CONJUGACY SEPARABILITY AND CYCLIC CONJUGACY SEPARABILITY OF CERTAIN HNN EXTENSIONS, GENERALISED FREE PRODUCTS AND TREE PRODUCTS

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FACULTY OF SCIENCE
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CONJUGACY SEPARABILITY AND CYCLIC CONJUGACY SEPARABILITY OF CERTAIN HNN EXTENSIONS, GENERALISED FREE PRODUCTS AND TREE PRODUCTS

ABSTRACT

In this thesis, we study two interrelated strong residually finite properties of groups, namely conjugacy separability and cyclic conjugacy separability. We extend them to certain HNN extensions, generalized free products and tree products where the associated subgroups and amalgamated subgroups are not necessarily cyclic. In the first part of the thesis, we consider HNN extensions. We begin by establishing two criteria, one for conjugacy separability and another for cyclic conjugacy separability. Using these two criteria we establish conditions for HNN extensions where the associated subgroups are central or they are a finite extension of a central subgroup or cyclic to be conjugacy separable and cyclic conjugacy separable. In the second part of the thesis, we consider generalized free products and tree products. We shall consider only cyclic conjugacy separability as results on conjugacy separability are already known. Again we begin by establishing a criterion for cyclic conjugacy separability. We then prove that certain generalized free products and tree products where the amalgamated subgroups are central or they are a finite extension of a central subgroup or cyclic are again cyclic conjugacy separable.

Keywords: Residually Finite, Conjugacy Separable, Cyclic Conjugacy Separable, Generalized Free Products, HNN Extensions.

KEBOLEHPISAHAN KONJUGASI DAN KEBOLEHPISAHAN KONJUGASI KITARAN UNTUK PERLUASAN HNN, HASIL DARAB TERITLAK DAN HASIL DARAB POKOK TERTENTU

ABSTRAK

Dalam tesis ini, kami mengkaji dua sifat kuat sisa terhingga yang saling terhubung, iaitu kebolehpisahan konjugasi dan kebolehpisahan konjugasi kitaran. Kami memperluaskannya ke perluasan HNN, hasil darab teritlak dan hasil darab pokok tertentu di mana subkumpulan-subkumpulan bersekutu atau bergabung tidak semestinya kitaran. Di bahagian pertama tesis ini, kami mempertimbangkan perluasan HNN. Kami mula dengan membuktikan dua kriteria, satu untuk kebolehpisahan konjugasi dan satu lagi untuk kebolehpisahan konjugasi kitaran. Dengan kedua-dua kriteria ini, kami menetapkan syarat-syarat bagi perluasan HNN di mana subkumpulan-subkumpulan bersekutu adalah memusat atau lanjutan terhingga subkumpulan memusat atau kitaran untuk menjadi konjugasi terpisah dan konjugasi kitaran terpisah. Di bahagian kedua tesis, kami mempertimbangkan hasil darab teritlak dan hasil darab pokok. Kami hanya akan mempertimbangkan kebolehpisahan konjugasi kitaran sebab kebolehpisahan konjugasi sudah diketahui. Sekali lagi kami mulakan dengan membuktikan kriteria untuk kebolehpisahan konjugasi kitaran. Kami kemudiannya membuktikan bahawa hasil darab teritlak dan hasil darab pokok di mana subkumpulan-subkumpulan yang bergabung adalah memusat atau lanjutan terhingga bagi subkumpulan memusat atau kitaran adalah sekali lagi konjugasi kitaran terpisah.

Kata Kunci: Sisa Terhingga, Kebolehpisahan Konjugasi, Kebolehpisahan Konjugasi Kitaran, Hasil Darab Teritlak, Perluasan HNN

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CHAPTER 1: INTRODUCTION

1.1 Introduction and Background

In this thesis, we shall study the related properties of conjugacy separability and cyclic conjugacy separability in HNN extensions and generalized free products. These two properties arise from certain fundamental problems in Combinatorial Group Theory.

Combinatorial Group Theory studies groups from the perspective of their presentations, that is, their generators and relations. It is useful where finiteness assumptions are satisfied, for example, finitely generated groups. Dehn (1911) raised three fundamental decision problems in 1911. The problems are the *word problem*, the *conjugacy problem* and the *isomorphism problem*. The word problem asks whether two words are effectively the same group element. This is similar to the problem that we were given any arbitrary word, is it the identity element in the group? The conjugacy problem asks whether two words are conjugate in a group. It is clear that if a group has solvable conjugacy problem, then the group will have solvable word problem but the converse is not necessarily true.

The property of residual finiteness is useful in order to study the word problem. A finitely presented residually finite group has solvable word problem (Mostowski, 1966). Philip Hall (1959) first introduced the term residually finite in 1955 whereas Gruenberg was first to do a systematic study on residually finite groups. G. Baumslag (1963) was the first to study the residual finiteness of generalized free products in detail. He proved that free products of residually finite groups amalgamating a finite subgroup or, under certain conditions, amalgamating a cyclic subgroup is again residually finite. Since then many mathematicians have done research on residual finiteness and its various extensions.

The first topic we study is conjugacy separability. Following Mostowski (1966), a group G is said to be conjugacy separable if for each pair of elements x, y in G such that x and y are not conjugate in G, then there exists a finite homomorphic image $\bar{G}$ of G such that $\bar{x}$ and $\bar{y}$, the images of x and y in $\bar{G}$ respectively, are again not conjugate. Clearly, a conjugacy separable group is residually finite. In 1966, Mostowski (1966) showed that finitely presented conjugacy separable groups have solvable conjugacy problem. It is well known that finitely generated torsion-free nilpotent groups and free groups are conjugacy separable (Blackburn, 1965; Stebe, 1970). Surface groups are also conjugacy separable (Scott, 1978). Building on these results, Dyer (1980), Formanek (1976) and Remeslennikov (1969) and Fine and Rosenberger (1990) respectively showed that polycyclic-by-finite groups, free-by-finite groups and Fuchsian groups (finite extension of surface groups) are conjugacy separable. Stebe (1970) was the first to do a systematic study of conjugacy separability on generalized free products. He proved that the free products of conjugacy separable groups are conjugacy separable. Since then many mathematicians have done research on conjugacy separability for generalized free products and various group extensions.

The second topic we study is cyclic conjugacy separability which was formally defined by Tang (1995). Following Tang (1995), a group G is called cyclic conjugacy separable if for each $x \in G$ and each cyclic subgroup $\langle y \rangle$ in G such that no conjugate of x in G belongs to $\langle y \rangle$, then there exists a finite homomorphic image $\bar{G}$ of G such that no conjugate of $\bar{x}$ in $\bar{G}$ belongs to $\langle \bar{y} \rangle$. Clearly a cyclic conjugacy separable group is residually finite. Dyer (1980) was first to prove that finitely generated nilpotent groups and free groups have this property without giving it a name. Tang (1995) proved that surface groups are cyclic conjugacy separable. Moldavansky (1993) showed that supersolvable groups are cyclic conjugacy separable. In 1995, Kim and Tang (1995) showed that conjugacy separable finite extensions of conjugacy separable residually finitely generated torsion-free nilpotent groups are cyclic conjugacy separable. As consequences, surface groups and finitely generated Fuchsian groups are cyclic conjugacy separable.

The importance of cyclic conjugacy separability lies in the fact that it is an essential condition in extending conjugacy separability to generalized free products amalgamating a cyclic subgroup. Indeed, Dyer (1980) first made use of this property to show that the generalized free products of two finitely generated nilpotent groups or two free groups amalgamating a cyclic subgroup are conjugacy separable. Tang (1995) similarly used cyclic conjugacy separability to show the conjugacy separability of the generalized free products of two surface groups. More recently, Kim and Tang in the papers (Kim & Tang, 1996; Kim & Tang, 1999) established the criteria for the conjugacy separability of generalized free products of two conjugacy separable groups with a cyclic amalgamated subgroup and for the conjugacy separability of HNN extensions of a conjugacy separable group with cyclic associated subgroups. One of the conditions in these criteria is that the factor groups in the generalized free product must be cyclic conjugacy separable relative to the amalgamated subgroup and the base group in the HNN extension must be cyclic conjugacy separable relative to the associated subgroups.

1.2 General Description of All Chapters

We now give a brief description of all the chapters in this thesis.

This thesis is divided into two parts. In the first part, we study HNN extensions of the form $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where A is the base group and H, K are the associated subgroups and ϕ is the isomorphism from H onto K.

In Chapter 2, we establish two criteria for $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ to be conjugacy separable and cyclic conjugacy separable in Theorem 2.12 and Theorem 2.14 respectively.

From Chapter 3 to Chapter 5, we will use these two criteria to investigate the conjugacy separability and cyclic conjugacy separability for $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where, A is conjugacy separable or cyclic conjugacy separable with

(i) $H = \langle h \rangle, K = \langle k \rangle$ are infinite cyclic and h^m is conjugate to $k^{\pm m}$ in A for some positive integer m; or

(ii) H, K are finite; or

(iii) $H \cap K$ is central in A and $H \cap K$ is a subgroup of finite index in H and in K; or

(iv) H, K are central in A.

Further conditions are imposed on the base group A and the associated subgroups H, K in order to obtain conjugacy separability and cyclic conjugacy separability in $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$. Some applications to finitely generated nilpotent groups are given.

In the second part, we study generalized free products of the form $G = A *_H B$ where A, B are the factor groups and H is the amalgamated subgroup. We will only study the cyclic conjugacy separability as the conjugacy separability were established by various authors.

In Chapter 6, we establish a criterion for $G = A *_H B$ to be cyclic conjugacy separable in Theorem 6.6. From Chapter 7 to Chapter 9, we will use this criterion to investigate the cyclic conjugacy separability for $G = A *_H B$ where A, B are cyclic conjugacy separable with

(i) H is cyclic; or

(ii) H is finite; or

(iii) D is a subgroup of finite index in H and D is central in both A and B; or

(iv) H is central in both A and B.

In Chapter 9, we extend cyclic conjugacy separability to finite tree products where the amalgamated subgroups are central. Again, further conditions are imposed on the factor groups A, B and the amalgamated subgroup H in order to obtain positive results.

Finally in the last chapter, Chapter 10, we state some topics for further research.

We next give a brief description of generalized free products, tree products and HNN extensions of groups.

1.3 Generalized Free Products

O. Schreier (1927) first introduced the concept of generalized free product in 1927. Now we describe the concept of generalized free product of two groups. Let A, B be two groups and H, K be subgroups of A and B respectively with $\phi : H \to K$ an isomorphism. Then the generalized free product G of A and B amalgamating the subgroups H of A and K of B via the isomorphism ϕ, is defined to be the group generated by the generators and relations of the groups A and B with the extra relations $\phi(h) = k$ where $h \in H, k \in K$. Then we can write G as follows.

$$G = \langle A, B | \phi(h) = k \rangle, h \in H, k \in K.$$

By abuse of notation, we shall write G in the commonly used form in this thesis.

$$G = A \mathop{*}_{H} B$$

A and B are called the factors of the group G and H is the amalgamated subgroup.

We let g be an element in G. We say that g is in reduced form if $g = g_1g_2\ldots g_n$ and no consecutive terms are from the same factor. The length of the reduced element $g = g_1g_2\ldots g_n$ is denoted by $||g||$ and is defined as follows:

$$||g|| = \begin{cases} 0, & \text{if } n = 1 \text{ and } g_1 \in H \\ 1, & \text{if } n = 1 \text{ and } g_1 \in (A \cup B)\backslash H \\ n, & \text{otherwise} \end{cases}$$

The reduced element $g = g_1g_2\ldots g_n$ is called cyclically reduced if each of its cyclic permutations $g_ig_{i+1}\ldots g_ng_1g_2\ldots g_{i-1}$ is reduced.

1.4 Tree Products

Let us give some facts about tree products. Tree products were first introduced by Karrass and Solitar (1970). A description of tree products was given by Kim and Tang (1998) as follows:

> " Let T be a tree. To each vertex v of T, assign a group G_v. To each edge e of T, assign a group G_e together with monomorphisms α_e, β_e embedding G_e into the two vertex groups at the end of the edge e. Then the tree product G is defined to be the group generated by the generators and relations of the vertex groups together with the additional relations $\alpha_e(g_e) = \beta_e(g_e)$ for each $g_e \in G_e$." (Kim & Tang, 1998, p. 323)

By abuse of notation, let G be a tree product of the vertex groups $A_1, A_2, \ldots, A_n$, $n \geq 2$, amalgamating the edge subgroups H_{ij} of A_i and H_{ji} of A_j. We shall denote $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$.

1.5 HNN Extensions

Let A be a group and let H and K be subgroups of A such that $\phi : H \rightarrow K$ is an isomorphism. The HNN extension G of A relative to the subgroups H and K with the isomorphism ϕ is defined to be the group generated by the generators and relations of the group A with an extra generator t and extra relation $t^{-1}ht = \phi(h)$ for each $h \in H$. We write

$$G = \langle t, A | t^{-1}ht = \phi(h), \forall h \in H \rangle$$

We shall write G in the more commonly used form in this thesis.

$$G = \langle t, A | t^{-1}Ht = K, \phi \rangle$$

The group A is called the base group and t is called the stable letter. H and K are called the associated subgroups and ϕ is the associated isomorphism of G.

Let $g \in G$. Then we have $g = g_0 t^{e_1} g_1 \ldots t^{e_n} g_n$ with $e_i = \pm 1$. The element g is said to be in reduced form if there are no consecutive terms $t^{-1} g_i t$ with $g_i \in H$ or $t g_n t^{-1}$ with $g_n \in K$. Note that each element of G can be written in reduced form. The length of a reduced element $g = g_0 t^{e_1} g_1 \ldots t^{e_n} g_n$ is denoted by $||g||$ and is defined as follows:

$$||g|| = \begin{cases} 0, & \text{if } g = g_0 \\ n, & \text{otherwise} \end{cases}$$

If $g \in G$ is cyclically reduced, we write $g = t^{e_1} g_1 t^{e_2} \ldots t^{e_n} g_n$ where $g_i \in A$ and $e_i = \pm 1, 1 \leq i \leq n$. Finally, if $x, y \in G$, then $x \sim_{A,t} y$ means $x, y \in A$ and either $x \sim_A y$ or $x \in H$ and $t^{-1}xt = y$ or $x \in K$ and $txt^{-1} = y$.

1.6 Notations

Standard notations will be used in this thesis. In addition, we shall use the following.

Let G be a group.

$N \lhd_f G$ means N is a normal subgroup of finite index in the group G.

$Z(G)$ denotes the center of G.

$x \sim_G y$ means x is conjugate to y in G for $x, y \in G$.

$\{x\}^G$ denotes the conjugacy class of x in G.

CHAPTER 2: CONJUGACY SEPARABILITY AND CYCLIC CONJUGACY SEPARABILITY OF HNN EXTENSIONS

2.1 Introduction

We begin this chapter by establishing two criteria which can be used to prove the conjugacy separability and cyclic conjugacy separability on HNN extensions of conjugacy separable and cyclic conjugacy separable groups respectively. These two criteria state the common basic core conditions that are sufficient to prove these conjugacy properties. These core conditions are simple and direct. However, to utilize them, we need to use the special and unique properties belonging to the base group of the HNN extensions as well as their relations to the associated subgroups.

The two criteria are given respectively in Theorem 2.12 and Theorem 2.14. They will be used extensively from Chapter 3 to Chapter 5. We begin with some definitions.

2.2 Definitions

In this section, we state all the definitions which we are going to use later in this thesis.

Definition 2.1. A group A is said to be residually finite if, for each nontrivial element $x \in A$, there exists $N \lhd_f A$ such that $x \notin N$.

Definition 2.2. A group A is called H-separable for the subgroup H of A if for each $x \in A$ such that $x \notin H$, there exists $N \lhd_f A$ such that $x \notin HN$. A is called subgroup separable if A is H-separable for every finitely generated subgroup H.

Definition 2.3. Let A be a group and H, K be subgroups of A. Then A is said to be HK-double coset separable if for each $x \in A$, A is HxK-separable. In particular, we say that A is H-double coset separable if A is HxH-separable for all $x \in A$. Suppose h, k are elements of infinite order in A. Then A is said to be $\{h, k\}$-double coset separable if, for each $x \in A$ and for each integer $\epsilon > 0$, A is $\langle h^{\epsilon}\rangle x\langle h^{\epsilon}\rangle$-separable, $\langle h^{\epsilon}\rangle x\langle k^{\epsilon}\rangle$-separable, and $\langle k^{\epsilon}\rangle x\langle k^{\epsilon}\rangle$-separable.

The well known subgroup separable groups are free groups and polycyclic groups (Hall, 1949; Mal'cev, 1983). Free-by-finite groups and polycyclic-by-finite groups are subgroup separable since a finite extension of a subgroup separable group is again subgroup separable. Note that A is (H, K)-double coset separable if and only if A is HK-separable (Kim et al., 1995).

Definition 2.4. (Tang, 1995) Let A be a group and h be an element of infinite order in A. Then A is said to be $\langle h\rangle$-weakly potent if we can find a positive integer r with the property that for every positive integer n, there exists a normal subgroup N of finite index in A such that hN has order exactly rn. A group A is called weakly potent if A is $\langle h\rangle$-weakly potent for all element h of infinite order in A.

For example, free groups and finitely generated nilpotent groups are weakly potent (Evans, 1974; Tang, 1995).

Definition 2.5. Let A be a group and $x \in A$. Then x is called self-conjugate if $x^i \sim_A x^j$ then $i = j$ for all integers i, j.

For each element x of infinite order in both free groups and finitely generated nilpotent groups, x is self-conjugate (Dyer, 1980).

Definition 2.6. (Mostowski, 1966) Let $x, y \in A$ such that $x \nsim_A y$. Then x, y are said to be conjugacy distinguishable if there exists $N \lhd_f A$ such that $\bar{x} \nsim_{\bar{A}} \bar{y}$ in $\bar{A} = A/N$. A is said to be conjugacy separable if A is conjugacy distinguishable for all $x \nsim_A y$.

It is well known that finitely generated torsion-free nilpotent groups, free groups and surface groups are conjugacy separable (Blackburn, 1965; Stebe, 1970; Scott, 1978). Building on these results, free-by-finite groups, polycyclic-by-finite groups and Fuchsian groups (finite extension of surface groups) are conjugacy separable by (Dyer, 1980; Formanek, 1976; Remeslennikov, 1969; Fine & Roserberger 1990).

Definition 2.7. (Kim & Tang, 1995) A group A is said to be subgroup conjugacy separable if for every $x \in A$ and subgroup H of A such that $\{x\}^A \cap H = \emptyset$, there exists $N \lhd_f A$ such that $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$ in $\bar{A} = A/N$. In particular if A is subgroup conjugacy separable for every cyclic subgroup of A, then A is said to be cyclic conjugacy separable.

Clearly a cyclic conjugacy separable group is residually finite. Tang (1995) proved that surface groups are cyclic conjugacy separable. Moldavansky (1993) showed that supersolvable groups are cyclic conjugacy separable.

2.3 Conjugacy Separability of HNN Extensions

In this section, we give the essential lemmas and then we prove the criterion for conjugacy separability of HNN extensions.

Lemma 2.8. *(Collins, 1969) Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension and x, y be cyclically reduced in G. Suppose $x \sim_G y$. Then $||x|| = ||y||$ and one of the following holds:*

(i) $||x|| = ||y|| = 0$ *and there is a finite sequence* $z_1, z_2, \ldots, z_n \in H \cup K$ *such that*

$$x \sim_A z_1 \sim_{A,t} z_2 \sim_{A,t} \ldots \sim_{A,t} z_n \sim_A y.$$

(ii) $||x|| = ||y|| \geq 1$ *and* $x' \sim_{H \cup K} y$ *where* x' *is a cyclic permutation of* x*.*

Definition 2.9. Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension and x, y be cyclically reduced in G with $||x|| = ||y|| = n \geq 1$. Suppose that $x = t^{e_1}x_1 \ldots t^{e_n}x_n$ and $y = t^{\epsilon_1}y_1 \ldots t^{\epsilon_n}y_n$ where $x_i, y_i \in A$ and $e_i = \epsilon_i = \pm 1, 1 \leq i \leq n$.

Consider the following set of equations:

$$\begin{aligned} x_{i+1} &= u_1^{-1} y_1 v_1 \\ x_{i+2} &= u_2^{-1} y_2 v_2 \\ &\vdots \qquad\qquad\qquad (1^*) \\ x_{i+n} &= u_n^{-1} y_n v_n \end{aligned}$$

A pair of elements ρ_j, σ_j of A is called an admissible solution of the j-th equation if and only if $x_{i+j} = \rho_j^{-1} y_j \sigma_j$, where $\rho_j, \sigma_j \in H \cup K$.

A set of admissible solutions $\sigma_0, \rho_1, \sigma_1, \ldots, \rho_n, \sigma_n \in H \cup K$ to (1^*) is said to be complete if $t^{-e_j}\sigma_{j-1}t^{e_j} = \rho_j$ for each j and $\sigma_0 = \sigma_n$. This is equivalent to $x' = \sigma_0^{-1} y \sigma_0$ where $x' = x_{i+1}x_{i+2} \ldots x_i$ is a cyclic permutation of x. So, $x \sim_G y$ if and only if the system of equations (1^*) has a set of complete admissible solutions for some $0 \leq i < n$.

For the case $H \cap K = 1, \rho_i \in H$ or K according as $e_i = -1$ or 1 respectively, and $\sigma_i \in H$ or K according as $e_{i+1} = 1$ or -1 respectively. (Here $e_{n+1} = e_1$.)

Lemma 2.10. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where A is finite. Then G is subgroup separable (Wong, 1993), conjugacy separable (Dyer, 1980) and cyclic conjugacy separable (Kim & Tang, 1995).*

The following lemma is easy to obtain from Baumslag & Tretkoff (1978).

Lemma 2.11. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension. Suppose*

(i) A is residually finite;

(ii) A is H-separable and K-separable;

(iii) For each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$.

Then G is residually finite.

We now state and prove a criterion for conjugacy separability which will be used in Chapter 3 to Chapter 4.

Theorem 2.12. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where*

(a) A is residually finite;

(b) A is H-separable and K-separable;

(c) For each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$;

(d) For $x, y \in G$ such that $||x|| = ||y|| \geq 0$ and $x \nsim_G y$, there exists $Q \lhd_f G$ such that $xQ \nsim_{G/Q} yQ$.

Then G is conjugacy separable.

Proof. Let $x, y \in G$ such that $x \nsim_G y$. Without loss of generality, we assume that x, y are cyclically reduced and have the minimum length in their conjugacy classes. Since G is residually finite by Lemma 2.11, we can assume $x \neq 1 \neq y$.

Case 1. Suppose $||x|| = n \geq 1, ||y|| = m \geq 1$. Let $x = t^{e_1}x_1t^{e_2}x_2 \dots t^{e_n}x_n, y = t^{E_1}y_1t^{E_2}y_2 \dots t^{E_m}y_m$ where $m, n \geq 1$ and $e_i, E_j = \pm 1, x_i, y_j \in A$ for $i = 1, \dots, n, j = 1, \dots, m$. Let p_r denote those $x_i, y_j \in A \backslash H$, q_s denote those $x_i, y_j \in A \backslash K$ and u_t denote those $x_i, y_j \in (H \cap K)\backslash\{1\}$. Since A is residually finite, H-separable and K-separable by (a) and (b), there exists $M \lhd_f A$ such that $p_r \notin HM, q_s \notin KM$ and $u_t \notin M$.

Subcase 1a. Suppose $n \neq m$. By (c), there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. We form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi}\rangle$ where $\bar{A} = A/N, \bar{H} = HN/N, \bar{K} = KN/N$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$. Then in $\bar{G}$, we have $||\bar{x}|| = n, ||\bar{y}|| = m$ where $n \neq m$. This implies that $||\bar{x}|| \neq ||\bar{y}||$ and hence by Lemma 2.8, we have $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable by Lemma 2.10, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{x}\bar{P} \nsim_{\bar{G}/\bar{P}} \bar{y}\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $xP \nsim_{G/P} yP$. Our result follows.

Subcase 1b. Suppose $n = m$. By (d), there exists $Q \lhd_f G$ such that $xQ \nsim_{G/Q} yQ$. Let $P = M \cap Q$. Then $P \lhd_f A$. By (c), there exists $N \lhd_f A$ such that $N \subseteq P$ and $\phi(N \cap H) = N \cap K$. We form $\bar{G}$ as in Subcase 1a. Then in $\bar{G}$, we have $||\bar{x}|| = n, ||\bar{y}|| = m, n = m$ and $\bar{x} \nsim_{\bar{G}} \bar{y}$. We proceed as in Subcase 1a and the result follows.

Case 2. Suppose $||x|| = 0, ||y|| \geq 1$ or $||x|| \geq 1, ||y|| = 0$. We consider the case $||x|| = 0, ||y|| \geq 1$. As in Case 1, we can form $\bar{G}$ such that in $\bar{G}$ we have $\bar{x} \neq \bar{1}, ||\bar{x}|| = 0$ and $||\bar{y}|| \geq 1$. By Lemma 2.8(i), any conjugate of $\bar{x}$ is either an element $\bar{x}'$ of $\bar{A}$ or an element of the form $\bar{u}^{-1}\bar{x}\bar{u}$ where $\bar{u}^{-1}\bar{x}\bar{u}$ and $\bar{u}$ are reduced in $\bar{G}$. Since $\bar{y}$ is cyclically reduced and $||\bar{y}|| \geq 1$, then by Lemma 2.8 we have $\bar{x} \nsim_{\bar{G}} \bar{y}$. We now proceed as in Subcase 1a and our result follows.

Case 3. Suppose $||x|| = ||y|| = 0$. This case follows from (d).

The proof for this theorem is now complete and hence G is conjugacy separable.

2.4 Cyclic Conjugacy Separability of HNN Extensions

In this section, we prove the criterion for cyclic conjugacy separability of HNN extensions.

Lemma 2.13. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension and $x, y \in G$ such that $||x|| = ||y^{\pm k}|| \geq 1$ for some positive integer k. Then $\{x\}^G \cap \langle y \rangle = \emptyset$ if and only if $x \nsim_G y^{\pm k}$.*

Proof. Suppose $\{x\}^G \cap \langle y \rangle = \emptyset$. Then $g^{-1}xg \notin \langle y \rangle$ implies $g^{-1}xg \neq y^{\pm r}$ for any $g \in G, r \in \mathbb{Z}^+$. Thus, $g^{-1}xg \neq y^{\pm k}$. Hence $x \nsim_G y^{\pm k}$. Conversely, suppose $x \nsim_G y^{\pm k}$. We have $g^{-1}xg \neq y^{\pm k}$ for any $g \in G$. Since $||x|| \neq ||y^{\pm n}||$ for all $n \in \mathbb{Z}^+ \backslash \{k\}$, $x \nsim_G y^{\pm n}$ by Lemma 2.8(ii). Thus, $x \nsim_G y^{\pm r}$ for all $r \in \mathbb{Z}^+$. Therefore, $\{x\}^G \cap \langle y \rangle = \emptyset$.

We now state and prove our main criterion for cyclic conjugacy separability of HNN extensions which will be used to prove our main results in Chapter 3, Chapter 4 and Chapter 5.

Theorem 2.14. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where*

(a) A is residually finite;

(b) A is H-separable and K-separable;

(c) For each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$;

(d) For $x, y \in G$ such that $||x|| = ||y|| \geq 1$ and $x \nsim_G y$, there exists $Q \lhd_f G$ such that $xQ \nsim_{G/Q} yQ$;

(e) For $x, y \in G$ such that $||x|| = ||y|| = 0$ and $\{x\}^G \cap \langle y \rangle = \emptyset$, there exists $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$.

Then G is cyclic conjugacy separable.

Proof. Let $x, y \in G$ such that $\{x\}^G \cap \langle y \rangle = \emptyset$. Without loss of generality, we assume that x, y are cyclically reduced and have the minimum length in their conjugacy classes. Since G is residually finite by Lemma 2.11, we can assume $x \neq 1 \neq y$.

Case 1. Suppose $||x|| = n \geq 1, ||y|| = m \geq 1$. Let $x = t^{e_1}x_1t^{e_2}x_2 \dots t^{e_n}x_n, y = t^{E_1}y_1t^{E_2}y_2 \dots t^{E_m}y_m$ where $m, n \geq 1$ and $e_i, E_j = \pm 1, x_i, y_j \in A$ for $i = 1, \dots, n, j = 1, \dots, m$. Let p_r denote those $x_i, y_j \in A \backslash H$, q_s denote those $x_i, y_j \in A \backslash K$ and u_t denote those $x_i, y_j \in (H \cap K)\backslash\{1\}$. Since A is residually finite, H-separable and K-separable by (a) and (b), there exists $M \lhd_f A$ such that $p_r \notin HM, q_s \notin KM$ and $u_t \notin M$.

Subcase 1a. Suppose $n \neq km$ for all positive integers k. By (c), there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. We form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi}\rangle$ where $\bar{A} = A/N, \bar{H} = HN/N, \bar{K} = KN/N$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$. Then in $\bar{G}$, we have $||\bar{x}|| = n, ||\bar{y}|| = m$ where $n \neq km$ for all positive integers k. This implies that $||\bar{x}|| \neq ||\bar{y}^{\pm k}||$ and hence by Lemma 2.8, $\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$ for all positive integers k. Hence by Lemma 2.13, we have $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 2.10, there exists $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P}\rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$. Our result follows.

Subcase 1b. Suppose $n = km$ for some positive integer k. By Lemma 2.13, $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $x \nsim_G y^{\pm k}$. By (d), there exists $Q \lhd_f G$ such that $xQ \nsim_{G/Q} y^{\pm k}Q$. Let $P = M \cap Q$. Then $P \lhd_f A$. By (c), there exists $N \lhd_f A$ such that $N \subseteq P$ and $\phi(N \cap H) = N \cap K$. We form $\bar{G}$ as in Subcase 1a. Then in $\bar{G}$, we have $||\bar{x}|| = n, ||\bar{y}|| = m, n = km$ and $\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$. Again by Lemma 2.13, we have $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. We proceed as in Subcase 1a and the result follows.

Case 2. Suppose $||x|| = 0, ||y|| \geq 1$. As in Case 1, we can form $\bar{G}$ such that in $\bar{G}$ we have $\bar{x} \neq \bar{1}, ||\bar{x}|| = 0$ and $||\bar{y}|| \geq 1$. By Lemma 2.8(i), any conjugate of $\bar{x}$ is either an element $\bar{x}'$ of $\bar{A}$ or an element of the form $\bar{u}^{-1}\bar{x}\bar{u}$ where $\bar{u}^{-1}\bar{x}\bar{u}$ and $\bar{u}$ are reduced in $\bar{G}$. Since $\bar{y}$ is cyclically reduced and $||\bar{y}|| \geq 1$, then $\bar{y}^{\pm k}$ is cyclically reduced and $||\bar{y}^{\pm k}|| \geq k$ for all positive integers k. Hence, by Lemma 2.8, we have $\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$ for all positive integers k. Thus this implies that $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. We now proceed as in Subcase 1a and our result follows.

Case 3. Suppose $||x|| \geq 1, ||y|| = 0$. As in Case 1, we can form $\bar{G}$ such that in $\bar{G}$, we have $||\bar{x}|| \geq 1, ||\bar{y}|| = 0$ and $\bar{y} \neq \bar{1}$. Since $\bar{x}$ is cyclically reduced and $||\bar{x}|| \geq 1$, then any conjugate of $\bar{x}$, say $\bar{z}$ has length $||\bar{z}|| \geq ||\bar{x}|| \geq 1$. Since $\bar{y} \in \bar{A}$, then $\bar{y}^{\pm k} \in \bar{A}$ for all positive integers k. Hence by Lemma 2.8, we have $\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$. Therefore, this implies that $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. We now proceed as in Subcase 1a and our result follows.

Case 4. Suppose $||x|| = ||y|| = 0$. The result follows from (e).

The proof for this theorem is now complete and hence G is cyclic conjugacy separable.

CHAPTER 3: HNN EXTENSIONS WITH CYCLIC ASSOCIATED SUBGROUPS

3.1 Introduction

In this chapter, we extend the conjugacy separability and cyclic conjugacy separability to certain HNN extensions with infinite cyclic associated subgroups. More precisely, we shall show that the HNN extension $G = \langle t, A | t^{-1}ht = k \rangle$ where h, k be elements of infinite order in A and $h^m \sim_A k^{\pm m}$ for some positive integer m is conjugacy separable and cyclic conjugacy separable when A is a free group or a finitely generated torsion-free nilpotent group.

Recall that the Baumslag-Solitar group $G = \langle t, a | t^{-1}a^2t = a^3 \rangle$ is an example of an HNN extension with cyclic associated subgroups which is not even residually finite.

This chapter is divided into three parts. In the first part, we gather all the lemmas needed to prove the main results. We prove the conjugacy separability in the second part and the cyclic conjugacy separability in the final part.

3.2 Lemmas Needed

In this section, we gather all the lemmas that we need to prove the main results later in this chapter. We begin with the following remark.

Remark A. *Let A be a group and h, k be elements of infinite order in A. If A is $\langle h \rangle$-weakly potent and $\langle k \rangle$-weakly potent, then we can find positive integers r_1, r_2 such that for each positive integer n, there exist $Q_1 \lhd_f A, Q_2 \lhd_f A$ such that $Q_1 \cap \langle h \rangle = \langle h^{r_1 n} \rangle$ and $Q_2 \cap \langle k \rangle = \langle k^{r_2 n} \rangle$.*

Lemma 3.1. *Let A be a group and h, k be elements of infinite order in A and $h^m \sim_A k^{\pm m}$ for some positive integer m. Suppose that $\phi : \langle h \rangle \to \langle k \rangle$ is an isomorphism such that $\phi(h) = k$. Suppose A is $\langle h \rangle$-weakly potent and $\langle k \rangle$-weakly potent. Then for each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap \langle h \rangle) = N \cap \langle k \rangle$.*

Proof. Let $M \lhd_f A$ be given. Suppose $M \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $M \cap \langle k \rangle = \langle k^{s_2} \rangle$ for some positive integers s_1, s_2. Since A is $\langle h \rangle$-weakly potent, $\langle k \rangle$-weakly potent and by Remark A, we can find $M_1 \lhd_f A, M_2 \lhd_f A$ such that $M_1 \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2 mn} \rangle$ and $M_2 \cap \langle k \rangle = \langle k^{r_1 r_2 s_1 s_2 mn} \rangle$. Let $N = M \cap M_1 \cap M_2$. Since $h^m = a^{-1} k^{\pm m} a$ for some $a \in A$, we have $N \lhd_f A$ such that

$$\begin{aligned}
N \cap \langle h \rangle &= M \cap M_1 \cap M_2 \cap \langle h \rangle \\
&= M_2 \cap \langle h^{r_1 r_2 s_1 s_2 mn} \rangle \\
&= M_2 \cap \langle a^{-1} k^{\pm r_1 r_2 s_1 s_2 mn} a \rangle \\
&= a^{-1} (M_2 \cap \langle k^{\pm r_1 r_2 s_1 s_2 mn} \rangle) a \\
&= a^{-1} \langle k^{\pm r_1 r_2 s_1 s_2 mn} \rangle a \\
&= \langle h^{r_1 r_2 s_1 s_2 mn} \rangle
\end{aligned}$$

Similarly, we have $N \cap \langle k \rangle = \langle k^{r_1 r_2 s_1 s_2 mn} \rangle$. Therefore, $\phi(N \cap \langle h \rangle) = N \cap \langle k \rangle$ as required.

Theorem 3.2. *(Wong & Gan, 1999) Let $G = \langle t, A | t^{-1} H t = K, \phi \rangle$ be an HNN extension where A is a cyclic subgroup separable group. Suppose that*

(a) A is H-separable and K-separable;

(b) for each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$,

Then G is cyclic subgroup separable.

Lemma 3.3. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ be an HNN extension where h, k be elements of infinite order in A and $h^m \sim_A k^{\pm m}$ for some positive integer m. Suppose*

(i) A is cyclic subgroup separable;

(ii) A is $\langle h \rangle$-weakly potent and $\langle k \rangle$-weakly potent.

Then G is cyclic subgroup separable.

Proof. Since A is cyclic subgroup separable, A is $\langle h \rangle$-separable and $\langle k \rangle$-separable. Thus, the result follows from Lemma 3.1 and Theorem 3.2.

Lemma 3.4. *(Kim & Tang, 1996) Let A be a group that is $\langle h^\epsilon \rangle x \langle k^\epsilon \rangle$-separable, where x, h, k are elements in A such that h, k have infinite orders. If $\langle x^{-1}hx \rangle \cap \langle k \rangle = 1$, then there exists $N \lhd_f A$ such that $\bar{x}^{-1}\bar{h}^i\bar{x} = \bar{k}^j$ only if $\epsilon | i, j$, where $\bar{x}, \bar{h}, \bar{k} \in \bar{A} = A/N$ and $\epsilon, i, j \in \mathbb{Z}$.*

Lemma 3.5. *Let A be a group where x, h, k are elements in A with h, k having finite orders. Let $\epsilon, i, j \in \mathbb{Z}$.*

(i) If $x^{-1}h^ix = k^j$ only if $\epsilon | i, j$, then $\epsilon || h|, |k|$.

(ii) If $x^{-1}h^ix = k^j$ and $\epsilon | i, |h|$, where $|h| = |k|$, then $\epsilon | j$.

Proof. (i) Let $i = |h|$ and $j = |k|$. Then $x^{-1}h^ix = 1 = k^j$. Then $\epsilon || h|, |k|$.

