COMMUTING ADDITIVE MAPS ON TENSOR PRODUCTS OF MATRIX ALGEBRAS

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FACULTY OF SCIENCE
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COMMUTING ADDITIVE MAPS ON TENSOR PRODUCTS OF MATRIX ALGEBRAS

ABSTRACT

Let $k \geqslant 1$ and $n_1, \ldots, n_k \geqslant 2$ be integers. Let $\mathbb{F}$ be a field and let $\mathcal{M}_{n_i}$ be the algebra of $n_i \times n_i$ matrices over $\mathbb{F}$ for $i = 1, \ldots, k$. Let $\bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ be the tensor product of $\mathcal{M}_{n_1}, \ldots, \mathcal{M}_{n_k}$. In this dissertation, we obtain a complete structural characterization of additive maps $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ satisfying

$$\psi(\otimes_{i=1}^{k} A_i)(\otimes_{i=1}^{k} A_i) = (\otimes_{i=1}^{k} A_i)\, \psi(\otimes_{i=1}^{k} A_i)$$

for all $A_1 \in \mathcal{S}_{1,n_1}, \ldots, A_k \in \mathcal{S}_{k,n_k}$, where

$$\mathcal{S}_{i,n_i} = \left\{ E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)} : \alpha \in \mathbb{F} \text{ and } 1 \leqslant p, q, s, t \leqslant n_i \text{ are not all distinct integers} \right\}$$

and $E_{st}^{(n_i)}$ is the standard matrix unit in $\mathcal{M}_{n_i}$ for $i = 1, \ldots, k$. In particular, we show that $\psi : \mathcal{M}_{n_1} \to \mathcal{M}_{n_1}$ is an additive map commuting on $\mathcal{S}_{1,n_1}$ if and only if there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : \mathcal{M}_{n_1} \to \mathbb{F}$ such that

$$\psi(A) = \lambda A + \mu(A) I_{n_1}$$

for all $A \in \mathcal{M}_{n_1}$, where $I_{n_1} \in \mathcal{M}_{n_1}$ is the identity matrix. As an application, we classify additive maps $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ satisfying $\psi(\otimes_{i=1}^{k} A_i)(\otimes_{i=1}^{k} A_i) = (\otimes_{i=1}^{k} A_i)\, \psi(\otimes_{i=1}^{k} A_i)$ for all $A_1 \in \mathcal{R}_{r_1}^{n_1}, \ldots, A_k \in \mathcal{R}_{r_k}^{n_k}$. Here, $\mathcal{R}_{r_i}^{n_i}$ denotes the set of rank r_i matrices in $\mathcal{M}_{n_i}$ and $1 < r_i \leqslant n_i$ is a fixed integer such that $r_i \neq n_i$ when $n_i = 2$ and $|\mathbb{F}| = 2$ for $i = 1, \ldots, k$.

Keywords: commuting map, tensor product of matrices, rank, functional identity, linear preserver problem.

PEMETAAN BERDAYA TAMBAH KALIS TUKAR TERTIB PADA HASIL DARAB TENSOR ALGEBRA MATRIKS

ABSTRAK

Biar $k \geqslant 1$ dan $n_1, \ldots, n_k \geqslant 2$ integer. Biar $\mathbb{F}$ suatu medan dan biar $\mathcal{M}_{n_i}$ algebra bagi matriks $n_i \times n_i$ terhadap $\mathbb{F}$ bagi $i = 1, \ldots, k$. Biar $\bigotimes_{i=1}^k \mathcal{M}_{n_i}$ menandakan hasil darab tensor bagi $\mathcal{M}_{n_1}, \ldots, \mathcal{M}_{n_k}$. Dalam disertasi ini, kami memperolehi suatu pencirian berstruktur lengkap bagi pemetaan berdaya tambah $\psi : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \bigotimes_{i=1}^k \mathcal{M}_{n_i}$ yang memenuhi

$$\psi(\otimes_{i=1}^k A_i)(\otimes_{i=1}^k A_i) = (\otimes_{i=1}^k A_i)\,\psi(\otimes_{i=1}^k A_i)$$

bagi semua $A_1 \in \mathcal{S}_{1,n_1}, \ldots, A_k \in \mathcal{S}_{k,n_k}$, di mana

$$\mathcal{S}_{i,n_i} = \left\{E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)} : \alpha \in \mathbb{F} \text{ dan } 1 \leqslant p,q,s,t \leqslant n_i \text{ bukan semua integer berbeza}\right\}$$

dan $E_{st}^{(n_i)}$ merupakan unit matriks piawai dalam $\mathcal{M}_{n_i}$ bagi $i = 1, \ldots, k$. Khususnya, kami membuktikan bahawa $\psi : \mathcal{M}_{n_1} \to \mathcal{M}_{n_1}$ merupakan pemetaan berdaya tambah kalis tukar tertib pada $\mathcal{S}_{1,n_1}$ jika dan hanya jika wujudnya suatu skalar $\lambda \in \mathbb{F}$ dan suatu pemetaan berdaya tambah $\mu : \mathcal{M}_{n_1} \to \mathbb{F}$ supaya

$$\psi(A) = \lambda A + \mu(A) I_{n_1}$$

bagi semua $A \in \mathcal{M}_{n_1}$, di mana $I_{n_1} \in \mathcal{M}_{n_1}$ adalah matriks identiti. Sebagai aplikasi, kami mengelaskan pemetaan berdaya tambah $\psi : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \bigotimes_{i=1}^k \mathcal{M}_{n_i}$ yang memenuhi $\psi(\otimes_{i=1}^k A_i)(\otimes_{i=1}^k A_i) = (\otimes_{i=1}^k A_i)\,\psi(\otimes_{i=1}^k A_i)$ bagi semua $A_1 \in \mathcal{R}_{r_1}^{n_1}, \ldots, A_k \in \mathcal{R}_{r_k}^{n_k}$. Di sini, $\mathcal{R}_{r_i}^{n_i}$ mewakili set bagi matriks berpangkat r_i dalam $\mathcal{M}_{n_i}$ dan $1 < r_i \leqslant n_i$ merupakan suatu integer tetap dengan $r_i \neq n_i$ apabila $n_i = 2$ dan $|\mathbb{F}| = 2$ bagi $i = 1, \ldots, k$.

Kata kunci: pemetaan kalis tukar tertib, hasil darab tensor matriks, pangkat, identiti fungsian, masalah pengekal linear.

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LIST OF SYMBOLS

$\mathcal{M}_n(\mathbb{F})$	:	the algebra of $n \times n$ matrices over the field $\mathbb{F}$
$\lvert\mathbb{F}\rvert$	:	the cardinality of the field $\mathbb{F}$
$[a, b]$	:	the commutator $ab - ba$ of the elements a and b
$f \circ g$	:	the composition of the maps f and g
$\det A$	:	the determinant of the matrix A
$\mathbb{C}$	:	the field of complex numbers
$\mathcal{A} \cong \mathcal{B}$	:	the isomorphic algebras $\mathcal{A}$ and $\mathcal{B}$
I_n	:	the $n \times n$ identity matrix
$E_{ij}^{(n)}$	:	the $n \times n$ standard matrix unit whose (i, j)th entry is one and zero elsewhere
$S \setminus T$	:	the set difference of the sets S and T
$\otimes_{i=1}^n v_i$	:	the tensor product of the elements $v_1, \ldots, v_n$
$\bigotimes_{i=1}^n \mathcal{V}_i$	:	the tensor product of the linear spaces or algebras $\mathcal{V}_1, \ldots, \mathcal{V}_n$
$\operatorname{tr} A$	:	the trace of the matrix A
A^t	:	the transpose of the matrix A

CHAPTER 1: INTRODUCTION

1.1 Background of the Study

Linear preserver problems have a long history in matrix mathematics. This research area began in 1897 when Frobenius first studied the determinant preservers. It poses new challenges to researchers and motivates a good source of intriguing research problems in matrix mathematics. Moreover, the solutions of the identified linear preserver problems are usually simple and elegant. This makes linear preserver problems to remain attractive for decades. Many interesting linear preserver problems have been attempted and some of the linear preserver results have been extended or generalized until today.

Lately, the study of linear preserver problems in quantum information science has been related to tensor products of matrices. The new research problems possess unique features which distinguish them from classical linear preserver problems and have now inspired a new line of active research in linear preserver problems on tensor products of matrices. Furthermore, Brešar (2016a) recently initiated the study of functional identities on tensor products of algebras. Motivated by the study, together with the inspiration of the study of linear preserver problems on tensor products of matrices from quantum information science, we study commuting additive maps on tensor products of matrix algebras.

1.2 Objectives of the Study

The main objectives of this study are:

(a) to characterize commuting additive maps on tensor products of matrices of the form $E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)}$, where $E_{st}^{(n_i)}$ is the $n_i \times n_i$ standard matrix unit, and

(b) to classify commuting additive maps on tensor products of fixed-rank matrices.

The following are the main questions in this study.

(a) Are commuting additive maps on tensor products of matrices of the form $E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)}$ necessarily of the standard form?

