow 0.1033938492|c| < h_3 < 0.2019180635|c|$$

and

$$h_4 = \frac{9 - 7\alpha + \Psi + \sqrt{\frac{-16\alpha^3+(17\Psi+148)\alpha^2-(74\Psi+328)\alpha+69\Psi+212-\Psi^3}{\Psi}}}{24(1-\alpha)}|c|$$

$$\Rightarrow 0.970886808|c| < h_4 < 1.325742818|c|.$$

Thus, it can be concluded that $H(k_6) > 0$ for $k_6 < h_3$ or $k_6 > h_4$. Since $k > \frac{2+\sqrt{2}}{6}|c|$, therefore

$$k_6 > h_4 = \frac{9 - 7\alpha + \Psi + \sqrt{\frac{-16\alpha^3+(17\Psi+148)\alpha^2-(74\Psi+328)\alpha+69\Psi+212-\Psi^3}{\Psi}}}{24(1-\alpha)}|c|$$

isatisfies (3.3.17) which implies that $T_{p,b,c}$, by Lemma 3.2.5 when $\beta = 0$, is starlike of order α in U. □

Theorem 3.3.7. *Let $T_{p,b,c}$ be defined by (3.1.6) and $f \in \mathcal{A}$. If $p, c, b \in \mathbb{C}$, $k_7 = p + \frac{b+2}{2}$ and*

$$k_7 > \frac{13 - 7\alpha + \sqrt{2}\sqrt{14\alpha^2 - 61\alpha + 65}}{12(1-\alpha)}|c|,$$

then $T_{p,b,c}$ is convex of order α where $0 \leq \alpha \leq \frac{1}{2}$ in U.

Proof. Let $f \in \mathcal{A}$. As with preceeding results, the goal is to find the range of k_7 such that $T_{p,b,c}$ is convex of order α, $0 \leq \alpha \leq \frac{1}{2}$ in U. To achieve this, replace k_3 with k_7 and 1 with

$1-\alpha$ in (3.3.12) which results in the following,

$$\begin{aligned}
&\frac{48k_7|c|-6|c|^2}{48k_7^2-56k_7|c|+7|c|^2} < 1-\alpha \\
&\Rightarrow 48(1-\alpha)k_7^2-(104-56\alpha)k_7|c|+(13-7\alpha)|c|^2 > 0 \qquad (3.3.18)\\
&\Rightarrow P(k_7) = k_7^2 - \frac{104-56\alpha}{48(1-\alpha)}k_7|c| + \frac{13-7\alpha}{48(1-\alpha)}|c|^2 > 0.
\end{aligned}$$

The zeros of P are

$$x_5 = \frac{13-7\alpha-\sqrt{2}\sqrt{14\alpha^2-61\alpha+65}}{12(1-\alpha)}|c| \qquad (3.3.19)$$

and

$$x_6 = \frac{13-7\alpha+\sqrt{2}\sqrt{14\alpha^2-61\alpha+65}}{12(1-\alpha)}|c|. \qquad (3.3.20)$$

Substituting $0 \leq \alpha \leq \frac{1}{2}$ into (3.3.19) and (3.3.20) results in the following approximations

$$\frac{19-4\sqrt{19}}{12}|c| \leq x_5 \leq \frac{13-\sqrt{130}}{12}|c| \Rightarrow 0.13067018|c| < x_5 < 0.13318747|c|$$

and

$$\frac{13+\sqrt{130}}{12}|c| \leq x_6 \leq \frac{19+4\sqrt{19}}{12}|c| \Rightarrow 2.03347951|c| < x_6 < 3.03629966|c|.$$

Since $k > \frac{7+2\sqrt{7}}{12}|c|$,

$$k_7 > x_6 = \frac{13-7\alpha+\sqrt{2}\sqrt{14\alpha^2-61\alpha+65}}{12(1-\alpha)}|c|$$

satisfies (3.3.18). As the hypothesis in Theorem 3.3.6 are satisfied by the range of k_7, then $zT'_{p,b,c}$ is starlike of order α in U which implies that $T_{p,b,c}$ is convex of order α in U as

$f \in \mathcal{C}(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$. $\square$

Remark 3.3.2. Obeserve that when $\alpha = 0$, the results in Theorem 3.3.6 and Theorem 3.3.7 equal to that in Theorem 3.3.2 and Theorem 3.3.3 respectively.

CHAPTER 4: DISCUSSION AND CONCLUSION

This section will discuss the results found in Chapter 2 and 3 for analyzing and improving purposes as well as to consider questions for future authors to ponder upon.

4.1 Future Works on Close-to-Convex Functions

In Chapter 2, we introduced a new subclass of close-to-convex functions denoted by $\mathcal{K}_s^{k,N}$ and a new subclass of close-to-convex p-valent functions denoted by $\mathcal{K}_{s,p}^{k,N}$. Properties for the class $\mathcal{K}_s^{k,N}$ and $\mathcal{K}_{s,p}^{k,N}$ have been investigated in 2.3 and 2.4 respectively, namely distortion and growth theorems as well as the coefficient estimates. Toeplitz determinant, Fekete-Szegö inequality and the radius of convexity R_c, Theorems 2.3.4, 2.3.5 and 2.3.6 respectively, have also been investigated as additional properties for the class $\mathcal{K}_s^{k,N}$.