(ii) Let $i = \epsilon\alpha$ and $|h| = |k| = \epsilon c$ where c is some positive integer. Since $1 = x^{-1}h^{\epsilon\alpha c}x = k^{jc}$, then $\epsilon c | jc$. Hence $\epsilon | j$.

Lemma 3.6. *Let A be a group and h, k be elements of infinite order in A and $h^m \sim_A k^{\pm m}$ for some positive integer m. Suppose h, k are self-conjugate in A. If $h^r \sim_A k^s$ for some integers r, s, then $r = \pm s$.*

Proof. First suppose $h^m \sim_A k^{-m}$. Let $r = \alpha m$ for some integer α. Then $k^{-r} = k^{-\alpha m} \sim_A h^{\alpha m} = h^r \sim_A k^s$. Since k is self-conjugate, we have $s = -r$. Suppose $r \neq \alpha m$ for all $\alpha \in \mathbb{Z}$. Then $k^{-rm} \sim_A h^{rm} \sim_A k^{sm}$. Again since k is self-conjugate, we have $s = -r$. For the case $h^m \sim_A k^m$, the proof is similar and we will get $s = r$.

3.3 Conjugacy Separability on Certain HNN Extensions

In this section, we prove the two main results on conjugacy separability, that is Lemma 3.7 and Theorem 3.8.

Lemma 3.7. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ be an HNN extension where h, k be elements of infinite order in A and $h^m \sim_A k^{\pm m}$ for some positive integer m. Let h, k be self-conjugate in A. Suppose*

(i) A is cyclic subgroup separable;

(ii) A is $\{h, k\}$-double coset separable;

(iii) A is $\langle h \rangle$-weakly potent and $\langle k \rangle$-weakly potent.

Then for each $x, y \in G$ such that $||x|| = ||y|| \geq 1$ and $x \nsim_G y$, there exists $Q \lhd_f G$ such that $xQ \nsim_{G/Q} yQ$ in G/Q.

Proof. We shall only consider the case $h^m \sim_A k^{-m}$. The other case is similar. Let x, y be cyclically reduced. We assume $x = t^{e_1}x_1t^{e_2}x_2 \ldots t^{e_n}x_n$ and $y = t^{\epsilon_1}y_1t^{\epsilon_2}y_2 \ldots t^{\epsilon_n}y_n$, where $x_i, y_i \in A, n \geq 1$ and $e_i, \epsilon_i = \pm 1$. Since $x \nsim_G y$, the system of equations (1^*) of Definition 2.9 has no set of complete admissible solutions for each i, $1 \leq i \leq n$. Therefore, we need to show that, for each i, there exists $P_i \lhd_f G$ in $\bar{G}_i = G/P_i$ such that the corresponding system of equations has no set of complete admissible solutions. Letting P to be the intersection of all normal subgroups P_i, we have $\bar{x} \nsim_{\bar{G}} \bar{y}$ in $\bar{G} = G/P$ and the result follows. Hence it is sufficient to show the case $i = 0$ in (1^*) of Definition 2.9.

Let u_i denote those $x_i, y_i \in A\backslash\langle h\rangle$ and v_i denote those $x_i, y_i \in A\backslash\langle k\rangle$. Since A is $\langle h\rangle$-separable and $\langle k\rangle$-separable, there exist $M_1 \lhd_f A, M_2 \lhd_f A$ such that $u_i \notin M_1\langle h\rangle$ and $v_i \notin M_2\langle k\rangle$ for all i. By Lemma 3.1, there exists $N \lhd_f A$ such that $N \subseteq M_1 \cap M_2$ and $\phi(N \cap \langle h\rangle) = N \cap \langle k\rangle$. Let $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{h}t = \bar{k}\rangle$ where $\bar{A} = A/N$ and $\bar{h} = hN, \bar{k} = kN$. Clearly $\bar{G}$ is a homomorphic image of G. Let $\bar{g}$ denote the image of any element $g \in G$ in $\bar{G}$. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ in $\bar{G}$.

Suppose there exists some i such that $e_i \neq \epsilon_i$. Then $\bar{x} \nsim_{\bar{G}} \bar{y}$ in $\bar{G}$. Since $\bar{G}$ is conjugacy separable by Lemma 2.10, there exists $\bar{Q} \lhd_f \bar{G}$ such that $\bar{x}\bar{Q} \nsim_{\bar{G}/\bar{Q}} \bar{y}\bar{Q}$. Let Q be the preimage of $\bar{Q}$ in G. Then $Q \lhd_f G$ such that $xQ \nsim_{G/Q} yQ$ and the result follows.

Suppose $e_i = \epsilon_i$ for all i. Since $x \nsim_G y$, either some equations in (1*) of Definition 2.9 has no admissible solution or every set of admissible solutions to (1*) of Definition 2.9 is incomplete.

First suppose there exists some $j, 1 \leq j \leq n$, such that the j-th equation has no admissible solution, that is, $x_j \notin \langle a_j\rangle y_j\langle b_j\rangle$ where $a_j, b_j \in \langle h\rangle \cup \langle k\rangle$. Since A is $\{h, k\}$-double coset separable, there exists $M_3 \lhd_f A$ such that $\tilde{x}_j \notin \langle\tilde{a}_j\rangle\tilde{y}_j\langle\tilde{b}_j\rangle$ in $\tilde{A} = A/M_3$. Let $M = M_1 \cap M_2 \cap M_3$. By Lemma 3.1, we can find $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap \langle h\rangle) = N \cap \langle k\rangle$. Again, we form $\bar{G}$ as above. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. Furthermore, $\bar{x}_j \notin \langle\bar{a}_j\rangle\bar{y}_j\langle\bar{b}_j\rangle$ where $\bar{a}_j, \bar{b}_j \in \langle\bar{h}\rangle \cup \langle\bar{k}\rangle$. Hence, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

Suppose $a_1^{c_1}, b_1^{d_1}, \ldots, a_n^{c_n}, b_n^{d_n} \in \langle h\rangle \cup \langle k\rangle$ is a set of incomplete admissible solutions to (1*) of Definition 2.9. Then we have the following:

$$
\begin{aligned}
x_1 &= a_1^{-c_1} y_1 b_1^{d_1} \\
x_2 &= a_2^{-c_2} y_2 b_2^{d_2} \\
&\vdots \qquad\qquad (2)
\end{aligned}
$$

$$x_n = a_n^{-c_n} y_n b_n^{d_n}$$

where $a_i = h$ or k accordingly as $e_i = -1$ or 1 respectively and $b_i = h$ or k accordingly as $e_{i+1} = 1$ or -1 respectively. Hence each equation $x_i = a_i^{-c_i} y_i b_i^{d_i}$ from (2) must take one of the following forms:

$$x_i = h^{-c_i} y_i h^{d_i} \text{ or,}$$

$$x_i = k^{-c_i} y_i k^{d_i} \text{ or,}$$

$$x_i = h^{-c_i} y_i k^{d_i} \text{ or,}$$

$$x_i = k^{-c_i} y_i h^{d_i}.$$

Before continuing with the proof, we analyze the various possible subcases. We first consider the equation $x_i = h^{-c_i} y_i k^{d_i}$. Then either $\langle y_i^{-1} h y_i \rangle \cap \langle k \rangle = 1$ or $\langle y_i^{-1} h y_i \rangle \cap \langle k \rangle = \langle k^{\gamma_i} \rangle$ where $\gamma_i > 0$.

Suppose $\langle y_i^{-1} h y_i \rangle \cap \langle k \rangle = 1$. If $x_i = h^{-p_i} y_i k^{q_i}$ for some other h^{-p_i}, k^{q_i}, then $y_i^{-1} h^{p_i - c_i} y_i = k^{q_i - d_i} \in \langle y_i^{-1} h y_i \rangle \cap \langle k \rangle = 1$ and hence $p_i - c_i = q_i - d_i = 0$, that is, $p_i = c_i, q_i = d_i$. This implies that c_i, d_i are uniquely determined in the equation $x_i = h^{-c_i} y_i k^{d_i}$.

Now suppose $\langle y_i^{-1} h y_i \rangle \cap \langle k \rangle = \langle k^{\gamma_i} \rangle$ where $\gamma_i > 0$. Let λ_i be the smallest positive integer such that $y_i^{-1} h^{\lambda_i} y_i = k^{\gamma_i}$. Since $h^m \sim_A k^{-m}$, then $y_i^{-1} h^{m\lambda_i} y_i = k^{m\gamma_i} \sim_A h^{-m\gamma_i}$. Since h is self-conjugate in A, we have $\gamma_i = -\lambda_i$ and hence $y_i^{-1} h^{\lambda_i} y_i = k^{-\lambda_i}$ and $y_i^{-1} h^q y_i \notin \langle k \rangle$ for all $1 \le q < \lambda_i$.

Similarly for the equation $x_i = h^{-c_i} y_i h^{d_i}$, either c_i, d_i are uniquely determined or $y_i^{-1} h^{m_i} y_i = h^{m_i}$ and $y_i^{-1} h^q y_i \notin \langle h \rangle$ for all $1 \le q < m_i$.

We can now proceed with the proof. We will only consider the case when $x = tx_1t^{e_2}x_2\dots t^{e_n}x_n$ and $y = ty_1t^{e_2}y_2\dots t^{e_n}y_n$. The proof for the other case when $x = t^{-1}x_1t^{e_2}x_2\dots t^{e_n}x_n$ and $y = t^{-1}y_1t^{e_2}y_2\dots t^{e_n}y_n$ is similar. By Lemma 2.8(ii), we have $x \nsim_G y$ if and only if $x \neq h^{-z}yh^z$ and $x \neq k^{-z}yk^z$ for all $z \in \mathbb{Z}$.

Case 1. Suppose

$$\langle y_i^{-1}hy_i\rangle \cap \langle h\rangle = \langle h^{\alpha_i}\rangle, \alpha_i > 0, \text{ for all equations } x_i = h^{-c_i}y_ih^{d_i},$$

$$\langle y_i^{-1}ky_i\rangle \cap \langle k\rangle = \langle k^{\beta_i}\rangle, \beta_i > 0, \text{ for all equations } x_i = k^{-c_i}y_ik^{d_i},$$

$$\langle y_i^{-1}hy_i\rangle \cap \langle k\rangle = \langle k^{\lambda_i}\rangle, \lambda_i > 0, \text{ for all equations } x_i = h^{-c_i}y_ik^{d_i},$$

$$\langle y_i^{-1}ky_i\rangle \cap \langle h\rangle = \langle h^{\rho_i}\rangle, \rho_i > 0, \text{ for all equations } x_i = k^{-c_i}y_ih^{d_i}.$$

Let $\gamma = m(lcm\{|\alpha_i|, |\beta_i|, |\lambda_i|, |\rho_i|\})$. Then

$$y_i^{-1}h^{\gamma}y_i = h^{\gamma} \text{ if } x_i = h^{-c_i}y_ih^{d_i},$$

$$y_i^{-1}k^{\gamma}y_i = k^{\gamma} \text{ if } x_i = k^{-c_i}y_ik^{d_i},$$

$$y_i^{-1}h^{\gamma}y_i = k^{-\gamma} \text{ if } x_i = h^{-c_i}y_ik^{d_i},$$

$$y_i^{-1}k^{\gamma}y_i = h^{-\gamma} \text{ if } x_i = k^{-c_i}y_ih^{d_i}.$$

Subcase 1a. Suppose in the system of equations (2), the total number of the equations $y_i^{-1}h^{\gamma}y_i = k^{-\gamma}$ together with the equations $y_i^{-1}k^{\gamma}y_i = h^{-\gamma}$ is zero or even. This implies that $y^{-1}h^{\gamma}y = y_n^{-1}t^{-e_n}\dots y_1^{-1}t^{-1}h^{\gamma}ty_1\dots t^{e_n}y_n = h^{\gamma}$, that is, $[y, h^{\gamma}] = 1$. Hence $h^{-z}yh^z = h^{-u}yh^u$ for some $0 \leq u < |\gamma|$. So, $x \neq h^{-z}yh^z$ for all $z \in \mathbb{Z}$ implies that $x \neq h^{-u}yh^u$ for all $0 \leq u < |\gamma|$, that is, $x^{-1}h^{-u}yh^u \neq 1$ for all $0 \leq u < |\gamma|$. Similarly, $yk^{\gamma}y^{-1} = k^{\gamma}$, that is $[y, k^{\gamma}] = 1$. So, $x \neq k^{-z}yk^z$ for all $z \in \mathbb{Z}$ implies that $x \neq k^{-u}yk^u$ for all $0 \leq u < |\gamma|$.

By Lemma 2.8(ii), $x \nsim_G y$ if and only if $x \neq h^{-u}yh^u$ and $x \neq k^{-u}yk^u$ for all $0 \leq u < |\gamma|$.

Since G is residually finite by Lemma 3.3, there exists $M_3 \lhd_f G$ such that $x^{-1}h^{-u}yh^u \notin M_3$ and $x^{-1}k^{-u}yk^u \notin M_3$ for all $0 \leq u < |\gamma|$. Let $M = M_1 \cap M_2 \cap M_3$. By Lemma 3.1, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. We now form $\bar{G}$ as above. Note that $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. Clearly, $\bar{x} \neq \bar{h}^{-u}\bar{y}\bar{h}^u$ and $\bar{x} \neq \bar{k}^{-u}\bar{y}\bar{k}^u$ for all $0 \leq u < |\gamma|$. Since $[\bar{y}, \bar{h}^\gamma] = 1$ and $[\bar{y}, \bar{k}^\gamma] = 1$, then by Lemma 2.8(ii), we have $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

Subcase 1b. Suppose in the system of equations (2) the total number of the equations $y_i^{-1}h^\gamma y_i = k^{-\gamma}$ together with equations $y_i^{-1}k^\gamma y_i = h^{-\gamma}$ is odd. Then arguing as in Subcase 1a, we have $y^{-1}h^\gamma y = h^{-\gamma}$ and $yk^\gamma y^{-1} = k^{-\gamma}$, that is, $h^{-\gamma}yh^\gamma = yh^{2\gamma}$ and $k^{-\gamma}yk^\gamma = yk^{2\gamma}$. So, $x \neq h^{-z}yh^z$ and $x \neq k^{-z}yk^z$ for all $z \in \mathbb{Z}$ implies that $x^{-1}h^{-u}yh^u \notin \langle h^{2\gamma} \rangle$ and $x^{-1}k^{-u}yk^u \notin \langle k^{2\gamma} \rangle$ for all $0 \leq u < |\gamma|$.

By Lemma 2.8(ii), $x \nsim_G y$ if and only if $x^{-1}h^{-u}yh^u \notin \langle h^{2\gamma} \rangle$ and $x^{-1}k^{-u}yk^u \notin \langle k^{2\gamma} \rangle$. Since G is $\langle h^{2\gamma} \rangle$-separable and $\langle k^{2\gamma} \rangle$-separable by Lemma 3.3, there exists $M_3 \lhd_f G$ such that $x^{-1}h^{-u}yh^u \notin \langle h^{2\gamma} \rangle M_3$ and $x^{-1}k^{-u}yk^u \notin \langle k^{2\gamma} \rangle M_3$. Let $M = M_1 \cap M_2 \cap M_3$. By Lemma 3.1, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. We now form $\bar{G}$ as above. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. Clearly, $\bar{x}^{-1}\bar{h}^{-u}\bar{y}\bar{h}^u \notin \langle \bar{h}^{2\gamma} \rangle$ and $\bar{x}^{-1}\bar{k}^{-u}\bar{y}\bar{k}^u \notin \langle \bar{k}^{2\gamma} \rangle$ for all $0 \leq u < |\gamma|$. Since $\bar{h}^{-\gamma}\bar{y}\bar{h}^\gamma = \bar{y}\bar{h}^{2\gamma}$ and $\bar{k}^{-\gamma}\bar{y}\bar{k}^\gamma = \bar{y}\bar{k}^{2\gamma}$, then by Lemma 2.8(ii), we have $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

Case 2. Suppose $\langle y_j^{-1}hy_j \rangle \cap \langle k \rangle = 1$ for one equation $x_j = h^{-c_j}y_jk^{d_j}$. Since $\langle y_j^{-1}hy_j \rangle \cap \langle k \rangle = 1$, then c_j, d_j are uniquely determined in the equation $x_j = h^{-c_j}y_jk^{d_j}$. Fixing this integer j, we consider the next equation $x_{j+1} = h^{-c_{j+1}}y_{j+1}k^{d_{j+1}}$. We arrange if possible so that $c_{j+1} = d_j$. Continuing this way, we see that this must eventually fail at some equation, say $x_r = h^{-c_r}y_rk^{d_r}$ where $c_r \neq d_{r-1}$. Here the integer r is taken modulo n and hence this may be the next equation or it may be the equation we started with.

Let $x' = t^{e_j}x_jt^{e_{j+1}} \dots t^{e_{r-1}}x_{r-1}t^{e_r}x_r$ and $y' = t^{e_j}y_jt^{e_{j+1}} \dots t^{e_{r-1}}y_{r-1}t^{e_r}y_r$.

Subcase 2a. $x_r = k^{-c_r} y_r h^{d_r}$. Then by substituting the values of $x_j, \ldots, x_{r-1}$ from (2) and x_r into x' and using the fact that we have arranged that $c_{i+1} = d_i$, $j \leq i \leq r-1$, we obtain $x' = k^{-c_j} t^{e_j} y_j t^{e_{j+1}} \ldots t^{e_{r-1}} y_{r-1} k^{d_{r-1}-c_r} t^{e_r} y_r h^{d_r}$ and hence $x' \notin \langle k \rangle y' \langle h \rangle$.

First we suppose $\langle y_r^{-1} k y_r \rangle \cap \langle h \rangle = 1$. Then c_r, d_r are uniquely determined in the equation $x_r = k^{-c_r} y_r h^{d_r}$. Let $\epsilon = 2|d_{r-1} - c_r|$. By Lemma 3.4, there exists $M_3 \lhd_f A$ such that in $\tilde{A} = A/M_3$, we have $\tilde{y}_j^{-1} \tilde{h}^{\sigma} \tilde{y}_j = \tilde{k}^{\rho}$ only if $\epsilon | \sigma, \rho$ and we have $\tilde{y}_r^{-1} \tilde{k}^p \tilde{y}_r = \tilde{h}^q$ only if $\epsilon | p, q$. By Lemma 3.1, we can find $N \lhd_f A$ such that $N \subseteq M_1 \cap M_2 \cap M_3$ and $\phi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. We now form $\bar{G}$. Note that $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. By the choice of M_3, we have $\bar{y}_j^{-1} \bar{h}^{\sigma} \bar{y}_j = \bar{k}^{\rho}$ only if $\epsilon | \sigma, \rho$ and we have $\bar{y}_r^{-1} \bar{k}^p \bar{y}_r = \bar{h}^q$ only if $\epsilon | p, q$ in $\bar{G}$.

First, we note that in $\bar{G}$, we have $\bar{x}' = \bar{k}^{-c_j} t^{e_j} \bar{y}_j t^{e_{j+1}} \ldots t^{e_{r-1}} \bar{y}_{r-1} k^{d_{r-1}-c_r}\ t^{e_r} \bar{y}_r \bar{h}^{d_r}$ and $\bar{y}' = t^{e_j} \bar{y}_j t^{e_{j+1}} \ldots t^{e_{r-1}} \bar{y}_{r-1} t^{e_r} \bar{y}_r$. Now we suppose $\bar{x}' \in \langle \bar{k} \rangle \bar{y}' \langle \bar{h} \rangle$. Then there exist integers $\delta_{j-1}, \delta_j, \ldots, \delta_r$ such that the following hold:

$$\begin{aligned} \bar{y}_j &= \bar{h}^{-\delta_{j-1}} \bar{y}_j \bar{k}^{\delta_j} \\ \bar{y}_{j+1} &= \bar{h}^{-\delta_j} \bar{y}_{j+1} \bar{b}_{j+1}^{\delta_{j+1}} \\ &\vdots \\ \bar{y}_{r-1} &= \bar{a}_{r-1}^{-\delta_{r-2}} \bar{y}_{r-1} \bar{h}^{\delta_{r-1}} \\ \bar{k}^{d_{r-1}-c_r} \bar{y}_r &= \bar{k}^{-\delta_{r-1}} \bar{y}_r \bar{h}^{\delta_r} \end{aligned} \tag{3}$$

From the first equation in (3), $\epsilon | \delta_{j-1}, \delta_j$ by the choice of M_3. Now by Lemma 3.5(i), $\epsilon || h |$. This together with $|h| = |k|$ in $\bar{G}$ in applying Lemma 3.5(ii), from the second equation to the second last equation, we have $\epsilon | \delta_k$ for all $j+1 \leq k \leq r-1$. Now from the last equation, we have $\bar{y}_r = \bar{k}^{-(d_{r-1}-c_r+\delta_{r-1})} \bar{y}_r \bar{h}^{\delta_r}$. Hence $\epsilon | d_{r-1} - c_r + \delta_{r-1}$ by the choice of M_3. Since $\epsilon | \delta_{r-1}$, we have $\epsilon | d_{r-1} - c_r$, which is a contradiction. Therefore $\bar{x}' \notin \langle \bar{k} \rangle \bar{y}' \langle \bar{h} \rangle$

in $\bar{G}$ and thus $\bar{x} \nsim_{\bar{G}} \bar{y}$.

Suppose $\langle y_r^{-1} k y_r \rangle \cap \langle h \rangle \neq 1$. Let s_r be the smallest positive integer such that $y_r^{-1} k^{s_r} y_r \in \langle h \rangle$. This implies that $y_r^{-1} k^q y_r \notin \langle h \rangle$ for all $1 \leq q < s_r$. Since A is $\langle h \rangle$-separable, there exists $M_3 \lhd_f A$ such that $y_r^{-1} k^q y_r \notin \langle h \rangle M_3$ for all $1 \leq q < s_r$. As the matching fails at the equation $x_r = k^{-c_r} y_r h^{d_r}$, we must have $k^{d_{r-1}-c_r} \notin \langle k^{s_r} \rangle$, that is, $d_{r-1} - c_r \neq z s_r$ for all $z \in \mathbb{Z}$. Since $\langle y_j^{-1} h y_j \rangle \cap \langle k \rangle = 1$ by Lemma 3.4, there exists $M_4 \lhd_f A$ such that in $\tilde{A} = A/M_4$, we have $\tilde{y}_j^{-1} \tilde{h}^u \tilde{y}_j = \tilde{k}^v$ only if $s_r | u, v$. By Lemma 3.1, we can find $N \lhd_f A$ such that $N \subseteq M_1 \cap M_2 \cap M_3 \cap M_4$ and $\phi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. We now form $\bar{G}$. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. Furthermore, by the choice of M_4, $\bar{y}_j^{-1} \bar{h}^u \bar{y}_j = \bar{k}^v$ only if $s_r | u, v$ whereas by the choice of M_3, $\bar{y}_r^{-1} \bar{k}^q \bar{y}_r \notin \langle \bar{h} \rangle$ for all $1 \leq q < s_r$ in $\bar{G}$.

Suppose $\bar{x}' \in \langle \bar{k} \rangle \bar{y}' \langle \bar{h} \rangle$. Then we again have the system of equations (3) as above. From the first equation in (3), $s_r | \delta_{j-1}, \delta_j$ by the choice of M_3. Now by Lemma 3.5(i), $s_r || h|$. This together with $|h| = |k|$ in $\bar{G}$ by applying Lemma 3.5(ii) from the second equation to the second last equation, we have $s_r | \delta_k$ for all $j+1 \leq k \leq r-1$. Now from the last equation we have $\bar{y}_r = \bar{k}^{-(d_{r-1}-c_r+\delta_{r-1})} \bar{y}_r \bar{h}^{s_r}$. Hence, $s_r | d_{r-1} - c_r + \delta_{r-1}$ by the choice of M_3. Since $s_r | \delta_{r-1}$, we have $s_r | d_{r-1} - c_r$. So, $d_{r-1} - c_r = v s_r$ for some $v \in \mathbb{Z}$. But this contradicts the fact that $d_{r-1} - c_r \notin z s_r$ for all $z \in \mathbb{Z}$. Therefore, $\bar{x}' \notin \langle \bar{k} \rangle \bar{y}' \langle \bar{h} \rangle$ in $\bar{G}$ and thus $\bar{x} \nsim_{\bar{G}} \bar{y}$.

The following subcases are similar and we can proceed as in Subcase 1a.

(i) $x_r = h^{-c_r} y_r k^{d_r}$,

(ii) $x_r = h^{-c_r} y_r h^{d_r}$,

(iii) $x_r = k^{-c_r} y_r k^{d_r}$.

Finally the following cases are similar and we can proceed as in Case 2.

Case 3. $\langle y_j^{-1} h y_j \rangle \cap \langle h \rangle = 1$, for one equation $x_j = h^{-c_j} y_j h^{d_j}$, or

Case 4. $\langle y_j^{-1} k y_j \rangle \cap \langle k \rangle = 1$, for one equation $x_j = k^{-c_j} y_j k^{d_j}$, or

Case 5. $\langle y_j^{-1} k y_j \rangle \cap \langle h \rangle = 1$, for one equation $x_j = k^{-c_j} y_j h^{d_j}$.

This completes the proof of this lemma.

Theorem 3.8. *Let $G = \langle t, A | t^{-1} h t = k \rangle$ be an HNN extension where h, k be elements of infinite order in A and $h^m \sim_A k^{\pm m}$ for some positive integer m. Let h, k be self-conjugate in A. Suppose*

(i) A is conjugacy separable;

(ii) A is cyclic subgroup separable;

(iii) A is $\langle h \rangle$-conjugacy separable and $\langle k \rangle$-conjugacy separable;

(iv) A is $\{h, k\}$-double coset separable;

(v) A is $\langle h \rangle$-weakly potent and $\langle k \rangle$-weakly potent;

(vi) for each integer $s > 0$, there exists $M_1 \lhd_f A$ such that $M_1 \cap \langle h \rangle = \langle h^s \rangle$ and $\tilde{h}^i \nsim_{\tilde{A}} \tilde{h}^j$ for all $\tilde{h}^i \neq \tilde{h}^j$ in $\tilde{A} = A/M_1$;

(vii) for each integer $s > 0$, there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle k \rangle = \langle k^s \rangle$ and $\hat{k}^i \nsim_{\hat{A}} \hat{k}^j$ for all $\hat{k}^i \neq \hat{k}^j$ in $\hat{A} = A/M_2$.

Then G is conjugacy separable.

Proof. We apply Theorem 2.12 here. We only prove for the case $h^m \sim_A k^{-m}$. The other case is similar. Since A is conjugacy separable, $\langle h \rangle$-separable and $\langle k \rangle$-separable, conditions (a) and (b) are satisfied. By Lemma 3.1, we have condition (c). We now show condition (d).

Let $x, y \in G$ such that $||x|| = ||y|| = 0$ and $x \nsim_G y$.

Case 1. Suppose $\{x\}^A \cap \langle h \rangle = \emptyset$ and $\{x\}^A \cap \langle k \rangle = \emptyset$. Note that $x \nsim_G y$ implies that $x \nsim_A y$. Since A is conjugacy separable, $\langle h \rangle$-conjugacy separable and $\langle k \rangle$-conjugacy separable, there exists $M \lhd_f A$ such that $xM \nsim_{A/M} yM$, $\{xM\}^{A/M} \cap \langle hM \rangle = \emptyset$ and $\{xM\}^{A/M} \cap \langle kM \rangle = \emptyset$. By Lemma 3.1, there exists $N \lhd_f A$ such that $N \subseteq M$ and

$\phi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. We now form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{h}t = \bar{k} \rangle$ where $\bar{A} = A/N, \bar{h} = hN$ and $\bar{k} = kN$. Then in $\bar{G}$, we have $\bar{x} \neq \bar{1}, \bar{y} \neq \bar{1}$, $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{h} \rangle = \emptyset$, $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{k} \rangle = \emptyset$ and $\bar{x} \nsim_{\bar{A}} \bar{y}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \langle \bar{h} \rangle \cup \langle \bar{k} \rangle$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \ldots \sim_{\bar{A},t} \bar{z}_w \sim_{\bar{A}} \bar{y}$. Since $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{h} \rangle = \emptyset$ and $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{k} \rangle = \emptyset$, the sequence reduces to $\bar{x} \sim_{\bar{A}} \bar{y}$ or $\bar{x} \sim_t \bar{y}$. Since $\bar{x} \notin \langle \bar{h} \rangle \cup \langle \bar{k} \rangle$, that is, $t^{-1}\bar{x}t, t\bar{x}t^{-1} \notin \langle \bar{h} \rangle \cup \langle \bar{k} \rangle$, the sequence further reduces to $\bar{x} \sim_{\bar{A}} \bar{y}$ which is a contradiction. Thus, $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable by Lemma 2.10, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{x}\bar{P} \nsim_{\bar{G}/\bar{P}} \bar{y}\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $xP \nsim_{G/P} yP$.

Case 2. Either $\{x\}^A \cap \langle h \rangle \neq \emptyset$ or $\{x\}^A \cap \langle k \rangle \neq \emptyset$. Suppose $\{x\}^A \cap \langle h \rangle \neq \emptyset$. Then $x \sim_A h^r$ for some integer r. Since h is self-conjugate in A, then r is uniquely determined. If $h^r \sim_A k^s$, then $s = \pm r$ by Lemma 3.6. Similarly if $k^r \sim_A h^s$, then $s = \pm r$. So in this case, $x \nsim_G y$ implies that $x \nsim_A y, h^{\pm r} \nsim_A y$ and $k^{\pm r} \nsim_A y$. Since A is conjugacy separable, there exists $M \lhd_f A$ such that $xM \nsim_{A/M} yM, h^{\pm r}M \nsim_{A/M} yM$ and $k^{\pm r}M \nsim_{A/M} yM$. Suppose $M \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $M \cap \langle k \rangle = \langle k^{s_2} \rangle$ for some positive integers s_1, s_2. Let $s = s_1 s_2 m$. By (vi), there exists $M_1 \lhd_f A$ such that $M_1 \cap \langle h \rangle = \langle h^s \rangle$ and $\tilde{h}^i \nsim_{\tilde{A}} \tilde{h}^j$ for all $\tilde{h}^i \neq \tilde{h}^j$ in $\tilde{A} = A/M_1$. Similarly with (vii), there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle k \rangle = \langle k^s \rangle$ and $\hat{k}^i \nsim_{\hat{A}} \hat{k}^j$ for all $\hat{k}^i \neq \hat{k}^j$ in $\hat{A} = A/M_2$. Let $N = M \cap M_1 \cap M_2$. Then $N \lhd_f A$ such that $N \cap \langle h \rangle = \langle h^s \rangle = \langle k^s \rangle = N \cap \langle k \rangle$. As above, we form $\bar{G}$ as in Case 1. Then in $\bar{G}$ we have $\bar{x} \nsim_{\bar{A}} \bar{y}, \bar{h}^{\pm r} \nsim_{\bar{A}} \bar{y}$ and $\bar{k}^{\pm r} \nsim_{\bar{A}} \bar{y}$. Also we have $\bar{h}^i \nsim_{\bar{A}} \bar{h}^j$ for all $\bar{h}^i \neq \bar{h}^j$ and $\bar{k}^i \nsim_{\bar{A}} \bar{k}^j$ for all $\bar{k}^i \neq \bar{k}^j$ in $\bar{G}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \langle \bar{h} \rangle \cup \langle \bar{k} \rangle$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \ldots \sim_{\bar{A},t} \bar{z}_w \sim_{\bar{A}} \bar{y}$. Consider the first conjugation relation $\bar{x} \sim_{\bar{A}} \bar{z}_1$. Recall that $x \sim_A h^r$ and hence $\bar{x} \sim_{\bar{A}} \bar{h}^r$. Note that the integer r is unique since $\bar{h}^i \nsim_{\bar{A}} \bar{h}^j$ for all $\bar{h}^i \neq \bar{h}^j$ in $\bar{A}$. Hence $\bar{z}_1 = \bar{h}^r$. Now consider the second conjugation relation $\bar{z}_1 \sim_{\bar{A},t} \bar{z}_2$. If $\bar{z}_2 = \bar{h}^s$, then $\bar{h}^s \sim_{\bar{A},t} \bar{z}_1 = \bar{h}^r$. Again since $\bar{h}^i \nsim_{\bar{A}} \bar{h}^j$ for all $\bar{h}^i \neq \bar{h}^j$

in $\bar{A}$, we have $\bar{z}_2 = \bar{h}^r$. Now suppose $\bar{z}_2 = \bar{k}^s$. Now if $h^r \sim_A k^s$, then by Lemma 3.6, $s = \pm r$. So, $h^r \sim_A k^{\pm r}$ and hence $\bar{h}^r \sim_{\bar{A}} \bar{k}^{\pm r}$. Again r is unique since $\bar{k}^i \nsim_{\bar{A}} \bar{k}^j$ for all $\bar{k}^i \neq \bar{k}^j$ in $\bar{A}$. Finally since $h \sim_t k$, we have $\bar{h} \sim_t \bar{k}$ and so $\bar{h}^r \sim_t \bar{k}^r$. Hence, $\bar{z}_2 = \bar{h}^{\pm r}$ or $\bar{z}_2 = \bar{k}^{\pm r}$. Continuing this way, we have $\bar{z}_i = \bar{h}^{\pm r}$ or $\bar{z}_i = \bar{k}^{\pm r}$ for $i = 3, \ldots, w$. Hence $\bar{z}_w = \bar{h}^{\pm r}$ or $\bar{z}_w = \bar{k}^{\pm r}$. This implies that $\bar{h}^{\pm r} \sim_{\bar{A}} \bar{y}$ or $\bar{k}^{\pm r} \sim_{\bar{A}} \bar{y}$. This is a contradiction since $\bar{h}^{\pm r} \nsim_{\bar{A}} \bar{y}$ and $\bar{k}^{\pm r} \nsim_{\bar{A}} \bar{y}$. Therefore $\bar{x} \nsim_{\bar{G}} \bar{y}$ and our result follows as in Case 1. The case when $\{x\}^A \cap \langle k \rangle \neq \emptyset$ is similarly proved.

Hence we have condition (d) from Case 1, Case 2 and Lemma 3.7. This completes the proof and thus G is conjugacy separable by Theorem 2.12.

Corollary 3.9. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ be an HNN extension where h, k be elements of infinite oder in A and $h^m \sim_A k^{\pm m}$ for some positive integer m. Suppose A is either finitely generated torsion-free nilpotent or free. Then G is conjugacy separable.*

Proof. Finitely generated torsion-free nilpotent groups and free groups are subgroup separable, conjugacy separable, cyclic conjugacy separable and weakly potent. Furthermore, both groups are H-double coset separable for each finitely generated subgroup H (Lennox & Wilson, 1979; Ribes & Zalesskii, 1993) and $\langle h \rangle$-self conjugate for all element h of infinite order (Dyer, 1980). Conditions (vi) and (vii) in Theorem 3.8 are straightforward from Corollary 2.2 of Tang (1997). Then we have G is conjugacy separable by Theorem 3.8.

3.4 Cyclic Conjugacy Separability on Certain HNN Extensions

In this section, we study the related property of cyclic conjugacy separability in HNN extensions. Recall that this property was used by Dyer (1980), Kim & Tang (1996), Kim & Tang (1999) and Tang (1997) to prove the conjugacy separability in certain generalized free products and HNN extensions. This property was formally defined by Tang (1997).

Recall that a group A is called cyclic conjugacy separable if for any pair of elements x, h of A such that $\{x\}^A \cap \langle h \rangle = \emptyset$, then there exists $N \lhd_f A$ such that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{h} \rangle = \emptyset$ in $\bar{A} = A/N$. The proof of Theorem 3.10 is similar to the proof of Theorem 3.8.

Theorem 3.10. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ be an HNN extension where h, k be elements of infinite order in A and $h^m \sim_A k^{\pm m}$ for some positive integer m. Let h, k be self-conjugate in A. Suppose*

(i) A is cyclic conjugacy separable;

(ii) A is cyclic subgroup separable;

(iii) A is $\{h, k\}$-double coset separable;

(iv) A is $\langle h \rangle$-weakly potent and $\langle k \rangle$-weakly potent;

(v) for each integer $s > 0$, there exists $M_1 \lhd_f A$ such that $M_1 \cap \langle h \rangle = \langle h^s \rangle$ and $\tilde{h}^i \nsim_{\tilde{A}} \tilde{h}^j$ for all $\tilde{h}^i \neq \tilde{h}^j$ in $\tilde{A} = A/M_1$;

(vi) for each integer $s > 0$, there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle k \rangle = \langle k^s \rangle$ and $\hat{k}^i \nsim_{\hat{A}} \hat{k}^j$ for all $\hat{k}^i \neq \hat{k}^j$ in $\hat{A} = A/M_2$.

Then G is cyclic conjugacy separable.

Proof. We apply Theorem 2.14 here. We prove the case when $h^m \sim_A k^{-m}$. Since A is cyclic conjugacy separable, $\langle h \rangle$-separable and $\langle k \rangle$-separable, conditions (a) and (b) are satisfied. By Lemma 3.1 and Lemma 3.7, we have conditions (c) and (d). We only need to show that condition (e) holds.

Let $x, y \in G$ such that $||x|| = ||y|| = 0$ and $\{x\}^G \cap \langle y \rangle = \emptyset$.