(b) Are commuting additive maps on tensor products of fixed-rank matrices necessarily of the standard form?

1.3 Significance of the Study

This study aims to facilitate the advancement of the existing knowledge in the study of linear preserver problems on tensor products of matrices which arises from quantum information science. With the results developed as well as the techniques implemented in this study, several existing results on commuting maps may be further extended or generalized. Some other relevant linear preserver problems may also be reduced and solved by applying the results and techniques established from this study.

1.4 Organisation of the Dissertation

In Chapter 2, we begin with some preliminaries on tensor products and Kronecker product of matrices which will be useful in the following chapters. We then continue with a literature review of this study and the methodology employed in this study.

Chapter 3 is devoted to the study of commuting additive maps on tensor products of matrices of the form $E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)}$. We first derive a few preliminary results before we prove the main results.

Chapter 4 is devoted to the study of commuting additive maps on tensor products of fixed-rank matrices. As in the preceding chapter, the main results will follow after the preliminary results.

In Chapter 5, we provide a summary of the findings in this study and suggest some potential open problems that may be considered for future research work.

CHAPTER 2: LITERATURE REVIEW AND METHODOLOGY

2.1 Tensor Products of Linear Spaces

Throughout this section, the linear spaces are always assumed to be finite dimensional.

Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ and $\mathcal{W}$ be linear spaces over $\mathbb{F}$. A map $\phi : \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{W}$ is called **multilinear** if for each integer $1 \leqslant j \leqslant n$, the following conditions are satisfied:

$$\phi(v_1, \ldots, u_j + v_j, \ldots, v_n) = \phi(v_1, \ldots, u_j, \ldots, v_n) + \phi(v_1, \ldots, v_j, \ldots, v_n),$$

$$\phi(v_1, \ldots, \lambda v_j, \ldots, v_n) = \lambda \phi(v_1, \ldots, v_j, \ldots, v_n)$$

for all $v_1 \in \mathcal{V}_1, \ldots, u_j, v_j \in \mathcal{V}_j, \ldots, v_n \in \mathcal{V}_n$ and $\lambda \in \mathbb{F}$. In particular, the map ϕ is called **bilinear** when $n = 2$.

Definition 2.1.1. Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be linear spaces over the same field $\mathbb{F}$. A **tensor product** of $\mathcal{V}_1, \ldots, \mathcal{V}_n$ is a pair $(\mathcal{T}, \otimes)$, consisting of a linear space $\mathcal{T}$ over $\mathbb{F}$ and a multilinear map $\otimes : \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{T}$, which satisfies the following condition:

(**Universal Factorization Property**) If $\mathcal{W}$ is any linear space over $\mathbb{F}$ and $\phi : \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{W}$ is any multilinear map, then there exists a unique linear map $\psi : \mathcal{T} \to \mathcal{W}$ such that $\phi = \psi \circ \otimes$.

The following results show the existence and uniqueness of tensor products.

Proposition 2.1.2. (Gallier & Quaintance, 2020, Theorem 2.6) *Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be linear spaces over the same field. Then a tensor product $(\mathcal{T}, \otimes)$ of $\mathcal{V}_1, \ldots, \mathcal{V}_n$ always exists.*

Proposition 2.1.3. (Gallier & Quaintance, 2020, Proposition 2.5) *Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be linear spaces over the same field. If $(\mathcal{T}_1, \otimes_1)$ and $(\mathcal{T}_2, \otimes_2)$ are tensor products of $\mathcal{V}_1, \ldots, \mathcal{V}_n$, then there exists a linear isomorphism $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ such that $\Phi \circ \otimes_1 = \otimes_2$.*

In view of Proposition 2.1.3, the tensor space $\mathcal{T}$ in any tensor product $(\mathcal{T}, \otimes)$ of linear spaces $\mathcal{V}_1, \ldots, \mathcal{V}_n$ over a field $\mathbb{F}$ is essentially unique up to isomorphism. We thus denote the tensor space $\mathcal{T}$ over $\mathbb{F}$ by

$$\bigotimes_{i=1}^{n} \mathcal{V}_i \quad \text{or} \quad \mathcal{V}_1 \bigotimes \cdots \bigotimes \mathcal{V}_n. \tag{2.1}$$

The tensor space $\bigotimes_{i=1}^{n} \mathcal{V}_i$ is also called the **tensor product of linear spaces** $\mathcal{V}_1, \ldots, \mathcal{V}_n$. The image of the multilinear map $\otimes(v_1, \ldots, v_n)$, with $v_1 \in \mathcal{V}_1, \ldots, v_n \in \mathcal{V}_n$, is denoted by

$$\otimes_{i=1}^{n} v_i \quad \text{or} \quad v_1 \otimes \cdots \otimes v_n. \tag{2.2}$$

The elements in $\bigotimes_{i=1}^{n} \mathcal{V}_i$ are called **tensors** and the tensors of the form $\otimes_{i=1}^{n} v_i$ are called **decomposable tensors**. We denote by $D(\bigotimes_{i=1}^{n} \mathcal{V}_i) = \{\otimes_{i=1}^{n} v_i : v_1 \in \mathcal{V}_1, \ldots, v_n \in \mathcal{V}_n\}$ the set of all decomposable tensors in $\bigotimes_{i=1}^{n} \mathcal{V}_i$.

The result below gives a basis of $\bigotimes_{i=1}^{n} \mathcal{V}_i$ which is formed by decomposable tensors.

Proposition 2.1.4. (Gallier & Quaintance, 2020, Proposition 2.12) *Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be linear spaces over the same field $\mathbb{F}$. If $\mathscr{B}_1, \ldots, \mathscr{B}_n$ are bases of $\mathcal{V}_1, \ldots, \mathcal{V}_n$, respectively, then $\{\otimes_{i=1}^{n} b_i : b_1 \in \mathscr{B}_1, \ldots, b_n \in \mathscr{B}_n\}$ forms a basis of $\bigotimes_{i=1}^{n} \mathcal{V}_i$ and $\dim(\bigotimes_{i=1}^{n} \mathcal{V}_i) = \dim \mathcal{V}_1 \times \cdots \times \dim \mathcal{V}_n$. Here, $\dim \mathcal{V}$ denotes the dimension of the linear space $\mathcal{V}$ over $\mathbb{F}$.*

It follows from Proposition 2.1.4 that $\bigotimes_{i=1}^{n} \mathcal{V}_i = \langle D(\bigotimes_{i=1}^{n} \mathcal{V}_i)\rangle$ the linear span of $D(\bigotimes_{i=1}^{n} \mathcal{V}_i)$. In other words, we have

$$\bigotimes_{i=1}^{n} \mathcal{V}_i = \langle \otimes_{i=1}^{n} v_i : v_1 \in \mathcal{V}_1, \ldots, v_n \in \mathcal{V}_n \rangle . \tag{2.3}$$

Therefore each tensor $X \in \bigotimes_{i=1}^{n} \mathcal{V}_i$ can be represented by a sum $X = \sum_p x_{1p} \otimes \cdots \otimes x_{np}$ of a finite number of decomposable tensors in $\bigotimes_{i=1}^{n} \mathcal{V}_i$.

The following propositions provide more basic results on tensor products.

Proposition 2.1.5. (Conrad, n.d., Theorems 5.11 and 5.15) *Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be linear spaces over the same field $\mathbb{F}$. Then the following hold.*

(a) *Let $v_1 \in \mathcal{V}_1, \ldots, v_n \in \mathcal{V}_n$. Then $\otimes_{i=1}^{n} v_i = 0$ if and only if $v_i = 0$ for some integer $1 \leqslant i \leqslant n$. Equivalently, $\otimes_{i=1}^{n} v_i \neq 0$ if and only if $v_i \neq 0$ for $i = 1, \ldots, n$.*

(b) *Let $u_1, v_1 \in \mathcal{V}_1, \ldots, u_n, v_n \in \mathcal{V}_n$ be nonzero. Then $\otimes_{i=1}^{n} u_i = \otimes_{i=1}^{n} v_i$ if and only if for each integer $1 \leqslant i \leqslant n$, $v_i = \lambda_i u_i$ for some nonzero $\lambda_i \in \mathbb{F}$ with $\lambda_1 \cdots \lambda_n = 1$.*

Proposition 2.1.6. (Brešar, 2014, Lemma 4.8 and Corollary 4.13) *Let m be a positive integer. Let $\mathcal{U}$ and $\mathcal{V}$ be linear spaces over the same field. Then the following hold.*

(a) *If $u_1, \ldots, u_m \in \mathcal{U}$ are linearly independent and $v_1, \ldots, v_m \in \mathcal{V}$, then $\sum_{i=1}^{m} u_i \otimes v_i = 0$ implies that $v_i = 0$ for $i = 1, \ldots, m$.*

(b) *If $\{u_1, \ldots, u_m\}$ is a basis of $\mathcal{U}$, then for each $w \in \mathcal{U} \bigotimes \mathcal{V}$, there exist unique vectors $v_1, \ldots, v_m$ in $\mathcal{V}$ such that $w = \sum_{i=1}^{m} u_i \otimes v_i$.*

2.2 Tensor Products of Algebras

Recall that an **associative algebra**, or simply **algebra**, over a field is a linear space over the field that is endowed with an associative bilinear multiplication. A **unital algebra** $\mathcal{A}$ is an algebra that contains an element 1 such that $1a = a = a1$ for all $a \in \mathcal{A}$. The unique element 1 is called the **unity** of $\mathcal{A}$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be algebras over the same field $\mathbb{F}$. The tensor space $\bigotimes_{i=1}^{n} \mathcal{A}_i$ can be turned into an algebra over $\mathbb{F}$ by defining multiplication in a simple and natural way.