It is interesting to note that when $N = k = 2$, the distortion and growth bounds and the coefficient estimates for the class $\mathcal{K}_s^{2,2}$ are exactly the same as those obtained by Gao & Zhou (2005) and Singh (1977) when $\alpha = 0$. In fact, when $N = k \in \mathbb{N}$ and when $k = 1$ for all $N \in \mathbb{N}$, the coefficient estimates found are similar to those obtained by Chand & Singh (1979). This is because $G_{k,k}$ has the same form as that of a k-fold symmetric function starlike in the the unit disc thus using properties belonging to those functions resulted in similar results. With that in mind, as to why the distortion and growth bounds are not the same as Chand and Singh is a question that requires further investigation.

The inequalities obtained in Theorem 2.3.2 and Theorem 2.3.3 are sharp with extremal functions in their respective theorems. On the other hand, Theorem 2.4.2 and Theorem 2.4.3 are sharp only when $N = k = 1$. This means that the coefficient estimates, distortion and growth bounds obtained for the class $\mathcal{K}_{s,p}^{k,N}$ are not best possible and require further investigation in order to find the correct bounds such that they are sharp for all k and N. To date, we have yet to determine N-fold p-valent starlike functions of order $\alpha \in [0, p)$ which can aid in attempts to find the sharp bounds for the properties of $\mathcal{K}_{s,p}^{k,N}$.

Another avenue of research is to apply these techniques to expand to other univalent and multivalent classes, such as the class $\mathcal{S}_c^*$ and the class $\mathcal{S}_{sc}^*$, to investigate their properties and expand the boundaries of geometric function theory.

4.2 Future Works on Functions Associated with Special Functions

Chapter 3 focuses on finding the range of $k = p + \frac{b+2}{2}$ in order for the function $T_{p,b,c}$ to be either univalent, starlike or convex in $|z| < \frac{1}{2}$ and the unit disc $|z| < 1$. In summary, we have the following cases.

$$\begin{cases} k_1 > 1.098143352|c| & \Rightarrow T_{p,b,c} \text{ is univalent in } U, \\ k_2 > 0.970886809|c| & \Rightarrow T_{p,b,c} \text{ is starlike in } U, \\ k_3 > 2.033479521|c| & \Rightarrow T_{p,b,c} \text{ is convex in } U, \\ k_4 > 0.569035593|c| & \Rightarrow T_{p,b,c} \text{ is starlike in } |z| < \frac{1}{2}, \\ k_5 > 1.193712943|c| & \Rightarrow T_{p,b,c} \text{ is convex in } |z| < \frac{1}{2}, \\ k_6 > h_4 & \Rightarrow T_{p,b,c} \text{ is starlike of order } \alpha \text{ for } 0 \le \alpha \le \frac{1}{2} \text{ in } U, \\ k_7 > x_6 & \Rightarrow T_{p,b,c} \text{ is convex of order } \alpha \text{ for } 0 \le \alpha \le \frac{1}{2} \text{ in } U, \end{cases}$$

where $h_4 \in (0.970886810|c|,\ 1.325742818|c|)$ and $x_6 \in (2.03347951|c|,\ 3.03629966|c|)$.

As it can be seen from the cases above, the bounds obtained for convexity and starlikenss of the function $T_{p,b,c}$ are in agreement since $k_2 < k_3$ and all convex functions are starlike.

The same could be said for the function to be convex and univalent as $k_1 < k_3$ and all conevx functions are univalent. However, the relationship between k_1 and k_2 indicates that all univalent functions are starlike but that not all starlike functions are univalent in the unit disc U which is not necessarily true as within the unit disc, all starlike functions are univalent. A possible reason as to why this occured is the bounds obtained may not be sharp.

Using the argument that all convex functions are starlike functions in $|z| < \frac{1}{2}$ and that $k_4 < k_5$, the relationship on the convexity and starlikeness of $T_{p,b,c}$ in $|z| < \frac{1}{2}$ remain intact.

Evidently, when $\alpha = 0$, the bounds obtained for k_6 and k_7 equal to k_2 and k_3 respectively. As α increases from 0 to $\frac{1}{2}$, the lower bounds for k_6 and k_7 varies depending on α as it can be seen in h_4 and x_6 respectively. Once again, it can be seen that the bounds for the function $T_{p,b,c}$ to be convex of order α and univalent are in agreement as all convex functions of order α are univalent in the unit disc U, and the relationship between the convexity and starlikeness of order α of $T_{p,b,c}$ is retained as all convex functions of order α are starlike functions of α in U and the bounds $k_6 < k_7$ for all α. However, with the current findings, it is not possible to conclude with definite accuracy the relationship between univalency and starlikeness of order α.

The results obtained are in accordance to the results obtained by Orhan & Yağmur (2014). As to why the current findings may not currently agree with already established facts, i.e. all starlike functions are univalent in the unit disc, it may be due to the method of finding them. Note that in establishing Theorem 3.3.3 and Theorem 3.3.6, Lemma 3.2.5 was utilized for both theorems (one with $\alpha = 0$ and another without respectively), which is only a necessary condition, as opposed to using their respective definitions, which is both necessary and sufficient. This opens up an avenue of research to achieve better bounds for k_1, k_2 and k_6 such that the values of k_6 are within the range of k_1 to signify that starlike functions of order α are univalent in U which in turn also show that the values of k_2 are in the range of k_1.

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LIST OF PUBLICATIONS

1. **Mosiun, J.A.A.** & Abdul Halim, S. (2019). Properties of Functions Involving Struve Function. *Mathematics*, *6*(11), 240. https://doi.org/10.3390/math6110240