Case 1. Suppose $\{x\}^A \cap \langle h \rangle = \emptyset$ and $\{x\}^A \cap \langle k \rangle = \emptyset$. Then $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$. Since A is cyclic conjugacy separable, there exists $M \lhd_f A$ such that $\{xM\}^{A/M} \cap \langle hM \rangle = \emptyset, \{xM\}^{A/M} \cap \langle kM \rangle = \emptyset$ and $\{xM\}^{A/M} \cap \langle yM \rangle = \emptyset$. By

Lemma 3.1, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. We now form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{h}t = \bar{k} \rangle$ where $\bar{A} = A/N, \bar{h} = hN$ and $\bar{k} = kN$. Then in $\bar{G}$, we have $\bar{x} \neq \bar{1}, \bar{y} \neq \bar{1}$, $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{h} \rangle = \emptyset, \{\bar{x}\}^{\bar{A}} \cap \langle \bar{k} \rangle = \emptyset$ and $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \langle \bar{h} \rangle \cup \langle \bar{k} \rangle$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \dots \sim_{\bar{A},t} \bar{z}_w \sim_{\bar{A}} \bar{y}^k$. Since $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{h} \rangle = \emptyset$ and $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{k} \rangle = \emptyset$, the sequence reduces to $\bar{x} \sim_{\bar{A}} \bar{y}^k$ or $\bar{x} \sim_t \bar{y}^k$. Since $\bar{x} \notin \langle \bar{h} \rangle \cup \langle \bar{k} \rangle$, that is, $t^{-1}\bar{x}t, t\bar{x}t^{-1} \notin \langle \bar{h} \rangle \cup \langle \bar{k} \rangle$, the sequence further reduces to $\bar{x} \sim_{\bar{A}} \bar{y}^k$. But this contradicts the fact that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Thus, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 2.10, there exists $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P} \rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$.

Case 2. Either $\{x\}^A \cap \langle h \rangle \neq \emptyset$ or $\{x\}^A \cap \langle k \rangle \neq \emptyset$. Suppose $\{x\}^A \cap \langle h \rangle \neq \emptyset$. Then $x \sim_A h^r$ for some integer r. Since h is self-conjugate in A, then r is uniquely determined. If $h^r \sim_A k^s$, then $s = \pm r$ by Lemma 3.6. Similarly if $k^r \sim_A h^s$, then $s = \pm r$. So in this case, $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset, \{h^{\pm r}\}^A \cap \langle y \rangle = \emptyset$ and $\{k^{\pm r}\}^A \cap \langle y \rangle = \emptyset$. Since A is cyclic conjugacy separable, there exists $M \lhd_f A$ such that $\{xM\}^{A/M} \cap \langle yM \rangle = \emptyset, \{h^{\pm r}M\}^{A/M} \cap \langle yM \rangle = \emptyset$ and $\{k^{\pm r}M\}^{A/M} \cap \langle yM \rangle = \emptyset$. Suppose $M \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $M \cap \langle k \rangle = \langle k^{s_2} \rangle$ for some positive integers s_1, s_2. Let $s = s_1 s_2 m$. By (v), there exists $M_1 \lhd_f A$ such that $M_1 \cap \langle h \rangle = \langle h^s \rangle$ and $\tilde{h}^i \nsim_{\tilde{A}} \tilde{h}^j$ for all $\tilde{h}^i \neq \tilde{h}^j$ in $\tilde{A} = A/M_1$. Similarly with (vi), there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle k \rangle = \langle k^s \rangle$ and $\hat{k}^i \nsim_{\hat{A}} \hat{k}^j$ for all $\hat{k}^i \neq \hat{k}^j$ in $\hat{A} = A/M_2$. Let $N = M \cap M_1 \cap M_2$. Then $N \lhd_f A$ such that $N \cap \langle h \rangle = \langle h^s \rangle = \langle k^s \rangle = N \cap \langle k \rangle$. As above, we form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{h}t = \bar{k} \rangle$ where $\bar{A} = A/N, \bar{h} = hN$ and $\bar{k} = kN$. Then in $\bar{G}$, we have $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, \{\bar{h}^{\pm r}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$ and $\{\bar{k}^{\pm r}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Also we have $\bar{h}^i \nsim_{\bar{A}} \bar{h}^j$ for all $\bar{h}^i \neq \bar{h}^j$ and $\bar{k}^i \nsim_{\bar{A}} \bar{k}^j$ for all $\bar{k}^i \neq \bar{k}^j$ in $\bar{G}$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \langle \bar{h} \rangle \cup \langle \bar{k} \rangle$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \dots \sim_{\bar{A},t} \bar{z}_w \sim_{\bar{A}} \bar{y}^k$. Since $x \sim_A h^r$, then $\bar{x} \sim_{\bar{A}} \bar{h}^r$. Note that the integer r is unique since $\bar{h}^i \nsim_{\bar{A}} \bar{h}^j$ for all $\bar{h}^i \neq \bar{h}^j$ in $\bar{A}$. Hence, $\bar{z}_1 = \bar{h}^r$. Next we determine $\bar{z}_2$. Using the same argument, if $\bar{h}^r \sim_{\bar{A}} \bar{h}^s$, then $\bar{h}^r = \bar{h}^s$. Now if $h^r \sim_A k^s$, then by Lemma 3.6, $s = \pm r$. Hence, $\bar{h}^r \sim_{\bar{A}} \bar{k}^{\pm r}$. Now r is unique since $\bar{k}^i \nsim_{\bar{A}} \bar{k}^j$ for all $\bar{k}^i \neq \bar{k}^j$ in $\bar{A}$. Finally since $h \sim_t k$, we have $\bar{h} \sim_t \bar{k}$ and so $\bar{h}^r \sim_t \bar{k}^r$. Hence, $\bar{z}_2 = \bar{h}^{\pm r}$ or $\bar{z}_2 = \bar{k}^{\pm r}$. Continuing this way, we have $\bar{z}_i = \bar{h}^{\pm r}$ or $\bar{z}_i = \bar{k}^{\pm r}$ for $i = 3, \dots, w$. Hence $\bar{z}_w = \bar{h}^{\pm r}$ or $\bar{z}_w = \bar{k}^{\pm r}$. This implies that $\bar{h}^{\pm r} \sim_{\bar{A}} \bar{y}^k$ or $\bar{k}^{\pm r} \sim_{\bar{A}} \bar{y}^k$. This is a contradiction since $\{\bar{h}^{\pm r}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$ and $\{\bar{k}^{\pm r}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Therefore $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$ and our result follows as in Case 1. This completes the proof and thus G is cyclic conjugacy separable by Theorem 2.14.

Corollary 3.11. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ be an HNN extension where h, k be elements of infinite oder in A and $h^m \sim_A k^{\pm m}$ for some positive integer m. Suppose A is either finitely generated torsion-free nilpotent or free. Then G is cyclic conjugacy separable.*

Proof. As in the proof of Corollary 3.9, we have G is cyclic conjugacy separable by Theorem 3.10.

CHAPTER 4: HNN EXTENSIONS OF FINITELY GENERATED NILPOTENT GROUPS

4.1 Introduction

It has been established that the HNN extension $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ is conjugacy separable and cyclic conjugacy separable if A is finite. Dyer (1980) and Kim & Tang (1995) showed that these HNN extensions are free-by-finite and free-by-finite groups are conjugacy separable and cyclic conjugacy separable.

Collins (1969) showed that $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ is conjugacy separable if A is conjugacy separable and H, K are finite.

In this chapter, we shall show that $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ is cyclic conjugacy separable if A is conjugacy separable and cyclic conjugacy separable and H, K are finite. This is given in Theorem 4.13.

By using these results, we shall study the conjugacy separability and cyclic conjugacy separability of $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where $H \cap K$ is a non-trivial finitely generated subgroup in the center of A and $H \cap K$ has finite index in H and in K and ϕ is the identity map on $H \cap K$. First we shall establish a criterion for such HNN extension to be conjugacy separable (Theorem 4.7).

We shall use the criterion to show the conjugacy separability for $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where H and K are each a direct product of a finite group and the subgroup $H \cap K$. Then we shall show the cyclic conjugacy separability of these HNN extensions and extend the result to finitely generated nilpotent groups.

4.2 Lemmas Needed

In this section, we gather and prove the lemmas needed in this chapter.

Lemma 4.1. *Let A be a group and H be a subgroup of A. Suppose that there exists a finitely generated subgroup $R \subseteq H$ such that $|H : R| < \infty$ and $R \lhd A$. If A is R-separable, then A is H-separable.*

Proof. Since A is R-separable, then $\bar{A} = A/R$ is residually finite. Let $x \in A \backslash H$. Then $\bar{x} \notin \bar{H} = H/R$. Since $\bar{H}$ is finite, there exists $\bar{N} \lhd_f \bar{A}$ such that $\bar{N} \cap \bar{x}\bar{H} = \emptyset$. Let N be the preimage of $\bar{N}$ in A. Then $N \lhd_f A$ such that $x \notin HN$.

Next, we prove Lemma 4.2 and Lemma 4.3. These two lemmas are technical in nature and they will facilitate the proofs in the later theorems.

Lemma 4.2. *Let A be a group and H, K be subgroups of A where $H \cap K \neq 1$. Let $\phi : H \to K$ be an isomorphism such that $\phi(H \cap K) = H \cap K$. Let $N \lhd_f A$ such that $\phi(N \cap H) = N \cap K$. Then in $\bar{A} = A/N$, we have $\bar{\phi}(\overline{H \cap K}) = \overline{H \cap K}$ where $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$.*

Proof. First we note that the induced isomorphism $\bar{\phi}(\bar{H}) = \bar{K}$ is defined by $\bar{\phi}(hN) = \phi(h)N$ for all $h \in H$. We will show that $\bar{\phi}(\overline{H \cap K}) = \overline{H \cap K}$.

Let $\alpha \in \bar{\phi}(\overline{H \cap K})$. Then $\alpha = \bar{\phi}(uN)$ where $u \in H \cap K$. So, $\alpha = \phi(u)N$. Since $u \in H \cap K$, we have $\phi(u) \in \phi(H \cap K) = H \cap K$. This implies that $\alpha \in (H \cap K)N/N$ and thus $\alpha \in \overline{H \cap K}$. So, $\bar{\phi}(\overline{H \cap K}) \subseteq \overline{H \cap K}$.

Let $\beta \in \overline{H \cap K}$. Then $\beta = vN$ where $v \in H \cap K$. Since $\phi(H \cap K) = H \cap K$, there exists $w \in H \cap K$ such that $\phi(w) = v$. So, $\beta = \phi(w)N$. This implies that $\beta = \bar{\phi}(wN)$ and thus $\beta \in \bar{\phi}(\overline{H \cap K})$. So, $\overline{H \cap K} \subseteq \bar{\phi}(\overline{H \cap K})$. Therefore, $\bar{\phi}(\overline{H \cap K}) = \overline{H \cap K}$.

Lemma 4.3. *Let A be a group and H, K be subgroups of A where $H \cap K \neq 1$. Let $\phi : H \to K$ be an isomorphism such that $\phi(H \cap K) = H \cap K$. Suppose $H \cap K$ is finitely generated and $|H : H \cap K| < \infty, |K : H \cap K| < \infty$. Let A be subgroup separable. Then for each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. Furthermore, in $\bar{A} = A/N$, we have $\overline{H \cap K} = \bar{H} \cap \bar{K}$ and $\bar{\phi}(\bar{H} \cap \bar{K}) = \bar{H} \cap \bar{K}$ where $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$.*

Proof. We let $H \cap K = S$ and $M \lhd_f A$ be given. Since $M \cap S$ has finite index in S and S is finitely generated, there exists $R \subseteq M \cap S$ such that R is a characteristic subgroup of finite index in S. Since ϕ is an automorphism of S, $\phi(R) = R$. Let $\bar{A} = A/R$. Since $\bar{A}$ is residually finite and $\overline{HK}$ is finite, there exists $\bar{N}_1 \lhd_f \bar{A}$ such that $\bar{N}_1 \cap \overline{HK} = \bar{1}$. Let N_1 be the preimage of $\bar{N}_1$ in A. Then $N_1 \cap HK = R$. Let $N = M \cap N_1$. Then $N \lhd_f A$. Next, we need to show $N \cap H = R$. Let $\alpha \in N \cap H$. Then $\alpha \in N_1 \cap H \subseteq N_1 \cap HK = R$ and hence $\alpha \in R$. Now let $\beta \in R$. Then $\beta \in N_1$ since $R \subseteq N_1$. Furthermore $\beta \in M \cap H$ since $R \subseteq M \cap S \subseteq M \cap H$. Therefore $\beta \in M \cap N_1 \cap H = N \cap H$. Thus $N \cap H = R$. Similarly, we can show that $N \cap K = R$. Hence $\phi(N \cap H) = N \cap K$.

Now let $\bar{A} = A/N$. Note that $\overline{H \cap K} = (H \cap K)N/N \subseteq (HN \cap KN)/N = (HN/N) \cap (KN/N) = \bar{H} \cap \bar{K}$. Since $N \cap H = N \cap K$, $\bar{H} \cap \bar{K} = (HN/N) \cap (KN/N) \cong (H/H \cap N) \cap (K/K \cap N) = (H \cap K)/(H \cap K \cap N) \cong (H \cap K)N/N = \overline{H \cap K}$. Note that $\bar{H}, \bar{K}$ and $\overline{H \cap K}$ are finite. Since $\overline{H \cap K} \subseteq \bar{H} \cap \bar{K}$ and $\bar{H} \cap \bar{K} \cong \overline{H \cap K}$, we have $\overline{H \cap K} = \bar{H} \cap \bar{K}$. By Lemma 4.2, we have $\bar{\phi}(\bar{H} \cap \bar{K}) = \bar{H} \cap \bar{K}$. Thus, N is the required subgroup.

Lemma 4.4. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where $H \cap K \neq 1$. Suppose $H \cap K$ is a finitely generated subgroup of $Z(A)$ and $|H : H \cap K| < \infty, |K : H \cap K| < \infty$ such that $\phi(H \cap K) = H \cap K$. Let A be subgroup separable. Then G is residually finite.*

Proof. By Lemma 4.3, for each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. Note that A is H-separable and K-separable by Lemma 4.1. Since subgroup separable groups are also residually finite, we have G is residually finite by Lemma 2.11.

Lemma 4.5. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where $H \cap K \neq 1$. Suppose $H \cap K$ is a finitely generated subgroup of $Z(A)$ and $|H : H \cap K| < \infty$, $|K : H \cap K| < \infty$. Further suppose $\phi(H \cap K) = H \cap K$ and $\phi(s) = s$ for all $s \in H \cap K$. Let A be subgroup separable. Then for each $x, y \in G$ such that $||x|| = ||y|| \geq 1$ and $x \nsim_G y$, there exists $P \lhd_f G$ such that $xP \nsim_{G/P} yP$.*

Proof. We assume $x = t^{e_1}x_1 \ldots t^{e_n}x_n$ and $y = t^{E_1}y_1 \ldots t^{E_n}y_n$ where $e_i, E_i = \pm 1, x_i, y_i \in A$ and $n \geq 2$. Let a_i denote those $x_i, y_i \in A \backslash H$, b_i denote those $x_i, y_i \in A \backslash K$ and c_i denote those $x_i, y_i \in (H \cap K) \backslash \{1\}$ for all i. Since A is H-separable, K-separable and residually finite, there exists $M_1 \lhd_f A$ such that $a_i \notin HM_1, b_i \notin KM_1$ and $c_i \notin M_1$. By Lemma 2.8(ii), $x \nsim_G y$ if and only if $x' \nsim_{H \cup K} y$ for all cyclic permutations x' of x. Let $X = \{u^{-1}x'u | u \in (H \cup K) \backslash (H \cap K)$ and x' is a cyclic permutation of $x\}$. Note that $|H : H \cap K| < \infty, |K : H \cap K| < \infty$ and $H \cap K \subset Z(A)$. Furthermore, $t^{-1}st = s$ since $\phi(s) = s$ for all $s \in H \cap K$. Hence it follows that X is finite and $y \notin X$. Since G is residually finite by Lemma 4.4, there exists $L \lhd_f G$ such that $yL \cap \{zL | z \in X\} = \emptyset$. Let $M = M_1 \cap L$. By Lemma 4.3, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. We now form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi} \rangle$ where $\bar{A} = A/N, \bar{H} = HN/N, \bar{K} = KN/N$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$. Note that in $\bar{G}$, we have $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ and $\bar{x}' \nsim_{\bar{H} \cup \bar{K}} \bar{y}$. Hence by Lemma 2.8(ii), we have $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable by Lemma 2.10, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{x}\bar{P} \nsim_{\bar{G}/\bar{P}} \bar{y}\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $xP \nsim_{G/P} yP$.

Theorem 4.6. *Let $G = \langle t, A|t^{-1}Ht = K, \phi\rangle$ be an HNN extension where H, K are finite. Suppose A is residually finite (conjugacy separable). Then G is residually finite (Baumslag & Tretkoff, 1978) (conjugacy separable (Collins, 1969)).*

4.3 Conjugacy Separability of Certain HNN Extensions of Finitely Generated Nilpotent Groups

In this section, we prove our criterion for the HNN extension to be conjugacy separable.

Theorem 4.7. *Let $G = \langle t, A|t^{-1}Ht = K, \phi\rangle$ be an HNN extension where $H \cap K \neq 1$. Suppose $H \cap K$ is a finitely generated subgroup of $Z(A)$ and $|H : H \cap K| < \infty, |K : H \cap K| < \infty$. Further suppose $\phi(H \cap K) = H \cap K$ and $\phi(s) = s$ for all $s \in H \cap K$. Let*

(i) A be subgroup separable;

(ii) A be conjugacy separable;

(iii) A be H-conjugacy separable and K-conjugacy separable;

(iv) $A/(H \cap K)$ be conjugacy separable.

Then G is conjugacy separable if and only if,

(Λ) *for $u \in H \cup K \backslash (H \cap K)$ and $c \in H \cap K$, if $u \nsim_G uc$, there exists $N \lhd_f A$ such that $\phi(N \cap H) = N \cap K$ and we have $\hat{u} \nsim_{\hat{G}} \hat{u}\hat{c}$, in $\hat{G} = \langle t, \hat{A}|t^{-1}\hat{H}t = \hat{K}, \hat{\phi}\rangle$ where $\hat{A} = A/N, \hat{H} = HN/N, \hat{K} = KN/N$ and $\hat{\phi}$ is the induced isomorphism from $\hat{H}$ to $\hat{K}$.*

Proof. Suppose G is conjugacy separable. Let $u \in H\backslash(H \cap K)$ and $c \in H \cap K$ such that $u \nsim_G uc$. Then there exists $P \lhd_f G$ such that in $\bar{G} = G/P$, $\bar{u} \nsim_{\bar{G}} \bar{u}\bar{c}$. Let $M = P \cap A$. By Lemma 4.3, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. We now form $\hat{G} = \langle t, \hat{A}|t^{-1}\hat{H}t = \hat{K}, \hat{\phi}\rangle$ where $\hat{A} = A/N, \hat{H} = HN/N, \hat{K} = KN/N$ and $\hat{\phi}$ is the induced isomorphism from $\hat{H}$ to $\hat{K}$. By Lemma 2.10, $\hat{G}$ is conjugacy separable. Since there is a natural homomorphism from $\bar{G}$ to $\hat{G}$, we have $\hat{u} \nsim_{\hat{G}} \hat{u}\hat{c}$.

Conversely, suppose condition (Λ) is satisfied.

We apply Theorem 2.12 here. Since A is conjugacy separable, condition (a) holds. By Lemma 4.1 and Lemma 4.3, we have conditions (b) and (c).

We now show condition (d). We only need to consider the case when $||x|| = 0 = ||y||$ and $x \nsim_G y$. The case when $||x|| = ||y|| \geq 1$ follows from Lemma 4.5.

Case 1. Suppose $\{x\}^A \cap H = \emptyset$ and $\{x\}^A \cap K = \emptyset$. Note that $x \nsim_G y$ implies that $x \nsim_A y$. Since A is conjugacy separable, H-conjugacy separable and K-conjugacy separable, there exists $M \lhd_f A$ such that $xM \nsim_{A/M} yM$, $\{xM\}^{A/M} \cap HM/M = \emptyset$ and $\{xM\}^{A/M} \cap KM/M = \emptyset$. By Lemma 4.3, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. We now form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi}\rangle$ where $\bar{A} = A/N, \bar{H} = HN/N, \bar{K} = KN/N$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$. Note that in $\bar{G}$, we have $\bar{x}, \bar{y} \neq \bar{1}$, $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$, $\{\bar{x}\}^{\bar{A}} \cap \bar{K} = \emptyset$ and $\bar{x} \nsim_{\bar{A}} \bar{y}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \bar{H} \cup \bar{K}$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_n \sim_{\bar{A}} \bar{y}$. Since $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$ and $\{\bar{x}\}^{\bar{A}} \cap \bar{K} = \emptyset$, the sequence reduces to $\bar{x} \sim_{\bar{A}} \bar{y}$ or $\bar{x} \sim_t \bar{y}$. Since $\bar{x} \notin \bar{H} \cup \bar{K}$, this further reduces to $\bar{x} \sim_{\bar{A}} \bar{y}$. But this contradicts the fact that $\bar{x} \nsim_{\bar{A}} \bar{y}$. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable by Lemma 2.10, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{x}\bar{P} \nsim_{\bar{G}/\bar{P}} \bar{y}\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $xP \nsim_{G/P} yP$.

Case 2. Suppose $\{x\}^A \cap (H \cup K) \neq \emptyset$ but $\{x\}^A \cap (H \cap K) = \emptyset$. Suppose $\{x\}^A \cap H \neq \emptyset$. The proof is similar for the other case. Let $S = H \cap K$. Then $S \subseteq Z(A)$ and $\phi(S \cap H) = \phi(S) = S = S \cap K$. We form $\tilde{G} = \langle t, \tilde{A} | t^{-1}\tilde{H}t = \tilde{K}, \tilde{\phi}\rangle$ where $\tilde{A} = A/S, \tilde{H} = H/S, \tilde{K} = K/S$ and $\tilde{\phi}$ is the induced isomorphism from $\tilde{H}$ to $\tilde{K}$. By assumption, $\tilde{A}$ is conjugacy separable. Then $\tilde{G}$ is conjugacy separable by Theorem 4.6. If $\tilde{x} \nsim_{\tilde{G}} \tilde{y}$, then we are done. Suppose $\tilde{x} \sim_{\tilde{G}} \tilde{y}$. Then $\tilde{y} = \tilde{g}^{-1}\tilde{x}\tilde{g}$ for some $g \in G$. Hence $y = g^{-1}xgc = g^{-1}xcg$ for some $c \in H \cap K \subset Z(A)$. Since $x \nsim_G y$, we have $x \nsim_G xc$. By Condition (Λ), there exists $N \lhd_f A$ such that $\phi(N \cap H) = N \cap K$ and we have $\bar{x} \nsim_{\bar{G}} \bar{x}\bar{c}$ in $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi}\rangle$

where $\bar{A} = A/N, \bar{H} = HN/N, \bar{K} = KN/N$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. Then $\bar{y} = \bar{g}_1^{-1}\bar{x}\bar{g}_1$ for some $\bar{g}_1 \in \bar{G}$. Since $y = g^{-1}xcg$, this implies that $\bar{g}^{-1}\bar{x}\bar{c}\bar{g} = \bar{g}_1^{-1}\bar{x}\bar{g}_1$. Hence, $\bar{x}\bar{c} = \bar{g}_2^{-1}\bar{x}\bar{g}_2$ where $\bar{g}_2 = \bar{g}_1\bar{g}^{-1}$. Thus, we have $\bar{x}\bar{c} \sim_{\bar{G}} \bar{x}$, a contradiction. Hence $\bar{x} \nsim_{\bar{G}} \bar{y}$. By Lemma 2.10, $\bar{G}$ is conjugacy separable. Thus, we have $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows as in Case 1.

Case 3. Suppose $\{x\}^A \cap (H \cap K) \neq \emptyset$. Since $H \cap K \subseteq Z(A)$, without loss of generality we can assume $x \in H \cap K$. Hence $x \nsim_G y$ implies that $x \neq y$ and $t^{-n}xt^n \neq y$ for all integers n. Since $\phi(H \cap K) = H \cap K$ with $\phi(s) = s$ for all $s \in H \cap K$, we have $t^{-n}xt^n = x \neq y$. Since A is residually finite, there exists $M \lhd_f A$ such that $xy^{-1} \notin M$. By Lemma 4.3, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. We now form $\bar{G}$ as in Case 1. Then in $\bar{G}$, we have $\bar{x}, \bar{y} \neq \bar{1}$ and $\bar{x} \neq \bar{y}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \bar{H} \cup \bar{K}$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_n \sim_{\bar{A}} \bar{y}$. From the first conjugation relation, since $\bar{x} \in \bar{H} \cap \bar{K} \subseteq Z(\bar{A})$, we have $\bar{x} = \bar{z}_1$. From the second conjugation relation, $\bar{x} = \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2$, we obtain $\bar{z}_2 = \bar{x}$ if $\bar{z}_2 \sim_{\bar{A}} \bar{z}_1$ and $\bar{z}_2 = \bar{x}$ if $\bar{z}_2 \sim_t \bar{z}_1$ since $\phi(s) = s$ for all $s \in H \cap K$. Thus, $\bar{z}_2 = \bar{x}$. Continuing this way, we have $\bar{z}_i = \bar{x}$ for $i = 3, \ldots, n$. Hence $\bar{z}_n = \bar{x}$. This implies that $\bar{x} \sim_{\bar{A}} \bar{y}$. Since $\bar{x} \in Z(\bar{A})$, we have $\bar{x} = \bar{y}$. This contradicts the fact that $\bar{x} \neq \bar{y}$. Hence $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows as in Case 1. Thus, we have (d).

The proof is now complete and hence G is conjugacy separable by Theorem 2.12.

Note that finitely generated nilpotent groups are subgroup separable, conjugacy separable and cyclic conjugacy separable. Furthermore, the quotient groups of nilpotent groups are again nilpotent. Thus, we extend our criterion, Theorem 4.7 to finitely generated nilpotent group in this section. We begin with the next lemma.

Lemma 4.8. *Let A be a finitely generated nilpotent group and $C \lhd A$. If $C \leq H \leq A$ and $|H/C|$ is finite, then A is H-conjugacy separable.*

Proof. Let $a \in A$ such that $\{a\}^A \cap H = \emptyset$. Let $\bar{A} = A/C$. Then $\{\bar{a}\}^{\bar{A}} \cap \bar{H} = \emptyset$. Since $\bar{A}$ is cyclic conjugacy separable and $\bar{H}$ is finite, there exists $\bar{M} \lhd_f \bar{A}$ such that $\{\bar{a}\bar{M}\}^{\bar{A}/\bar{M}} \cap \langle \bar{h} \rangle = \emptyset$ for all $\bar{h} \in \bar{H}$. Let M be the preimage of $\bar{M}$ in A. Then we have $\{aM\}^{A/M} \cap HM/M = \emptyset$.

By Lemma 4.8, the following corollary is straightforward from Theorem 4.7.

Corollary 4.9. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where $H \cap K \neq 1$. Suppose $H \cap K$ is a finitely generated subgroup of $Z(A)$ and $|H : H \cap K| < \infty, |K : H \cap K| < \infty$. Further suppose $\phi(H \cap K) = H \cap K$ and $\phi(s) = s$ for all $s \in H \cap K$. Let A be finitely generated nilpotent. Then G is conjugacy separable if and only if,*

(Λ) for $u \in (H \cup K) \backslash (H \cap K)$ and $c \in H \cap K$, if $u \nsim_G uc$, there exists $M \lhd_f A$ such that $\phi(M \cap H) = M \cap K$ and we have $\hat{u} \nsim_{\hat{G}} \hat{u}\hat{c}$, in $\hat{G} = \langle t, \hat{A} | t^{-1}\hat{H}t = \hat{K}, \hat{\phi} \rangle$ where $\hat{A} = A/M, \hat{H} = HM/M, \hat{K} = KM/M$ and $\hat{\phi}$ is the induced isomorphism from $\hat{H}$ to $\hat{K}$.

Next we can have the following application.

Theorem 4.10. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where $H = P \times C, K = Q \times C$ with P, Q being finite and $C \subseteq Z(A)$ is finitely generated. Suppose $P \cap Q = 1$ and $\phi(P) = Q, \phi(C) = C$ with $\phi(c) = c$ for all $c \in C$. Let A be finitely generated nilpotent. Then G is conjugacy separable.*

Proof. Since C is a finitely generated abelian group, $C = K_1 \times C_1$ where K_1 is finite and C_1 is torsion-free. Hence we may assume that C is torsion-free.

We apply Theorem 4.7. here. Let $u \in H \backslash C$ and $c \in C$ such that $u \nsim_G uc$. The proof is similar when $u \in K \backslash C$. Since $H = P \times C$, then $u = p_0 c_0$ uniquely where $p_0 \in P, p_0 \neq 1$

and $c_0 \in C$. So, we have $p_0c_0 \sim_A p_0c_0c$ and thus $p_0c_0 = a^{-1}p_0c_0ca$ where $a \in A$. Since $c_0 \in C \subseteq Z(A)$, we then have $p_0 = a^{-1}p_0ca$. Hence, without loss of generality, we may assume $u \in P$. Since C is residually finite, there exists $R \lhd_f C$ such that $c \notin R$. Since R has finite index in the finitely generated subgroup C, we can find a subgroup $S \subseteq R$ such that S is characteristic and finite index in C. Furthermore, note that $\phi(S) = S$ with $\phi(s) = s$ for all $s \in S$ and $c \notin S$. We now form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi}\rangle$ where $\bar{A} = A/S, \bar{H} = H/S, \bar{K} = K/S$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$.

Suppose $\bar{u} \sim_{\bar{G}} \bar{u}\bar{c}$. By Lemma 2.8(i), there exists a finite sequence $\bar{z}_1, \ldots, \bar{z}_n \in \bar{H} \cup \bar{K}$ such that $\bar{u} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_n \sim_{\bar{A}} \bar{u}\bar{c}$. Since $\bar{H} = \bar{P} \times \bar{C}, \bar{K} = \bar{Q} \times \bar{C}$, we have $\bar{z}_i = \bar{p}_i\bar{c}_i$ uniquely where $\bar{p}_i \in \bar{P} \cup \bar{Q}, \bar{c}_i \in \bar{C}$. Hence, $\bar{u} \sim_{\bar{A}} \bar{p}_1\bar{c}_1 \sim_{\bar{A},t} \bar{p}_2\bar{c}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{p}_n\bar{c}_n \sim_{\bar{A}} \bar{u}\bar{c}$. From the first conjugation relation $\bar{u} \sim_{\bar{A}} \bar{p}_1\bar{c}_1$, we have $\bar{p}_1 \in \bar{P} \cup \bar{Q}, \bar{c}_1 \in \bar{C}$. So, $\bar{u} = \bar{a}_1^{-1}\bar{p}_1\bar{c}_1\bar{a}_1$ for some $a_1 \in A$ and thus $\bar{u}^{-1}\bar{a}_1^{-1}\bar{p}_1\bar{a}_1 = \bar{c}_1^{-1}$. It follows that $u^{-1}a_1^{-1}p_1a_1S = c_1^{-1}S$ and this implies that $u^{-1}a_1^{-1}p_1a_1 \in C$ since $S \lhd_f C$. We let $u^{-1}a_1^{-1}p_1a_1 = w_1 \in C$, then $uw_1 = a_1^{-1}p_1a_1$. Since P and Q are finite, we let $m_1 = lcm\{|u|, |p_1|\}$. Then $(uw_1)^{m_1} = (a_1^{-1}p_1a_1)^{m_1}$ which implies that $w_1^{m_1} = 1$. Since C is torsion-free, we have $w_1 = 1$. It follows that $u^{-1}a_1^{-1}p_1a_1 = 1$, thus $\bar{c}_1 = 1$ and $\bar{z}_1 = \bar{p}_1$. So, now we can write $\bar{u} \sim_{\bar{A}} \bar{p}_1 \sim_{\bar{A},t} \bar{p}_2\bar{c}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{p}_n\bar{c}_n \sim_{\bar{A}} \bar{u}\bar{c}$.

From the second conjugation relation, we have $\bar{p}_1 \sim_{\bar{A},t} \bar{p}_2\bar{c}_2$. If $\bar{p}_1 \sim_{\bar{A}} \bar{p}_2\bar{c}_2$, we have $\bar{c}_2 = 1$ and $\bar{z}_2 = \bar{p}_2$ as above. Now suppose $\bar{p}_1 \sim_t \bar{p}_2\bar{c}_2$. Then $\bar{p}_1 = t^{-r}\bar{p}_2\bar{c}_2t^r$ for some integer r and so we have $\bar{p}_2^{-1}t^r\bar{p}_1t^{-r} = \bar{c}_2$. Hence $p_2^{-1}t^rp_1t^{-r}S = c_2S$. This implies that $p_2^{-1}t^rp_1t^{-r} = w_2 \in C$. Since $\bar{P}$ and $\bar{Q}$ are finite, we let $m_2 = lcm\{|p_1|, |p_2|\}$. Then $(t^rp_1t^{-r})^{m_2} = (p_2w_2)^{m_2}$ which implies that $w_2^{m_2} = 1$ and thus $w_2 = 1$. Hence, we have $\bar{c}_2 = 1$ and $\bar{z}_2 = \bar{p}_2$ for this case as well.

Proceeding in this way, we have $\bar{c}_i = 1, \bar{z}_i = \bar{p}_i$ where $\bar{p}_i \in \bar{P} \cup \bar{Q}$ for all $i = 3, \ldots, n$. Now we can write $\bar{u} \sim_{\bar{A}} \bar{p}_1 \sim_{\bar{A},t} \bar{p}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{p}_n \sim_{\bar{A}} \bar{u}\bar{c}$. From the last conjugation relation $\bar{p}_n \sim_{\bar{A}} \bar{u}\bar{c}$, we have $\bar{c} = 1$ and $\bar{p}_n \sim_{\bar{A}} \bar{u}$ as above. This is a contradiction since $c \notin S$. Therefore, we have $\bar{u} \nsim_{\bar{G}} \bar{u}\bar{c}$ in $\bar{G}$.

Since $\bar{u} \nsim_{\bar{G}} \bar{u}\bar{c}$ and $\bar{G}$ is conjugacy separable, there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{u}\bar{L} \nsim_{\bar{G}/\bar{L}} \bar{u}\bar{c}\bar{L}$. Let L be the preimage of $\bar{L}$ in G. Let $M = L \cap A$. As in Lemma 4.3, there exists $N \lhd_f A$ such that $N \subseteq M$. We now form $\hat{G} = \langle t, \hat{A} | t^{-1}\hat{H}t = \hat{K}, \hat{\phi}\rangle$ where $\hat{A} = A/N, \hat{H} = HN/N, \hat{K} = KN/N$ and $\hat{\phi}$ is the induced isomorphism from $\hat{H}$ to $\hat{K}$. It is clear that $\hat{u} \nsim_{\hat{G}} \hat{u}\hat{c}$. Therefore, G is conjugacy separable by Corollary 4.9.

4.4 Cyclic Conjugacy Separability of Certain HNN Extension with Finite Associated Subgroups

In this section, we study the cyclic conjugacy separability of HNN extensions with finite associated subgroups. In Collins (1969), Collins has shown the conjugacy separability of such HNN extensions. We shall now show the cyclic conjugacy separability of such groups. We begin with the following lemma.

Lemma 4.11. *Let A be a group and H be a finite subgroup of A. If A is residually finite, then A is H-separable.*

Proof. Let $x \in A \backslash H$. Since A is residually finite and H is finite, there exists $M \lhd_f A$ such that $M \cap xH = \emptyset$. Therefore, A is H-separable.

Lemma 4.12. *Let A be a group and H be a finite subgroup of A. If A is cyclic conjugacy separable, then A is H-conjugacy separable.*

Proof. Let $x \in A$ such that $\{x\}^A \cap H = \emptyset$. Since A is residually finite and H is finite, there exists $M_0 \lhd_f A$ such that $M_0 \cap H = 1$. It is clear that $\{x\}^A \cap \langle h_i\rangle = \emptyset$ for all $h_i \in H$. Since A is cyclic conjugacy separable, there exists $M_i \lhd_f A$ such that $\{xM_i\}^{A/M_i} \cap \langle h_i\rangle M_i = \emptyset$ for all i. Let $M = \bigcap_i M_i \cap M_0$. Then $M \lhd_f A$ such that $\{xM\}^{A/M} \cap HM/M = \emptyset$.

Theorem 4.13. *Let $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$ be an HNN extension where H, K are finite. Suppose A is conjugacy separable and cyclic conjugacy separable. Then G is cyclic conjugacy separable.*

Proof. We apply Theorem 2.14 here. By assumption, we have condition (a). By Lemma 4.11 and Theorem 4.6, we have conditions (b) and (d) respectively.

Let $M \lhd_f A$ be given. Since A is residually finite and H, K are finite subgroups, there exists $M_0 \lhd_f A$ such that $M_0 \cap H = 1 = M_0 \cap K$. Let $N = M \cap M_0$. Then $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. Thus, we have condition (c).

Now we only need to prove condition (e). Let $x, y \in G$ such that $||x|| = ||y|| = 0$ and $\{x\}^G \cap \langle y\rangle = \emptyset$.