We believe the following theorem is known. Nevertheless, a proof is included for completeness. It is a generalisation of Proposition 2.22 in Gallier and Quaintance (2020).

Theorem 2.2.1. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be algebras over the same field $\mathbb{F}$. Then $\bigotimes_{i=1}^{n} \mathcal{A}_i$ forms an algebra over $\mathbb{F}$ relative to the multiplication determined by*

$$(\otimes_{i=1}^{n} x_i)(\otimes_{i=1}^{n} y_i) = \otimes_{i=1}^{n} x_i y_i \tag{2.4}$$

for every $x_1, y_1 \in \mathcal{A}_1, \ldots, x_n, y_n \in \mathcal{A}_n$. Moreover, if each $\mathcal{A}_i$ is a unital algebra with unity u_i, then $\bigotimes_{i=1}^{n} \mathcal{A}_i$ is a unital algebra with unity $\otimes_{i=1}^{n} u_i$.

Proof. Let $\Psi : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \times \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \bigotimes_{i=1}^{n} \mathcal{A}_i$ be the function defined by

$$\Psi(x_1, \ldots, x_n, y_1, \ldots, y_n) = \otimes_{i=1}^{n} x_i y_i$$

for every $x_1, y_1 \in \mathcal{A}_1, \ldots, x_n, y_n \in \mathcal{A}_n$. Then Ψ is a multilinear map as a result of the multilinearity of $\otimes$. By the universal factorization property, there exists a linear map

$\xi : \mathcal{A}_1 \bigotimes \cdots \bigotimes \mathcal{A}_n \bigotimes \mathcal{A}_1 \bigotimes \cdots \bigotimes \mathcal{A}_n \to \bigotimes_{i=1}^n \mathcal{A}_i$ such that $\xi \circ \otimes = \Psi$. Therefore

$$\xi(x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n) = \otimes_{i=1}^n x_i y_i$$

for all $x_1, y_1 \in \mathcal{A}_1, \ldots, x_n, y_n \in \mathcal{A}_n$. By virtue of the associativity isomorphism in Proposition 2.13 of Gallier and Quaintance (2020), there exists a linear isomorphism $\phi : (\bigotimes_{i=1}^n \mathcal{A}_i) \bigotimes (\bigotimes_{i=1}^n \mathcal{A}_i) \to \mathcal{A}_1 \bigotimes \cdots \bigotimes \mathcal{A}_n \bigotimes \mathcal{A}_1 \bigotimes \cdots \bigotimes \mathcal{A}_n$ such that

$$\phi((\otimes_{i=1}^n x_i) \otimes (\otimes_{i=1}^n y_i)) = x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n$$

for all $x_1, y_1 \in \mathcal{A}_1, \ldots, x_n, y_n \in \mathcal{A}_n$. It follows that

$$\xi'((\otimes_{i=1}^n x_i) \otimes (\otimes_{i=1}^n y_i)) = \otimes_{i=1}^n x_i y_i$$

for all $x_1, y_1 \in \mathcal{A}_1, \ldots, x_n, y_n \in \mathcal{A}_n$, where $\xi' = \xi \circ \phi$ is a linear map from $(\bigotimes_{i=1}^n \mathcal{A}_i) \bigotimes (\bigotimes_{i=1}^n \mathcal{A}_i)$ into $\bigotimes_{i=1}^n \mathcal{A}_i$. By (2.2), we obtain

$$(\xi' \circ \otimes)(\otimes_{i=1}^n x_i, \otimes_{i=1}^n y_i) = \otimes_{i=1}^n x_i y_i$$

for all $x_1, y_1 \in \mathcal{A}_1, \ldots, x_n, y_n \in \mathcal{A}_n$. Here, the map $\otimes : \bigotimes_{i=1}^n \mathcal{A}_i \times \bigotimes_{i=1}^n \mathcal{A}_i \to (\bigotimes_{i=1}^n \mathcal{A}_i) \bigotimes (\bigotimes_{i=1}^n \mathcal{A}_i)$ is a bilinear map. By defining $\Phi = \xi' \circ \otimes$, we see that $\Phi : \bigotimes_{i=1}^n \mathcal{A}_i \times \bigotimes_{i=1}^n \mathcal{A}_i \to \bigotimes_{i=1}^n \mathcal{A}_i$ is the bilinear map satisfying

$$\Phi(\otimes_{i=1}^n x_i, \otimes_{i=1}^n y_i) = \otimes_{i=1}^n x_i y_i$$

for all $x_1, y_1 \in \mathcal{A}_1, \ldots, x_n, y_n \in \mathcal{A}_n$. We define the product of X and Y in $\bigotimes_{i=1}^n \mathcal{A}_i$ by

$$XY = \Phi(X, Y). \tag{2.5}$$

Obviously, (2.4) is satisfied and $\bigotimes_{i=1}^n \mathcal{A}_i$ is closed under the multiplication in (2.5). We claim that the multiplication in (2.5) is bilinear. Let $X, Y_1, Y_2 \in \bigotimes_{i=1}^n \mathcal{A}_i$ and $\alpha \in \mathbb{F}$. Then

$$X(\alpha Y_1 + Y_2) = \Phi(X, \alpha Y_1 + Y_2) = \alpha XY_1 + XY_2.$$

Let $X_1, X_2, Y \in \bigotimes_{i=1}^n \mathcal{A}_i$ and $\beta \in \mathbb{F}$. We see that

$$(\beta X_1 + X_2)Y = \Phi(\beta X_1 + X_2, Y) = \beta X_1 Y + X_2 Y.$$

Next, we show that the multiplication in (2.5) is associative. Let $X = \sum_p x_{1p} \otimes \cdots \otimes x_{np}$, $Y = \sum_q y_{1q} \otimes \cdots \otimes y_{nq}$, $Z = \sum_r z_{1r} \otimes \cdots \otimes z_{nr} \in \bigotimes_{i=1}^n \mathcal{A}_i$. Then

$$\begin{aligned}
(XY)Z &= \left(\sum_p \sum_q (x_{1p}y_{1q}) \otimes \cdots \otimes (x_{np}y_{nq})\right)\left(\sum_r z_{1r} \otimes \cdots \otimes z_{nr}\right) \\
&= \sum_p \sum_q \sum_r (x_{1p}y_{1q})z_{1r} \otimes \cdots \otimes (x_{np}y_{nq})z_{nr} \\
&= \sum_p \sum_q \sum_r x_{1p}(y_{1q}z_{1r}) \otimes \cdots \otimes x_{np}(y_{nq}z_{nr}) \\
&= \left(\sum_p x_{1p} \otimes \cdots \otimes x_{np}\right)\left(\sum_q \sum_r (y_{1q}z_{1r}) \otimes \cdots \otimes (y_{nq}z_{nr})\right) \\
&= X(YZ).
\end{aligned}$$

Thus $\bigotimes_{i=1}^n \mathcal{A}_i$ forms an algebra over $\mathbb{F}$. Finally, we show that if $\mathcal{A}_i$ is a unital algebra with unity u_i for $i = 1, \ldots, n$, then $\otimes_{i=1}^n u_i$ is the unity of $\bigotimes_{i=1}^n \mathcal{A}_i$. Let $X = \sum_p x_{1p} \otimes \cdots \otimes x_{np} \in \bigotimes_{i=1}^n \mathcal{A}_i$. We see that $X(\otimes_{i=1}^n u_i) = \sum_p x_{1p}u_1 \otimes \cdots \otimes x_{np}u_n = X$. Likewise, we obtain $(\otimes_{i=1}^n u_i)X = X$. This completes the proof. $\square$

The algebra $\bigotimes_{i=1}^n \mathcal{A}_i$ with the multiplication in (2.4) is known as the **tensor product of algebras** $\mathcal{A}_1, \ldots, \mathcal{A}_n$.