Case 1. Suppose $\{x\}^A \cap H = \emptyset$ and $\{x\}^A \cap K = \emptyset$. Now $\{x\}^G \cap \langle y\rangle = \emptyset$ implies that $\{x\}^A \cap \langle y\rangle = \emptyset$. Since A is cyclic conjugacy separable, H-conjugacy separable and K-conjugacy separable by Lemma 4.12, there exists $M_1 \lhd_f A$ such that $\{xM_1\}^{A/M_1} \cap \langle yM_1\rangle = \emptyset$, $\{xM_1\}^{A/M_1} \cap HM_1/M_1 = \emptyset$ and $\{xM_1\}^{A/M_1} \cap KM_1/M_1 = \emptyset$. Since A is residually finite and H, K are finite, there exists $M_2 \lhd_f A$ such that $M_2 \cap H = 1 = M_2 \cap K$. Let $N = M_1 \cap M_2$. Then $N \lhd_f A$ and $\phi(N \cap H) = \phi(1) = 1 = N \cap K$. We now form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi}\rangle$ where $\bar{A} = A/N$, $\bar{H} = HN/N$, $\bar{K} = KN/N$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$. Then in $\bar{G}$, we have $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y}\rangle = \emptyset$, $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$ and $\{\bar{x}\}^{\bar{A}} \cap \bar{K} = \emptyset$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y}\rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 2.8(i), there exists a finite sequence $\bar{z}_i \in \bar{H} \cup \bar{K}$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_n \sim_{\bar{A}} \bar{y}^k$. Since $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$ and $\{\bar{x}\}^{\bar{A}} \cap \bar{K} = \emptyset$, the sequence reduces to $\bar{x} \sim_{\bar{A}} \bar{y}^k$ or $\bar{x} \sim_t \bar{y}^k$. Since $\bar{x} \notin \bar{H} \cup \bar{K}$, this further reduces to $\bar{x} \sim_{\bar{A}} \bar{y}^k$. But this contradicts the fact that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y}\rangle = \emptyset$. Therefore $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y}\rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 2.10, there exists $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P}\rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then

$P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$. Our result follows.

Case 2. Either $\{x\}^A \cap H \neq \emptyset$ or $\{x\}^A \cap K \neq \emptyset$. Suppose $\{x\}^A \cap H \neq \emptyset$. The proof is similar for $\{x\}^A \cap K \neq \emptyset$. Let $u_i \in H \cup K, i = 1, \ldots, m$ be all the elements in $H \cup K$ such that $u_i \sim_A x$. Since $\{x\}^A \cap \langle y \rangle = \emptyset$, we have $\{u_i\}^A \cap \langle y \rangle = \emptyset, i = 1, \ldots, m$. Since A is cyclic conjugacy separable, there exists $M_1 \lhd_f A$ such that $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$ and $\{u_iM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset, i = 1, \ldots, m$. Since A is conjugacy separable, there exists $M_2 \lhd_f A$ such that $p_rM_2 \nsim_{A/M_2} p_sM_2$ for all pairs of elements $p_r, p_s \in H \cup K$ where $p_r \nsim_A p_s$. Since G is residually finite by Theorem 4.6, there exists $Q \lhd_f G$ such that $u_vQ \nsim_t u_wQ$ for all pairs of elements $u_v, u_w \in H \cup K$ where $u_v \nsim_t u_w$. Again since H, K are finite and A is residually finite, there exists $M_3 \lhd_f A$ such that $M_3 \cap H = 1 = M_3 \cap K$. Let $N = M_1 \cap M_2 \cap M_3 \cap Q$. Then $N \lhd_f A$ and $\phi(N \cap H) = \phi(1) = 1 = N \cap K$. We now form $\bar{G}$ as in Case 1. Then in $\bar{G}$, we have $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, \{\bar{u}_i\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$ for all $u_i, i = 1, \ldots, m$, $\bar{p}_r \nsim_{\bar{A}} \bar{p}_s$ for all pairs $p_r, p_s \in H \cup K, p_r \nsim_A p_s$ and $\bar{u}_v \nsim_t \bar{u}_w$ for all pairs $u_v, u_w \in H \cup K, u_v \nsim_t u_w$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 2.8(i), there exists a finite sequence $\bar{z}_i \in \bar{H} \cup \bar{K}$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_n \sim_{\bar{A}} \bar{y}^k$. From the first conjugation relation $\bar{x} \sim_{\bar{A}} \bar{z}_1$, we have $xN \sim_{A/N} z_1N$. Now suppose $x \nsim_A z_1$. Then we have $xM_1 \nsim_{A/M_1} z_1M_1$. But this contradicts to $xN \sim_{A/N} z_1N$. Hence $x \sim_A z_1$. Since $\{x\}^A \cap \langle y \rangle = \emptyset$, we have $\{z_1\}^A \cap \langle y \rangle = \emptyset$. So in $\bar{G}$, we have $\{\bar{z}_1\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Without loss of generality, we assume $z_1 \in H$.

From the second conjugation relation $\bar{z}_1 \sim_{\bar{A},t} \bar{z}_2$, we have $\bar{z}_2 \sim_{\bar{A}} \bar{z}_1$ or $\bar{z}_2 \sim_t \bar{z}_1$. If $\bar{z}_2 \sim_{\bar{A}} \bar{z}_1$, then $\bar{z}_2 \sim_{\bar{A}} \bar{x}$. Arguing as above, we have $x \sim_A z_2$. Hence $\{z_2\}^A \cap \langle y \rangle = \emptyset$. Now suppose $\bar{z}_2 \sim_t \bar{z}_1$. Then $\bar{z}_2 \sim_t \bar{z}_1 \sim_A \bar{x}$. Again arguing as above, we have $z_2 \sim_t z_1 \sim_A x$. We now show that $\{z_2\}^A \cap \langle y \rangle = \emptyset$. Suppose $\{z_2\}^A \cap \langle y \rangle \neq \emptyset$. Then $z_2 \sim_A y^l$ for some integer l. Hence $y^l \sim_A z_2 \sim_t z_1 \sim_A x$. This implies that $\{x\}^G \cap \langle y \rangle \neq \emptyset$, a contradiction. Hence

$\{z_2\}^A \cap \langle y \rangle = \emptyset$. So in both cases, $\{z_2\}^A \cap \langle y \rangle = \emptyset$. Hence in $\bar{G}$, we have $\{\bar{z}_2\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

Proceeding from the third to the second last conjugation relation, we obtain $z_i \in H \cup K$ such that $\{z_i\}^A \cap \langle y \rangle = \emptyset, i = 3, \ldots, n$. So in $\bar{G}$, we have $\{\bar{z}_i\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, i = 3, \ldots, n$. From the last conjugation relation $\bar{z}_n \sim_{\bar{A}} \bar{y}^k$, we have $\{\bar{z}_n\}^{\bar{A}} \cap \langle \bar{y} \rangle \neq \emptyset$. This contradicts the fact that $\{\bar{z}_n\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$ and the result follows as in Case 1.

This completes the proof and hence G is cyclic conjugacy separable by Theorem 2.14.

4.5 Cyclic Conjugacy Separability of Certain HNN Extensions of Finitely Generated Nilpotent Groups

In this section, we give two criteria on cyclic conjugacy separability of HNN extensions. Then apply these two criteria to finitely generated nilpotent groups. We apply Theorem 4.13 in Theorem 4.14.

Theorem 4.14. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where $H = P \times C, K = Q \times C$ with P, Q being finite and $C \subseteq Z(A)$ is finitely generated torsion-free. Suppose $P \cap Q = 1$ and $\phi(P) = Q, \phi(C) = C$ with $\phi(c) = c$ for all $c \in C$. Suppose*

(i) *A is subgroup separable;*

(ii) *A is conjugacy separable;*

(iii) *A is cyclic conjugacy separable;*

(iv) *A is H-conjugacy separable and K-conjugacy separable;*

(v) *A/S is conjugacy separable and cyclic conjugacy separable for any $S \lhd_f C$.*

Then G is cyclic conjugacy separable.

Proof. We apply Theorem 2.14 here. Since A is subgroup separable, we have condition (a). By Lemma 4.1, Lemma 4.3 and Lemma 4.5, we have conditions (b), (c) and (d). Now we only need to prove condition (e).

Let $x, y \in G$ such that $||x|| = ||y|| = 0$ and $\{x\}^G \cap \langle y \rangle = \emptyset$.

Case 1. Suppose $\{x\}^A \cap H = \emptyset$ and $\{x\}^A \cap K = \emptyset$. Note that $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$. Since A is cyclic conjugacy separable, H-conjugacy separable and K-conjugacy separable, there exists $M \lhd_f A$ such that $\{xM\}^{A/M} \cap \langle yM \rangle = \emptyset$, $\{xM\}^{A/M} \cap HM/M = \emptyset$ and $\{xM\}^{A/M} \cap KM/M = \emptyset$. By Lemma 4.3, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. We now form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi} \rangle$ where $\bar{A} = A/N, \bar{H} = HN/N, \bar{K} = KN/N$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$. Note that in $\bar{G}$, we have $\bar{x}, \bar{y} \neq \bar{1}$, $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$, $\{\bar{x}\}^{\bar{A}} \cap \bar{K} = \emptyset$ and $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \bar{H} \cup \bar{K}$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_n \sim_{\bar{A}} \bar{y}^k$. Since $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$ and $\{\bar{x}\}^{\bar{A}} \cap \bar{K} = \emptyset$, the sequence reduces to $\bar{x} \sim_{\bar{A}} \bar{y}^k$ or $\bar{x} \sim_t \bar{y}^k$. Since $\bar{x} \notin \bar{H} \cup \bar{K}$, this further reduces to $\bar{x} \sim_{\bar{A}} \bar{y}^k$. But this contradicts the fact that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Therefore, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 2.10, there exists $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P} \rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$.

Case 2. Suppose $\{x\}^A \cap (H \cup K) \neq \emptyset$ but $\{x\}^A \cap C = \emptyset$.

Subcase 2a. Suppose $\{x\}^A \cap H \neq \emptyset$, $\{x\}^A \cap K = \emptyset$. This implies that $\{x\}^A \cap C = \emptyset$. Since A is cyclic conjugacy separable, K-conjugacy separable and subgroup separable, there exists $M_1 \lhd_f A$ such that $x \notin CM_1$, $\{xM_1\}^{A/M_1} \cap KM_1/M_1 = \emptyset$ and $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$. Let $r_i \in P \cup Q$ such that $x \sim_A r_i$. Since $\{x\}^A \cap \langle y \rangle = \emptyset$, we have $\{r_i\}^A \cap \langle y \rangle = \emptyset$. Since A is cyclic conjugacy separable, there exists $M_2 \lhd_f A$ such that $\{r_i M_2\}^{A/M_2} \cap \langle yM_2 \rangle = \emptyset$ for all $r_i \in P \cup Q$. Since A is conjugacy separable, there exists $M_3 \lhd_f A$ such that

$u_rM_3 \nsim_{A/M_3} u_sM_3$ for all pairs of elements $u_r, u_s \in P \cup Q$ where $u_r \nsim_A u_s$. Since G is residually finite by Lemma 4.4, there exists $L \lhd_f G$ such that $v_jL \nsim_t v_kL$ for all pairs of elements $v_j, v_k \in P \cup Q$ where $v_j \nsim_t v_k$. Let $S = M_1 \cap M_2 \cap M_3 \cap L \cap C$. Then $S \lhd_f C$ and thus $S \lhd_f H, S \lhd_f K$ with $\phi(S) = S$. We now form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi}\rangle$ where $\bar{A} = A/S, \bar{H} = H/S, \bar{K} = K/S$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$. Then in $\bar{G}$, we have $\bar{x} \notin \bar{C}, \{\bar{x}\}^{\bar{A}} \cap \bar{K} = \emptyset, \{\bar{x}\}^{\bar{A}} \cap \langle\bar{y}\rangle = \emptyset, \{\bar{r}_i\}^{\bar{A}} \cap \langle\bar{y}\rangle = \emptyset$ for all $r_i \in P \cup Q, x \sim_A r_i, \bar{u}_r \nsim_{\bar{A}} \bar{u}_s$ for all pairs $u_r, u_s \in P \cup Q, u_r \nsim_A u_s$ and $\bar{v}_j \nsim_t \bar{v}_k$ for all pairs $v_j, v_k \in P \cup Q, v_j \nsim_t v_k$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle\bar{y}\rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \bar{H} \cup \bar{K}$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_n \sim_{\bar{A}} \bar{y}^k$. Since $\bar{H} = \bar{P} \times \bar{C}, \bar{K} = \bar{Q} \times \bar{C}$, we have $\bar{z}_i = \bar{p}_i\bar{c}_i$ uniquely where $\bar{p}_i \in \bar{P} \cup \bar{Q}, \bar{c}_i \in \bar{C}$. Hence, $\bar{x} \sim_{\bar{A}} \bar{p}_1\bar{c}_1 \sim_{\bar{A},t} \bar{p}_2\bar{c}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{p}_n\bar{c}_n \sim_{\bar{A}} \bar{y}^k$. From the first conjugation relation $\bar{x} \sim_{\bar{A}} \bar{p}_1\bar{c}_1$, we have $\bar{p}_1 \in \bar{P}, \bar{c}_1 \in \bar{C}$ since $\{\bar{x}\}^{\bar{A}} \cap \bar{K} = \emptyset$. So, $\bar{x} = \bar{a}_1^{-1}\bar{p}_1\bar{c}_1\bar{a}_1$ for some $a_1 \in A$ and thus $\bar{x}^{-1}\bar{a}_1^{-1}\bar{p}_1\bar{a}_1 = \bar{c}_1^{-1}$. It follows that $x^{-1}a_1^{-1}p_1a_1S = c_1^{-1}S$ and this implies that $x^{-1}a_1^{-1}p_1a_1 \in C$ since $S \lhd_f C$. We let $x^{-1}a_1^{-1}p_1a_1 = w_0 \in C$, then $xw_0 = a_1^{-1}p_1a_1$.

Since $\{x\}^A \cap H \neq \emptyset$, we have $a^{-1}xa = p_0c_0$ for some $a \in A, p_0 \in P, c_0 \in C$ and thus $ap_0c_0a^{-1}w_0 = a_1^{-1}p_1a_1$, that is $ap_0a^{-1}c_0w_0 = a_1^{-1}p_1a_1$. Since P is finite, we let $m_1 = lcm\{|p_0|, |p_1|\}$. Then $(xw_0)^{m_1} = (ap_0c_0a^{-1}w_0)^{m_1} = (a_1^{-1}p_1a_1)^{m_1}$ which implies that $w_0^{m_1} = 1$. Since C is torsion-free, we have $w_0 = 1$. It follows that $x^{-1}a_1^{-1}p_1a_1 = 1$, thus $\bar{c}_1 = 1$ and $\bar{z}_1 = \bar{p}_1$. So, now we can write $\bar{x} \sim_{\bar{A}} \bar{p}_1 \sim_{\bar{A},t} \bar{p}_2\bar{c}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{p}_n\bar{c}_n \sim_{\bar{A}} \bar{y}^k$.

From the second conjugation relation, we have $\bar{p}_1 \sim_{\bar{A},t} \bar{p}_2\bar{c}_2$. If $\bar{p}_1 \sim_{\bar{A}} \bar{p}_2\bar{c}_2$, we have $\bar{c}_2 = 1$ and $\bar{z}_2 = \bar{p}_2$ as above. Now suppose $\bar{p}_1 \sim_t \bar{p}_2\bar{c}_2$. Then $\bar{p}_1 = t^{-r}\bar{p}_2\bar{c}_2t^r$ for some integer r and so we have $\bar{p}_2^{-1}t^r\bar{p}_1t^{-r} = \bar{c}_2$. Hence $p_2^{-1}t^rp_1t^{-r}S = c_2S$. This implies that $p_2^{-1}t^rp_1t^{-r} = w_2 \in C$. Since $\bar{P}$ and $\bar{Q}$ are finite, we let $m_2 = lcm\{|p_1|, |p_2|\}$. Then $(t^rp_1t^{-r})^{m_2} = (p_2w_2)^{m_2}$ which implies that $w_2^{m_2} = 1$ and thus $w_2 = 1$. Hence, we have

$\bar{c}_2 = 1$ and $\bar{z}_2 = \bar{p}_2$ for this case as well.

Proceeding in this way, we have $\bar{c}_i = 1, \bar{z}_i = \bar{p}_i$ where $\bar{p}_i \in \bar{P} \cup \bar{Q}$ for all $i = 3, \ldots, n$. Now we can write $\bar{x} \sim_{\bar{A}} \bar{p}_1 \sim_{\bar{A},t} \bar{p}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{p}_n \sim_{\bar{A}} \bar{y}^k$. From the first conjugation relation $\bar{x} \sim_{\bar{A}} \bar{p}_1$, we have $xS \sim_{A/S} p_1S$. Suppose $x \nsim_A p_1$, then $xM_3 \nsim_{A/M_3} p_1M_3$. But this contradicts to $xS \sim_{A/S} p_1S$. Hence $x \sim_A p_1$. Since $\{x\}^A \cap \langle y \rangle = \emptyset$, we have $\{p_1\}^A \cap \langle y \rangle = \emptyset$. So in $\bar{G}$, we have $\{\bar{p}_1\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

From the second conjugation relation $\bar{p}_1 \sim_{\bar{A},t} \bar{p}_2$, we have $\bar{p}_1 \sim_{\bar{A}} \bar{p}_2$ or $\bar{p}_1 \sim_t \bar{p}_2$. If $\bar{p}_1 \sim_{\bar{A}} \bar{p}_2$, then $\bar{x} \sim_{\bar{A}} \bar{p}_2$. Arguing as above, we have $x \sim_A p_2$. Hence $\{p_2\}^A \cap \langle y \rangle = \emptyset$. Now suppose $\bar{p}_1 \sim_t \bar{p}_2$. Then $\bar{p}_2 \sim_t \bar{p}_1 \sim_A \bar{x}$. Again arguing as above, we have $p_2 \sim_t p_1 \sim_A x$. We now show that $\{p_2\}^A \cap \langle y \rangle = \emptyset$. Suppose $\{p_2\}^A \cap \langle y \rangle \neq \emptyset$. Then $p_2 \sim_A y^l$ for some integer l. Hence $y^l \sim_A p_2 \sim_t p_1 \sim_A x$. This implies that $\{x\}^G \cap \langle y \rangle \neq \emptyset$, a contradiction. Hence $\{p_2\}^A \cap \langle y \rangle = \emptyset$. So in both cases, $\{p_2\}^A \cap \langle y \rangle = \emptyset$. Hence in $\bar{G}$, we have $\{\bar{p}_2\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

Proceeding from the third to the second last conjugation relation, we have $\{\bar{p}_i\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, i = 3, \ldots, n$. From the last conjugation relation $\bar{p}_n \sim_{\bar{A}} \bar{y}^k$, we have $\{\bar{p}_n\}^{\bar{A}} \cap \langle \bar{y} \rangle \neq \emptyset$. This contradicts the fact that $\{\bar{p}_n\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Theorem 4.13, the result follows as in Case 1.

The following subcases can be proved similarly.

Subcase 2b. $\{x\}^A \cap H = \emptyset, \{x\}^A \cap K \neq \emptyset$ and $\{x\}^A \cap C = \emptyset$.

Subcase 2c. $\{x\}^A \cap H \neq \emptyset, \{x\}^A \cap K \neq \emptyset$ and $\{x\}^A \cap C = \emptyset$.

Case 3. Suppose $\{x\}^A \cap C \neq \emptyset$. Since $C \subseteq Z(A)$, we can assume $x \in C$. Hence we have $x \notin \langle y \rangle, t^{-n}xt^n \notin \langle y \rangle$ for all integers n. Since $\phi(c) = c$ for all $c \in C$, we have $t^{-n}xt^n = x \notin \langle y \rangle$. Since A is subgroup separable, there exists $M \lhd_f A$ such that $x \notin \langle y \rangle M$. As in Case 1, we can find $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$. We now form $\bar{G}$ as in Case 1. Then $\bar{x} \notin \langle \bar{y} \rangle$ in $\bar{G}$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \bar{H} \cup \bar{K}$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_n \sim_{\bar{A}} \bar{y}^k$. From the first conjugation relation, since $x \in C \subseteq Z(A)$, we have $x = z_1$ and thus $\bar{x} = \bar{z}_1$. From the second conjugation relation, $\bar{x} = \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2$, we obtain $\bar{z}_2 = \bar{x}$ if $\bar{z}_2 \sim_{\bar{A}} \bar{z}_1$. Since $\phi(c) = c$ for all $c \in C$, we have $\bar{z}_2 = \bar{x}$ if $\bar{z}_2 \sim_t \bar{z}_1$. Thus, $\bar{z}_2 = \bar{x}$. Continuing this way, we have $\bar{z}_i = \bar{x}$ for all $i = 3, \ldots, n$. Hence $\bar{z}_n = \bar{x}$. This implies that $\bar{x} \sim_{\bar{A}} \bar{y}^k$. Since $\bar{x} \in Z(\bar{A})$, we have $\bar{x} = \bar{y}^k$. This contradicts the fact that $\bar{x} \notin \langle \bar{y} \rangle$. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$ and the result follows as in Case 1.

This completes the proof and thus condition (e) is satisfied. Therefore, G is cyclic conjugacy separable by Theorem 2.14.

We now apply the results to finitely generated nilpotent groups. By Lemma 4.8 and Theorem 4.14, we can obtain the following result.

Corollary 4.15. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where $H = P \times C, K = Q \times C$ with P, Q are finite and $C \subseteq Z(A)$ is finitely generated. Suppose $P \cap Q = 1, \phi(P) = Q, \phi(C) = C$ with $\phi(c) = c$ for all $c \in C$. Let A be finitely generated nilpotent. Then G is cyclic conjugacy separable.*

Proof. Since C is a finitely generated abelian group, $C = K_1 \times C_1$ where K_1 is finite and C_1 is torsion-free. Hence we may assume C is torsion-free. Then G is cyclic conjugacy separable by Theorem 4.14.

CHAPTER 5: HNN EXTENSIONS WITH CENTRAL ASSOCIATED SUBGROUPS

5.1 Introduction

The HNN extensions $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where A is polycyclic-by-finite and H and K are finitely generated subgroups in the center of A are conjugacy separable where $H \cap K = 1$ or when $H \cap K$ has finite index in H and in K and $\phi(H \cap K) = H \cap K$. This was proven in Wong & Tang (2000).

In this chapter, we explore the cyclic conjugacy separability of these HNN extensions where A is cyclic conjugacy separable and subgroup separable. First, we show that G is cyclic conjugacy separable when $H \cap K = 1$. When $H \cap K \neq 1$, but $H \cap K$ has finite index in H and in K, then we can show G is cyclic conjugacy separable for the case when $\phi(H \cap K) = H \cap K$ and $\phi(s) = s^{\pm 1}$ for all $s \in H \cap K$.

5.2 HNN Extensions with Trivial Intersection Associated Subgroups

In this section, we shall discuss HNN extensions $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where $H \cap K = 1$. We apply Theorem 2.14 to prove the main theorem (Theorem 5.3) here.

Lemma 5.1. *Let A be a group and H, K be finitely generated subgroups of $Z(A)$ such that $H \cap K = 1$. Let $\phi : H \to K$ be an isomorphism. Let A be subgroup separable. Then for each $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M, \phi(N \cap H) = N \cap K$ and $NH \cap NK = N$.*

Proof. Let $M \lhd_f A$ be given. Let $R = M \cap H$ and $S = M \cap K$. Let $R_1 = R \cap \phi^{-1}(S)$ and $S_1 = \phi(R) \cap S$. Then $R_1 \lhd_f H$, $S_1 \lhd_f K$ and $\phi(R_1) = S_1$. Since $R, S \subseteq Z(A)$, we can form $\bar{A} = A/R_1S_1$. Since A is subgroup separable and R_1, S_1 are finitely generated, then $\bar{A}$ is residually finite. Since $\bar{H} = H/R_1$ and $\bar{K} = K/S_1$ are finite, there exists $\bar{M}_1 \lhd_f \bar{A}$ such that $\bar{M}_1 \cap \bar{H}\bar{K} = \bar{1}$. This implies that $\bar{M}_1 \cap \bar{H} = \bar{1}, \bar{M}_1 \cap \bar{K} = \bar{1}$ and $\bar{M}_1\bar{H} \cap \bar{M}_1\bar{K} = \bar{M}_1$. Let M_1 be the preimage of $\bar{M}_1$ in A. Then $M_1 \lhd_f A$ such that $M_1 \cap H = R_1, M_1 \cap K = S_1$ and $M_1H \cap M_1K = M_1$. Let $N = M \cap M_1$. Then $N \lhd_f A$ such that $N \subseteq M, \phi(N \cap H) = \phi(R_1) = S_1 = N \cap K$ and $NH \cap NK = N$. Therefore, N is the required subgroup.

The proof of the following lemma is modified from Theorem 3.2 of Wong & Tang (2000).

Lemma 5.2. *Let $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$ be an HNN extension where H and K are finitely generated subgroups in $Z(A)$ such that $H \cap K = 1$. Let A be subgroup separable. Then for each $x, y \in G$ such that $||x|| = ||y|| \geq 1$ and $x \nsim_G y$, there exists $Q \lhd_f G$ such that $\bar{x} \nsim_{\bar{G}} \bar{y}$ in $\bar{G} = G/Q$.*

Proof. We let $x = t^{E_1}x_1t^{E_2}x_2 \dots t^{E_m}x_m, y = t^{e_1}y_1t^{e_2}y_2 \dots t^{e_m}y_m$ where $x_i, y_i \in A, m \geq 1$ and $E_i, e_i = \pm 1$. Let a_i denote those $x_i, y_i \in A \backslash H$ and b_i denote those $x_i, y_i \in A \backslash K$. Since A is subgroup separable, we can find $M_0 \lhd_f A$ such that $a_i \notin HM_0, b_i \notin KM_0$ for all i. By Lemma 5.1, there exists $N \lhd_f A$ such that $N \subseteq M_0, \phi(N \cap H) = N \cap K$ and $NH \cap NK = N$. Let $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi}\rangle$ where $\bar{A} = A/N, \bar{H} = HN/N, \bar{K} = KN/N$ with $\bar{H} \cap \bar{K} = 1$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$. Clearly, $\bar{G}$ is a homomorphic image of G. Let $\bar{g}$ denote the image of any element g of G in $\bar{G}$. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$.

Suppose $E_i \neq e_i$ for some i. Then $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable by Lemma 2.10, there exists $\bar{Q} \lhd_f \bar{G}$ such that $\bar{x}\bar{Q} \nsim_{\bar{G}} \bar{y}\bar{Q}$. Let Q be the preimage of $\bar{Q}$. Then $Q \lhd_f G$ such that $xQ \nsim_{G/Q} yQ$ and the result follows.

Now suppose $E_i = e_i$ for all i. Since $x \nsim_G y$, either some equations in (1*) of Definition 2.9 has no admissible solution or every set of admissible solutions to (1*) of Definition 2.9 is incomplete.

Suppose that the equation $x_i = u_i^{-1} y_i v_i$ has no admissible solution in A. This implies that $x_i \notin L_1 y_i L_2$ where L_i is either H or K. Since $H, K \subseteq Z(A)$, we have $x_i y_i^{-1} \notin L_1 L_2$. Since A is subgroup separable, there exists $M_1 \lhd_f A$ such that $x_i y_i^{-1} \notin L_1 L_2 M_1$. Let $M = M_0 \cap M_1$. By Lemma 5.1, there exists $N \lhd_f A$ such that $N \subseteq M$, $\phi(N \cap H) = N \cap K$ and $NH \cap NK = N$. We now form $\bar{G}$. Note that $\bar{x}_i \notin \bar{L}_1 \bar{y}_i \bar{L}_2$ and $\bar{H} \cap \bar{K} = \bar{1}$. Thus, we have $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows.

Now suppose $r_1, s_1, \ldots, r_m, s_m \in H \cup K$ is a set of incomplete solution to (1*) of Definition 2.9.

(a) Now suppose $x_i = r_i^{-1} y_i s_i$ where $r_i \in H, s_i \in K$ for some $1 \leq i \leq m$. Since $H \cap K = 1$, r_i, s_i are unique in the equation $x_i = r_i^{-1} y_i s_i$.

Since the set of admissible solutions is incomplete, we have $t^{-E_i} s_{i-1} t^{E_i} \neq r_i$ for some i. We let $z = t^{-E_i} s_{i-1} t^{E_i} r_i^{-1}$. Since G is residually finite by Lemma 2.11, there exists $P \lhd_f G$ such that $z \notin P$. By Lemma 5.1, there exists $N \subseteq M_0 \cap P$ such that $\phi(N \cap H) = N \cap K$ and $NH \cap NK = N$. We form $\bar{G}$ now. Then $t^{-E_i} \bar{s}_{i-1} t^{E_i} \neq \bar{r}_i$ and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{G}$. Since $\bar{H} \cap \bar{K} = \bar{1}$, the expression $\bar{x}_i = \bar{r}_i^{-1} \bar{y}_i \bar{s}_i$ is unique in $\bar{A}$. Thus, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows.

(b) Suppose $x_i = r_i^{-1} y_i s_i$ where $r_i, s_i \in H$ or $r_i, s_i \in K$ for all i. Note that this case only occur when $E_i = -E_{i+1}$ and $e_i = -e_{i+1}$ for all i. We assume $r_1, s_1 \in H$. Then $r_2, s_2 \in K, r_3, s_3 \in H$, and so on, that is $E_1, e_1 = -1, E_2, e_2 = 1, \ldots$. Hence, $x = t^{-1} x_1 t x_2 \ldots t^{-1} x_m, y = t^{-1} y_1 t y_2 \ldots t^{-1} y_m$ if m is odd or $x = t^{-1} x_1 t x_2 \ldots t x_m, y = t^{-1} y_1 t y_2 \ldots t y_m$ if m is even.

Suppose $||x|| = ||y|| = m$ is odd. Then $ts_mt^{-1} \neq r_1$ since $ts_mt^{-1} \notin H$ and $r_1 \in H$. We can proceed as above and the result follows.

Suppose $||x|| = ||y|| = m$ is even. We have the following from (1*) of Definition 2.9:

$$\begin{aligned} x_1 &= r_1^{-1} y_1 s_1 \\ x_2 &= r_2^{-1} y_2 s_2 \\ &\vdots \\ x_m &= r_m^{-1} y_m s_m \end{aligned} \qquad (2)$$

Since $H, K \subseteq Z(A)$, we assume $t^{-1}s_1t = r_2, ts_2t^{-1} = r_3, \ldots, t^{-1}s_{m-1}t = r_m$ but $ts_mt^{-1} \neq r_1$ since $x \nsim_G y$.

Recall that $x_1 = r_1^{-1}y_1s_1$ where $r_1, s_1 \in H$. Since $H \subseteq Z(A)$, then r_1, s_1 are not unique. If we replace r_1 by $\hat{r}_1 = r_1a$, for some $a \in A$, then s_1 is replaced by $\hat{s}_1 = s_1a$. This implies that r_2, s_2 are replaced by $\hat{r}_2 = t^{-1}\hat{s}_1t = r_2t^{-1}at, \hat{s}_2 = s_2t^{-1}at$ and so on. Continuing in this way, we obtain the following from (2):

$$\begin{aligned} x_1 &= \hat{r}_1^{-1} y_1 \hat{s}_1 = r_1^{-1} a^{-1} y_1 s_1 a \\ x_2 &= \hat{r}_2^{-1} y_2 \hat{s}_2 = r_2^{-1} (t^{-1}a^{-1}t) y_2 s_2 (t^{-1}at) \\ &\vdots \\ x_m &= \hat{r}_m^{-1} y_m \hat{s}_m = r_m^{-1} (t^{-1}a^{-1}t) y_m s_m (t^{-1}at) \end{aligned} \qquad (3)$$

This implies that $t^{-1}\hat{s}_1t = \hat{r}_2, t\hat{s}_2t^{-1} = \hat{r}_3, \ldots, t^{-1}\hat{s}_{m-1}t = \hat{r}_m$ but $t\hat{s}_mt^{-1}\hat{r}_1^{-1} = ts_m(t^{-1}at)t^{-1}r_1^{-1}a^{-1} = ts_mt^{-1}r_1^{-1} \neq 1$. Let $z = ts_mt^{-1}r_1^{-1}$. Since G is residually finite by Lemma 2.11, there exists $P \lhd_f G$ such that $z \notin P$. By Lemma 5.1, there exists $N \subseteq M_0 \cap P$ such that $\phi(N \cap H) = N \cap K$ and $NH \cap NK = N$. We form $\bar{G}$ now. Then $\bar{z} \neq \bar{1}$ and

$\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{G}$. Thus, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows.

Theorem 5.3. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where H and K are finitely generated subgroups in $Z(A)$ such that $H \cap K = 1$. Let A be cyclic conjugacy separable and subgroup separable. Then G is cyclic conjugacy separable.*

Proof. We apply Theorem 2.14 here. Since A is cyclic conjugacy separable, H-separable and K-separable, we have conditions (a) and (b). By Lemma 5.1 and Lemma 5.2, we have conditions (c) and (d).

We now prove condition (e). Let $x, y \in G$ such that $||x|| = ||y|| = 0$ and $\{x\}^G \cap \langle y \rangle = \emptyset$.

Case 1. Suppose $x \notin H \cup K$. Note that $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$. Since A is cyclic conjugacy separable, there exists $M_1 \lhd_f A$ such that $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$. Now since A is subgroup separable, there exists $M_2 \lhd_f A$ such that $x \notin (H \cup K)M_2$. Let $M = M_1 \cap M_2$. By Lemma 5.1, there exists $N \lhd_f A$ such that $N \subseteq M$, $\phi(N \cap H) = N \cap K$ and $NH \cap NK = N$. As before, we form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi} \rangle$ where $\bar{A} = A/N$, $\bar{H} = HN/N$, $\bar{K} = KN/N$ and $\bar{\phi}$ is the induced isomorphism from $\bar{H}$ to $\bar{K}$. Then in $\bar{G}$, we have $\bar{H}, \bar{K} \subseteq Z(\bar{A})$, $\bar{H} \cap \bar{K} = \bar{1}$, $\bar{x} \notin \bar{H} \cup \bar{K}$ and $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 2.8(i), there exists a finite sequence of $\bar{z}_i$ where $\bar{z}_i \in \bar{H} \cup \bar{K}$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_w \sim_{\bar{A}} \bar{y}^k$. Since $\bar{x} \notin \bar{H} \cup \bar{K}$ and $\bar{H}, \bar{K} \subseteq Z(\bar{A})$, the sequence reduces to $\bar{x} \sim_{\bar{A}} \bar{y}^k$ or $\bar{x} \sim_t \bar{y}^k$. Since $\bar{x} \notin \bar{H} \cup \bar{K}$, that is, $t^{-1}\bar{x}t, t\bar{x}t^{-1} \notin \bar{H} \cup \bar{K}$, this further reduces to $\bar{x} \sim_{\bar{A}} \bar{y}^k$. But this contradicts the fact that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 2.10, there exists $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P} \rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$.

Case 2. Suppose $x \in H \cup K$. Note that $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$, $t^{-1}xt \notin \langle y \rangle$ if $x \in H$ or $txt^{-1} \notin \langle y \rangle$ if $x \in K$. Given $H, K \subseteq Z(A)$ and $H \cap K = 1$, we have $x, t^{-1}xt \notin \langle y \rangle$ or $x, txt^{-1} \notin \langle y \rangle$. Since A is subgroup separable, there exists $M \lhd_f A$

such that $x \notin \langle y\rangle M, t^{-1}xt \notin \langle y\rangle M$ and $txt^{-1} \notin \langle y\rangle M$. As in Case 1, we can find $N \lhd_f A$ such that $N \subseteq M, \phi(N \cap H) = N \cap K$ and $NH \cap NK = N$ and thus we can form $\bar{G}$. Then $\bar{H}, \bar{K} \subseteq Z(\bar{A}), \bar{H} \cap \bar{K} = \bar{1}, \bar{x} \in \bar{H} \cup \bar{K}$ and $\bar{x}, t^{-1}\bar{x}t, t\bar{x}t^{-1} \notin \langle \bar{y}\rangle$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y}\rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 2.8(i), there exists a finite sequence $\bar{z}_i$ where $\bar{z}_i \in \bar{H} \cup \bar{K}$ such that $\bar{x} \sim_{\bar{A}} \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2 \sim_{\bar{A},t} \cdots \sim_{\bar{A},t} \bar{z}_w \sim_{\bar{A}} \bar{y}^k$. From the first conjugation relation $\bar{x} \sim_{\bar{A}} \bar{z}_1$, we obtain $\bar{z}_1 = \bar{x}$ since $\bar{x} \in \bar{H} \cup \bar{K} \subseteq Z(\bar{A})$. From the second conjugation relation $\bar{x} = \bar{z}_1 \sim_{\bar{A},t} \bar{z}_2$, we obtain $\bar{z}_2 = \bar{x}$ or $\bar{z}_2 = t^{-1}\bar{x}t$ or $\bar{z}_2 = t\bar{x}t^{-1}$. Continuing this way, we have $\bar{z}_i = \bar{x}$ or $t^{-1}\bar{x}t$ or $t\bar{x}t^{-1}$ for $i = 3, \ldots, w$. Hence $\bar{z}_w = \bar{x}$ or $t^{-1}\bar{x}t$ or $t\bar{x}t^{-1}$. This implies that $\bar{x} \sim_{\bar{A}} \bar{y}^k$ or $t^{-1}\bar{x}t \sim_{\bar{A}} \bar{y}^k$ or $t\bar{x}t^{-1} \sim_{\bar{A}} \bar{y}^k$. Since $\bar{H}, \bar{K} \subseteq Z(\bar{A})$, we obtain $\bar{x}, t^{-1}\bar{x}t, t\bar{x}t^{-1} \in \langle \bar{y}\rangle$. This is a contradiction since $\bar{x}, t^{-1}\bar{x}t, t\bar{x}t^{-1} \notin \langle \bar{y}\rangle$. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y}\rangle = \emptyset$ and the result follows as in Case 1.

Therefore we have condition (e) and this completes the proof. Thus, G is cyclic conjugacy separable by Theorem 2.14.

5.3 Cyclic Conjugacy Separability on Certain HNN Extensions

In this section, we show our main result (Theorem 5.6) by using Theorem 2.14. Here we study the HNN extensions with $H \cap K \neq 1$ and $\phi(H \cap K) = H \cap K$.

Lemma 5.4. *(Wong & Wong, 2008) Let $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$ be an HNN extension. Suppose H and K are finitely generated subgroups in $Z(A)$ and $H \neq A \neq K$. Then G is subgroup separable if and only if A is subgroup separable and $H \cap K$ is a subgroup of finite index in H and K and there exists a finitely generated subgroup S such that S has finite index in $H \cap K$ and $\phi(S) = S$.*

The proof of the following lemma is modified from Theorem 3.5 of Wong & Tang (2000).

Lemma 5.5. *Let $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$ be an HNN extension where $H \cap K \neq 1$. Suppose H and K are finitely generated subgroups in $Z(A)$ such that $|H : H \cap K| < \infty, |K : H \cap K| < \infty$. Further suppose $\phi(H \cap K) = H \cap K$. Let A be subgroup separable. Then for each $x, y \in G$ such that $||x|| = ||y|| \geq 1$ and $x \nsim_G y$, there exists $Q \lhd_f G$ such that $\bar{x} \nsim_{\bar{G}} \bar{y}$ in $\bar{G} = G/Q$.*

Proof. We let $x = t^{E_1}x_1t^{E_2}x_2 \dots t^{E_m}x_m, y = t^{e_1}y_1t^{e_2}y_2 \dots t^{e_m}y_m$ where $x_i, y_i \in A, m \geq 1$ and $E_i, e_i = \pm 1$. Let a_i denote those $x_i, y_i \in A\backslash H, b_i$ denote those $x_i, y_i \in A\backslash K$ and c_i denote those $x_i, y_i \in H \cap K\backslash\{1\}$. Since A is subgroup separable, we can find $N_0 \lhd_f A$ such that $a_i \notin HN_0, b_i \notin KN_0$ and $c_i \notin N_0$.

As in Lemma 5.2, we may assume $E_i = e_i$ for all i. Since $x \nsim_G y$, either some equations in (1*) of Definition 2.9 has no admissible solution in A or every set of admissible solutions to (1*) of Definition 2.9 is incomplete. If one of the equations in (1*) of Definition 2.9 has no admissible solution in A, the proof may proceed as in Lemma 5.2. So suppose $s_0, r_1, \dots, r_m, s_m \in H \cup K$ is a set of incomplete admissible solutions to (1*) of Definition 2.9.

Let $u_i = t^{e_1}x_1 \dots x_{i-1}t^{e_i}$ and $v_i = t^{e_1}y_1 \dots y_{i-1}t^{e_i}, 1 \leq i \leq m$. If $x \sim_G y$, then there exists an element $z \in H \cup K$ such that $z^{-1}xz = y$, that is $x^{-1}zy = z$. This implies $u_1^{-1}zv_1, u_2^{-1}zv_2, \dots, u_m^{-1}zv_m \in H \cup K$ and $x_m^{-1}u_m^{-1}zv_my_m = z$. Since $x \nsim_G y$, then for each element $w \in H \cup K$, either there exists an integer $j, 1 \leq j < m$, such that $u_j^{-1}wv_j \in H \cup K$ but $u_{j+1}^{-1}wv_{j+1} \notin A$ or $u_j^{-1}wv_j \in H \cup K$ but $x_m^{-1}u_m^{-1}wv_my_m \neq w$. We have the following cases.

Case 1. Suppose for each element $w \in H \cup K$, we can find the largest integer n such that $1 \leq n < m$ and $u_n^{-1}wv_n \in H \cup K$ but $u_{n+1}^{-1}wv_{n+1} \notin A$. Then $u_{n+1}^{-1}hv_{n+1} \notin A$ for all

$h \in H \cup K$.

Subcase 1a. Suppose $e_n = 1 = e_{n+1}$. Since $x_n = r_n^{-1} y_n s_n$ where $r_n, s_n \in H \cup K$ and $H, K \subseteq Z(A)$ such that $H \cap K \neq 1$, we assume that $s_n \in H$ and $r_n \in K$. We first show that $u_n r_n^{-1} v_n^{-1} \notin H \cup K$. Suppose $u_n r_n^{-1} v_n^{-1} = w'$ for some $w' \in H \cup K$. Then $u_{n+1}^{-1} w' v_{n+1} = t^{-1} x_n^{-1} u_n^{-1} w' v_n y_n t = t^{-1} x_n^{-1} r_n^{-1} y_n t = t^{-1} s_n^{-1} t \in A$, a contradiction since $u_{n+1}^{-1} h v_{n+1} \notin A$ for all $h \in H \cup K$. So, $u_n r_n^{-1} v_n^{-1} \notin H \cup K$. Since G is H-separable and K-separable by Lemma 5.4, there exists $P \lhd_f G$ such that $u_n r_n^{-1} v_n^{-1} \notin HP \cup KP$. Let $N^* = N_0 \cap P \cap A$. By Lemma 4.3, there exists $N \lhd_f A$ such that $N \subseteq N^*$ and $\phi(N \cap H) = N \cap K$. Now we form $\bar{G}$. Then $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ and $\bar{u}_n \bar{r}_n^{-1} \bar{v}_n^{-1} \notin \bar{H} \cup \bar{K}$ in $\bar{G}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. Then there exists a complete set of admissible solutions $\bar{p}_0, \bar{q}_1, \ldots, \bar{q}_m, \bar{p}_m \in \bar{H} \cup \bar{K}$ to (1*) of Definition 2.9 such that $\bar{x}_k = \bar{q}_k^{-1} \bar{y}_k \bar{p}_k$ and $t^{-e_k} \bar{p}_{k-1} t^{e_k} = \bar{q}_k$ for $1 \leq k \leq m$, where $\bar{p}_0 = \bar{p}_m$. Let $\bar{u}_k = t^{e_1} \bar{x}_1 \ldots \bar{x}_{k-1} t^{e_k}$ and $\bar{v}_k = t^{e_1} \bar{y}_1 \ldots \bar{y}_{k-1} t^{e_k}$. Then $\bar{u}_k^{-1} \bar{p}_0^{-1} \bar{v}_k = \bar{q}_k$. Furthermore, $\bar{v}_k^{-1} \bar{s} \bar{v}_k \in \bar{H} \cap \bar{K}$ for all $\bar{s} \in \bar{H} \cap \bar{K}$ since $\bar{\phi}(\bar{H} \cap \bar{K}) = \bar{H} \cap \bar{K}$ and $\bar{H}, \bar{K} \subset Z(\bar{A})$. Since $t^{-e_n} \bar{p}_{n-1} t^{e_n} = \bar{q}_n$ and $t^{-e_{n+1}} \bar{p}_n t^{e_{n+1}} = \bar{q}_{n+1}$ where $e_n = 1 = e_{n+1}$, we have $\bar{p}_n \in \bar{H}$ and $\bar{q}_n \in \bar{K}$. Since $\bar{x}_n = \bar{r}_n^{-1} \bar{y}_n \bar{s}_n = \bar{q}_n^{-1} \bar{y}_n \bar{p}_n$, we have $\bar{p}_n \bar{s}_n^{-1} = \bar{q}_n \bar{r}_n^{-1} \in \bar{H} \cap \bar{K}$. This gives us $\bar{r}_n^{-1} = \bar{q}_n^{-1} \bar{s}$ for some $\bar{s} \in \bar{H} \cap \bar{K}$. It follows that $\bar{u}_n \bar{r}_n^{-1} \bar{v}_n^{-1} = \bar{u}_n \bar{q}_n^{-1} \bar{s} \bar{v}_n^{-1} = (\bar{u}_n \bar{q}_n^{-1} \bar{v}_n^{-1})(\bar{v}_n \bar{s} \bar{v}_n^{-1}) = \bar{p}_0^{-1} \bar{s}' \in \bar{H} \cup \bar{K}$ for some $\bar{s}' \in \bar{H} \cap \bar{K}$, a contradiction. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows. Similarly if $e_n = -1 = e_{n+1}$.

Subcase 1b. Suppose $e_n = 1 = -e_{n+1}$. Now let w be an element, where $u_n^{-1} w v_n \in H \cup K$ but $u_{n+1}^{-1} w v_{n+1} \notin A$. Since $u_n^{-1} w v_n \in H \cup K$, and $u_n^{-1} w v_n$ has the form $u_n^{-1} w v_n = t^{-1} x_{n-1}^{-1} u_{n-1}^{-1} w v_{n-1} y_{n-1} t$, we have $u_n^{-1} w v_n \in K$. Since $u_{n+1}^{-1} w v_{n+1} = t x_n^{-1} u_n^{-1} w v_n y_n t^{-1} \notin A$, we have $x_n^{-1} u_n^{-1} w v_n y_n \notin K$. This implies that $x_n^{-1} y_n \notin K$ since $K \subseteq Z(A)$. Since A is subgroup separable, there exists $N_1 \lhd_f A$ such that $x_n^{-1} y_n \notin K N_1$. As above, there exists

$N \lhd_f A$ such that $N \subseteq N_0 \cap N_1$ and $\phi(N \cap H) = N \cap K$. We can form $\bar{G}$ now. Clearly $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ and $\bar{x}_n^{-1}\bar{y}_n \notin \bar{K}$ in $\bar{G}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. Then there exists a complete set of admissible solutions $\bar{p}_0, \bar{q}_1, \ldots, \bar{q}_m, \bar{p}_m \in \bar{H} \cup \bar{K}$ to (1*) of Definition 2.9 such that $\bar{x}_k = \bar{q}_k^{-1}\bar{y}_k\bar{p}_k$ and $t^{-e_k}\bar{p}_{k-1}t^{e_k} = \bar{q}_k$ for $1 \leq k \leq m$, where $\bar{p}_0 = \bar{p}_m$. Since $t^{-e_n}\bar{p}_{n-1}t^{e_n} = \bar{q}_n$ and $t^{-e_{n+1}}\bar{p}_n t^{e_{n+1}} = \bar{q}_{n+1}$ where $e_n = 1 = -e_{n+1}$, we have $\bar{p}_n \in \bar{K}$ and $\bar{q}_n \in \bar{K}$. Since $\bar{x}_n = \bar{q}_n^{-1}\bar{y}_n\bar{p}_n$, $\bar{x}_n^{-1}\bar{y}_n = \bar{q}_n\bar{p}_n^{-1} \in \bar{K}$, a contradiction. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows. Similarly if $-e_n = 1 = e_{n+1}$.

Case 2. Suppose there exists an element $w \in H \cup K$ such that $u_j^{-1}wv_j \in H \cup K$ but $x_m^{-1}u_m^{-1}wv_my_m \neq w$.

Subcase 2a. Let $e_1 = 1 = e_m$. Note that $x_m = r_m^{-1}y_ms_m$, where $r_m, s_m \in H \cup K$. Since $H, K \subseteq Z(A)$ and $H \cap K \neq 1$, we assume $s_m \in H$ and $r_m \in K$. Suppose $u_mr_m^{-1}v_m^{-1} \notin H$. Since G is H-separable, there exists $P \lhd_f G$ such that $u_mr_m^{-1}v_m^{-1} \notin HP$. Let $N^* = N_0 \cap P \cap A$. Then $N^* \lhd_f A$. As above, there exists $N \lhd_f A$ such that $N \subseteq N^*$ and $\phi(N \cap H) = N \cap K$. We now form $\bar{G}$. Then $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ and $\bar{u}_m\bar{r}_m^{-1}\bar{v}_m^{-1} \notin \bar{H}$ in $\bar{G}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. Then there exists a complete set of admissible solutions $\bar{p}_0, \bar{q}_1, \ldots, \bar{q}_m, \bar{p}_m \in \bar{H} \cup \bar{K}$ to (1*) of Definition 2.9 such that $\bar{x}_k = \bar{q}_k^{-1}\bar{y}_k\bar{p}_k$ and $t^{-e_k}\bar{p}_{k-1}t^{e_k} = \bar{q}_k$ for $1 \leq k \leq m$, where $\bar{p}_0 = \bar{p}_m$. Since $t^{-e_1}\bar{p}_0t^{e_1} = \bar{q}_1$ and $t^{-e_m}\bar{p}_{m-1}t^{e_m} = \bar{q}_m$, we must have $\bar{p}_0 \in \bar{H}$ and $\bar{q}_m \in \bar{K}$. Since $\bar{x}_m = \bar{r}_m^{-1}\bar{y}_m\bar{s}_m = \bar{q}_m^{-1}\bar{y}_m\bar{p}_m$, $\bar{p}_m\bar{s}_m^{-1} = \bar{q}_m\bar{r}_m^{-1} \in \bar{H} \cap \bar{K}$, which implies that $\bar{r}_m^{-1} = \bar{q}_m^{-1}\bar{s}$ for some $\bar{s} \in \bar{H} \cap \bar{K}$. Thus, $\bar{u}_m\bar{r}_m^{-1}\bar{v}_m^{-1} = \bar{u}_m\bar{q}_m^{-1}\bar{s}\bar{v}_m^{-1} = (\bar{u}_m\bar{q}_m^{-1}\bar{v}_m^{-1})(\bar{v}_m\bar{s}\bar{v}_m^{-1}) = \bar{p}_m^{-1}\bar{s}'$ for some $\bar{s}' \in \bar{H} \cap \bar{K}$. We can see that $\bar{u}_m\bar{r}_m^{-1}\bar{v}_m^{-1} = \bar{p}_m^{-1}\bar{s}' \in \bar{H}$, a contradiction. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows.

Now suppose $u_mr_m^{-1}v_m^{-1} \in H$, say $u_mr_m^{-1}v_m^{-1} = h_1^{-1}$ for some $h_1 \in H$. Hence $x^{-1}h_1^{-1}y = x_m^{-1}u_m^{-1}h_1^{-1}v_my_m = s_m^{-1}$, and therefore $x = h_1^{-1}ys_m$. Since $x \nsim_G y$, we have $s_m \neq h_1$.

Suppose $x = h_2^{-1} y h_3$ is another expression of x, where $h_2, h_3 \in H$. Then $y^{-1} h_2 h_1^{-1} y = h_3 s_m^{-1}$, i.e., $x_m^{-1} t^{-e_m} \ldots x_1^{-1} t^{-e_1} h_2 h_1^{-1} t^{e_1} x_1 \ldots t^{e_m} x_m = h_3 s_m^{-1}$. This implies that $t^{-e_1} h_2 h_1^{-1} t^{e_1} \in H \cup K$. Since $H, K \subseteq Z(A)$, we have $x_m^{-1} t^{-e_m} \ldots x_1^{-1} t^{-e_1} h_2 h_1^{-1} t^{e_1} x_1 \ldots t^{e_m} x_m = h_3 s_m^{-1}$. We must obtain $t^{-(e_1+e_2)} h_2 h_1^{-1} t^{(e_1+e_2)} \in H \cup K$. Continuing this, we have $t^{-e} h_2 h_1^{-1} t^{e} = h_3 s_m^{-1}$, where $e = e_1 + \ldots + e_m$.

If $e = 0$, then $h_2 h_1^{-1} = h_3 s_m^{-1}$. Thus, $h_1^{-1} s_m = h_2^{-1} h_3$. Let $z = h_1^{-1} s_m$. Since $z \neq 1$ and A is residually finite, there exists $N_1 \lhd_f A$ such that $z \notin N_1$. As above, there exists $N \lhd_f A$ such that $N \subseteq N_0 \cap N_1$ and $\phi(N \cap H) = N \cap K$. We now form $\bar{G}$. Then $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ and $\bar{z} \neq \bar{1}$ in $\bar{G}$. It follows that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

If $e \neq 0$, from $t^{-e} h_2 h_1^{-1} t^{e} = h_3 s_m^{-1}$, we have $t^{-e} h_1^{-1} t^{e} s_m = t^{-e} h_2^{-1} t^{e} h_3$. Furthermore, $h_2 h_1^{-1} \in H \cap K$ and $h_3 s_m^{-1} \in H \cap K$ since $h_2 h_1^{-1}, h_3 s_m^{-1} \in H$ and $\phi(H \cap K) = H \cap K$. Since $x \nsim_G y, x \neq h^{-1} y a$ for all $h \in H$. This implies that $t^{-e} h_1^{-1} t^{e} s_m \neq t^{-e} h^{-1} t^{e} h$, for all $h \in H$, where $h h_1^{-1} \in H \cap K$. Let $z = t^{-e} h_1^{-1} t^{e} s_m$ and $L = \{t^{-e} h^{-1} t^{e} h | h h_1^{-1} \in H \cap K\}$. Then $z \notin L$.

Let $L' = \{t^{-e} h^{-1} t^{e} h | h \in H \cap K\}$ and $w = t^{-e} h_1^{-1} t^{e} h_1$. Then $L = wL'$. To see this, let $x \in L$. Then $x = t^{-e} h^{-1} t^{e} h$, where $h h_1^{-1} = s \in H \cap K$. So $x = t^{-e} h_1^{-1} s^{-1} t^{e} h_1 s = t^{-e} h_1^{-1} t^{e} t^{-e} s^{-1} t^{e} h_1 s = t^{-e} h_1^{-1} t^{e} h_1 t^{-e} s^{-1} t^{e} s \in wL'$. Hence, $L \subseteq wL'$, and similarly, $wL' \subseteq L$.

Note that L' is a finitely generated subgroup of A since $L' \subseteq H \cap K$. Since $z \notin L$, we have $w^{-1} z \notin L'$. Since A is subgroup separable, there exists $N_1 \lhd_f A$ such that $w^{-1} z \notin L' N_1$. As above, there exists $N \lhd_f A$ such that $N \subseteq N_0 \cap N_1$ and $\phi(N \cap H) = N \cap K$. We now form $\bar{G}$. Clearly $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ and $\bar{w}^{-1} \bar{z} \notin \bar{L}'$ in $\bar{G}$. This implies that $\bar{z} \notin \bar{L}$ and hence $\bar{x} \nsim_{\bar{G}} \bar{y}$. The result follows. Similarly if $e_1 = -1 = e_m$.

Subcase 2b. Suppose $e_1 = 1 = -e_m$. We assume $x^{-1} w y \notin H$. Since $u_m^{-1} w v_m \in H \cup K$ and $u_m^{-1} w v_m$ has the form $u_m^{-1} w v_m = t x_{m-1}^{-1} u_{m-1}^{-1} w v_{m-1} y_{m-1} t^{-1}$, we have $u_m^{-1} w v_m \in H$.

Since $x_m^{-1}u_m^{-1}wv_my_m \notin H$ and $H \subseteq Z(A)$, we have $x_m^{-1}y_m \notin H$. Since A is subgroup separable, there exists $N_1 \lhd_f A$ such that $x_m^{-1}y_m \notin HN_1$. As above, there exists $N \lhd_f A$ such that $N \subseteq N_0 \cap N_1$ and $\phi(N \cap H) = N \cap K$. We now form $\bar{G}$. Clearly $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ and $\bar{x}_m^{-1}\bar{y}_m \notin \bar{H}$ in $\bar{G}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. Then there exists a complete set of admissible solutions $\bar{p}_0, \bar{q}_1, \ldots, \bar{q}_m, \bar{p}_m \in \bar{H} \cup \bar{K}$ to (1*) of Definition 2.9 such that $\bar{x}_k = \bar{q}_k^{-1}\bar{y}_k\bar{p}_k$ and $t^{-e_k}\bar{p}_{k-1}t^{e_k} = \bar{q}_k$ for $1 \leq k \leq m$, where $\bar{p}_0 = \bar{p}_m$. Since $t^{-e_m}\bar{p}_{m-1}t^{e_m} = \bar{q}_m$ and $t^{-e_1}\bar{p}_mt^{e_1} = \bar{q}_1$, we have $\bar{p}_0 \in \bar{H}$ and $\bar{q}_m \in \bar{H}$. Since $\bar{x}_m = \bar{q}_m^{-1}\bar{y}_m\bar{p}_m$, $\bar{x}_m^{-1}\bar{y}_m = \bar{q}_m\bar{p}_m^{-1} \in \bar{H}$, a contradiction. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows.

Now suppose $x^{-1}wy \in H$. Then $x^{-1}wy = h$ for some $h \in H$. Thus $x = wyh^{-1}$. Since $x \nsim_G y$, we have $w \neq h$. The result now follows as in Subcase 2a.

Theorem 5.6. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where $H \cap K \neq 1$. Suppose H and K are finitely generated subgroups in $Z(A)$ such that $|H : H \cap K| < \infty$, $|K : H \cap K| < \infty$. Further suppose $\phi(H \cap K) = H \cap K$ with $\phi(s) = s^{\pm 1}$ for all $s \in H \cap K$. Let A be cyclic conjugacy separable and subgroup separable. Then G is cyclic conjugacy separable.*

Proof. We apply Theorem 2.14 here. Since A is cyclic conjugacy separable and subgroup separable, we have conditions (a) and (b). By Lemma 4.3 and Lemma 5.5, conditions (c) and (d) are satisfied.

We now prove for condition (e). Let $x, y \in G$ such that $||x|| = ||y|| = 0$ and $\{x\}^G \cap \langle y \rangle = \emptyset$.

Case 1. Suppose $x \notin H \cup K$. The proof for this case is similar to Case 1 of Theorem 5.3 except that in this case we use Lemma 4.3 instead of Lemma 5.1 where $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi} \rangle$ with $\bar{H}, \bar{K} \subseteq Z(\bar{A})$, $\overline{H \cap K} = \bar{H} \cap \bar{K}$, $\bar{x} \notin \bar{H} \cup \bar{K}$ and $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. The proof follows as in Case 1 of Theorem 5.3.

Case 2. Suppose $x \in H \cup K \backslash (H \cap K)$. The proof of this case is similar to Case 2 of Theorem 5.3. As in Case 1 above, we use Lemma 4.3 instead of Lemma 5.1 where $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{K}, \bar{\phi} \rangle$ where $\bar{H}, \bar{K} \subseteq Z(\bar{A}), \overline{H \cap K} = \bar{H} \cap \bar{K}$. Furthermore, $\bar{x} \notin \bar{H} \cap \bar{K}$ and $\bar{x}, t^{-1}\bar{x}t \notin \langle \bar{y} \rangle$ if $\bar{x} \in \bar{H}$, $\bar{x}, t\bar{x}t^{-1} \notin \langle \bar{y} \rangle$ if $\bar{x} \in \bar{K}$.

Case 3. Suppose $x \in H \cap K$. Note that $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$ and $t^{-n}xt^n \notin \langle y \rangle$ for all integers n. Since $H, K \subseteq Z(A)$ and $\phi(s) = s^{\pm 1}$ for all $s \in H \cap K$, then $\{x\}^A \cap \langle y \rangle = \emptyset$ implies that $x, x^{-1} \notin \langle y \rangle$. We can now proceed as in Case 2 above and the result follows.

Therefore we have condition (e) and this completes the proof. Thus, G is cyclic conjugacy separable by Theorem 2.14.

5.4 Applications

We extend Theorem 5.3 and Theorem 5.6 here.

Corollary 5.7. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where H, K are finitely generated in $Z(A)$. Let A be free-by-finite. Suppose one of the following holds:*

(a) *$H \cap K = 1$;*

(b) *$H = K$;*

(c) *there exists $H \cap K \lhd_f H, H \cap K \lhd_f K$ and $\phi(H \cap K) = H \cap K$ where $\phi(s) = s$ or $\phi(s) = s^{-1}$ for all $s \in H \cap K$.*

Then G is cyclic conjugacy separable.

Proof. Free-by-finite groups are cyclic conjugacy separable and subgroup separable. Thus, (a) and (c) are straightforward from Theorem 5.3 and Theorem 5.6 respectively. As for (b), when $H = K$, we have $\phi(H) = H$ is an automorphism and the result follows from Theorem 5.6.

Corollary 5.8. *Let $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$ be an HNN extension. Let A be finitely generated abelian. Suppose one of the following holds:*

(a) $H \cap K = 1$;

(b) $H = K$;

(c) *there exists $H \cap K \lhd_f H, H \cap K \lhd_f K$ and $\phi(H \cap K) = H \cap K$ where $\phi(s) = s$ or $\phi(s) = s^{-1}$ for all $s \in H \cap K$.*

Then G is cyclic conjugacy separable.

Theorem 5.9. *(Meskin, 1972) The group $G = \langle a, b | a^{-\alpha} b^{\beta} a^{\alpha} b^{-\gamma}\rangle$ is residually finite if and only if $\beta \pm \gamma = 0$ or $|\beta| = 1$ or $|\gamma| = 1$.*

Theorem 5.10. *Let $G = \langle t, A | t^{-1}at = b\rangle$ be an HNN extension where $\langle a\rangle, \langle b\rangle$ are infinite cyclic subgroups of $Z(A)$. Let A be non-cyclic, cyclic conjugacy separable and subgroup separable. Then G is cyclic conjugacy separable if and only if $\langle a\rangle \cap \langle b\rangle = 1$ or $a^m = b^{\pm m}$ for some positive integer m.*

Proof. Note that G is an HNN extension with base group A, associated subgroups $\langle a\rangle, \langle b\rangle$ and $\phi : \langle a\rangle \to \langle b\rangle$ where $\phi(a) = b$. If $\langle a\rangle \cap \langle b\rangle = 1$, then G is cyclic conjugacy separable by Theorem 5.3. If $a^m = b^{\pm m}$ for some positive integer m, then $\langle a\rangle \cap \langle b\rangle = \langle a^m\rangle = \langle b^{\pm m}\rangle$. Thus, the result follows from Theorem 5.6.

Conversely, suppose G is cyclic conjugacy separable. Then G is residually finite. Then the result follows from Theorem 5.9.

Corollary 5.11. *Let $G = \langle t, A | t^{-1}at = b\rangle$ be an HNN extension where $\langle a\rangle$ and $\langle b\rangle$ are infinite cyclic subgroups in $Z(A)$. Let A be non-cyclic, free-by-finite. Then G is cyclic conjugacy separable if and only if $\langle a\rangle \cap \langle b\rangle = 1$ or $a^m = b^{\pm m}$ for some positive integer m.*

Proof. Since free-by-finite groups are cyclic conjugacy separable and subgroup separable, the result follows from Theorem 5.10.

Corollary 5.12. *Let* $G = \langle t, A | t^{-1}at = b\rangle$ *be an* HNN *extension where* $\langle a\rangle$ *and* $\langle b\rangle$ *are infinite cyclic subgroups in* $Z(A)$. *Let A be non-cyclic finitely generated abelian. Then G is cyclic conjugacy separable if and only if* $\langle a\rangle \cap \langle b\rangle = 1$ *or* $a^m = b^{\pm m}$ *for some positive integer m.*

CHAPTER 6: CYCLIC CONJUGACY SEPARABILITY OF GENERALIZED FREE PRODUCTS

6.1 Introduction

In the second part of this thesis, we will study the cyclic conjugacy separability of generalized free products and tree products with various amalgamated subgroups. As in Chapter 2, we establish a criterion which state the basic core conditions in order to be applied to the generalized free products. This criterion is given in Theorem 6.6.

6.2 Cyclic Conjugacy Separability of Generalized Free Products

We give the essential lemmas needed before we prove the criterion.

Lemma 6.1. *(Magnus et al., 1966) Let $G = A \ast_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced and $y \sim_G x$.*

(*a*) *If $||x|| = 0$, then $||y|| \leq 1$ and if $y \in A$ say, there exists a finite sequence $h_1, h_2, \ldots, h_n$ of elements of H such that $y \sim_A h_1 \sim_B h_2 \sim_A \ldots \sim_{B(A)} h_n \sim_{A(B)} x$.*

(*b*) *If $||x|| = 1$, then $||y|| = 1$ and either $x, y \in A$ and $x \sim_A y$ or else $x, y \in B$ and $x \sim_B y$.*

(*c*) *If $||x|| \geq 2$, then $||y|| = ||x||$ and $y \sim_H x'$ where x' is a cyclic permutation of x.*

Definition 6.2. Let $G = A \ast_H B$ and x, y are cyclically reduced elements in G with $||x|| = ||y|| = n \geq 2$. Suppose $x = x_1 x_2 \ldots x_n$, $y = y_1 y_2 \ldots y_n$.

Consider the following set of equations:

$$
\begin{aligned}
x_{i+1} &= k_0^{-1} y_1 k_1 \\
x_{i+2} &= k_1^{-1} y_1 k_2 \\
&\vdots \\
x_{i+n} &= k_{n-1}^{-1} y_n k_n
\end{aligned}
\tag{1}
$$

where $0 \leq i < n$ and the integer $i + j$ is taken modulo n.

A pair of elements h_{j-1}, h_j of H is called an admissible solution of the j-th equation if and only if $x_{i+j} = h_{j-1}^{-1} y_j h_j$. A set of admissible solutions $h_0, h_1, \ldots, h_n$ of H to (1) is said to be complete if and only if $h_0, h_1, \ldots, h_n$ satisfy (1) simultaneously and $h_0 = h_n$.

This is equivalent to $x' = h_0^{-1} y h_0$ where $x' = x_{i+1} x_{i+2} \ldots x_i$ is a cyclic permutation of x. So $x \sim_G y$ if and only if the system of equations (1) has a set of complete admissible solutions for some $0 \leq i < n$.

Lemma 6.3. *Let $G = A \ast_H B$ where A and B are finite. Then G is subgroup separable (Allenby & Gregorac, 1973), conjugacy separable (Dyer, 1980) and cyclic conjugacy separable (Tang, 1997).*

Lemma 6.4. *Let $G = A \ast_H B$ and $x, y \in G$ where x has minimal length in its conjugacy class and y is cyclically reduced. Suppose $||x|| = ||y^{\pm k}|| \geq 2$ for some positive integer k. Then $\{x\}^G \cap \langle y \rangle = \emptyset$ if and only if $x \nsim_G y^{\pm k}$.*

Proof. Suppose $\{x\}^G \cap \langle y \rangle = \emptyset$. Then $g^{-1}xg \notin \langle y \rangle$ for any $g \in G$. This implies that $g^{-1}xg \neq y^{\pm r}$ for all $r \in \mathbb{Z}^+$. Thus, $g^{-1}xg \neq y^{\pm k}$. Hence $x \nsim_G y^{\pm k}$. Conversely, suppose $x \nsim_G y^{\pm k}$. We have $g^{-1}xg \neq y^{\pm k}$ for any $g \in G$. Since $||x|| \neq ||y^{\pm n}||$ for all $n \in \mathbb{Z}^+ \backslash \{k\}$, $x \nsim_G y^{\pm n}$ by Lemma 6.1(c). Thus, $x \nsim_G y^{\pm r}$ for all $r \in \mathbb{Z}^+$. Therefore, $\{x\}^G \cap \langle y \rangle = \emptyset$.

We simplify the results in Baumslag (1963) and obtain the following lemma.

Lemma 6.5. *Let $G = A *_H B$. Suppose*

(i) *A and B are residually finite;*

(ii) *A and B are H-separable;*

(iii) *For each $R \lhd_f H$, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H \subseteq R$.*

Then G is residually finite.

We now state and prove our main criterion which will be used to prove our main results in Chapter 7, Chapter 8 and Chapter 9.

Theorem 6.6. *Let $G = A *_H B$. Suppose*

(a) *A and B are cyclic conjugacy separable;*

(b) *A and B are H-conjugacy separable;*

(c) *A and B are H-separable;*

(d) *For each $R \lhd_f H$, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H \subseteq R$;*

(e) *For $x, y \in G$ such that $||x|| = ||y|| \geq 2$ and $x \nsim_G y$, there exists $Q \lhd_f G$ such that $xQ \nsim_{G/Q} yQ$;*

(f) *For $x, y \in G$ such that $||x|| = 0, ||y|| \leq 1$ and $\{x\}^G \cap \langle y \rangle = \emptyset$, there exists $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$.*

Then G is cyclic conjugacy separable.

Proof. Let $x, y \in G$ such that $\{x\}^G \cap \langle y \rangle = \emptyset$. Without loss of generality, we assume that x has minimal length in its conjugacy class and y is cyclically reduced in G. Since G is residually finite by Lemma 6.5, we can assume $x \neq 1 \neq y$.

Case 1. Suppose $||x|| \geq 2, ||y|| \geq 2$. Let $x = a_1 a_2 \ldots a_n, y = b_1 b_2 \ldots b_m$ where $n \geq 2, m \geq 2$. Let u_r denote those $a_i, b_j \in A \backslash H$ and v_s denote those $a_i, b_j \in B \backslash H$. Since

A and B are H-separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $u_r \notin HM_1$ and $v_s \notin HN_1$ for all u_r, v_s.

Subcase 1a. Suppose $n \neq km$ for all positive integers k. Let $R = M_1 \cap N_1 \cap H$. Then $R \lhd_f H$. By (d), there exist $M_2 \lhd_f A, N_2 \lhd_f B$ such that $M_2 \cap H = N_2 \cap H = R_1 \subseteq R$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Then $M \lhd_f A$ and $N \lhd_f B$.

We now show that $M \cap H = N \cap H \subseteq R$. Recall that $R = M_1 \cap N_1 \cap H$ and $M_2 \cap H = N_2 \cap H = R_1 \subseteq R$. Let $M_1 \cap H = R_2, N_1 \cap H = R_3$ where $R \subseteq R_2 \cap R_3$. Then $M \cap H = M_1 \cap M_2 \cap H = (M_1 \cap H) \cap (M_2 \cap H) = R_2 \cap R_1 = R_1$. Similarly, we have $N \cap H = R_1$.

Thus $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H$. We form $\bar{G} = \bar{A} \underset{\bar{H}}{*} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. Then in $\bar{G}$, we have $||\bar{x}|| = ||x|| \neq k||y|| = k||\bar{y}|| = ||\bar{y}^{\pm k}||$ for all positive integers k. Hence by Lemma 6.1(c), $\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$ for all positive integers k and so $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 6.3, we can find $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P} \rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$. The result now follows.

Subcase 1b. Suppose $n = km$ for some positive integer k. Since $\{x\}^G \cap \langle y \rangle = \emptyset$, then by Lemma 6.4, $x \nsim_G y^{\pm k}$. By (e), there exists $Q \lhd_f G$ such that $xQ \nsim_{G/Q} y^{\pm k}Q$. Let $R = Q \cap M_1 \cap N_1 \cap H$. Then $R \lhd_f H$. By (d), there exist $M_2 \lhd_f A, N_2 \lhd_f B$ such that $M_2 \cap H = N_2 \cap H = R_1 \subseteq R$. Let $M = Q \cap M_1 \cap M_2$ and $N = Q \cap N_1 \cap N_2$.

We now show that $M \cap H = N \cap H \subseteq R$. Recall that $R = Q \cap M_1 \cap N_1 \cap H$ and $M_2 \cap H = N_2 \cap H = R_1 \subseteq R$. Let $Q \cap M_1 \cap H = R_2, Q \cap N_1 \cap H = R_3$ where $R \subseteq R_2 \cap R_3$. Then $M \cap H = Q \cap M_1 \cap M_2 \cap H = (Q \cap M_1 \cap H) \cap (M_2 \cap H) = R_2 \cap R_1 = R_1$. Similarly, we have $N \cap H = R_1$.

Thus $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H$. We form $\bar{G}$ as in Subcase 1a. Then in $\bar{G}$, we have $||\bar{x}|| = ||x|| = k||y|| = k||\bar{y}|| = ||\bar{y}^{\pm k}||$ for some positive integer k and also

$\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$. By Lemma 6.4, we have $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 6.3, our result follows as in Subcase 1a.

Case 2. Suppose $||x|| \le 1, ||y|| \ge 2$. Let $x \in A$ and $y = b_1 b_2 \ldots b_m$ where $m \ge 2$. As in Case 1, let u_r denote those $b_j \in A \backslash H$ and v_s denote those $b_j \in B \backslash H$. Since A is residually finite and A, B are H-separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $x \notin M_1, u_r \notin HM_1$ and $v_s \notin HN_1$ for all u_r, v_s. Let $R = M_1 \cap N_1 \cap H$. Then $R \lhd_f H$. By (d), there exist $M_2 \lhd_f A, N_2 \lhd_f B$ such that $M_2 \cap H = N_2 \cap H = R_1 \subseteq R$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Then $M \lhd_f A, N \lhd_f B$ and $M \cap H = N \cap H$ as in Subcase 1a.

As in Subcase 1a, we form $\bar{G}$. Then in $\bar{G}$ we have $\bar{x} \neq \bar{1}, ||\bar{x}|| \le 1$ and $||\bar{y}|| \ge 2$. By Lemma 6.1(a), any conjugate of $\bar{x}$ in $\bar{G}$ is either any element $\bar{x}' \in \bar{A} \cup \bar{B}$ or of the form $\bar{u}^{-1}\bar{x}'\bar{u}$ where $\bar{u}^{-1}\bar{x}'\bar{u}$ and $\bar{u}$ are the reduced words with $\bar{x}' \in \bar{A} \cup \bar{B}$. Since $\bar{y}$ is cyclically reduced and $||\bar{y}|| \ge 2$, then $\bar{y}^{\pm k}$ is cyclically reduced and $||\bar{y}^{\pm k}|| \ge 2$ for all positive integers k. Hence by Lemma 6.1(c), we have $\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$ for all positive integers k. So, we have $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. We now proceed as in Subcase 1a and the result follows.

Case 3. Suppose $||x|| \ge 2, ||y|| \le 1$. Let $y \in A$. As in Case 2, we can form $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ such that $\bar{1} \neq \bar{y} \in \bar{A}, ||\bar{y}|| \le 1$ and $||\bar{x}|| \ge 2$. As $\bar{y} \in \bar{A}$, then $\bar{y}^{\pm k} \in \bar{A}$ for all positive integers k. Since $||\bar{x}|| \ge 2$, then by Lemma 6.1(c), $\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$ for all positive integers k. Hence, we again proceed as in Subcase 1a and the result follows.

Case 4. Suppose $||x|| = 1, ||y|| \le 1$.