2.3 Kronecker Product of Matrices

Let $\mathbb{F}$ be a field and let $k, n_1, \ldots, n_k \geqslant 2$ be integers. We denote by $\mathcal{M}_{n_i}(\mathbb{F})$ the algebra of $n_i \times n_i$ matrices over $\mathbb{F}$ and abbreviate $\mathcal{M}_{n_i}(\mathbb{F})$ to $\mathcal{M}_{n_i}$ when no confusion can arise for $i = 1, \ldots, k$. Let $A = (a_{ij}) \in \mathcal{M}_{n_1}$ and $B \in \mathcal{M}_{n_2}$. The **Kronecker product** of two matrices A and B is defined by

$$A \otimes B = (a_{ij}B) := \begin{pmatrix} a_{11}B & \cdots & a_{1n_1}B \\ \vdots & \ddots & \vdots \\ a_{n_11}B & \cdots & a_{n_1n_1}B \end{pmatrix} \in \mathcal{M}_{n_1n_2}. \tag{2.6}$$

The Kronecker product is also called the **tensor product**. It follows from (2.6) that

(K1) $(\alpha A_1 + A_2) \otimes B = \alpha(A_1 \otimes B) + A_2 \otimes B$ for all $A_1, A_2 \in \mathcal{M}_{n_1}, B \in \mathcal{M}_{n_2}, \alpha \in \mathbb{F}$.
(K2) $A \otimes (\alpha B_1 + B_2) = \alpha(A \otimes B_1) + A \otimes B_2$ for all $A \in \mathcal{M}_{n_1}, B_1, B_2 \in \mathcal{M}_{n_2}, \alpha \in \mathbb{F}$.
(K3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all $A \in \mathcal{M}_{n_1}$, $B \in \mathcal{M}_{n_2}$ and $C \in \mathcal{M}_{n_3}$.
(K4) $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$ for all $A_1, A_2 \in \mathcal{M}_{n_1}$ and $B_1, B_2 \in \mathcal{M}_{n_2}$.

Let $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$. The tensor product of k matrices $A_1, \ldots, A_k$ with $k \geqslant 3$ is defined inductively as follows:

$$\otimes_{i=1}^{k} A_i = A_1 \otimes (\otimes_{i=2}^{k} A_i). \tag{2.7}$$

The notation $\otimes_{i=1}^{k} A_i$ is unambiguous due to (K3). It is not difficult to see that $\otimes_{i=1}^{k} A_i$ is a matrix in $\mathcal{M}_{n_1 \cdots n_k}$. Notice also that

$$\otimes_{i=1}^{k} I_{n_i} = I_{n_1 \cdots n_k}, \tag{2.8}$$

where I_n denotes the $n \times n$ identity matrix. By (K4) and (2.7), $(\otimes_{i=1}^{k} A_i)(\otimes_{i=1}^{k} B_i) = (A_1 \otimes (\otimes_{i=2}^{k} A_i))(B_1 \otimes (\otimes_{i=2}^{k} B_i)) = (A_1 B_1) \otimes (\otimes_{i=2}^{k} A_i)(\otimes_{i=2}^{k} B_i)$. It can be shown inductively that

$$(\otimes_{i=1}^{k} A_i)(\otimes_{i=1}^{k} B_i) = \otimes_{i=1}^{k} A_i B_i \tag{2.9}$$

for all matrices $A_1, B_1 \in \mathcal{M}_{n_1}, \ldots, A_k, B_k \in \mathcal{M}_{n_k}$.

Let $1 \leqslant s \leqslant k$ be an integer. We denote by $E_{ij}^{(n_s)}$ the **standard matrix unit** in $\mathcal{M}_{n_s}$ whose (i, j)th entry is one and zero elsewhere. Then $\{E_{ij}^{(n_s)} : 1 \leqslant i, j \leqslant n_s\}$ forms the **standard basis** of $\mathcal{M}_{n_s}$ and

$$E_{ij}^{(n_s)} E_{pq}^{(n_s)} = \begin{cases} E_{iq}^{(n_s)} & \text{if } j = p, \\ 0_{n_s} & \text{if } j \neq p \end{cases}$$

for all integers $1 \leqslant i, j, p, q \leqslant n_s$, where 0_{n_s} denotes the $n_s \times n_s$ zero matrix.

We continue with the following elementary lemma.

Lemma 2.3.1. *The following results hold.*

(a) $E_{ij}^{(n_1)} \otimes E_{st}^{(n_2)} = E_{n_2(i-1)+s,\, n_2(j-1)+t}^{(n_1n_2)}$ *for each* $1 \leqslant i, j \leqslant n_1$ *and* $1 \leqslant s, t \leqslant n_2$.

(b) $\left\{E_{ij}^{(n_1)} \otimes E_{st}^{(n_2)} : 1 \leqslant i, j \leqslant n_1,\ 1 \leqslant s, t \leqslant n_2\right\}$ *forms the standard basis of* $\mathcal{M}_{n_1n_2}$.

Proof. (a) Let $1 \leqslant i, j \leqslant n_1$ and $1 \leqslant s, t \leqslant n_2$ be integers. By (2.6), we have

$$E_{ij}^{(n_1)} \otimes E_{st}^{(n_2)} = \begin{pmatrix} 0_{n_2(i-1),n_2(j-1)} & 0 & 0 \\ 0 & E_{st}^{(n_2)} & 0 \\ 0 & 0 & 0_{n_2(n_1-i),n_2(n_1-j)} \end{pmatrix} = E_{n_2(i-1)+s,\, n_2(j-1)+t}^{(n_1n_2)},$$

where $0_{m,n}$ denotes the $m \times n$ zero matrix.

(b) The result follows immediately from (a) by observing that

$$\left\{E_{n_2(i-1)+s,n_2(j-1)+t}^{(n_1n_2)} : 1 \leqslant i, j \leqslant n_1,\ 1 \leqslant s, t \leqslant n_2\right\} = \left\{E_{pq}^{(n_1n_2)} : 1 \leqslant p, q \leqslant n_1n_2\right\}.$$

□

Let $\Theta : \mathcal{M}_{n_1} \times \mathcal{M}_{n_2} \to \mathcal{M}_{n_1n_2}$ be the function defined by

$$\Theta(A, B) = A \otimes B \tag{2.10}$$

for all matrices $A \in \mathcal{M}_{n_1}$ and $B \in \mathcal{M}_{n_2}$. Then Θ is a bilinear map by (K1) and (K2).

Proposition 2.3.2. *Let* $\Theta : \mathcal{M}_{n_1} \times \mathcal{M}_{n_2} \to \mathcal{M}_{n_1n_2}$ *be the bilinear map defined in* (2.10)*. If* $\mathcal{W}$ *is a linear space over* $\mathbb{F}$ *and* $\upsilon : \mathcal{M}_{n_1} \times \mathcal{M}_{n_2} \to \mathcal{W}$ *is a bilinear map, then there exits a unique linear map* $\tau : \mathcal{M}_{n_1n_2} \to \mathcal{W}$ *such that* $\upsilon = \tau \circ \Theta$.

Proof. By Lemma 2.3.1 (b), $\left\{E_{ij}^{(n_1)} \otimes E_{st}^{(n_2)} : 1 \leqslant i, j \leqslant n_1,\ 1 \leqslant s, t \leqslant n_2\right\}$ is a basis of $\mathcal{M}_{n_1n_2}$. Then there exists a unique linear map $\tau : \mathcal{M}_{n_1n_2} \to \mathcal{W}$ such that

$$\tau(E_{ij}^{(n_1)} \otimes E_{st}^{(n_2)}) = \upsilon(E_{ij}^{(n_1)}, E_{st}^{(n_2)})$$

for all $1 \leqslant i, j \leqslant n_1$ and $1 \leqslant s, t \leqslant n_2$. It follows from (2.10) that

$$\upsilon(E_{ij}^{(n_1)}, E_{st}^{(n_2)}) = \tau(E_{ij}^{(n_1)} \otimes E_{st}^{(n_2)}) = (\tau \circ \Theta)(E_{ij}^{(n_1)}, E_{st}^{(n_2)})$$

for all $1 \leqslant i, j \leqslant n_1$ and $1 \leqslant s, t \leqslant n_2$. By the bilinearity of υ and $\tau \circ \Theta$, we conclude that $\upsilon = \tau \circ \Theta$ as desired. $\square$

As an immediate consequence of Proposition 2.3.2, $(\mathcal{M}_{n_1 n_2}, \Theta)$ is a tensor product of $\mathcal{M}_{n_1}$ and $\mathcal{M}_{n_2}$ by Definition 2.1.1. In similar fashion, using Lemma 2.3.1, it can be proved inductively for $k \geqslant 3$ that

$$\left\{ \otimes_{i=1}^{k} E_{s_i,t_i}^{(n_i)} : E_{s_i,t_i}^{(n_i)} \in \mathscr{B}_i,\ i = 1, \ldots, k \right\}$$

constitutes the standard basis of $\mathcal{M}_{n_1 \cdots n_k}$, where $\mathscr{B}_1, \ldots, \mathscr{B}_k$ are the standard bases of $\mathcal{M}_{n_1}, \ldots, \mathcal{M}_{n_k}$, respectively. Furthermore, if $\Theta' : \mathcal{M}_{n_1} \times \cdots \times \mathcal{M}_{n_k} \to \mathcal{M}_{n_1 \cdots n_k}$ is the multilinear map defined by

$$\Theta'(A_1, \ldots, A_k) = \otimes_{i=1}^{k} A_i$$

for all matrices $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$, then $(\mathcal{M}_{n_1 \cdots n_k}, \Theta')$ is a tensor product of $\mathcal{M}_{n_1}, \ldots, \mathcal{M}_{n_k}$ with a similar argument as in the proof of Proposition 2.3.2. By virtue of (2.1), (2.8), (2.9) and Theorem 2.2.1, we may view the algebras

$$\bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \cong \mathcal{M}_{n_1 \cdots n_k} \tag{2.11}$$

as identical and conclude that $\bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ turns into an algebra over $\mathbb{F}$ with unity $\otimes_{i=1}^{k} I_{n_i} = I_{n_1 \cdots n_k}$ relative to the multiplication in (2.9). Consequently, the algebra $\bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ with the multiplication in (2.9) is the **tensor product of matrix algebras** $\mathcal{M}_{n_1}, \ldots, \mathcal{M}_{n_k}$.