Subcase 4a. Suppose $x \in A \backslash H, y \in A$. Note that $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$. Since A is cyclic conjugacy separable, there exists $M_1 \lhd_f A$ such that $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$. Given that x is of minimal length in its conjugacy class, we have $\{x\}^A \cap H = \emptyset$. Since A is H-conjugacy separable, there exists $M_2 \lhd_f A$ such that $\{xM_2\}^{A/M_2} \cap HM_2/M_2 = \emptyset$. Let $R = M_1 \cap M_2 \cap H$. Then $R \lhd_f H$. By (d), there exist $M_3 \lhd_f A, N_1 \lhd_f B$ such that $M_3 \cap H = N_1 \cap H = R_1 \subseteq R$. Let $M = M_1 \cap M_2 \cap M_3$ and

$N = N_1$. Then $M \lhd_f A$ and $N \lhd_f B$.

We now show that $M \cap H = N \cap H \subseteq R$. Recall that $R = M_1 \cap M_2 \cap H$. Then $M \cap H = M_1 \cap M_2 \cap M_3 \cap H = (M_1 \cap M_2 \cap H) \cap (M_3 \cap H) = R \cap R_1 = R_1$. Similarly, $N \cap H = N_1 \cap H = R_1$.

Thus $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H$. We form $\bar{G}$ as in Subcase 1a. Then in $\bar{G}$, we have $||\bar{x}|| = 1, ||\bar{y}|| \leq 1$ where $\bar{x} \in \bar{A} \backslash \bar{H}, \bar{y} \in \bar{A}$ and also $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, \{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$. Since $\bar{y}^{\pm k} \in \bar{A}$ for all positive integers k and $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$, then by Lemma 6.1(a) and (b), we have $\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$. So, this implies that $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. We now proceed as in Subcase 1a and our result follows.

Subcase 4b. Suppose $x \in A \backslash H, y \in B \backslash H$. Given that x is of minimal length in its conjugacy class, we have $\{x\}^A \cap H = \emptyset$. Since A is H-conjugacy separable, there exists $M_1 \lhd_f A$ such that $\{xM_1\}^{A/M_1} \cap HM_1/M_1 = \emptyset$. Since B is H-separable, there exists $N_1 \lhd_f B$ such that $y \notin HN_1$. Let $R = M_1 \cap N_1 \cap H$. Then $R \lhd_f H$. By (d), there exist $M_2 \lhd_f A, N_2 \lhd_f B$ such that $M_2 \cap H = N_2 \cap H = R_1 \subseteq R$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Then $M \lhd_f A, N \lhd_f B$ and $M \cap H = N \cap H$ as in Subcase 1a.

As in Subcase 1a, we form $\bar{G}$. Then in $\bar{G}$, $||\bar{x}|| = 1 = ||\bar{y}||$ where $\bar{x} \in \bar{A} \backslash \bar{H}, \bar{y} \in \bar{B} \backslash \bar{H}$ and also we have $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$. Since $\bar{y}^{\pm k} \in \bar{B}$ and $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$, then by Lemma 6.1(a) and (b), we have $\bar{x} \nsim_{\bar{G}} \bar{y}^{\pm k}$ for all positive integers k. So, this implies that $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. We now proceed as in Subcase 1a and the result follows.

Case 5. Suppose $||x|| = 0, ||y|| \leq 1$. The result follows from (f).

The proof of the theorem is now completed.

CHAPTER 7: FREE PRODUCTS AMALGAMATING CYCLIC SUBGROUPS

7.1 Introduction

The conjugacy separability of the generalized free products of two finitely generated nilpotent groups amalgamating a cyclic group was shown by Dyer (1980). One of the requirement in that proof was that each of the factor group is cyclic conjugacy separable. So, the first step to extend conjugacy separability to tree products amalgamating cyclic subgroups, we will need the requirement that such generalized free products are cyclic conjugacy separable. Thus, this is the main result (Theorem 7.6) of this chapter.

7.2 Lemmas Needed

We now state and prove the lemmas needed in this chapter.

Lemma 7.1. *Let A be a group and h an element of infinite order in A. Suppose A is $\langle h\rangle$-weakly potent. If $h^s \sim_A h^t$ for some integers s, t, then $s = \pm t$.*

Proof. Since A is $\langle h\rangle$-weakly potent, we can find a positive integer r such that for each positive integer n, there exists $M \lhd_f A$ such that $M \cap \langle h\rangle = \langle h^{rn}\rangle$. We choose $n = |s||t|$ and denote the image of h in $\bar{A} = A/M$ by $\bar{h}$. Then $|\bar{h}^s| = r|t|$ and $|\bar{h}^t| = r|s|$. Since $\bar{h}^s \sim_{\bar{A}} \bar{h}^t$, we have $r|s| = r|t|$. This implies $|s| = |t|$ and hence $s = \pm t$.

Lemma 7.2. *Let $G = A \ast_{\langle h\rangle} B$ where h has infinite order. Suppose A and B are $\langle h\rangle$-weakly potent. Then for any $R \lhd_f \langle h\rangle$, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap \langle h\rangle = N \cap \langle h\rangle \subseteq R$.*

Proof. Let $R \lhd_f \langle h\rangle$ be given. Then $R = \langle h^s\rangle$ for some positive integer s. Since A and B are $\langle h\rangle$-weakly potent, we can find some positive integers r_1, r_2 such that for each positive integer n, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $M_1 \cap \langle h\rangle = \langle h^{r_1 n}\rangle$ and $N_1 \cap \langle h\rangle = \langle h^{r_2 n}\rangle$.

Thus, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap \langle h \rangle = \langle h^{r_1 r_2 s} \rangle$ and $N \cap \langle h \rangle = \langle h^{r_1 r_2 s} \rangle$. Furthermore, $M \cap \langle h \rangle = N \cap \langle h \rangle = \langle h^{r_1 r_2 s} \rangle \subseteq R$.

Theorem 7.3. *(Kim & Tang, 1993) Let $G = A \underset{H}{*} B$. Suppose that*

(a) A and B are H-separable;

(b) for each $R \lhd_f H$, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H \subset R$.

Then G is cyclic subgroup separable.

Lemma 7.4. *Let $G = A \underset{\langle h \rangle}{*} B$ where h has infinite order.*

(i) A and B are cyclic subgroup separable;

(ii) A and B are $\langle h \rangle$-weakly potent.

Then G is cyclic subgroup separable.

Proof. By Lemma 7.2, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap \langle h \rangle = N \cap \langle h \rangle$. Since A and B are cyclic subgroup separable and $\langle h \rangle$-separable, we have G is cyclic subgroup separable by Theorem 7.3.

Lemma 7.5. *Let $G = A \underset{\langle h \rangle}{*} B$ where h has infinite order. Suppose*

(i) A and B are cyclic subgroup separable;

(ii) A and B are $\langle h \rangle$-double coset separable;

(iii) A and B are $\langle h \rangle$-weakly potent.

Then for each $x, y \in G$ such that $||x|| = ||y|| \geq 2$ and $x \nsim_G y$, there exists $P \lhd_f G$ such that $xP \nsim_{G/P} yP$.

Proof. Let $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_n$ be cyclically reduced in G where $n \geq 2$. Since $x \nsim_G y$, the system of equations (1) of Definition 6.2 has no solution for all i. Therefore we need to show that, for each i, there exists $P_i \lhd_f G$ such that in $\bar{G}_i = G/P_i$, the corresponding system of equations has no solution. Letting P be the intersection of all the normal subgroups P_i, we have $\bar{x} \nsim_{\bar{G}} \bar{y}$ in $\bar{G} = G/P$ and the result follows. Hence it is sufficient to show the case $i = 0$ in (1) of Definition 6.2.

Since A and B are $\langle h \rangle$-separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $x_i, y_i \notin \langle h \rangle M_1$ for those $x_i, y_i \in A \backslash \langle h \rangle$ and $x_i, y_i \notin \langle h \rangle N_1$ for those $x_i, y_i \in B \backslash \langle h \rangle$.

Since $x \nsim_G y$, either some equations in (1) of Definition 6.2 has no admissible solution in $\langle h \rangle$ or every set of admissible solutions to (1) of Definition 6.2 is incomplete. First suppose there exists some $k, 1 \leq k \leq n$, such that the k-th equation has no admissible solution, that is, $x_k \notin \langle h \rangle y_k \langle h \rangle$ where $x_k, y_k \in A \backslash \langle h \rangle$. The proof for the case where $x_k, y_k \in B \backslash \langle h \rangle$ is similar. Since A is $\langle h \rangle$-double coset separable, there exists $M_2 \lhd_f A$ such that $x_k \notin \langle h \rangle y_k \langle h \rangle M_2$. Let $R = M_1 \cap M_2 \cap N_1 \cap \langle h \rangle$. By Lemma 7.2, we can find $M_3 \lhd_f A, N_2 \lhd_f B$ such that $M_3 \cap \langle h \rangle = N_2 \cap \langle h \rangle \subseteq R$. Let $M = M_1 \cap M_2 \cap M_3$ and $N = N_1 \cap N_2$. Then $M \lhd_f A, N \lhd_f B$ and $M \cap \langle h \rangle = N \cap \langle h \rangle$. We now form $\bar{G} = \bar{A} \underset{\langle \bar{h} \rangle}{*} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{h} = hM = hN$. Clearly $\bar{G}$ is a homomorphic image of G. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. Furthermore, $\bar{x}_k \notin \langle \bar{h} \rangle \bar{y}_k \langle \bar{h} \rangle$ in $\bar{G}$. Hence $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{A} = A/M, \bar{B} = B/N$ are finite, then $\bar{G}$ is conjugacy separable by Lemma 6.3 and the result follows.

Suppose $h^{a_1}, h^{b_1}, \ldots, h^{a_r}, h^{b_r}$ is a set of incomplete admissible solutions to (1) of Definition 6.2. Then we have the following.

$$\begin{aligned} x_1 &= h^{-a_1} y_1 h^{b_1} \\ x_2 &= h^{-a_2} y_2 h^{b_2} \\ &\vdots \\ x_r &= h^{-a_r} y_r h^{b_r} \end{aligned} \qquad (2)$$

Suppose $y_i^{-1} \langle h \rangle y_i \cap \langle h \rangle = \langle h^{\gamma_i} \rangle$ where each $\gamma_i \geq 0, 1 \leq i \leq r$.

Case 1. Suppose $\gamma_i \neq 0$ for all i. Suppose $y_i^{-1} h^{\delta_i} y_i = h^{\gamma_i}$ for some integer δ_i. Since A and B are $\langle h \rangle$-weakly potent, we have $\gamma_i = \pm\delta_i$ by Lemma 7.1. Hence we have $y_i^{-1} h^{\gamma_i} y_i = h^{\pm\gamma_i}$ for all i. Let $\gamma = lcm\{\gamma_i | 1 \leq i \leq r\}$. Clearly, $y_i^{-1} h^{\gamma} y_i = h^{\pm\gamma}$ for all i.

Subcase 1a. Suppose in the system of equations (2), the total number of equations $y_i^{-1} h^{\gamma} y_i = h^{-\gamma}$ is zero or is even. This implies that $y^{-1} h^{\gamma} y = y_r^{-1} \dots y_1^{-1} h^{\gamma} y_1 \dots y_r = h^{\gamma}$, that is $[y, h^{\gamma}] = 1$. Hence $h^{-z} y h^{z} = h^{-j} y h^{j}$ for some $0 \leq j < |\gamma|$. So, $x \neq h^{-z} y h^{z}$ for all $z \in \mathbb{Z}$ implies that $x \neq h^{-j} y h^{j}$ for all $0 \leq j < |\gamma|$. Hence by Lemma 6.1(c), $x \nsim_{\langle h \rangle} y$ if and only if $x \neq h^{-j} y h^{j}$ for all $0 \leq j < |\gamma|$. Since G is residually finite by Lemma 7.4, there exists $Q \lhd_f G$ such that $x^{-1} h^{-j} y h^{j} \notin Q$ for all $0 \leq j < |\gamma|$. Let $R = M_1 \cap N_1 \cap Q \cap \langle h \rangle$. By Lemma 7.2, we can find $M_2 \lhd_f A, N_2 \lhd_f B$ such that $M_2 \cap \langle h \rangle = N_2 \cap \langle h \rangle \subseteq R$. Let $M = M_1 \cap M_2 \cap Q$ and $N = N_1 \cap N_2 \cap Q$. Then $M \lhd_f A, N \lhd_f B$ and $M \cap \langle h \rangle = N \cap \langle h \rangle$. We now form $\bar{G}$ as above. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. Furthermore, $\bar{x} \neq \bar{h}^{-j} \bar{y} \bar{h}^{j}$ for all $0 \leq j < |\gamma|$. Since $\bar{h}^{-\gamma} \bar{y} \bar{h}^{\gamma} = \bar{y}$, then $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows as above.

Subcase 1b. Suppose in the system of equations (2), the total number of equations $y_i^{-1} h^{\gamma} y_i = h^{-\gamma}$ is odd. Then arguing as in Subcase 1a, we have $y^{-1} h^{\gamma} y = h^{-\gamma}$, that is $h^{-\gamma} y h^{\gamma} = y h^{2\gamma}$. Hence, $x \neq h^{-z} y h^{z}$ for all $z \in \mathbb{Z}$ implies that $x^{-1} h^{-j} y h^{j} \notin \langle h^{2\gamma} \rangle$ for all $0 \leq j < |\gamma|$. Hence by Lemma 6.1(c), $x \nsim_{\langle h \rangle} y$ if and only if $x^{-1} h^{-j} y h^{j} \notin \langle h^{2\gamma} \rangle$ for all $0 \leq j < |\gamma|$. Since G is cyclic subgroup separable by Lemma 7.4, there exists $Q \lhd_f G$ such that $x^{-1} h^{-j} y h^{j} \notin \langle h^{2\gamma} \rangle Q$ for all $0 \leq j < |\gamma|$. As for Subcase 1a, we can find $M \lhd_f A, N \lhd_f B$ and $M \cap \langle h \rangle = N \cap \langle h \rangle$. We now form $\bar{G}$ as above. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. Furthermore, $\bar{x}^{-1} \bar{h}^{-j} \bar{y} \bar{h}^{j} \notin \langle \bar{h}^{2\gamma} \rangle$ for all $0 \leq j < |\gamma|$. Since $\bar{h}^{-\gamma} \bar{y} \bar{h}^{\gamma} = \bar{y} \bar{h}^{2\gamma}$, then $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows as above.

Case 2. Suppose $\gamma_i = 0$ for some i, that is $y_i^{-1} \langle h \rangle y_i \cap \langle h \rangle = 1$. Recall that $x_i = h^{-a_i} y_i h^{b_i}$ and suppose $x_i = h^{-\alpha} y_i h^{\beta}$ for some other h^{α}, h^{β}. Then $y_i^{-1} h^{\alpha - a_i} y_i = h^{\beta - b_i} \in y_i^{-1} \langle h \rangle y_i \cap$

$\langle h\rangle = 1$ and hence $\alpha = a_i, \beta = b_i$. This implies that the equation $x_i = h^{-a_i} y_i h^{b_i}$ has unique solution h^{-a_i}, h^{b_i}. Fixing i, we consider the next equation $x_{i+1} = h^{-a_{i+1}} y_{i+1} h^{b_{i+1}}$ and arrange if possible, so that $a_{i+1} = b_i$. Continuing this way, since $x \nsim_G y$, we see that this matching must eventually fail at some equation, say $x_j = h^{-a_j} y_j h^{b_j}$ where $a_j \neq b_{j-1}$. Here the integer j is taken modulo r and hence this equation may be the next equation or the equation we started with. Let $x' = x_i x_{i+1} \dots x_{j-1} x_j$ and $y' = y_i y_{i+1} \dots y_{j-1} y_j$. Then, by substituting the value of each x_i from (2) into x', we obtain $x' = h^{-a_i} y_i y_{i+1} \dots y_{j-1} h^{b_{j-1}-a_j} y_j h^{b_j}$. Hence $x' \notin \langle h\rangle y' \langle h\rangle$. We shall only consider the case $y_i \in A\backslash\langle h\rangle$ and $y_j \in B\backslash\langle h\rangle$. The other cases are similar.

Subcase 2a. Suppose $\gamma_j = 0$, that is $y_j^{-1}\langle h\rangle y_j \cap \langle h\rangle = 1$. Then as above, the equation $x_j = h^{-a_j} y_j h^{b_j}$ has unique solution h^{-a_j}, h^{b_j}. Let $\epsilon = 2|a_j - b_{j-1}|$. Since A is $\langle h\rangle$-double coset separable, then by Lemma 3.4, there exists $M_2 \lhd_f A$ such that $y_i^{-1} h^a y_i M_2 = h^b M_2$ only if $\epsilon | a, b$. Similarly, there exists $N_2 \lhd_f B$ such that $y_j^{-1} h^c y_j N_2 = h^d N_2$ only if $\epsilon | c, d$. Let $R = M_1 \cap M_2 \cap N_1 \cap N_2 \cap \langle h\rangle$. By Lemma 7.2, we can find $M_3 \lhd_f A, N_3 \lhd_f B$ such that $M_3 \cap \langle h\rangle = N_3 \cap \langle h\rangle \subseteq R$. Let $M = M_1 \cap M_2 \cap M_3$ and $N = N_1 \cap N_2 \cap N_3$. Then $M \lhd_f A, N \lhd_f B$ and $M \cap \langle h\rangle = N \cap \langle h\rangle$. We can now form $\bar{G}$ as above. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. Furthermore, by the choice of M_2, $\bar{y}_i^{-1} \bar{h}^a \bar{y}_i = \bar{h}^b$ only if $\epsilon | a, b$ whereas, by the choice of N_2, $\bar{y}_j^{-1} \bar{h}^c \bar{y}_j = \bar{h}^d$ only if $\epsilon | c, d$.

Note that $\bar{x}' = \bar{x}_i \bar{x}_{i+1} \dots \bar{x}_{j-1} \bar{x}_j = \bar{h}^{-a_i} \bar{y}_i \bar{y}_{i+1} \dots \bar{y}_{j-1} \bar{h}^{b_{j-1}-a_j} \bar{y}_j \bar{h}^{b_j}$ and $\bar{y}' = \bar{y}_i \bar{y}_{i+1} \dots \bar{y}_{j-1} \bar{y}_j$. Suppose $\bar{x}' \in \langle \bar{h}\rangle \bar{y}' \langle \bar{h}\rangle$. Then there exist integers α, β and $\delta_i, \dots, \delta_{j-1}$ such that the following holds:

$$\begin{aligned} \bar{y}_i &= \bar{h}^{\alpha} \bar{y}_i \bar{h}^{\delta_i} \\ \bar{y}_{i+1} &= \bar{h}^{-\delta_i} \bar{y}_{i+1} \bar{h}^{\delta_{i+1}} \\ &\vdots \end{aligned} \qquad (3)$$

$$\bar{y}_{j-1} = \bar{h}^{-\delta_{j-2}}\bar{y}_{j-1}\bar{h}^{\delta_{j-1}}$$

$$\bar{h}^{b_{j-1}-a_j}\bar{y}_j = \bar{h}^{-\delta_{j-1}}\bar{y}_j\bar{h}^{\beta}$$

From the first equation in (3), $\epsilon|\alpha, \delta_i$ by the choice of M_2. Now by Lemma 3.5(i), $\epsilon||h|$. By applying Lemma 3.5(ii), from the second equation to the second last equation, we have $\epsilon|\delta_k$ for all $i+1 \leq k \leq j-1$. Now from the last equation, we have $\bar{y}_j = \bar{h}^{-(b_{j-1}-a_j+\delta_{j-1})}\bar{y}_j\bar{h}^{\beta}$. Hence $\epsilon|b_{j-1} - a_j + \delta_{j-1}$ by the choice of N_2. Since $\epsilon|\delta_{j-1}$, we have $\epsilon| - (a_j - b_{j-1})$, which is a contradiction. Therefore $\bar{x}' \notin \langle\bar{h}\rangle\bar{y}'\langle\bar{h}\rangle$ and thus $\bar{x} \nsim_{\bar{G}} \bar{y}$.

Subcase 2b. Suppose $\gamma_j \neq 0$, that is $y_j^{-1}\langle h\rangle y_j \cap \langle h\rangle = \langle h^{r_j}\rangle$. Since B is $\langle h\rangle$-weakly potent, then by Lemma 7.1, $y_j^{-1}h^{\gamma_j}y_j = h^{\pm\gamma_j}$ and $y_j^{-1}h^q y_j \notin \langle h\rangle$ for all $1 \leq q < \gamma_j$. Since B is $\langle h\rangle$-separable and $y_j^{-1}h^q y_j \notin \langle h\rangle$ for all $1 \leq q < \gamma_j$, there exists $N_2 \lhd_f B$ such that $y_j^{-1}h^q y_j \notin \langle h\rangle N_2$ for all $1 \leq q < \gamma_j$. Since the matching fails at the equation $x_j = h^{-a_j}y_j h^{b_j}$, we further must have $h^{b_{j-1}-a_j} \notin \langle h^{\gamma_j}\rangle$, that is $a_j \neq b_{j-1}+n\gamma_j$ for all $n \in \mathbb{Z}$. Since A is $\langle h\rangle$-double coset separable, then by Lemma 3.4, there exists $M_2 \lhd_f A$ such that $y_i^{-1}h^a y_i M_2 = h^b M_2$ only if $\gamma_j|a, b$. As before, we form $\bar{G}$. Then $\bar{x}, \bar{y}$ are cyclically reduced and $||\bar{x}|| = ||x||, ||\bar{y}|| = ||y||$ in $\bar{G}$. Furthermore, by the choice of M_2, $\bar{y}_i^{-1}\bar{h}^a\bar{y}_i = \bar{h}^b$ only if $\gamma_j|a, b$ whereas, by the choice of N_2, $\bar{y}_j^{-1}\bar{h}^q\bar{y}_j \notin \langle\bar{h}\rangle$ for all $1 \leq q < \gamma_j$.

Suppose that $\bar{x}' \in \langle\bar{h}\rangle\bar{y}'\langle\bar{h}\rangle$. Then we have the system of equations (3) as above. From the first equation in (3), $\gamma_j|\alpha, \delta_i$ by the choice of M_2. Now by Lemma 3.5(i), $\gamma_j||h|$. By applying Lemma 3.5(ii) from the second equation to the second last equation, we have $\gamma_j|\delta_k$ for all $i+1 \leq k \leq j-1$. Now from the last equation we have $\bar{y}_j = \bar{h}^{a_j-b_{j-1}-\delta_{j-1}}\bar{y}_j\bar{h}^{\beta}$. Hence, $\gamma_j|a_j - b_{j-1} - \delta_{j-1}$ by the choice of N_2. Since $\gamma_j|\delta_{j-1}$, we have $\gamma_j|a_j - b_{j-1}$. So, $a_j - b_{j-1} = v\gamma_j$ for some $v \in \mathbb{Z}$. But this contradicts the fact that $a_j - b_{j-1} \notin z\gamma_j$ for all $z \in \mathbb{Z}$. Therefore, $\bar{x}' \notin \langle\bar{h}\rangle\bar{y}'\langle\bar{h}\rangle$ in $\bar{G}$ and thus $\bar{x} \nsim_{\bar{G}} \bar{y}$.

7.3 Free Products with Cyclic Amalgamated Subgroups

In this section, we first prove our main result Theorem 7.6 and then extend the result to free groups and finitely generated torsion-free nilpotent groups.

Theorem 7.6. *Let $G = A \underset{\langle h \rangle}{*} B$ where h has infinite order. Suppose*

(i) *A and B are cyclic conjugacy separable;*

(ii) *A and B are cyclic subgroup separable;*

(iii) *A and B are $\langle h \rangle$-double coset separable;*

(iv) *A and B are $\langle h \rangle$-weakly potent;*

(v) *For each integer $s > 0$, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap \langle h \rangle = \langle h^s \rangle = N \cap \langle h \rangle$ with $\tilde{h}^i \nsim_{\tilde{A}} \tilde{h}^j$ for all $\tilde{h}^i \neq \tilde{h}^j$ in $\tilde{A} = A/M$ and $\hat{h}^u \nsim_{\hat{B}} \hat{h}^v$ for all $\hat{h}^u \neq \hat{h}^v$ in $\hat{B} = B/N$.*

Then G is cyclic conjugacy separable.

Proof. We apply Theorem 6.6 here. Since A and B are cyclic conjugacy separable and $\langle h \rangle$-double coset separable, then conditions (a), (b) and (c) are satisfied. By Lemma 7.2 and Lemma 7.5, we have conditions (d) and (e). Now we show condition (f). Let $x, y \in G$ such that $||x|| = 0, ||y|| \leq 1$ and $\{x\}^G \cap \langle y \rangle = \emptyset$.

Case 1. Suppose $||x|| = 0 = ||y||$. In this case, $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$ and $\{x\}^B \cap \langle y \rangle = \emptyset$. Since A and B are cyclic conjugacy separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$ and $\{xN_1\}^{B/N_1} \cap \langle yN_1 \rangle = \emptyset$. Let $M_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $N_1 \cap \langle h \rangle = \langle h^{s_2} \rangle$ for some positive integers s_1, s_2. Let $s = s_1 s_2$. By the hypothesis, there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle h \rangle = \langle h^s \rangle$ and $\tilde{h}^i \nsim_{\tilde{A}} \tilde{h}^j$ for all $\tilde{h}^i \neq \tilde{h}^j$ in $\tilde{A} = A/M_2$. Similarly, there exists $N_2 \lhd_f B$ such that $N_2 \cap \langle h \rangle = \langle h^s \rangle$ and $\hat{h}^u \nsim_{\hat{B}} \hat{h}^v$ for all $\hat{h}^u \neq \hat{h}^v$ in $\hat{B} = B/N_2$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Then $M \lhd_f A, N \lhd_f B$ such that $M \subseteq M_1, N \subseteq N_1$ and $M \cap \langle h \rangle = N \cap \langle h \rangle$. We now form

$\bar{G} = \bar{A} \underset{\langle \bar{h} \rangle}{*} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{h} = hM = hN$. Note that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$ and $\{\bar{x}\}^{\bar{B}} \cap \langle \bar{y} \rangle = \emptyset$. Furthermore, we have $\bar{h}^i \nsim_{\bar{A}} \bar{h}^j$ for all $\bar{h}^i \neq \bar{h}^j$ in $\bar{A} = A/M$ and $\bar{h}^u \nsim_{\bar{B}} \bar{h}^v$ for all $\bar{h}^u \neq \bar{h}^v$ in $\bar{B} = B/N$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 6.1(a), there exists a finite sequence $\bar{h}^{r_1}, \bar{h}^{r_2}, \ldots, \bar{h}^{r_n}$ where $\bar{h}^{r_i} \in \langle \bar{h} \rangle$ for all i such that $\bar{x} \sim_{\bar{A}} \bar{h}^{r_1} \sim_{\bar{B}} \bar{h}^{r_2} \sim_{\bar{A}} \ldots \sim_{\bar{A}(\bar{B})} \bar{h}^{r_n} \sim_{\bar{B}(\bar{A})} \bar{y}^k$. Without loss of generality, we can assume that $\bar{x} \sim_{\bar{A}} \bar{h}^{r_1} \sim_{\bar{B}} \bar{h}^{r_2} \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{h}^{r_n} \sim_{\bar{A}} \bar{y}^k$. Note that each r_i is uniquely determined since $\bar{h}^i \nsim_{\bar{A}} \bar{h}^j$ for all $\bar{h}^i \neq \bar{h}^j$ in $\bar{A}$ and $\bar{h}^u \nsim_{\bar{B}} \bar{h}^v$ for all $\bar{h}^u \neq \bar{h}^v$ in $\bar{B} = B/N$. Thus from the first conjugation relation, $\bar{x} \sim_{\bar{A}} \bar{h}^{r_1}$ implies that $\bar{x} = \bar{h}^{r_1}$ in $\bar{A}$ and hence $xM = h^{r_1}M$. This implies that $xM = h^{r_1}M, h^{r_1}N = h^{r_2}N, \ldots, h^{r_{n-1}}N = h^{r_n}N, h^{r_n}M = y^kM$. From the second equality $h^{r_1}N = h^{r_2}N$, we have $h^{r_1 - r_2} \in N \cap \langle h \rangle = M \cap \langle h \rangle$ and thus $h^{r_1}M = h^{r_2}M$. Continuing in this way, we have $xM = h^{r_1}M = h^{r_2}M = \ldots = h^{r_{n-1}}M = h^{r_n}M = y^kM$ and this implies that $\bar{x} = \bar{h}^{r_1} = \bar{h}^{r_2} = \ldots = \bar{h}^{r_n} = \bar{y}^k$ in $\bar{A}$. So, $\bar{x} = \bar{y}^k$, that is $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle \neq \emptyset$, a contradiction. Hence, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 6.3, we can find $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P} \rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$.

Case 2. Suppose $||x|| = 0, ||y|| = 1$. Let $y \in A \backslash \langle h \rangle$. The proof for case $y \in B \backslash \langle h \rangle$ is similar. In this case, $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$. Since A is cyclic conjugacy separable and $\langle h \rangle$-separable, there exists $M_1 \lhd_f A$ such that $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$ and $y \notin \langle h \rangle M_1$. As in Case 1, there exist $M \lhd_f A, N \lhd_f B$ such that $M \subseteq M_1$ and $M \cap \langle h \rangle = N \cap \langle h \rangle$. We now form $\bar{G}$ as in Case 1. Note that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$ and $\bar{y} \in \bar{A} \backslash \langle \bar{h} \rangle$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 6.1(a), there exists a finite sequence $\bar{h}^{r_1}, \bar{h}^{r_2}, \ldots, \bar{h}^{r_n}$ where $\bar{h}^{r_i} \in \langle \bar{h} \rangle$ for all i such that $\bar{x} \sim_{\bar{A}} \bar{h}^{r_1} \sim_{\bar{B}} \bar{h}^{r_2} \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{h}^{r_n} \sim_{\bar{A}} \bar{y}^k$. Arguing as in Case 1, we have $\bar{x} = \bar{h}^{r_1} = \bar{h}^{r_2} = \ldots = \bar{h}^{r_n}$ in $\bar{A}$ since $\bar{x}, \bar{h}^{r_i} \in \langle \bar{h} \rangle$. From the last conjugation relation, we only have $\bar{h}^{r_n} \sim_{\bar{A}} \bar{y}^k$. Hence

$\bar{x} = \bar{h}^{r_n} \sim_{\bar{A}} \bar{y}^k$, that is $\bar{x} \sim_{\bar{A}} \bar{y}^k$. This implies that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle \neq \emptyset$, a contradiction. So, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$ and the result follows as in Case 1. Thus, we have condition (f).

This completes the proof. Hence G is cyclic conjugacy separable by Theorem 6.6.

Free groups and finitely generated nilpotent groups are cyclic subgroup separable and cyclic conjugacy separable. Furthermore, we have both free and finitely generated nilpotent groups are H-double coset separable for each finitely generated subgroup H (Lennox & Wilson, 1979; Ribes & Zalesskii, 1993) and weakly potent (Evans, 1974; Tang, 1995). We can see that free groups and finitely generated torsion-free nilpotent groups satisfy Condition (v) of Theorem 7.6 by Corollary 2.2 of Tang (1997). Thus, we extend Theorem 7.6 to free groups and finitely generated torsion-free nilpotent groups, then obtaining the following results by Tang (1997).

Corollary 7.7. *(Tang, 1997) Let $G = A \ast_{\langle h \rangle} B$ where h has infinite order in both A and B. Suppose A and B are free groups. Then G is cyclic conjugacy separable.*

Corollary 7.8. *(Tang, 1997) Let $G = A \ast_{\langle h \rangle} B$ where h has infinite order in both A and B. Suppose A and B are finitely generated torsion-free nilpotent groups. Then G is cyclic conjugacy separable.*

CHAPTER 8: GENERALIZED FREE PRODUCTS OF FINITELY GENERATED NILPOTENT GROUPS

8.1 Introduction

It has been established that the generalized free product $G = A \underset{H}{*} B$ is conjugacy separable and cyclic conjugacy separable if A and B are finite. Dyer (1980) and Tang (1997) showed that these generalized free products are free-by-finite and free-by-finite groups are conjugacy separable and cyclic conjugacy separable.

Dyer (1980) also showed that $G = A \underset{H}{*} B$ is conjugacy separable if A and B are conjugacy separable and H is finite.

In this chapter, we shall show that $G = A \underset{H}{*} B$ is cyclic conjugacy separable if A and B are conjugacy separable and cyclic conjugacy separable and H is finite. This is given in Theorem 8.2. We then apply our result to show that $G = A \underset{H}{*} B$ is cyclic conjugacy separable when $H = K \times D$ where K is finite and D is central in A and in B. We further apply our result to free products of finitely generated nilpotent groups.

8.2 Free Products with Finite Amalgamated Subgroups

In this section, we show that the free products of two cyclic conjugacy separable and conjugacy separable groups amalgamating a finite subgroup are cyclic conjugacy separable (Theorem 8.2).

Theorem 8.1. *(Dyer, 1980) Let $G = A \underset{H}{*} B$ where H is finite. Suppose A and B are conjugacy separable (residually finite). Then G is conjugacy separable (residually finite).*

Theorem 8.2. *Let $G = A *_H B$ where H is finite. Suppose A, B are conjugacy separable and cyclic conjugacy separable. Then G is cyclic conjugacy separable.*

Proof. We apply Theorem 6.6 here. By assumption, we have condition (a). Next by Lemma 4.11 and Lemma 4.12, we obtain conditions (b) and (c) respectively. By Theorem 8.1, we have condition (e).

Suppose $R \lhd_f H$ be given. Since A, B are residually finite and H is finite, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H = 1 \subseteq R$. Thus, we have condition (d).

Now we only need to prove condition (f). Let $x, y \in G$ such that $\{x\}^G \cap \langle y \rangle = \emptyset$ and $||x|| = 0, ||y|| \leq 1$. Let $y \in A$. The proof for the case $y \in B$ is similar. Now $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$ and $\{x\}^B \cap \langle y \rangle = \emptyset$. Since A and B are cyclic conjugacy separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$ and $\{xN_1\}^{B/N_1} \cap \langle yN_1 \rangle = \emptyset$.

Now let $h_i \in H, i = 1, \ldots, p$ be all the elements in $H \subseteq A$ such that $h_i \sim_A x$. Since $\{x\}^A \cap \langle y \rangle = \emptyset$, we have $\{h_i\}^A \cap \langle y \rangle = \emptyset, i = 1, \ldots, p$. Since A is cyclic conjugacy separable, there exists $M_2 \lhd_f A$ such that $\{h_i M_2\}^{A/M_2} \cap \langle yM_2 \rangle = \emptyset, i = 1, \ldots, p$. Now let $k_j \in H, j = 1, 2, \ldots, q$ be all the elements in $H \subseteq B$ such that $k_j \sim_B x$. Since $\{x\}^B \cap \langle y \rangle = \emptyset$, we have $\{k_j\}^B \cap \langle y \rangle = \emptyset, j = 1, 2, \ldots q$. Since B is cyclic conjugacy separable, there exists $N_2 \lhd_f B$ such that $\{k_j N_2\}^{B/N_2} \cap \langle yN_2 \rangle = \emptyset, j = 1, 2, \ldots, q$. Now since A is conjugacy separable, for all pairs of $a_u \nsim_A a_v$ where $a_u, a_v \in H \subseteq A$, there exists $M_3 \lhd_f A$ such that $a_u M_3 \nsim_{A/M_3} a_v M_3$. Similarly, since B is conjugacy separable, for all pairs of $b_r \nsim_B b_s$ where $b_r, b_s \in H \subseteq B$, there exists $N_3 \lhd_f B$ such that $b_r N_3 \nsim_{B/N_3} b_s N_3$. Furthermore, since A, B are residually finite and H is finite, there exist $M_4 \lhd_f A, N_4 \lhd_f B$ such that $M_4 \cap H = 1 = N_4 \cap H$. Let $M = M_1 \cap M_2 \cap M_3 \cap M_4$ and $N = N_1 \cap N_2 \cap N_3 \cap N_4$. Then $M \lhd_f A, N \lhd_f B$ such that $M \cap H = 1 = N \cap H$.

We now form $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$.

Then in $\bar{G}$, we have $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, \{\bar{x}\}^{\bar{B}} \cap \langle \bar{y} \rangle = \emptyset, \{\bar{h}_i\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, h_i \sim_A x, i = 1, \ldots, p, \{\bar{k}_j\}^{\bar{B}} \cap \langle \bar{y} \rangle = \emptyset, k_j \sim_B x, j = 1, \ldots, q$ and $\bar{a}_u \nsim_{\bar{A}} \bar{a}_v$ for all pairs of $a_u, a_v \in H, a_u \nsim_A a_v, \bar{b}_r \nsim_{\bar{B}} \bar{b}_s$ for all pairs of $b_r, b_s \in H, b_r \nsim_B b_s$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$, say $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 6.1(a), there exists a finite sequence $\bar{h}_i \in \bar{H}$ such that $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{B}} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{A}(\bar{B})} \bar{h}_n \sim_{\bar{B}(\bar{A})} \bar{y}^k$. Without loss of generality, we assume $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{B}} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{h}_n \sim_{\bar{A}} \bar{y}^k$. From the first conjugation relation $\bar{x} \sim_{\bar{A}} \bar{h}_1$, we have $xM \sim_{A/M} h_1M$. Suppose x and h_1 are not conjugate in A. Then $xM_3 \nsim_{A/M_3} h_1M_3$. But this contradicts the fact that $xM \sim_{A/M} h_1M$. Hence $x \sim_A h_1$. Since $\{x\}^A \cap \langle y \rangle = \emptyset$, we have $\{h_1\}^A \cap \langle y \rangle = \emptyset$. So in $\bar{G}$, we have $\{\bar{h}_1\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

From the second conjugation relation $\bar{h}_1 \sim_{\bar{B}} \bar{h}_2$, we have $h_1N \sim_{B/N} h_2N$. Arguing as above, we obtain $h_1 \sim_B h_2$. Suppose $\{h_2\}^A \cap \langle y \rangle \neq \emptyset$, say $h_2 \sim_A y^l$ for some integer l. Hence $y^l \sim_A h_2 \sim_B h_1 \sim_A x$. This implies that $\{x\}^G \cap \langle y \rangle \neq \emptyset$, a contradiction. Hence $\{h_2\}^A \cap \langle y \rangle = \emptyset$. So in $\bar{G}$, we have $\{\bar{h}_2\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

Proceeding from the third to the second last conjugation relation, we obtain $h_i \in H$ such that $\{h_i\}^A \cap \langle y \rangle = \emptyset, i = 3, \ldots, n$. So in $\bar{G}$, we have $\{\bar{h}_i\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, i = 3, \ldots, n$. From the last conjugation relation $\bar{h}_n \sim_{\bar{A}} \bar{y}^k$, we obtain $\{\bar{h}_n\}^{\bar{A}} \cap \langle \bar{y} \rangle \neq \emptyset$. But this contradicts the fact that $\{\bar{h}_n\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 6.3, we can find $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P} \rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$. Thus we have (f).