By (Brešar, 2014, p. 88), if $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$ are algebras over the same field such that $\mathcal{A} \cong \mathcal{X}$ and $\mathcal{B} \cong \mathcal{Y}$, then $\mathcal{A} \bigotimes \mathcal{B} \cong \mathcal{X} \bigotimes \mathcal{Y}$. It follows from (2.11) that $\bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \cong \mathcal{M}_{n_1 \cdots n_k} \cong \mathcal{M}_{n_1} \bigotimes \mathcal{M}_{n_2 \cdots n_k} \cong \mathcal{M}_{n_1} \bigotimes (\bigotimes_{i=2}^{k} \mathcal{M}_{n_i})$. Thus we may view the algebras

$$\bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \cong \mathcal{M}_{n_1} \bigotimes \left(\bigotimes_{i=2}^{k} \mathcal{M}_{n_i} \right) \tag{2.12}$$

as identical.

2.4 Literature Review

The study of linear preserver problems has been one of the most active research areas in matrix theory, operator theory and multilinear algebra in recent times. This study involves the characterization of linear maps on matrices, operators or tensors which leave certain functions, properties, subsets, relations or functional identities invariant. Possibly the earliest paper on the study of linear preserver problems such as Frobenius (1897) dates back to a century ago. In this paper, Frobenius obtained a characterization of linear maps $\psi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ preserving the determinant function, i.e., $\det(\psi(A)) = \det A$ for all $A \in \mathcal{M}_n(\mathbb{C})$, and showed that there exist invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{C})$ with $\det(PQ) = 1$ such that ψ is of the form

$$\psi(A) = PAQ$$

for all $A \in \mathcal{M}_n(\mathbb{C})$, or

$$\psi(A) = PA^tQ$$

for all $A \in \mathcal{M}_n(\mathbb{C})$. Since then, these problems have been studied extensively and many elegant results have been discovered. For an expository survey of the subject and its historic developments, we refer to the survey papers Li and Pierce (2001) and Pierce et al. (1992) or the book Molnár (2007) and their references therein.

Recently, there have been numerous studies of linear preserver problems related to the study of quantum information science, see for instance Fošner et al. (2013), Friedland et al. (2011), Hou and Qi (2013) and Lim (2012). In Friedland et al. (2011), the authors obtained a characterization of linear preservers of pure states. Let $\mathcal{H}_n(\mathbb{C})$ be the linear space of $n \times n$ complex Hermitian matrices and let $\mathcal{P}_n$ be the compact subset of $\mathcal{H}_n(\mathbb{C})$ consisting of rank one Hermitian matrices of trace one. They showed that if $m, n \geqslant 2$ are distinct integers and $\psi : \mathcal{H}_m(\mathbb{C}) \bigotimes \mathcal{H}_n(\mathbb{C}) \to \mathcal{H}_m(\mathbb{C}) \bigotimes \mathcal{H}_n(\mathbb{C})$ is a linear map satisfying $\psi(\mathcal{P}_m \otimes \mathcal{P}_n) = \mathcal{P}_m \otimes \mathcal{P}_n$, then there exist unitary matrices $U \in \mathcal{M}_m(\mathbb{C})$ and $V \in \mathcal{M}_n(\mathbb{C})$ such that

$$\psi(A \otimes B) = \phi_1(A) \otimes \phi_2(B)$$

for all $A \otimes B \in \mathcal{H}_m(\mathbb{C}) \bigotimes \mathcal{H}_n(\mathbb{C})$, where $\phi_1 : \mathcal{H}_m(\mathbb{C}) \to \mathcal{H}_m(\mathbb{C})$ and $\phi_2 : \mathcal{H}_n(\mathbb{C}) \to$

$\mathcal{H}_n(\mathbb{C})$ are linear maps of the forms

$$\phi_1(A) = UAU^*$$

for all $A \in \mathcal{H}_m(\mathbb{C})$ and

$$\phi_2(B) = VBV^*$$

for all $B \in \mathcal{H}_n(\mathbb{C})$, respectively. Here, U^* denotes the conjugate transpose of U. We refer to Fošner et al. (2013) for a survey of linear preserver problems on tensor products of matrices arising from quantum information science.

The study of functional identities on algebras of matrices or operators is often related to linear preserver problems. A functional identity can be informally described as an equation with its functions appearing as unknowns. The functions satisfying a functional identity are referred as the solutions to that functional identity. The goal of this study is to determine the general forms and the classifications of all solutions for each functional identity. This theory provides an effective tool for solving a variety of problems in many areas such as prime rings, Lie algebras, Poisson algebras and preserver problems. For an extensive survey of the subject and a full account on the theory of functional identities, the reader is referred to the book Brešar et al. (2007). A map $\psi : \mathcal{R} \to \mathcal{R}$, with $\mathcal{R}$ being a ring, on a nonempty subset $\mathcal{S}$ of $\mathcal{R}$ is called **commuting** on $\mathcal{S}$ if

$$[\psi(a), a] = 0 \tag{2.13}$$

for all $a, b \in \mathcal{S}$, where $[a, b]$ is the commutator $ab - ba$ of elements $a, b \in \mathcal{R}$. In this dissertation, if $\psi : \mathcal{R} \to \mathcal{R}$ is an additive map commuting on $\mathcal{S}$, then we simply say ψ is a commuting additive map on $\mathcal{S}$ even if $\mathcal{S}$ has no additive structure. The study of commuting maps on rings or algebras is one of the most essential topics of functional identities. It was initiated by Posner in 1957. He proved that a prime ring admitting a nonzero commuting derivation is commutative, see (Posner, 1957, Theorem 2). In 1993, Brešar obtained a structural result for commuting additive maps on a prime ring and showed that additive maps $\psi : \mathcal{R} \to \mathcal{R}$ commuting on a prime ring $\mathcal{R}$ are of the **standard form**

$$\psi(a) = \lambda a + \mu(a)$$

for all $a \in \mathcal{R}$, where λ is an element in the extended centroid $\mathcal{C}$ of $\mathcal{R}$ and $\mu : \mathcal{R} \to \mathcal{C}$ is an additive map, see (Brešar, 1993, Theorem 3.2). This result has been extremely influential and initiated considerable interest in commuting maps on various rings and algebras, see for example Beidar (1998), Beidar et al. (2000), Cheung (2001), Chou and Liu (2021), P.-H. Lee and Lee (1997) or T.-K. Lee and Lee (1996). For a survey of the subject, the reader is referred to the survey paper Brešar (2004) and the book Brešar et al. (2007).

Recently, Brešar (2016a) initiated the study of functional identities on tensor products of algebras. More results on Jordan derivations, derivations and biderivations on tensor products of algebras have been obtained in Brešar (2016b, 2017) and Eremita (2018). Motivated by these results, together with the inspiration of the study of linear preserver problems on tensor products of matrices arising from quantum information science, we study commuting additive maps on tensor products of matrix algebras.

Let $k \geqslant 1$ and $n_1, \ldots, n_k \geqslant 2$ be integers. Let $\mathbb{F}$ be a field and let $\mathcal{M}_{n_i}$ be the algebra of $n_i \times n_i$ matrices over $\mathbb{F}$ for $i = 1, \ldots, k$. In this dissertation, we obtain a complete structural characterization of additive maps $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ satisfying

$$[\psi(\otimes_{i=1}^{k} A_i), \otimes_{i=1}^{k} A_i] = 0 \tag{2.14}$$

for all $A_1 \in \mathcal{S}_{1,n_1}, \ldots, A_k \in \mathcal{S}_{k,n_k}$, where

$$\mathcal{S}_{i,n_i} = \left\{ E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)} : \alpha \in \mathbb{F} \text{ and } 1 \leqslant p, q, s, t \leqslant n_i \text{ are not all distinct integers} \right\}$$

and $E_{st}^{(n_i)}$ is the standard matrix unit in $\mathcal{M}_{n_i}$ for $i = 1, \ldots, k$. In particular, we show that commuting additive maps on $\mathcal{S}_{1,n_1}$ are of the standard form.

We next deduce from the obtained result a characterization of commuting additive maps on tensor products of fixed-rank matrices. More precisely, we classify all additive maps $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ satisfying $[\psi(\otimes_{i=1}^{k} A_i), \otimes_{i=1}^{k} A_i] = 0$ for all $A_1 \in \mathcal{R}_{r_1}^{n_1}, \ldots, A_k \in \mathcal{R}_{r_k}^{n_k}$, where $\mathcal{R}_{r_i}^{n_i}$ denotes the set of rank r_i matrices in $\mathcal{M}_{n_i}$ and $1 < r_i \leqslant n_i$

is a fixed integer such that $r_i \neq n_i$ when $n_i = 2$ and $|\mathbb{F}| = 2$ for $i = 1, \ldots, k$.

2.5 Methodology

The methodology of this research study comprises the four components as follows.

1. **Literature Review**

   This research study began with an in-depth study on relevant research materials to acquire basic knowledge in tensor products and to review the latest development in the study of linear preserver problems on tensor products of matrices. The materials included recent survey papers and books on commuting maps on rings, functional identities on tensor products and linear preserver problems on tensor products of matrices. A proper literature review of the classical results and the latest articles on the research topics was also conducted to ensure the research problems were still open and to review useful ideas and techniques before attempting the problems.

2. **Main Study**

   Having been equipped with the basic knowledge and techniques, useful preliminary results were first established before obtaining a necessary and sufficient condition for commuting additive maps on tensor products of matrices of the form $E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)}$. Next, a characterization was deduced for the identified commuting additive maps using induction. As an application, the characterization was used to classify commuting additive maps on tensor products of fixed-rank matrices with the help of some other preliminary lemmas.

3. **Findings Refinement**

   All established characterizations were examined carefully and justified by rigorous mathematical arguments to verify their validity. In addition, the structural results of the identified commuting additive maps were analyzed to ensure the simplest possible forms were obtained. Evaluation, discussion and refinement of findings were also carried out at this stage.

4. **Dissertation Writing**

   All findings of this research study were finally reported in detail in this dissertation.

CHAPTER 3: COMMUTING ADDITIVE MAPS ON TENSOR PRODUCTS OF MATRICES OF THE FORM $E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)}$

Throughout this chapter, unless stated otherwise, let $\mathbb{F}$ be a field and let $k, n_1, \ldots, n_k$ be positive integers such that $n_i \geqslant 2$ for $i = 1, \ldots, k$. Let $\mathcal{M}_{n_i}$ be the algebra of $n_i \times n_i$ matrices over $\mathbb{F}$ with unity I_{n_i} for $i = 1, \ldots, k$. For each integer $1 \leqslant h \leqslant k$, we denote

$$\mathcal{S}_h^k = \Big\{ \otimes_{i=h}^k A_i : A_h \in \mathcal{S}_{h,n_h}, \ldots, A_k \in \mathcal{S}_{k,n_k} \Big\},$$

where $\mathcal{S}_{i,n_i} = \{E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)} \in \mathcal{M}_{n_i} : \alpha \in \mathbb{F},\ 1 \leqslant p, q, s, t \leqslant n_i$ are not all distinct$\}$ and $E_{st}^{(n_i)}$ is the standard matrix unit in $\mathcal{M}_{n_i}$ for $i = 1, \ldots, k$. We write $E_{st}^{(n_i)}$ simply E_{st} when there is no danger of ambiguity. We denote by $D(\bigotimes_{i=h}^k \mathcal{M}_{n_i})$ the set of all decomposable tensors in $\bigotimes_{i=h}^k \mathcal{M}_{n_i}$. As an abuse of notation, $\otimes_{i=h}^h A_i \in \bigotimes_{i=h}^h \mathcal{M}_{n_i}$ is taken to be $A_h \in \mathcal{M}_{n_h}$.

3.1 Preliminary Results

We begin our discussion by giving a necessary and sufficient condition for additive maps $\psi_1, \psi_2 : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \bigotimes_{i=1}^k \mathcal{M}_{n_i}$ satisfying

$$\psi_1(\otimes_{i=1}^k A_i)(\otimes_{i=1}^k A_i) = \psi_2(\otimes_{i=1}^k A_i)(\otimes_{i=1}^k A_i)$$

for all $A_1 \in \mathcal{S}_{1,n_1}, \ldots, A_k \in \mathcal{S}_{k,n_k}$. More precisely, we prove the following result.

Lemma 3.1.1. *Let $k \geqslant 1$ and $n_1, \ldots, n_k \geqslant 2$ be integers and let $\alpha \in \{0, 1\}$ be a fixed scalar. Let $\psi_1, \psi_2 : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \bigotimes_{i=1}^k \mathcal{M}_{n_i}$ be additive maps. Then*

$$\alpha\psi_1(A)A + (1-\alpha)A\psi_1(A) = \alpha\psi_2(A)A + (1-\alpha)A\psi_2(A) \tag{3.1}$$

for all $A \in \mathcal{S}_1^k$ if and only if $\psi_1 = \psi_2$.

Proof. The sufficiency is clear. For the necessity, we note that the hypothesis in (3.1) is equivalent to

$$\alpha\varphi(A)A + (1-\alpha)A\varphi(A) = 0$$

for all $A \in \mathcal{S}_1^k$, where $\varphi := \psi_1 - \psi_2$. We only consider the case $\alpha = 1$ as the case $\alpha = 0$ can be shown analogously. We argue by induction on k. Consider $k = 1$. For abbreviation, we write E_{ij} for $E_{ij}^{(n_1)}$. Let $1 \leqslant s, p, q \leqslant n_1$ be integers and let $a \in \mathbb{F}$. Since $\varphi(X)X = 0$ for $X \in \{E_{ss} + aE_{pq}, E_{ss}, aE_{pq}\} \subseteq \mathcal{S}_{1,n_1}$, it follows that

$$\begin{aligned}
&\varphi(E_{ss} + aE_{pq})(E_{ss} + aE_{pq}) = 0, \quad \varphi(E_{ss})E_{ss} = 0, \quad \varphi(aE_{pq})aE_{pq} = 0 \\
&\implies \varphi(E_{ss})E_{ss} + \varphi(aE_{pq})E_{ss} + \varphi(E_{ss})aE_{pq} + \varphi(aE_{pq})aE_{pq} = 0 \\
&\implies \varphi(aE_{pq})E_{ss} + \varphi(E_{ss})aE_{pq} = 0 \\
&\implies \varphi(aE_{pq})E_{ss} = -\varphi(E_{ss})aE_{pq}. \qquad (3.2)
\end{aligned}$$

In particular, when $a = 1$ and $p = q$, we have $\varphi(E_{pp})E_{ss} = -\varphi(E_{ss})E_{pp}$. Then $\varphi(E_{ss}) = \varphi(E_{ss})I_{n_1} = \sum_{j=1}^{n_1} \varphi(E_{ss})E_{jj} = -\sum_{j=1}^{n_1} \varphi(E_{jj})E_{ss} = -\varphi(I_{n_1})E_{ss}$. Thus $\varphi(E_{ss})E_{jj} = 0$ for all $j = 1, \ldots, n_1$, and so $\varphi(E_{ss}) = \varphi(E_{ss})I_{n_1} = \sum_{j=1}^{n_1} \varphi(E_{ss})E_{jj} = 0$. Hence $\varphi(I_{n_1}) = 0$. By (3.2),

$$\varphi(aE_{pq}) = \varphi(aE_{pq})I_{n_1} = \sum_{j=1}^{n_1} \varphi(aE_{pq})E_{jj} = -\sum_{j=1}^{n_1} \varphi(E_{jj})aE_{pq} = -\varphi(I_{n_1})aE_{pq} = 0$$

for all integers $1 \leqslant p, q \leqslant n_1$ and $a \in \mathbb{F}$. It follows from the additivity of φ that $\varphi = 0$. This validates the base step $k = 1$.

Suppose that $k \geqslant 2$ and that the result holds for $k - 1$. By (2.12), $\bigotimes_{i=1}^k \mathcal{M}_{n_i} \cong \mathcal{M}_{n_1} \otimes \left(\bigotimes_{i=2}^k \mathcal{M}_{n_i}\right)$. By abuse of notation, we abbreviate $\bigotimes_{i=2}^k \mathcal{M}_{n_i}$ to $\mathcal{M}$. In view of Proposition 2.1.6 (b) and since $\{E_{ij} : i, j = 1, \ldots, n_1\}$ is a basis of $\mathcal{M}_{n_1}$, it follows that for each pair of integers $1 \leqslant s, t \leqslant n_1$, the map φ induces maps $\phi_{ij}^{(st)} : \mathcal{M} \to \mathcal{M}$, $i, j = 1, \ldots, n_1$, such that

$$\varphi(E_{st} \otimes A) = \sum_{i,j=1}^{n_1} E_{ij} \otimes \phi_{ij}^{(st)}(A)$$

for all $A \in \mathcal{M}$. Then $\sum_{i,j=1}^{n_1} E_{ij} \otimes \phi_{ij}^{(st)}(A + B) = \varphi(E_{st} \otimes (A + B)) = \varphi(E_{st} \otimes A) + \varphi(E_{st} \otimes B) = \sum_{i,j=1}^{n_1} E_{ij} \otimes \phi_{ij}^{(st)}(A) + \sum_{i,j=1}^{n_1} E_{ij} \otimes \phi_{ij}^{(st)}(B) = \sum_{i,j=1}^{n_1} E_{ij} \otimes (\phi_{ij}^{(st)}(A) + \phi_{ij}^{(st)}(B))$ yields $\sum_{i,j=1}^{n_1} E_{ij} \otimes (\phi_{ij}^{(st)}(A+B) - (\phi_{ij}^{(st)}(A) + \phi_{ij}^{(st)}(B))) = 0$ for all $A, B \in \mathcal{M}$.