This completes the proof and thus G is cyclic conjugacy separable by Theorem 6.6.

8.3 Generalized Free Products of Finitely Generated Nilpotent Groups

In this section, we apply our result(Theorem 8.2) to Theorem 8.7. We start with some lemmas.

Lemma 8.3. *Let A be a group and H be a subgroup of A. Suppose that there exists a* *finitely generated subgroup $R \subseteq H$ such that $|H : R| < \infty$ and $R \lhd A$. If A is R-separable, then*

(i) A is H-separable; and

(ii) there exists $N \lhd_f A$ such that $N \cap H = R$.

Proof. Since A is R-separable, then $\bar{A} = A/R$ is residually finite. Let $x \in A \backslash H$. Then $\bar{x} \notin \bar{H} = H/R$. Since $\bar{H}$ is finite, there exists $\bar{N} \lhd_f \bar{A}$ such that $\bar{N} \cap \bar{x}\bar{H} = \emptyset$ and $\bar{N} \cap \bar{H} = \bar{1}$. Let N be the preimage of $\bar{N}$ in A. Then $N \lhd_f A$ such that $x \notin HN$ and $N \cap H = R$.

Lemma 8.4. *Let $G = A *_H B$ and let $D \subseteq H$ such that D is a finitely generated normal subgroup of A and B with $|H : D| < \infty$. Suppose A and B are subgroup separable. Then for each $R \lhd_f H$, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H \subseteq R$.*

Proof. Let $R \lhd_f H$ be given. Let $S = R \cap D$. Then $S \lhd_f D$. Suppose S has finite index r in D. Since D is finitely generated, then there exist only a finite number of subgroups of finite index r in D. Let E be the intersection of all these subgroups. Then $E \subset D$ and E is a characteristic subgroup of finite index in D. Since $E \lhd_f D$, we have $E \lhd_f H$. Since A is E-separable, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H = E \subseteq R$ by Lemma 8.3(ii).

The following lemma is modified from Lemma 2.6 of Zhou et al. (2010).

Lemma 8.5. *Let $G = A *_H B$ and let $D \subseteq H$ such that $D \subset Z(G)$ is finitely generated with $|H : D| < \infty$. Suppose A and B are subgroup separable groups. Then G is residually finite.*

Proof. Let $g \in G$ be nontrivial.

Case 1. $g \in D$. There exists $S \lhd_f D$ such that $g \notin S$. Let $\bar{G} = \bar{A} \underset{\bar{H}}{*} \bar{B}$ where $\bar{A} = A/S, \bar{B} = B/S$ and $\bar{H} = H/S$. Then $\bar{g} \neq 1$. Since $A/S, B/S$ are residually finite and H/S is finite, $\bar{G}$ is residually finite by Theorem 8.1. Then there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{g} \notin \bar{L}$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ such that $g \notin L$.

Case 2. $g \notin D$. Then we form $\bar{G} = \bar{A} \underset{\bar{H}}{*} \bar{B}$ where $\bar{A} = A/D, \bar{B} = B/D$ and $\bar{H} = H/D$. Note that $\bar{g} \neq 1$. Since $A/D, B/D$ are residually finite and H/D is finite, $\bar{G}$ is residually finite by Theorem 8.1. As before, we can find $L \lhd_f G$ such that $g \notin L$.

Lemma 8.6. *Let $G = A \underset{H}{*} B$ and let $D \subseteq H$ such that $D \subset Z(G)$ is finitely generated with $|H : D| < \infty$. Suppose A and B are subgroup separable groups. Suppose $x, y \in G$ such that $||x|| = ||y|| \geq 2$ and $x \nsim_G y$. Then there exists $P \lhd_f G$ such that $xP \nsim_{G/P} yP$.*

Proof. Let x, y be cyclically reduced. We assume $x = a_1 b_1 \ldots a_n b_n$ and $y = c_1 d_1 \ldots c_n d_n$ where $a_i, c_i \in A \backslash H, b_i, d_i \in B \backslash H$ for $1 \leq i \leq n$ and $n \geq 2$. Since A and B are H-separable by Lemma 8.3(i), there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $a_i, c_i \notin HM_1, b_i, d_i \notin HN_1$ for all i. Now, by Lemma 6.1(c), $x \nsim_G y$ if and only if $x^* \nsim_H y$, for all cyclic permutation x^* of x. Let $X = \{h^{-1} x^* h | h \in H$ and x^* is a cyclic permutation of $x\}$. Since $|H : D| < \infty$ and $D \subset Z(G)$, it follows that X is finite and $y \notin X$. Since G is residually finite by Lemma 8.5, there exists $L \lhd_f G$ such that $yL \cap \{zL | z \in X\} = \emptyset$. Let $R = M_1 \cap N_1 \cap L \cap H$. Then $R \lhd_f H$. By Lemma 8.4, there exist $M_2 \lhd_f A, N_2 \lhd_f B$ such that $M_2 \cap H = N_2 \cap H \subseteq R$. Let $M = M_1 \cap M_2 \cap L$ and $N = N_1 \cap N_2 \cap L$. Then $M \lhd_f A, N \lhd_f B$ and $M \cap H = N \cap H$. We now form $\bar{G} = \bar{A} \underset{\bar{H}}{*} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. Note that $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ and $\bar{x}^* \nsim_{\bar{H}} \bar{y}$ for all cyclic permutation $\bar{x}^*$ of $\bar{x}$. This implies that $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable by Lemma 6.3, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{x}\bar{P} \nsim_{\bar{G}/\bar{P}} \bar{y}\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $xP \nsim_{G/P} yP$.

Theorem 8.7. *Let $G = A *_H B$ where $H = K \times D$ such that K is finite and $D \subseteq Z(G)$ is finitely generated torsion-free. Suppose*

(i) A and B are subgroup separable;

(ii) A and B are conjugacy separable;

(iii) A and B are cyclic conjugacy separable;

(iv) A and B are H-conjugacy separable;

(v) A/S and B/S are conjugacy separable and cyclic conjugacy separable for any $S \lhd_f D$.

Then G is cyclic conjugacy separable.

Proof. We apply Theorem 6.6 here. Since A and B are both cyclic conjugacy separable and H-conjugacy separable, conditions (a) and (b) are satisfied. By Lemma 8.3, Lemma 8.4 and Lemma 8.6, conditions (c), (d) and (e) are satisfied.

We only need to prove condition (f). Let $x, y \in G$ such that $||x|| = 0, ||y|| \leq 1$ and $\{x\}^G \cap \langle y \rangle = \emptyset$.

Case 1. Suppose $x \in D, y \in A \cup B$. Let $x \in D, y \in A$. The proof is similar if $x \in D, y \in B$. Since $x \in D \subseteq Z(G)$, then $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $x \notin \langle y \rangle$. Since A and B are subgroup separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $x \notin \langle y \rangle M_1$ and $x \notin \langle y \rangle N_1$. Let $R = M_1 \cap N_1 \cap H$. Then by Lemma 8.4, we can find $M_2 \lhd_f A, N_2 \lhd_f B$ such that $M_2 \cap H = N_2 \cap H \subseteq R$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Then $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H$. Now we form $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. Note that $\bar{x} \notin \langle \bar{y} \rangle$ in $\bar{G}$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 6.1(a), there exists a finite sequence $\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n \in \bar{H}$ such that $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{B}} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{A}(\bar{B})} \bar{h}_n \sim_{\bar{B}(\bar{A})} \bar{y}^k$. Without loss of generality, we assume $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{B}} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{h}_n \sim_{\bar{A}} \bar{y}^k$. Since $\bar{x} \in Z(\bar{G})$, we have $\bar{x} = \bar{h}_1 = \bar{h}_2 = \ldots = \bar{h}_n = \bar{y}^k$. So $\bar{x} = \bar{y}^k$, contradicting the fact that $\bar{x} \notin \langle \bar{y} \rangle$. Hence, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 6.3, there exists $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P} \rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$.

Case 2. Suppose $x \in K \backslash D, y \in D$. In this case, $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$ and $\{x\}^B \cap \langle y \rangle = \emptyset$. Since A, B are subgroup separable and cyclic conjugacy separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $x \notin DM_1, x \notin DN_1$ and $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset, \{xN_1\}^{B/N_1} \cap \langle yN_1 \rangle = \emptyset$. By Lemma 8.4, we can find $M \lhd_f A, N \lhd_f B$ such that $M \subseteq M_1, N \subseteq N_1$ and $M \cap H = N \cap H$. We now form $\bar{G}$ as above. Then in $\bar{G}$, we have $\bar{x} \notin \bar{D}$ and $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, \{\bar{x}\}^{\bar{B}} \cap \langle \bar{y} \rangle = \emptyset$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. As in Case 1, we can assume $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{B}} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{h}_n \sim_{\bar{A}} \bar{y}^k$. Since $\bar{y} \in Z(\bar{G})$, we have $\bar{x} = \bar{h}_1 = \bar{h}_2 = \ldots = \bar{h}_n = \bar{y}^k$. So $\bar{x} = \bar{y}^k$, a contradiction. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 6.3, we again obtain our result as in Case 1.

Case 3. Suppose $x, y \in K \backslash D$. In this case, $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$ and $\{x\}^B \cap \langle y \rangle = \emptyset$. Since A, B are subgroup separable and cyclic conjugacy separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $x, y \notin DM_1, x, y \notin DN_1, \{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$ and $\{xN_1\}^{B/N_1} \cap \langle yN_1 \rangle = \emptyset$.

Now let $p_i \in K, i = 1, \ldots, r$ be all the elements in K such that $p_i \sim_A x$. Since $\{x\}^A \cap \langle y \rangle = \emptyset$, we have $\{p_i\}^A \cap \langle y \rangle = \emptyset, i = 1, \ldots, r$. Since A is cyclic conjugacy separable, there exists $M_2 \lhd_f A$ such that $\{p_i M_2\}^{A/M_2} \cap \langle yM_2 \rangle = \emptyset, i = 1, \ldots, r$. Now let $q_j \in K, j = 1, \ldots, s$ be all the elements in K such that $q_j \sim_B x$. Since $\{x\}^B \cap \langle y \rangle = \emptyset$, we have $\{q_j\}^B \cap \langle y \rangle = \emptyset, j = 1, \ldots, s$. Since B is cyclic conjugacy separable, there exists $N_2 \lhd_f B$ such that $\{q_j N_2\}^{B/N_2} \cap \langle yN_2 \rangle = \emptyset, j = 1, \ldots, s$.

Now since A is conjugacy separable, for all pairs of $a_u \nsim_A a_v$ where $a_u, a_v \in K$, there exists $M_3 \lhd_f A$ such that $a_u M_3 \nsim_{A/M_3} a_v M_3$. Similarly, since B is conjugacy separable, for all pairs of $b_r \nsim_B b_s$ where $b_r, b_s \in K$, there exists $N_3 \lhd_f B$ such that $b_r N_3 \nsim_{B/N_3} b_s N_3$. Let $S = \bigcap_{i=1}^3 M_i \cap \bigcap_{i=1}^3 N_i \cap D$. Then $S \lhd_f D$ and thus $S \lhd_f H$. We now form $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$ where $\bar{A} = A/S, \bar{B} = B/S$ and $\bar{H} = H/S$. Then in $\bar{G}$, we have $\bar{x}, \bar{y} \notin \bar{D}, \{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, \{\bar{x}\}^{\bar{B}} \cap \langle \bar{y} \rangle = \emptyset, \{\bar{p}_i\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, p_i \sim_A x, i = 1, \ldots, r, \{\bar{q}_j\}^{\bar{B}} \cap \langle \bar{y} \rangle = \emptyset, q_j \sim_B x, j = 1, \ldots, s$, and $\bar{a}_u \nsim_{\bar{A}} \bar{a}_v$ for all pairs of $a_u, a_v \in K, a_u \nsim_A a_v$ and $\bar{b}_r \nsim_{\bar{B}} \bar{b}_s$ for all pairs of $b_r, b_s \in K, b_r \nsim_B b_s$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. As in Case 1, we can assume $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{B}} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{h}_n \sim_{\bar{A}} \bar{y}^k$. Since $H = K \times D$, we can write $\bar{h}_i = \bar{k}_i \bar{d}_i$ uniquely, where $k_i \in K, d_i \in D$ for $i = 1, 2, \ldots, n$. Then we have $\bar{x} \sim_{\bar{A}} \bar{k}_1 \bar{d}_1 \sim_{\bar{B}} \bar{k}_2 \bar{d}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{k}_n \bar{d}_n \sim_{\bar{A}} \bar{y}^k$. From the first conjugation relation $\bar{x} \sim_{\bar{A}} \bar{k}_1 \bar{d}_1$, we have $\bar{x} = \bar{a}_1^{-1} \bar{k}_1 \bar{d}_1 \bar{a}_1$ for some $a_1 \in A$ and thus $\bar{x}^{-1} \bar{a}_1^{-1} \bar{k}_1 \bar{a}_1 = \bar{d}_1^{-1}$. It follows that $x^{-1} a_1^{-1} k_1 a_1 S = d_1^{-1} S$ and this implies that $x^{-1} a_1^{-1} k_1 a_1 \in D$ since $S \lhd_f D$. We let $x^{-1} a_1^{-1} k_1 a_1 = z \in D$, then $xz = a_1^{-1} k_1 a_1$. Since K is finite, we let $m = lcm\{|k_1|, |x|\}$. Thus, $(xz)^m = (a_1^{-1} k_1 a_1)^m$ implies that $z^m = 1$. Since D is torsion-free, we have $z = 1$. It follows that $x^{-1} a_1^{-1} k_1 a_1 = 1$, thus $\bar{d}_1 = 1$ and $\bar{h}_1 = \bar{k}_1$. So, now we have $\bar{x} \sim_{\bar{A}} \bar{k}_1 \sim_{\bar{B}} \bar{k}_2 \bar{d}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{k}_n \bar{d}_n \sim_{\bar{A}} \bar{y}^k$. Similarly, since now we have $\bar{k}_1 \sim_{\bar{B}} \bar{k}_2 \bar{d}_2$, then $\bar{d}_2 = 1$ and $\bar{h}_2 = \bar{k}_2$. Continuing this way, we can write $\bar{x} \sim_{\bar{A}} \bar{k}_1 \sim_{\bar{B}} \bar{k}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{k}_n \sim_{\bar{A}} \bar{y}^k$.

From the first conjugation relation $\bar{x} \sim_{\bar{A}} \bar{k}_1$, we have $xS \sim_{A/S} k_1 S$. Suppose x and k_1 are not conjugate in A. Then $xM_3 \nsim_{A/M_3} k_1 M_3$. But this contradicts the fact that $xS \sim_{A/S} k_1 S$. Hence $x \sim_A k_1$. Since $\{x\}^A \cap \langle y \rangle = \emptyset$, we have $\{k_1\}^A \cap \langle y \rangle = \emptyset$. So in $\bar{G}$, we have $\{\bar{k}_1\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

From the second conjugation relation $\bar{k}_1 \sim_{\bar{B}} \bar{k}_2$, we have $k_1 S \sim_{B/S} k_2 S$. Arguing as above, we obtain $k_1 \sim_B k_2$. Suppose $\{k_2\}^A \cap \langle y \rangle \neq \emptyset$, say $k_2 \sim_A y^l$ for some integer l. Hence $y^l \sim_A k_2 \sim_B k_1 \sim_A x$. This implies that $\{x\}^G \cap \langle y \rangle \neq \emptyset$, a contradiction. Hence $\{k_2\}^A \cap \langle y \rangle = \emptyset$. So in $\bar{G}$, we have $\{\bar{k}_2\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$.

Proceeding from the third to the second last conjugation relation, we obtain $k_i \in K$ such that $\{k_i\}^A \cap \langle y\rangle = \emptyset, i = 3, \ldots, n$. So in $\bar{G}$, we have $\{\bar{k}_i\}^{\bar{A}} \cap \langle \bar{y}\rangle = \emptyset, i = 3, \ldots, n$. From the last conjugation relation $\bar{k}_n \sim_{\bar{A}} \bar{y}^k$, we obtain $\{\bar{k}_n\}^{\bar{A}} \cap \langle \bar{y}\rangle \neq \emptyset$. But this contradicts the fact that $\{\bar{k}_n\}^{\bar{A}} \cap \langle \bar{y}\rangle = \emptyset$. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y}\rangle = \emptyset$. Since $A/S, B/S$ are both conjugacy separable and cyclic conjugacy separable with H/S is finite, we have $\bar{G}$ is cyclic conjugacy separable by Theorem 8.2. Then the result follows as in Case 1.

Case 4. Suppose $x \in K\backslash D, y \in A\backslash H$ or $x \in K\backslash D, y \in B\backslash H$. We shall prove for the case $x \in K\backslash D, y \in A\backslash H$. The proof is similar to the proof in Case 3. For completeness, we shall write the proof in full.

In this case, $\{x\}^G \cap \langle y\rangle = \emptyset$ implies that $\{x\}^A \cap \langle y\rangle = \emptyset$. Since A, B are subgroup separable and cyclic conjugacy separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $x \notin DM_1, x \notin DN_1, y \notin HM_1$ and $\{xM_1\}^{A/M_1} \cap \langle yM_1\rangle = \emptyset$. Now let $p_i \in K, i = 1, \ldots, r$ be all the elements in K such that $p_i \sim_A x$. Since $\{x\}^A \cap \langle y\rangle = \emptyset$, we have $\{p_i\}^A \cap \langle y\rangle = \emptyset, i = 1, \ldots, r$. Since A is cyclic conjugacy separable, there exists $M_2 \lhd_f A$ such that $\{p_iM_2\}^{A/M_2} \cap \langle yM_2\rangle = \emptyset, i = 1, \ldots, r$.

As in Case 3, for all pairs of $a_u \nsim_A a_v$ where $a_u, a_v \in K$, there exists $M_3 \lhd_f A$ such that $a_uM_3 \nsim_{A/M_3} a_vM_3$. Similarly, for all pairs of $b_r \nsim_B b_s$ where $b_r, b_s \in K$, there exists $N_2 \lhd_f B$ such that $b_rN_2 \nsim_{B/N_2} b_sN_2$. Let $S = \bigcap_{i=1}^{3} M_i \bigcap_{i=1}^{2} N_i \cap D$. Then $S \lhd_f D$ and thus $S \lhd_f H$. We now form $\bar{G} = \bar{A} \underset{\bar{H}}{*} \bar{B}$ where $\bar{A} = A/S, \bar{B} = B/S$ and $\bar{H} = H/S$. Then in $\bar{G}$, we have $\bar{x} \notin \bar{D}, \bar{y} \notin \bar{H}, \{\bar{x}\}^{\bar{A}} \cap \langle \bar{y}\rangle = \emptyset, \{\bar{p}_i\}^{\bar{A}} \cap \langle \bar{y}\rangle = \emptyset, p_i \sim_A x, i = 1, \ldots, r$, and $\bar{a}_u \nsim_{\bar{A}} \bar{a}_v$ for all pairs of $a_u, a_v \in K, a_u \nsim_A a_v$ and $\bar{b}_r \nsim_{\bar{B}} \bar{b}_s$ for all pairs of $b_r, b_s \in K, b_r \nsim_B b_s$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y}\rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. As in Case 1, we can assume $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{B}} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{h}_n \sim_{\bar{A}} \bar{y}^k$. Since $H = K \times D$, we can write $\bar{h}_i = \bar{k}_i\bar{d}_i$ uniquely, where $k_i \in K, d_i \in D$ for $i = 1, 2, \ldots, n$. Then we have $\bar{x} \sim_{\bar{A}} \bar{k}_1\bar{d}_1 \sim_{\bar{B}} \bar{k}_2\bar{d}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{k}_n\bar{d}_n \sim_{\bar{A}} \bar{y}^k$. As in Case 3, we can write $\bar{x} \sim_{\bar{A}} \bar{k}_1 \sim_{\bar{B}} \bar{k}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{k}_n \sim_{\bar{A}} \bar{y}^k$

since $\bar{d}_i = 1$ and $\bar{h}_i = \bar{k}_i$ for all i. Again as in Case 3, we obtain $k_i \in K$ such that $\{k_i\}^A \cap \langle y \rangle = \emptyset, i = 1, 2, \ldots, n$. So in $\bar{G}$, we have $\{\bar{k}_i\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset, i = 1, 2, \ldots, n$. From the last conjugation relation $\bar{k}_n \sim_{\bar{A}} \bar{y}^k$, we obtain $\{\bar{k}_n\}^{\bar{A}} \cap \langle \bar{y} \rangle \neq \emptyset$. But this contradicts the fact that $\{\bar{k}_n\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Hence, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$ and the result follows as in Case 3.

From all the cases above, we obtain condition (f). This completes the proof and hence G is cyclic conjugacy separable by Theorem 6.6.

Note that finitely generated nilpotent groups are subgroup separable, conjugacy separable and cyclic conjugacy separable. Furthermore, the quotient groups of nilpotent groups are again nilpotent. Thus, we can apply Lemma 4.8 and Theorem 8.7 to finitely generated nilpotent groups. We have the next result.

Corollary 8.8. *Let $G = A *_H B$ where $H = K \times D$ such that K is finite and $D \subseteq Z(G)$. Suppose A and B are finitely generated nilpotent groups. Then G is cyclic conjugacy separable.*

Proof. Since D is a finitely generated abelian group, $D = K_1 \times D_1$ where K_1 is finite and D_1 is torsion-free. Hence we may assume D is torsion-free. Then G is cyclic conjugacy separable by Theorem 8.7.

CHAPTER 9: TREE PRODUCTS AMALGAMATING CENTRAL SUBGROUPS

9.1 Introduction

The tree products of finitely many free groups or surface groups or polycyclic-by-finite groups amalgamating cyclic subgroups are conjugacy separable (Kim & Tang, 1996; Kim & Tang, 1999; Ribes et al., 1998). Also the tree products of finitely many conjugacy separable and (central) subgroup separable groups amalgamating central subgroups are conjugacy separable (Wong & Tang, 1999; Kim & Tang, 2002). Hence the tree product of polycyclic-by-finite groups amalgamating central subgroups are conjugacy separable.

In this chapter, we shall study the cyclic conjugacy separability of tree product of finitely many cyclic conjugacy separable and subgroup separable groups amalgamating central subgroups (Theorem 9.18). We then apply our result to tree products of free-by-finite groups and finitely generated abelian groups.

9.2 Free Products Amalgamating Central Subgroups

In this section, we show that the generalized free product of two cyclic conjugacy separable and subgroup separable groups amalgamating central subgroups are cyclic conjugacy separable (Theorem 9.3).

Lemma 9.1. *Let A be subgroup separable. Suppose $R \lhd_f H \subseteq Z(A)$. Then there exists $M \lhd_f A$ such that $M \cap H = R$.*

Proof. Let $R \lhd_f H \subseteq Z(A)$ be given. Since A is subgroup separable, we have $\bar{A} = A/R$ is residually finite. Since $\bar{H}$ is finite, there exists $\bar{M} \lhd_f \bar{A}$ such that $\bar{M} \cap \bar{H} = \bar{1}$. Let M be the preimage of $\bar{M}$ in A. Then $M \cap H = R$ as required.

Lemma 9.2. *Let $G = A \ast_H B$ where $H \subseteq Z(A) \cap Z(B)$ is a finitely generated subgroup. Suppose A and B are subgroup separable. Then for each $x, y \in G$ such that $||x|| = ||y|| \geq 2$ and $x \nsim_G y$, there exists $Q \lhd_f G$ such that $xQ \nsim_{G/Q} yQ$.*

Proof. Let $x, y \in G$ be cyclically reduced. We assume $x = a_1b_1 \ldots a_nb_n$ and $y = c_1d_1 \ldots c_nd_n$ where $a_i, c_i \in A \backslash H$, $b_i, d_i \in B \backslash H$ for $1 \leq i \leq n$ and $n \geq 2$. Since A and B are H-separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $a_i, c_i \notin HM_1, b_i, d_i \notin HN_1$ for all i. Now, by Lemma 6.1(c), $x \nsim_G y$ if and only if $x \nsim_H y^*$, for all cyclic permutations y^* of y. Since $H \subseteq Z(A) \cap Z(B)$, we have $xy^{*-1} \neq 1$ for all cyclic permutations y^* of y. By Lemma 6.5, G is residually finite. Since there is only a finite number of cyclic permutations y^* of y, we can find $L \lhd_f G$ such that $xy^{*-1} \notin L$ for all cyclic permutations y^* of y. Let $R = M_1 \cap N_1 \cap L \cap H$. Then $R \lhd_f H$. By Lemma 9.1, there exist $M_2 \lhd_f A, N_2 \lhd_f B$ such that $M_2 \cap H = R = N_2 \cap H$. Let $M = L \cap M_1 \cap M_2$ and $N = L \cap N_1 \cap N_2$. Thus $M \cap H = N \cap H$. We now form $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. Note that $||\bar{x}|| = ||x|| = ||y|| = ||\bar{y}||$ and $\bar{x}\bar{y}^{*-1} \neq \bar{1}$ for all cyclic permutations y^* of y. This implies that $\bar{x} \nsim_{\bar{H}} \bar{y}^*$ and thus $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable by Lemma 6.3, there exists $\bar{Q} \lhd_f \bar{G}$ such that $\bar{x}\bar{Q} \nsim_{\bar{G}/\bar{Q}} \bar{y}\bar{Q}$. Let Q be the preimage of $\bar{Q}$. Then $Q \lhd_f G$ such that $xQ \nsim_{G/Q} yQ$.

Theorem 9.3. *Let $G = A \ast_H B$ where $H \subseteq Z(A) \cap Z(B)$ is a finitely generated subgroup. Suppose A and B are cyclic conjugacy separable and subgroup separable. Then G is cyclic conjugacy separable.*

Proof. We apply Theorem 6.6 here. Since A and B are cyclic conjugacy separable and subgroup separable with H finitely generated, the conditions (a) and (c) are satisfied. By Lemma 9.1 and Lemma 9.2, conditions (d) and (e) are satisfied.

To show (b), we let $x \in A$ such that $\{x\}^A \cap H = \emptyset$. Since $H \subseteq Z(A)$, we have $\{x\}^A \cap H = \emptyset$ if and only if $x \notin H$. Now, since A is subgroup separable, there exists $M \lhd_f A$ such that $x \notin HM$. Let $R = M \cap H$. By Lemma 9.1, we can find $N \lhd_f B$ such that $N \cap H = R$. We now form $\bar{G} = \bar{A} \underset{\bar{H}}{*} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. Since $\bar{x} \notin \bar{H}$ and $\bar{H} \subseteq Z(\bar{A})$ in $\bar{G}$, we have $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$. Therefore, A is H-conjugacy separable and similarly for B. Thus, we have condition (b).

We show (f) now. Let $x, y \in G$ such that $||x|| = 0, ||y|| \leq 1$ and $\{x\}^G \cap \langle y \rangle = \emptyset$. Suppose $y \in A$. The proof is similar if $y \in B$. For this case $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $x \notin \langle y \rangle$ since $H \subseteq Z(A) \cap Z(B)$. Since A is subgroup separable, there exists $M \lhd_f A$ such that $x \notin \langle y \rangle M$. Let $R = M \cap H$. Then $R \lhd_f H$. By Lemma 9.1, there exists $N \lhd_f B$ such that $N \cap H = R$. Now we can form $\bar{G}$ as above.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$, say $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 6.1(a), there exists a finite sequence of $\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n \in \bar{H}$ such that $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{B}} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{A}(\bar{B})} \bar{h}_n \sim_{\bar{B}(\bar{A})} \bar{y}^k$. Since $y \in A$, without loss of generality, we assume $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{B}} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{B}} \bar{h}_n \sim_{\bar{A}} \bar{y}^k$. Since $\bar{H} \subseteq Z(\bar{A}) \cap Z(\bar{B})$, we have $\bar{x} = \bar{h}_1 = \bar{h}_2 = \ldots = \bar{h}_n = \bar{y}^k$. So, $\bar{x} = \bar{y}^k$, contradicting $\bar{x} \notin \langle \bar{y} \rangle$. Therefore, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 6.3, there exists $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P} \rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$.

This completes the proof and therefore G is cyclic conjugacy separable by Theorem 6.6.

Corollary 9.4. *Let $G = A \underset{H}{*} B$ where $H \subseteq Z(A) \cap Z(B)$ is a finitely generated subgroup. Suppose A and B are free-by-finite groups. Then G is cyclic conjugacy separable.*

Corollary 9.5. *Let $G = A \underset{H}{*} B$ where A and B are finitely generated abelian groups. Then G is cyclic conjugacy separable.*

9.3 Lemmas Needed

Now we extend Theorem 9.3 to tree products of finitely many subgroup separable and cyclic conjugacy separable groups. We note that Lemma 9.6, Lemma 9.7, Lemma 9.8, Lemma 9.9 and Lemma 9.17 are modified from Wong & Tang (1999).

Lemma 9.6. *Let $G = A *_H B$ and $K < B$. Suppose*

(i) *for each $R \lhd_f H$, there exists $M \lhd_f A$ such that $M \cap H = R$;*

(ii) *for each $S \lhd_f K$, there exists $N \lhd_f B$ such that $N \cap K = S$.*

Then there exists $P \lhd_f G$ such that $P \cap K = S$.

Proof. Let $S \lhd_f K < B$ be given such that there exists $N \lhd_f B$ with $N \cap K = S$. Let $R = N \cap H$. By (i), there exists $M \lhd_f A$ such that $M \cap H = R$. Now we can form $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. Since $\bar{G}$ is residually finite by Lemma 6.3 and $\bar{K} = KN/N$ is finite, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{P} \cap \bar{K} = \bar{1}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $P \cap K = S$.

Lemma 9.7. *Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating the subgroups H_{ij} of A_i and H_{ji} of A_j. Let $K < A_r$. Suppose*

(i) *for each $R_{ij} \lhd_f H_{ij}$, there exists $M_i \lhd_f A_i$ such that $M_i \cap H_{ij} = R_{ij}$;*

(ii) *for each $S \lhd_f K$, there exists $N \lhd_f A_r$ such that $N \cap K = S$.*

Then there exists $P \lhd_f G$ such that $P \cap K = S$.

Proof. We use induction on n. The case when $n = 2$ follows from Lemma 9.6. Let $n > 2$. The tree product G has an extremal vertex, say A_n, which is joined to a unique vertex, say A_{n-1}. The subgroup of G generated by $A_1, A_2, \ldots, A_{n-1}$ is just the tree product of themselves. Let A denote this subgroup. Then we write $G = A *_H A_n$ where $H = H_{(n-1)n} = H_{n(n-1)}$.

Case 1. $K < A_r < A$. By induction, for each $S \lhd_f K$, there exists $M \lhd_f A$ such that $M \cap K = S$. By (i), for each $R \lhd_f H_{n(n-1)}$, there exists $N \lhd_f A_n$ such that $N \cap H_{n(n-1)} = R$. Then the result follows from Lemma 9.6.

Case 2. $K < A_n$. By (ii), for each $S \lhd_f K$, there exists $N \lhd_f A_n$ such that $N \cap K = S$. By induction, for each $R \lhd_f H_{(n-1)n}$, there exists $M \lhd_f A$ such that $M \cap H = R$. We now form $\bar{G}$ as above and the result follows from Lemma 9.6.

Lemma 9.8. *Let $G = A \ast_H B$ and $K < B$. Suppose*

(*i*) *A and B are H-separable;*

(*ii*) *for each $R \lhd_f H$, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap H = R = N \cap H$;*

(*iii*) *B is K-separable.*

Then G is K-separable.

Proof. Let $g \in G \backslash K$.

Case 1. $g \in A \backslash H$. Since A is H-separable, we can find $M \lhd_f A$ such that $g \notin HM$. Let $M \cap H = R$. By (ii), we can find $N \lhd_f B$ such that $N \cap H = R$. We now form $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. Note that $\bar{K} < \bar{B}$ and $\bar{g} \in \bar{A} \backslash \bar{H}$. Thus, $\bar{g} \notin \bar{K}$. Since $\bar{G}$ is subgroup separable by Lemma 6.3, we can find $\bar{P} \lhd_f \bar{G}$ such that $\bar{g} \notin \bar{K}\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $g \notin KP$.

Case 2. $g \in B \backslash K$. Since B is K-separable, there exists $N \lhd_f B$ such that $g \notin KN$. Let $N \cap H = R$. As above, we can find $M \lhd_f A$ such that $M \cap H = R$. We now form $\bar{G}$ as in Case 1. It is clear that $\bar{g} \notin \bar{K}$. The result follows as in Case 1.

Case 3. $||g|| \geq 2$. WLOG, we assume $g = a_1 b_1 a_2 b_2 \ldots a_n b_n$ where $a_i \in A \backslash H, b_i \in B \backslash H$ and $n \geq 2$. Since A and B are H-separable, there exist $M_1 \lhd_f A, N_1 \lhd_f B$ such that $a_i \notin HM_1$ and $b_i \notin HN_1$. Suppose $M_1 \cap H = R_1$ and $N_1 \cap H = R_2$. Then we can find $M_2 \lhd_f A, N_2 \lhd_f B$ such that $M_2 \cap H = R_1 \cap R_2 = N_2 \cap H$. Let $M = M_1 \cap M_2$ and

$N = N_1 \cap N_2$. We now form $\bar{G}$ as in Case 1. Since $||\bar{g}|| = ||g||$, it is clear that $\bar{g} \notin \bar{K}$. The result follows as in Case 1.

Lemma 9.9. *Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating the subgroups H_{ij} of A_i and H_{ji} of A_j. Let $K < A_r$. Suppose*

(i) A_i is H_{ij}-separable;

(ii) for each $R_{ij} \lhd_f H_{ij}$, there exists $M_i \lhd_f A_i$ such that $M_i \cap H_{ij} = R_{ij}$;

(iii) A_r is K-separable.

Then G is K-separable.

Proof. We use induction on n. The case when $n = 2$ follows from Lemma 9.8. Let $n > 2$. The tree product G has an extremal vertex, say A_n, which is joined to a unique vertex, say A_{n-1}. The subgroup of G generated by $A_1, A_2, \ldots, A_{n-1}$ is just the tree product of themselves. Let A denote this subgroup. Then we write $G = A \ast_H A_n$ where $H = H_{(n-1)n} = H_{n(n-1)}$. By induction, A is $H_{(n-1)n}$-separable. By Lemma 9.7, for each $R \lhd_f H_{(n-1)n}$, there exists $M \lhd_f A$ such that $M \cap H_{(n-1)n} = R$. By (i), A_n is $H_{n(n-1)}$-separable. By (ii), for each $S \lhd_f H_{n(n-1)}$, there exists $N \lhd_f A_n$ such that $N \cap H_{n(n-1)} = S$.

Case 1. $K < A_r < A$. By induction, A is K-separable. Thus, G is K-separable by Lemma 9.8.

Case 2. $K < A_n$. Since A_n is K-separable, the result follows from Lemma 9.8.

Lemma 9.10. *Let $G = A \ast_H B$ and $K < B$. Suppose*

(i) for each $R \lhd_f H$, there exists $M \lhd_f A$ such that $M \cap H = R$ and $\bar{h}_i \nsim_{\bar{A}} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}$ of $\bar{A} = A/M$;

(ii) for each $S \lhd_f K$, there exists $N \lhd_f B$ such that $N \cap K = S$ and $\bar{k}_i \nsim_{\bar{B}} \bar{k}_j$ for $\bar{k}_i \neq \bar{k}_j$ in $\bar{K}$ of $\bar{B} = B/N$.

Then there exists $P \lhd_f G$ such that $P \cap K = S$ and $\bar{k}_i \nsim_{\bar{G}} \bar{k}_j$ for $\bar{k}_i \neq \bar{k}_j$ in $\bar{G} = G/P$.

Proof. Let $S \lhd_f K$ be given. By (ii), there exists $N \lhd_f B$ such that $N \cap K = S$ and $\bar{k}_i \nsim_{\bar{B}} \bar{k}_j$ for $\bar{k}_i \neq \bar{k}_j$ in $\bar{K}$ of $\bar{B} = B/N$. Let $N \cap H = R$. By (i), there exists $M \lhd_f A$ such that $M \cap H = R$ and $\bar{h}_i \nsim_{\bar{A}} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}$ of $\bar{A} = A/M$. We now form $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. Note that $\bar{k}_i \nsim_{\bar{G}} \bar{k}_j$ for $\bar{k}_i \neq \bar{k}_j$. Since $\bar{G}$ is conjugacy separable by Lemma 6.3 and $\bar{K}$ is finite, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{k}_i\bar{P} \nsim_{\bar{G}/\bar{P}} \bar{k}_j\bar{P}$ and $\bar{P} \cap \bar{K} = 1$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $P \cap K = S$ and $k_iP \nsim_{G/P} k_jP$ for $k_i \neq k_j$.