By Proposition 2.1.6 (a), $\phi_{ij}^{(st)}(A+B) = \phi_{ij}^{(st)}(A) + \phi_{ij}^{(st)}(B)$ for all integers $1 \leqslant i, j \leqslant n_1$ and $A, B \in \mathcal{M}$, so $\phi_{ij}^{(st)}$, $i, j = 1, \ldots, n_1$, are additive. In particular, for each pair of integers $1 \leqslant s, t \leqslant n_1$, there exist additive maps $\phi_{ij}^{(st)} : \mathcal{M} \to \mathcal{M}$, $i, j = 1, \ldots, n_1$, such that

$$\varphi(E_{st} \otimes A) = \sum_{i,j=1}^{n_1} E_{ij} \otimes \phi_{ij}^{(st)}(A) \tag{3.3}$$

for all decomposable tensors $A \in D(\mathcal{M})$. Let $1 \leqslant s, t \leqslant n_1$ be arbitrary but fixed integers. By the hypothesis and (3.3) together with (2.9), we see that

$$0 = \varphi(E_{st} \otimes A)(E_{st} \otimes A) = \sum_{i,j=1}^{n_1} E_{ij}E_{st} \otimes \phi_{ij}^{(st)}(A)A = \sum_{i=1}^{n_1} E_{it} \otimes \phi_{is}^{(st)}(A)A$$

for all $A \in \mathcal{S}_2^k$. Then for each $1 \leqslant i \leqslant n_1$, by Proposition 2.1.6 (a), we get $\phi_{is}^{(st)}(A)A = 0$ for all $A \in \mathcal{S}_2^k$. By the induction hypothesis, we obtain

$$\phi_{is}^{(st)} = 0 \quad \text{for all } i = 1, \ldots, n_1. \tag{3.4}$$

Let $p \in \{1, \ldots, n_1\} \backslash \{t\}$ and let $A \in \mathcal{S}_2^k$. Since $\varphi(X)X = 0$ for $X \in \{(E_{pp} + E_{st}) \otimes A, E_{pp} \otimes A, E_{st} \otimes A\}$, it follows that

$$\begin{aligned}
&\varphi((E_{pp} + E_{st}) \otimes A)((E_{pp} + E_{st}) \otimes A) = 0, \quad \varphi(E_{pp} \otimes A)(E_{pp} \otimes A) = 0, \\
&\qquad \varphi(E_{st} \otimes A)(E_{st} \otimes A) = 0 \\
\implies\ &\varphi(E_{pp} \otimes A)(E_{pp} \otimes A) + \varphi(E_{st} \otimes A)(E_{pp} \otimes A) + \varphi(E_{pp} \otimes A)(E_{st} \otimes A) + \\
&\qquad \varphi(E_{st} \otimes A)(E_{st} \otimes A) = 0 \\
\implies\ &\varphi(E_{st} \otimes A)(E_{pp} \otimes A) + \varphi(E_{pp} \otimes A)(E_{st} \otimes A) = 0.
\end{aligned}$$

By virtue of (3.3), we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{n_1} E_{ij}E_{pp} \otimes \phi_{ij}^{(st)}(A)A + \sum_{i,j=1}^{n_1} E_{ij}E_{st} \otimes \phi_{ij}^{(pp)}(A)A = 0 \\
\implies\ &\sum_{i=1}^{n_1} E_{ip} \otimes \phi_{ip}^{(st)}(A)A + \sum_{i=1}^{n_1} E_{it} \otimes \phi_{is}^{(pp)}(A)A = 0
\end{aligned}$$

for all $A \in \mathcal{S}_2^k$. Since $p \neq t$, it follows from Proposition 2.1.6 (a) that for each $1 \leqslant i \leqslant n_1$,

$\phi_{ip}^{(st)}(A)A = 0$ for all $A \in \mathcal{S}_2^k$. By the induction hypothesis, for each $p \in \{1, \ldots, n_1\} \backslash \{t\}$, we have

$$\phi_{ip}^{(st)} = 0 \quad \text{for all} \quad i = 1, \ldots, n_1. \tag{3.5}$$

When $s = t$, in view of (3.4) and (3.5), we conclude that

$$\phi_{ij}^{(ss)} = 0 \quad \text{for all} \quad i, j = 1, \ldots, n_1. \tag{3.6}$$

Consider now $s \neq t$. Let $A \in \mathcal{S}_2^k$. Using the fact that $\varphi(X)X = 0$ for $X \in \{(E_{tt} + E_{st}) \otimes A, E_{tt} \otimes A, E_{st} \otimes A\}$, we obtain $\varphi(E_{tt} \otimes A)(E_{st} \otimes A) + \varphi(E_{st} \otimes A)(E_{tt} \otimes A) = 0$. It follows from (3.3) that

$$\begin{aligned} &\sum_{i,j=1}^{n_1} E_{ij}E_{st} \otimes \phi_{ij}^{(tt)}(A)A + \sum_{i,j=1}^{n_1} E_{ij}E_{tt} \otimes \phi_{ij}^{(st)}(A)A = 0 \\ &\implies \sum_{i=1}^{n_1} E_{it} \otimes \phi_{is}^{(tt)}(A)A + \sum_{i=1}^{n_1} E_{it} \otimes \phi_{it}^{(st)}(A)A = 0 \end{aligned}$$

for all $A \in \mathcal{S}_2^k$. By virtue of (3.6), $\phi_{is}^{(tt)} = 0$ for $i = 1, \ldots, n_1$. Thus $\sum_{i=1}^{n_1} E_{it} \otimes \phi_{it}^{(st)}(A)A = 0$ for all $A \in \mathcal{S}_2^k$. Then by Proposition 2.1.6 (a), $\phi_{it}^{(st)}(A)A = 0$ for all $A \in \mathcal{S}_2^k$, $i = 1, \ldots, n_1$. By the induction hypothesis,

$$\phi_{it}^{(st)} = 0 \quad \text{for all} \quad i = 1, \ldots, n_1. \tag{3.7}$$

In view of (3.5) and (3.7), we conclude that

$$\phi_{ij}^{(st)} = 0 \quad \text{for all} \quad i, j = 1, \ldots, n_1. \tag{3.8}$$

It follows from (3.3), (3.6) and (3.8) that $\varphi(E_{st} \otimes A) = 0$ for all $A \in D(\mathcal{M})$ and integers $1 \leqslant s, t \leqslant n_1$. By the additivity of φ, we get

$$\varphi(X) = \sum_{s,t=1}^{n_1} \varphi(E_{st} \otimes \alpha_{st} Z) = 0$$

for all $X = Y \otimes Z \in D(\bigotimes_{i=1}^k \mathcal{M}_{n_i})$, where $Y = \sum_{s,t=1}^{n_1} \alpha_{st} E_{st} \in \mathcal{M}_{n_1}$, $\alpha_{st} \in \mathbb{F}$ and $Z \in D(\mathcal{M})$. Consequently, $\varphi = 0$ by (2.3). Hence $\psi_1 = \psi_2$ as desired. $\square$

As an immediate consequence of defining $\psi = \psi_1 - \psi_2$ in Lemma 3.1.1, we obtain:

Theorem 3.1.2. *Let $k \geqslant 1$ and $n_1, \ldots, n_k \geqslant 2$ be integers and let $\alpha \in \{0, 1\}$ be a fixed scalar. Then $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ is an additive map satisfying*

$$\alpha\psi(A)A + (1-\alpha)A\psi(A) = 0$$

for all $A \in \mathcal{S}_1^k$ if and only if $\psi = 0$.

We now obtain a characterization of additive maps satisfying condition (2.14) for $k = 1$.

Theorem 3.1.3. *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Then $\psi : \mathcal{M}_n \to \mathcal{M}_n$ is a commuting additive map on $\mathcal{S}_{1,n} = \{E_{st} + \alpha E_{pq} : \alpha \in \mathbb{F}$* and $1 \leqslant p, q, s, t \leqslant n$ are not all distinct integers$\}$ *if and only if there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : \mathcal{M}_n \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n$$

for all $A \in \mathcal{M}_n$.