Lemma 9.11. *Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating the subgroups H_{ij} of A_i and H_{ji} of A_j. Let $K < A_r$. Suppose*

(*i*) *for each $R_{ij} \lhd_f H_{ij}$, there exists $M_i \lhd_f A_i$ such that $M_i \cap H_{ij} = R_{ij}$ and $\bar{h}_i \nsim_{\bar{A}_i} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}_{ij}$ of $\bar{A}_i = A_i/M_i$;*

(*ii*) *for each $S \lhd_f K$, there exists $N \lhd_f A_r$ such that $N \cap K = S$ and $\bar{k}_i \nsim_{\bar{A}_r} \bar{k}_j$ for $\bar{k}_i \neq \bar{k}_j$ in $\bar{K}$ of $\bar{A}_r = A_r/N$.*

Then there exists $P \lhd_f G$ such that $P \cap K = S$ and $\bar{k}_i \nsim_{\bar{G}} \bar{k}_j$ for $\bar{k}_i \neq \bar{k}_j$ in $\bar{G} = G/P$.

Proof. We use induction on n. The case when $n = 2$ follows from Lemma 9.10. Let $n > 2$. The tree product G has an extremal vertex, say A_n, which is joined to a unique vertex, say A_{n-1}. The subgroup of G generated by $A_1, A_2, \ldots, A_{n-1}$ is just the tree product of themselves. Let A denote this subgroup. Then we write $G = A \ast_H A_n$ where $H = H_{(n-1)n} = H_{n(n-1)}$.

Case 1. $K < A_r < A$. By induction, for each $S \lhd_f K$, there exists $Q \lhd_f A$ such that $Q \cap K = S$ and $\bar{k}_i \nsim_{\bar{A}} \bar{k}_j$ for $\bar{k}_i \neq \bar{k}_j$ in $\bar{K}$ where $\bar{A} = A/Q$. By (i), for each $R \lhd_f H_{n(n-1)}$, there exists $N \lhd_f A_n$ such that $N \cap H_{n(n-1)n} = R$ and $\bar{h}_i \nsim_{\bar{A}_n} \bar{h}_i$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}_{n(n-1)}$ where $\bar{A}_n = A_n/N$. Then the result follows from Lemma 9.10.

Case 2. $K < A_n$. By (ii), for each $S \lhd_f K$, there exists $N \lhd_f A_n$ such that $N \cap K = S$ and $\bar{k}_i \nsim_{\bar{A}_n} \bar{k}_j$ for $\bar{k}_i \neq \bar{k}_j$ in $\bar{K}$ where $\bar{A}_n = A_n/N$. By induction, for each $R \lhd_f H_{(n-1)n}$, there exists $M \lhd_f A$ such that $M \cap H_{(n-1)n} = R$ and $\bar{h}_i \nsim_{\bar{A}} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}_{(n-1)n}$ where $\bar{A} = A/M$. The result now follows from Lemma 9.10.

Lemma 9.12. *Let $G = A *_H B$ and $K < B$. Suppose*

(*i*) *A and B are H-separable;*

(*ii*) *for each $R \lhd_f H$, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap H = R = N \cap H$ and $\bar{h}_i \nsim_{\bar{A}} \bar{h}_j, \bar{h}_i \nsim_{\bar{B}} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}$ of $\bar{A} = A/M, \bar{B} = B/N$;*

(*iii*) *A and B are H-conjugacy separable;*

(*iv*) *B is K-conjugacy separable.*

Then G is K-conjugacy separable.

Proof. Let $x \in G$ be of minimal length in its conjugacy class and $\{x\}^G \cap K = \emptyset$.

Case 1. $x \in A \backslash H$. Since x is of minimal length in its conjugacy class, we have $\{x\}^A \cap H = \emptyset$. Since A is H-conjugacy separable, there exists $M \lhd_f A$ such that $\{xM\}^{A/M} \cap HM/M = \emptyset$. By (ii), we can find $N \lhd_f B$ such that $N \cap H = M \cap H$. We now form $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M = HN/N$. It is clear that $\{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$ and $\bar{K} < \bar{B}$. Thus, $\{\bar{x}\}^{\bar{G}} \cap \bar{K} = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 6.3 and $\bar{K}$ is finite, there exists $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \bar{K}\bar{P}/\bar{P} = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap KP/P = \emptyset$.

Case 2. $x \in B$. Since $\{x\}^G \cap K = \emptyset$, we have $\{x\}^B \cap K = \emptyset$. By (iv), there exists $N \lhd_f B$ such that $\{xN\}^{B/N} \cap KN/N = \emptyset$. By (ii), we can find $M \lhd_f A$ such that $M \cap H = N \cap H$ and $\bar{h}_i \nsim_{\bar{A}} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}$ of $\bar{A} = A/M$. We form $\bar{G}$ as in Case 1. Since $\{\bar{x}\}^{\bar{B}} \cap \bar{K} = \emptyset$ and $\bar{h}_i \nsim_{\bar{A}} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}$, we have $\{\bar{x}\}^{\bar{G}} \cap \bar{K} = \emptyset$ and the result follows as in Case 1.

Case 3. $x \notin A \cup B$. WLOG we assume $x = a_1 b_1 \ldots a_n b_n$ where $a_i \in A \backslash H, b_i \in B \backslash H$ for $1 \leq i \leq n$. By (i) and (ii), there exist $M \lhd_f A, N \lhd_f B$ such that $a_i \notin HM, b_i \notin HN$ for all i and $M \cap H = N \cap H$. We form $\bar{G}$ as in Case 1. Since $||\bar{x}|| = ||x||$, we have $\{\bar{x}\}^{\bar{G}} \cap \bar{K} = \emptyset$ and the result follows as in Case 1.

Lemma 9.13. *Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji}\rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating the subgroups H_{ij} of A_i and H_{ji} of A_j. Let $K < A_r$. Suppose*

(i) A_i is H_{ij}-separable;

(ii) for each $R_{ij} \lhd_f H_{ij}$, there exists $M_i \lhd_f A_i$ such that $M_i \cap H_{ij} = R_{ij}$ and $\bar{h}_i \nsim_{\bar{A}_i} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}_{ij}$ of $\bar{A}_i = A_i/M_i$;

(iii) A_i is H_{ij}-conjugacy separable;

(iv) A_r is K-conjugacy separable.

Then G is K-conjugacy separable.

Proof. We use induction on n. The case when $n = 2$ follows from Lemma 9.12. Let $n > 2$. The tree product G has an extremal vertex, say A_n, which is joined to a unique vertex, say A_{n-1}. The subgroup of G generated by $A_1, A_2, \ldots, A_{n-1}$ is just the tree product of themselves. Let A denote this subgroup. Then we write $G = A *_H A_n$ where $H = H_{(n-1)n} = H_{n(n-1)}$. By Lemma 9.9, we have A is $H_{(n-1)n}$-separable. By Lemma 9.11, for each $R \lhd_f H_{(n-1)n}$, there exists $M \lhd_f A$ such that $M \cap H_{(n-1)n} = R$ and $\bar{h}_i \nsim_{\bar{A}} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}_{(n-1)n}$ in $\bar{A} = A/M$. By induction, A is $H_{(n-1)n}$-conjugacy separable. By (i), A_n is $H_{n(n-1)}$-separable. By (ii), for each $S \lhd_f H_{n(n-1)}$, there exists $N \lhd_f A_n$ such that $N \cap H_{n(n-1)} = S$ and $\bar{h}_i \nsim_{\bar{A}_n} \bar{h}_j$ for $\bar{h}_i \neq \bar{h}_j$ in $\bar{H}_{n(n-1)}$ where $\bar{A}_n = A_n/N$. By (iii), A_n is $H_{n(n-1)}$-conjugacy separable.

Case 1. $K < A_r < A$. By the induction hypothesis, A is K-conjugacy separable. Then G is K-conjugacy separable by Lemma 9.12.

Case 2. $K < A_n$. By (iv), A_n is K-conjugacy separable. Then G is K-conjugacy separable by Lemma 9.12.

Lemma 9.14. *Let A be a subgroup separable group and H, K be finitely generated subgroups of $Z(A)$ such that $H \cap K = 1$. Then for each $R \lhd_f H$, there exists $M \lhd_f A$ such that $M \cap H = R$ and $HM \cap KM = M$.*

Proof. Let $R \lhd_f H$ be given. Since A is subgroup separable and R is finitely generated in $Z(A)$, we have $\bar{A} = A/R$ is residually finite. Since $\bar{H} = H/R$ is finite and $\bar{H} \cap \bar{K} = \bar{1}$, there exists $\bar{M} \lhd_f \bar{A}$ such that $\bar{M} \cap \bar{H} = \bar{1}$ and $\bar{H}\bar{M} \cap \bar{K}\bar{M} = \bar{M}$. Let M be the preimage of $\bar{M}$ in A. Then $M \lhd_f A$ such that $M \cap H = R$ and $HM \cap KM = M$ as required.

Lemma 9.15. *(Wong & Tang, 1999) Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating the subgroups H_{ij} of $Z(A_i)$ and H_{ji} of $Z(A_j)$, where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Let $A_1, A_2, \ldots, A_n$ be residually finite groups and K_1, K_2 be finitely generated subgroups of $Z(A_r), Z(A_s)$, respectively, such that $K_1 \cap H_{ri} = 1 = K_2 \cap H_{si}$. Suppose that*

(a) *A_i is H_{ij}-separable, A_r is K_1-separable, and A_s is K_2-separable;*

(b) *for each $R_{ijk} \lhd_f H_{ij}$, there exists $M_{ijk} \lhd_f A_i$ such that $M_{ijk} \cap H_{ij} = R_{ijk}$ and $H_{ij}M_{ijk} \cap H_{ik}M_{ijk} = M_{ijk}$;*

(c) *for each $R_{ri} \lhd_f H_{ri}$, there exists $M_{ri} \lhd_f A_r$ such that $M_{ri} \cap H_{ri} = R_{ri}$ and $K_1M_{ri} \cap H_{ri}M_{ri} = M_{ri}$;*

(d) *for each $R_{si} \lhd_f H_{si}$, there exists $M_{si} \lhd_f A_s$ such that $M_{si} \cap H_{si} = R_{si}$ and $K_2M_{si} \cap H_{si}M_{si} = M_{si}$;*

(e) *for $u \notin J_1vJ_2$ where $u, v \in A_i$ and $J_1, J_2 \subseteq Z(A_i)$, there exists $L_i \lhd_f A_i$ such that $\bar{u} \notin \bar{J_1}\bar{v}\bar{J_2}$ in $\bar{A_i} = A_i/L_i$;*

(f) *for each $R_{ij} \lhd_f H_{ij}$, there exists $N_i \lhd_f A_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Let $x \notin K_1yK_2$, where $x, y \in G$. Then there exists $P \lhd_f G$ such that $\bar{x} \notin \bar{K_1}\bar{y}\bar{K_2}$ in $\bar{G} = G/P$.

Lemma 9.16. *(Wong & Tang, 1999) Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating subgroups H_{ij} of $Z(A_i)$ and H_{ji} of $Z(A_j)$. Let K be a subgroup of $Z(A_r)$ such that $H_{ri} \cap K = 1$. Suppose that*

(a) A_i *is* H_{ij}*-separable;*

(b) *for each* $S \lhd_f K$*, there exists* $M \lhd_f A_r$ *such that* $M \cap K = S$ *and* $H_{ri}M \cap KM = M$*;*

(c) *for each* $R_{ij} \lhd_f H_{ij}$*, there exists* $M_i \lhd_f A_i$ *such that* $M_i \cap H_{ij} = R_{ij}$*.*

Then $K^z \cap K = 1$ *for all* $z \in G\backslash A_r$ *and for each* $S \lhd_f K$ *and* $x, y \in G\backslash A_r$*, there exists* $P \lhd_f G$ *such that* $P \cap K \subseteq S$ *and* $K^x P \cap KP = P, K^y P \cap KP = P$*.*

Lemma 9.17. *Let* $G = \langle A_1, A_2, \ldots A_n | H_{ij} = H_{ji}\rangle$ *be a tree product of subgroup separable groups* $A_1, A_2, \ldots, A_n$ *amalgamating finitely generated subgroups* H_{ij} *of* $Z(A_i)$ *and* H_{ji} *of* $Z(A_j)$ *where* $H_{ij} \cap H_{ik} = 1$*. Let* $x \nsim_G y$ *where* $x, y \in G$ *are cyclically reduced and* $||x|| = ||y|| \geq 2$*. Then there exists* $P \lhd_f G$ *such that* $\bar{x} \nsim_{\bar{G}} \bar{y}$ *in* $\bar{G} = G/P$*.*

Proof. We use induction on n. The case $n = 2$ follows from Lemma 9.2. Let $n > 2$. The tree product G has an extremal vertex, say A_n, which is joined to a unique vertex, say A_{n-1}. The subgroup of G generated by $A_1, A_2, \ldots, A_{n-1}$ is just their tree product. Let A denote this subgroup. Then we write $G = A *_H A_n$, where $H = H_{(n-1)n} = H_{n(n-1)}$. By induction, for each $a_1 \nsim_A a_2$ for any $a_1, a_2 \in A$, there exists $M \lhd_f A$ such that $a_1 M \nsim_{A/M} a_2 M$.

Let $x = x_1 x_2 \ldots x_r$ and $y = y_1 y_2 \ldots y_r$ be cyclically reduced in G, $r \geq 2$. Since $x \nsim_G y$, the system of equations (1) of Definition 6.2 has no solution in H for all $0 \leq i < r$. Therefore we need to show that, for each i, there exists $N_i \lhd_f G$ such that in $\bar{G}_i = G/N_i$, the corresponding system of equations (1) has no solution in H. Letting N be the intersection of the normal subgroups N_i in G, we have $\bar{x} \nsim_{\bar{G}} \bar{y}$ in $\bar{G} = G/N$ and the result follows. Hence it is sufficient to show the case $i = 0$ in (1) of Definition 6.2.

Since A is $H_{(n-1)n}$-separable and A_n is $H_{n(n-1)}$-separable, there exist $M_1 \lhd_f A$, $M_2 \lhd_f A_n$ such that $x_i, y_i \notin H_{(n-1)n}M_1$ if $x_i, y_i \in A\backslash H_{(n-1)n}$ and $x_j, y_j \notin H_{n(n-1)}M_2$ if $x_j, y_j \in A_n\backslash H_{n(n-1)}$.

Since $x \nsim_G y$, either some equations in (1) of Definition 6.2 has no admissible solution in H or every set of admissible solutions to (1) of Definition 6.2 is incomplete. First suppose there exists some $t, 1 \leq t \leq r$, such that the t-th equation has no admissible solution, that is, $x_t \notin Hy_tH$, where $x_t, y_t \in A$ or $x_t, y_t \in A_n$.

First suppose $x_t, y_t \in A$. By Lemma 9.14 and Lemma 9.15, there exists $T_1 \lhd_f A$ such that $\bar{x}_t \notin \bar{H}_{(n-1)n}\bar{y}_t\bar{H}_{(n-1)n}$ in $\bar{A} = A/T_1$. By Lemma 9.1 and Lemma 9.7, there exist $N_1 \lhd_f A, N_2 \lhd_f A_n$ such that $N_1 \subseteq M_1 \cap T_1, N_2 \subseteq M_2$ and $N_1 \cap H_{(n-1)n} = N_2 \cap H_{n(n-1)}$. Now we form $\bar{G} = \bar{A} *_{\bar{H}} \bar{A}_n$, where $\bar{A} = A/N_1, \bar{A}_n = A_n/N_2$ and $\bar{H} = H_{(n-1)n}N_1/N_1 = H_{n(n-1)}N_2/N_2$. Clearly $\bar{G}$ is a homomorphic image of G. Then $\bar{x}_t \notin \bar{H}_{(n-1)n}\bar{y}_t\bar{H}_{(n-1)n}$ in $\bar{G}$ and hence $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable by Lemma 6.3, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{x}\bar{P} \nsim_{\bar{G}/\bar{P}} \bar{y}\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $xP \nsim_{G/P} yP$.

Now suppose $x_t, y_t \in A_n$. Since $H_{n(n-1)} \subseteq Z(A_n)$, we have $x_t y_t^{-1} \notin H_{n(n-1)}$. Since A_n is $H_{n(n-1)}$-separable, there exists $T_2 \lhd_f A_n$ such that $\bar{x}_t\bar{y}_t^{-1} \notin \bar{H}_{n(n-1)}$ in $\bar{A}_n = A_n/T_2$. This implies that $\bar{x}_t \notin \bar{H}_{n(n-1)}\bar{y}_t\bar{H}_{n(n-1)}$ and thus $\bar{x} \nsim_{\bar{G}} \bar{y}$. Therefore, the result follows as above.

Suppose $a_1, b_1, \ldots, a_r, b_r \in H$ is a set of incomplete admissible solutions to (1) of Definition 6.2. Then we have the following.

$$\begin{aligned} x_1 &= a_1^{-1}y_1b_1 \\ x_2 &= a_2^{-1}y_2b_2 \\ &\vdots \\ x_r &= a_r^{-1}y_rb_r \end{aligned}$$

Recall that $y = y_1y_2\ldots y_r$, where $y_i \in A\backslash H_{(n-1)n}$ or $y_i \in A_n\backslash H_{n(n-1)}$. First suppose that all the y_i from $A\backslash H_{(n-1)n}$ are actually in $A_{n-1}\backslash H_{(n-1)n}$. So each of these y_i commutes with every element of $H_{(n-1)n}$ since $H_{(n-1)n} \subseteq Z(A_{n-1})$. In this case, $x \sim_G y$ if and only if $x = y$. As before, we can find $N_1 \lhd_f A, N_2 \lhd_f A_n$ such that in $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{A}_n$, where $\bar{A} = A/N_1, \bar{A}_n = A_n/N_2$ and $\bar{H} = H_{(n-1)n}N_1/N_1 = H_{n(n-1)}N_2/N_2$, we have $||\bar{x}|| = ||\bar{y}||$ and $\bar{x} \neq \bar{y}$. Hence, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

So we may assume that for at least one i, the y_i from $A \backslash H_{(n-1)n}$ is not in $A_{n-1} \backslash H_{(n-1)n}$. Then by Lemma 9.14 and Lemma 9.16, we have $H^{y_i}_{(n-1)n} \cap H_{(n-1)n} = 1$ and hence the equation $x_i = a_i^{-1} y_i b_i$ has unique solutions a_i^{-1}, b_i. Fixing this i, we consider the next equation $x_{i+1} = a_{i+1}^{-1} y_{i+1} b_{i+1}$ and arrange, if possible, so that $a_{i+1} = b_i$. Continuing this way, we see that this matching must eventually fail at some equation, say $x_j = a_j^{-1} y_j b_j$, where $a_j \neq b_{j-1}$. This equation may be the equation we started with. Furthermore, $y_j \notin A_{n-1} \backslash H_{(n-1)n}$, or otherwise, this y_j commutes with every element of $H_{(n-1)n}$ and we can match $a_j = b_{j-1}$. Again by Lemma 9.14 and Lemma 9.16, we have $H^{y_j}_{(n-1)n} \cap H_{(n-1)n} = 1$ and the equation $x_j = a_j^{-1} y_j b_j$ has unique solutions a_j^{-1}, b_j. Now since H is residually finite, there exists $L \lhd_f H$ such that $a_j b_{j-1}^{-1} \notin L$. Let $R = M_1 \cap M_2 \cap L$. Then $R \lhd_f H$. By Lemma 9.14 and Lemma 9.16, there exists $T_1 \lhd_f A$ such that $T_1 \cap H = R_1 \subseteq R$ and $H^{y_j}_{(n-1)n} T_1 \cap H_{(n-1)n} T_1 = T_1, H^{y_i}_{(n-1)n} T_1 \cap H_{(n-1)n} T_1 = T_1$. Let $N_1 = M_1 \cap T_1$. Then $N_1 \lhd_f A$ such that $N_1 \cap H_{(n-1)n} = R_1$ and $H^{y_j}_{(n-1)n} N_1 \cap H_{(n-1)n} N_1 = N_1, H^{y_i}_{(n-1)n} N_1 \cap H_{(n-1)n} N_1 = N_1$. Now, by Lemma 9.1, we can find $T_2 \lhd_f A_n$ such that $T_2 \cap H_{n(n-1)} = R_1$. Let $N_2 = M_2 \cap T_2$. Then $N_2 \lhd_f A_n$ and $N_2 \cap H_{n(n-1)} = R_1$. As above, we form $\bar{G}$. Then $\bar{H}^{\bar{y}_j}_{(n-1)n} \cap \bar{H}_{(n-1)n} = \bar{1}$, $\bar{H}^{\bar{y}_i}_{(n-1)n} \cap \bar{H}_{(n-1)n} = \bar{1}$ in $\bar{G}$. This implies that both the equations $\bar{x}_j = \bar{a}_j^{-1} \bar{y}_j \bar{b}_j$ and $\bar{x}_i = \bar{a}_i^{-1} \bar{y}_i \bar{b}_i$ have unique solutions. Since $\bar{a}_j \neq \bar{b}_{j-1}$ in $\bar{G}$, the matching of $\bar{a}_j$ with $\bar{b}_{j-1}$ fail at the equation $\bar{x}_j = \bar{a}_j^{-1} \bar{y}_j \bar{b}_j$. Therefore $\bar{x} \nsim_{\bar{G}} \bar{y}$ and our result follows.

9.4 Cyclic Conjugacy Separability of Tree Products

In this section, we show that the tree products of finitely many cyclic conjugacy separable and subgroup separable groups amalgamating finitely generated central subgroups are cyclic conjugacy separable.

Theorem 9.18. *Let $G = \langle A_1, A_2 \ldots A_n | H_{ij} = H_{ji} \rangle$ be a tree product of cyclic conjugacy separable and subgroup separable groups $A_1, A_2, \ldots, A_n$ amalgamating finitely generated subgroups H_{ij} of $Z(A_i)$ and H_{ji} of $Z(A_j)$ where $H_{ij} \cap H_{ik} = 1$. Then G is cyclic conjugacy separable.*

Proof. We use induction on n. The case when $n = 2$ follows from Theorem 9.3. Let $n > 2$. The tree product G has an extremal vertex, say A_n, which is joined to a unique vertex, say A_{n-1}. The subgroup of G generated by $A_1, A_2, \ldots, A_{n-1}$ is just the tree product of themselves. Let A denote this subgroup. Then we write $G = A \ast_H A_n$ where $H = H_{(n-1)n} = H_{n(n-1)}$.

We prove this theorem by using Theorem 6.6. By the induction hypothesis, A is cyclic conjugacy separable and by assumption, A_n is cyclic conjugacy separable. Since $H_{n(n-1)} \subseteq Z(A_n)$ and A_n is $H_{n(n-1)}$-separable, we have A_n is also $H_{n(n-1)}$-conjugacy separable. By Lemma 9.9 and Lemma 9.13, we have A is $H_{(n-1)n}$-separable and $H_{(n-1)n}$-conjugacy separable. Thus, conditions (a), (b) and (c) hold. By Lemma 9.1, Lemma 9.7 and Lemma 9.17, conditions (d) and (e) are satisfied.

Now we only need to prove for condition (f). Let $x, y \in G$ such that $||x|| = 0, ||y|| \leq 1$ and $\{x\}^G \cap \langle y \rangle = \emptyset$.

Case 1. $||x|| = 0 = ||y||$. Clearly, $\{x\}^A \cap \langle y \rangle = \emptyset$ and $\{x\}^{A_n} \cap \langle y \rangle = \emptyset$. Since A and A_n are cyclic conjugacy separable, there exist $M_1 \lhd_f A, M_2 \lhd_f A_n$ such that $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$ and $\{xM_2\}^{A_n/M_2} \cap \langle yM_2 \rangle = \emptyset$. Let $R_{(n-1)n} = R_{n(n-1)} =$

$M_1 \cap M_2 \cap H$. Since $A_1, A_2, \ldots, A_{n-1}$ are subgroup separable and $H_{ij} \subseteq Z(A_{ij})$, by Lemma 9.11, for each $R_{(n-1)n} \lhd_f H_{(n-1)n}$, there exists $Q_1 \lhd_f A$ such that $Q_1 \cap H_{(n-1)n} = R_{(n-1)n}$ and $\hat{h}_i \nsim_{A/Q_1} \hat{h}_j$ for $\hat{h}_i \neq \hat{h}_j$ in $\bar{H}_{(n-1)n}$ of $\hat{A} = A/Q_1$. Since A_n is subgroup separable and $H_{n(n-1)} \subseteq Z(A_n)$ is finitely generated, by Lemma 9.10, for each $R_{n(n-1)} \lhd_f H_{n(n-1)}$, there exists $Q_2 \lhd_f A_n$ such that $Q_2 \cap H_{n(n-1)} = R_{n(n-1)}$ and $\tilde{h}_i \nsim_{A_n/Q_2} \tilde{h}_j$ for $\tilde{h}_i \neq \tilde{h}_j$ in $\bar{H}_{n(n-1)}$ of $\tilde{A}_n = A_n/Q_2$. By Lemma 9.1 and Lemma 9.7, there exist $N_1 \lhd_f A, N_2 \lhd_f A_n$ such that $N_1 \subseteq M_1 \cap Q_1, N_2 \subseteq M_2 \cap Q_2$ and $N_1 \cap H_{(n-1)n} = N_2 \cap H_{n(n-1)}$. Now we form $\bar{G} = \bar{A} \underset{\bar{H}}{*} \bar{A}_n$, where $\bar{A} = A/N_1, \bar{A}_n = A_n/N_2$ and $\bar{H} = H_{(n-1)n}N_1/N_1 = H_{n(n-1)}N_2/N_2$. Note that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$ and $\{\bar{x}\}^{\bar{A}_n} \cap \langle \bar{y} \rangle = \emptyset$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 6.1(a), there exists a finite sequence $\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_w$ where $\bar{h}_i \in \bar{H} = \bar{H}_{(n-1)n} = \bar{H}_{n(n-1)}$ such that $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{A}_n} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{A}(\bar{A}_n)} \bar{h}_w \sim_{\bar{A}_n(\bar{A})} \bar{y}^k$. Without loss of generality, we assume $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{A}_n} \bar{h}_2 \sim_{\bar{A}} \ldots \sim_{\bar{A}_n} \bar{h}_w \sim_{\bar{A}} \bar{y}^k$. Note that each h_i is uniquely determined since $\bar{h}_i \nsim_{\bar{A}} \bar{h}_j$ for all $\bar{h}_i \neq \bar{h}_j$ in $\bar{A}$ and $\bar{h}_u \nsim_{\bar{A}_n} \bar{h}_v$ for all $\bar{h}_u \neq \bar{h}_v$ in $\bar{A}_n$. From the first conjugation relation $\bar{x} \sim_{\bar{A}} \bar{h}_1$ implies that $\bar{x} = \bar{h}_1$ in $\bar{A}$ and hence $xN_1 = h_1N_1$. This implies that $xN_1 = h_1N_1, h_1N_2 = h_2N_2, \ldots, h_wN_1 = y^kN_1$. From the first equality $xN_1 = h_1N_1$, we have $xh_1^{-1} \in N_1 \cap H = N_2 \cap H$ and thus $xN_2 = h_1N_2$. Continuing in this way, we have $xN_2 = h_1N_2 = h_2N_2 = \ldots = h_wN_2 = y^kN_2$ and this implies that $\bar{x} = \bar{h}_1 = \bar{h}_2 = \ldots = \bar{h}_w = \bar{y}^k$. So, $\bar{x} = \bar{y}^k$, that is $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle \neq \emptyset, \{\bar{x}\}^{\bar{A}_n} \cap \langle \bar{y} \rangle \neq \emptyset$, a contradiction. Hence, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$. Since $\bar{G}$ is cyclic conjugacy separable by Lemma 6.3, we can find $\bar{P} \lhd_f \bar{G}$ such that $\{\bar{x}\bar{P}\}^{\bar{G}/\bar{P}} \cap \langle \bar{y}\bar{P} \rangle = \emptyset$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\{xP\}^{G/P} \cap \langle yP \rangle = \emptyset$.

Case 2. $||x|| = 0, ||y|| = 1$. We assume $y \in A \backslash H_{(n-1)n}$. The other case is similar. In this case $\{x\}^G \cap \langle y \rangle = \emptyset$ implies that $\{x\}^A \cap \langle y \rangle = \emptyset$. By the induction hypothesis, we have A is cyclic conjugacy separable. Then there exists $M_1 \lhd_f A$ such that $\{xM_1\}^{A/M_1} \cap \langle yM_1 \rangle = \emptyset$.

Since A is $H_{(n-1)n}$-separable by Lemma 9.9, there exists $L_1 \lhd_f A$ such that $y \notin H_{(n-1)n}L_1$. Let $R_{(n-1)n} = R_{n(n-1)} = M_1 \cap L_1 \cap H_{(n-1)n}$. Then $R_{(n-1)n} \lhd_f H_{(n-1)n}$ and $R_{n(n-1)} \lhd_f H$. Since $A_1, A_2, \ldots, A_{n-1}$ are subgroup separable and $H_{ij} \subseteq Z(A_{ij})$, by Lemma 9.11, for each $R_{(n-1)n} \lhd_f H_{(n-1)n}$, there exists $Q_1 \lhd_f A$ such that $Q_1 \cap H_{(n-1)n} = R_{(n-1)n}$ and $\hat{h}_i \not\sim_{A/Q_1} \hat{h}_j$ for $\hat{h}_i \neq \hat{h}_j$ in $\bar{H}_{(n-1)n}$ of $\hat{A} = A/Q_1$. Since A_n is subgroup separable and $H_{n(n-1)} \subseteq Z(A_n)$, there exists $Q_2 \lhd_f A_n$ such that $Q_2 \cap H_{n(n-1)} = R_{n(n-1)}$ and $\tilde{h}_i \not\sim_{A_n/Q_2} \tilde{h}_j$ for $\tilde{h}_i \neq \tilde{h}_j$ in $\bar{H}_{n(n-1)}$ of $\tilde{A}_n = A_n/Q_2$. By Lemma 9.1 and Lemma 9.7 there exist $N_1 \lhd_f A, N_2 \lhd_f A_n$ such that $N_1 \subseteq M_1 \cap L_1 \cap Q_1, N_2 \subseteq Q_2$ and $N_1 \cap H_{(n-1)n} = N_2 \cap H_{n(n-1)}$. Now we form $\bar{G}$ as above. Note that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$ and $\bar{y} \notin \bar{H}$.

Suppose $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle \neq \emptyset$. Then $\bar{x} \sim_{\bar{G}} \bar{y}^k$ for some integer k. By Lemma 6.1(a), there exists a finite sequence $\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_w$ where $\bar{h}_i \in \bar{H} = \bar{H}_{(n-1)n} = \bar{H}_{n(n-1)}$ such that $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{A}_n} \bar{h}_2 \sim_{\bar{A}} \cdots \sim_{\bar{A}(\bar{A}_n)} \bar{h}_w \sim_{\bar{A}_n(\bar{A})} \bar{y}^k$. Since $y \in A \backslash H$, we have $\bar{x} \sim_{\bar{A}} \bar{h}_1 \sim_{\bar{A}_n} \bar{h}_2 \sim_{\bar{A}} \cdots \sim_{\bar{A}_n} \bar{h}_w \sim_{\bar{A}} \bar{y}^k$. Arguing as in Case 1, we have $\bar{x} = \bar{h}_1 = \bar{h}_2 = \ldots = \bar{h}_w$. Thus, $\bar{x} \sim_{\bar{A}} \bar{y}^k$, a contradiction to the fact that $\{\bar{x}\}^{\bar{A}} \cap \langle \bar{y} \rangle = \emptyset$. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{y} \rangle = \emptyset$ and the result follows as in Case 1.

This completes the proof and hence G is cyclic conjugacy separable by Theorem 6.6.

Corollary 9.19. *Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of free-by-finite groups $A_1, A_2, \ldots, A_n$ amalgamating finitely generated subgroups H_{ij} of $Z(A_i)$ and H_{ji} of $Z(A_j)$ where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Then G is cyclic conjugacy separable.*

Corollary 9.20. *Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of finitely generated abelian groups $A_1, A_2, \ldots, A_n$ amalgamating subgroups H_{ij} of A_i and H_{ji} of A_j where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Then G is cyclic conjugacy separable.*

CHAPTER 10: CONCLUSION

10.1 Conclusion and Brief Summary

In this thesis, we have extended conjugacy separability and cyclic conjugacy separability to generalized free products and HNN extensions subject to certain conditions. These results are new although the conditions, though some what restrictive, are present in finitely generated nilpotent groups.

The property of cyclic conjugacy separability played an important role in the proof of the conjugacy separability in generalized free products and HNN extensions by Dyer (1980), Kim & Tang (1996), Kim & Tang (1999) and others. We feel that cyclic conjugacy separability deserves more attention as this property is the starting point in the study of H-conjugacy separability for finitely generated subgroups H. We have used this property in Chapter 4 and Chapter 8. The property of H-conjugacy separability is like subgroup separability, is difficult to be proved.

To put our research in perspective, we give an example of a generalized free product cum HNN extension which is not even residually finite. Let $A = \langle t, a | t^{-1}at = a^2 \rangle$ and $B = \langle b \rangle$. Then A and B are conjugacy separable groups (Kim et al., 1995). Let G be the generalized free products of A and B amalgamating the cyclic subgroup $\langle a \rangle$ of A and $\langle b^2 \rangle$ of B. Then $G = A \underset{a=b^2}{*} B = \langle t, b | t^{-1}b^2t = b^4 \rangle$ is not even residually finite (Meskin, 1972).

10.2 Some Ongoing Work

In Chapter 4 and Chapter 8, we have studied the groups $G_1 = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where $H = P \times C, K = Q \times C$ where P, Q are finite, $P \cap Q = 1$ and $C \subseteq Z(A)$ and $G_2 = A \underset{H}{*} B$ where $H = K \times D$ where K is finite and $D \subseteq Z(A) \cap Z(B)$.

In 2013, Kim and Tang (2013) showed that $G = A \ast_H B$ with $H = \langle h \rangle \times C$ where $|h| = \infty$ and $C \subseteq Z(A) \cap Z(B)$, C is finite, is conjugacy separable. We propose to show that G is also cyclic conjugacy separable. The following two lemmas have been proved by Asri et al., (2019) and Asri et al., (2020).

Lemma 10.1. *(Asri et al., 2019) Let A be a group with subgroup $H = \langle h \rangle \times C$ such that $|h| = \infty$ and C is finite. If A is $\langle h \rangle$-separable, then A is H-separable.*

Lemma 10.2. *(Asri et al., 2020) Let $G = A \ast_H B$ where $H = \langle h \rangle \times C$ such that $|h| = \infty$ and C is finite. Suppose A and B are $\langle h \rangle$-separable and $\langle h \rangle$-weakly potent. Then for each $R \lhd_f H$, there exist $M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H \subseteq R$.*

We note that Lemma 10.1 and Lemma 10.2 satisfy conditions (c) and (d) of Theorem 6.6. If we assume that A and B are both cyclic conjugacy separable, then we have condition (a). Since G is conjugacy separable, condition (e) can be easily shown. We only need to work on conditions (b) and (f) here. This will complete the proof.

Similarly, we can further investigate on the HNN extension, $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where $H = \langle h \rangle \times C, K = \langle k \rangle \times D, C, D \subseteq Z(A), C \cap D = 1$ such that $|h| = \infty, |k| = \infty$ and C, D are finite. We can apply Theorem 2.12 and Theorem 2.14 to investigate if this type of HNN extension is both conjugacy separable and cyclic conjugacy separable. Again we have the following from Asri et al., (2019).

Lemma 10.3. *(Asri et al., 2019) Let A be a group with subgroups $H = \langle h \rangle \times C$ and $K = \langle k \rangle \times D$ such that $|h| = \infty, |k| = \infty$ and C, D are finite subgroups. Suppose that A is $\langle h \rangle$-separable and $\langle k \rangle$-separable, and $\phi : H \to K$ is an isomorphism such that $\phi(h) = k$ and $\phi(C) = D$. Suppose*

(1) $h \sim_A k$; or

(2) A is $\langle h \rangle$-weakly potent, $\langle k \rangle$-weakly potent and $h^m = k^{\pm m}$ for some $m \in \mathbb{Z}^+$.

Then for any $M \lhd_f A$, there exists $N \lhd_f A$ such that $N \subseteq M$ and $\phi(N \cap H) = N \cap K$.

Lemma 10.1 and Lemma 10.3 satisfy conditions (b) and (c) of both Theorem 2.12 and Theorem 2.14. Again we assume that A is both conjugacy separable and cyclic conjugacy separable. Then, we have condition (a). We first need to prove condition (d) of Theorem 2.12 and Theorem 2.14, then finally we work on condition (e) of Theorem 2.14. This will complete the proof.

The proof given in Kim & Tang (2013) are difficult and complex. They showed many interesting and deep properties in those groups in Kim & Tang, (2013). We are currently studying the techniques developed in this and other papers.

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2. **Lim, H. M.**, Wong, K. B., & Wong, P. C. (2015). Criteria for conjugacy separability and residual finiteness. *The 22nd National Symposium on Mathematical Sciences (SKSM22): Strengthening Research and Collaboration of Mathematical Sciences in Malaysia, 1682*, Article♯040001.

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