Proof. The sufficiency is obvious since $\psi(A)A = (\lambda A)A + (\mu(A)I_n)A = A(\lambda A) + A(\mu(A)I_n) = A\psi(A)$ for all $A \in \mathcal{S}_{1,n}$. For the necessity, we first note that for each pair of integers $1 \leqslant s, t \leqslant n$, there exist additive maps $\phi_{ij}^{(st)} : \mathbb{F} \to \mathbb{F}$, $i, j = 1, \ldots, n$, such that

$$\psi(aE_{st}) = \sum_{i,j=1}^{n} \phi_{ij}^{(st)}(a)E_{ij} \tag{3.9}$$

for all $a \in \mathbb{F}$. Since $[\psi(aE_{st}), aE_{st}] := \psi(aE_{st})aE_{st} - aE_{st}\psi(aE_{st}) = 0$ for all $a \in \mathbb{F}$, it follows from (3.9) that

$$\begin{aligned}
&\sum_{i,j=1}^{n} a\phi_{ij}^{(st)}(a)E_{ij}E_{st} - \sum_{i,j=1}^{n} a\phi_{ij}^{(st)}(a)E_{st}E_{ij} = 0 \\
&\implies \sum_{i=1}^{n} a\phi_{is}^{(st)}(a)E_{it} - \sum_{j=1}^{n} a\phi_{tj}^{(st)}(a)E_{sj} = 0 \\
&\implies a(\phi_{ss}^{(st)}(a) - \phi_{tt}^{(st)}(a))E_{st} + \sum_{i=1, i\neq s}^{n} a\phi_{is}^{(st)}(a)E_{it} - \sum_{j=1, j\neq t}^{n} a\phi_{tj}^{(st)}(a)E_{sj} = 0
\end{aligned}$$

for all $a \in \mathbb{F}$. For any integers $1 \leqslant s, t \leqslant n$, we have

$$\phi_{is}^{(st)} = 0 \quad \text{for all } \ i \in \{1, \ldots, n\} \backslash \{s\}, \tag{3.10}$$

$$\phi_{tj}^{(st)} = 0 \quad \text{for all } \ j \in \{1, \ldots, n\} \backslash \{t\}, \tag{3.11}$$

$$\phi_{ss}^{(st)} = \phi_{tt}^{(st)}. \tag{3.12}$$

In particular, when $s = t$, it follows from (3.10) and (3.11) that

$$\phi_{is}^{(ss)} = \phi_{si}^{(ss)} = 0 \quad \text{for all } \ i \in \{1, \ldots, n\} \backslash \{s\}. \tag{3.13}$$

Let $1 \leqslant s, t \leqslant n$ be integers, $p \in \{1, \ldots, n\} \backslash \{s, t\}$ and $a \in \mathbb{F}$. Since $[\psi(X), X] = 0$ for all $X \in \{E_{pp} + aE_{st}, E_{pp}, aE_{st}\}$, we obtain

$$\begin{aligned}
0 &= [\psi(E_{pp} + aE_{st}), E_{pp} + aE_{st}] \\
&= [\psi(E_{pp}), E_{pp}] + [\psi(aE_{st}), E_{pp}] + [\psi(E_{pp}), aE_{st}] + [\psi(aE_{st}), aE_{st}] \\
&= [\psi(aE_{st}), E_{pp}] + [\psi(E_{pp}), aE_{st}] \\
&= \psi(aE_{st})E_{pp} - E_{pp}\psi(aE_{st}) + \psi(E_{pp})aE_{st} - aE_{st}\psi(E_{pp}),
\end{aligned}$$

so $\psi(aE_{st})E_{pp} + \psi(E_{pp})aE_{st} = E_{pp}\psi(aE_{st}) + aE_{st}\psi(E_{pp})$. By (3.9), we get

$$\begin{aligned}
&\sum_{i,j=1}^{n} \phi_{ij}^{(st)}(a)E_{ij}E_{pp} + \sum_{i,j=1}^{n} \phi_{ij}^{(pp)}(1)aE_{ij}E_{st} = \sum_{i,j=1}^{n} \phi_{ij}^{(st)}(a)E_{pp}E_{ij} + \sum_{i,j=1}^{n} \phi_{ij}^{(pp)}(1)aE_{st}E_{ij} \\
&\implies \sum_{i=1}^{n} \phi_{ip}^{(st)}(a)E_{ip} + \sum_{i=1}^{n} \phi_{is}^{(pp)}(1)aE_{it} = \sum_{j=1}^{n} \phi_{pj}^{(st)}(a)E_{pj} + \sum_{j=1}^{n} \phi_{tj}^{(pp)}(1)aE_{sj}
\end{aligned} \tag{3.14}$$

for all $a \in \mathbb{F}$. Since $p \neq s, t$, it follows from (3.14) that

$$\begin{aligned}
(\phi_{pp}^{(st)}(a) - \phi_{pp}^{(st)}(a))E_{pp} &+ \sum_{i=1, i\neq p,s}^{n} \phi_{ip}^{(st)}(a)E_{ip} \\
&+ (\phi_{ps}^{(pp)}(1)a - \phi_{pt}^{(st)}(a))E_{pt} - \sum_{j=1, j\neq t,p}^{n} \phi_{pj}^{(st)}(a)E_{pj}
\end{aligned}$$

$$+ (\phi_{sp}^{(st)}(a) - \phi_{tp}^{(pp)}(1)a)E_{sp} - \sum_{j=1, j\neq p,t}^{n} \phi_{tj}^{(pp)}(1)aE_{sj}$$
$$+ (\phi_{ss}^{(pp)}(1) - \phi_{tt}^{(pp)}(1))aE_{st} + \sum_{i=1, i\neq s,p}^{n} \phi_{is}^{(pp)}(1)aE_{it} = 0$$

for all $a \in \mathbb{F}$. We conclude that

(A) for each $j \in \{1,\ldots,n\}\backslash\{s,t\}$, $i \in \{1,\ldots,n\}\backslash\{j,s\}$, $\phi_{ij}^{(st)}(a) = 0$ for all $a \in \mathbb{F}$,

(B) for each $i \in \{1,\ldots,n\}\backslash\{s,t\}$, $\phi_{it}^{(st)}(a) = \phi_{is}^{(ii)}(1)a$ for all $a \in \mathbb{F}$,

(C) for each $j \in \{1,\ldots,n\}\backslash\{s,t\}$, $\phi_{sj}^{(st)}(a) = \phi_{tj}^{(jj)}(1)a$ for all $a \in \mathbb{F}$.

From (A), we get

$$\phi_{ij}^{(st)} = 0 \quad \text{for all } j \in \{1,\ldots,n\}\backslash\{s,t\} \text{ and } i \in \{1,\ldots,n\}\backslash\{j,s\}. \tag{3.15}$$

From (B), together with (3.13), we have

$$\phi_{it}^{(st)} = 0 \quad \text{for all } i \in \{1,\ldots,n\}\backslash\{s,t\}. \tag{3.16}$$

Likewise, by (C), together with (3.13), we obtain

$$\phi_{sj}^{(st)} = 0 \quad \text{for all } j \in \{1,\ldots,n\}\backslash\{s,t\}. \tag{3.17}$$

It follows from (3.15) that when $s = t$,

$$\phi_{ij}^{(ss)} = 0 \quad \text{for all distinct } i,j \in \{1,\ldots,n\}\backslash\{s\}. \tag{3.18}$$

Consequently, when $s = t$, it follows from the observations in (3.9), (3.13) and (3.18) that

$$\psi(aE_{ss}) = \sum_{i=1}^{n} \phi_{ii}^{(ss)}(a)E_{ii} \tag{3.19}$$

for all $a \in \mathbb{F}$. When $s \neq t$, in view of (3.9), (3.10), (3.15), (3.16) and (3.17), we obtain

$$\psi(aE_{st}) = \phi_{st}^{(st)}(a)E_{st} + \sum_{i=1}^{n} \phi_{ii}^{(st)}(a)E_{ii} \tag{3.20}$$

for all $a \in \mathbb{F}$. By (3.19) and (3.20), we conclude that for any integers $1 \leqslant s, t \leqslant n$,

$$\psi(aE_{st}) = \begin{cases} \sum_{i=1}^{n} \phi_{ii}^{(ss)}(a)E_{ii} & \text{when } s = t, \\ \phi_{st}^{(st)}(a)E_{st} + \sum_{i=1}^{n} \phi_{ii}^{(st)}(a)E_{ii} & \text{when } s \neq t \end{cases} \tag{3.21}$$

for all $a \in \mathbb{F}$. We next claim that for each pair of distinct integers $1 \leqslant s, t \leqslant n$, there exists an additive map $\varphi_{st} : \mathbb{F} \to \mathbb{F}$ such that

$$\phi_{ii}^{(st)} = \varphi_{st} \ \text{ for all } i = 1, \ldots, n. \tag{3.22}$$

By (3.12), we see that claim (3.22) is shown when $n = 2$. Consider $n \geqslant 3$. Let $p \in \{1, \ldots, n\} \backslash \{s, t\}$ and let $a \in \mathbb{F}$. By (3.21), we have

$$\begin{aligned} &[\psi(aE_{st}), E_{sp}] := \psi(aE_{st})E_{sp} - E_{sp}\psi(aE_{st}) \\ &= \phi_{st}^{(st)}(a)E_{st}E_{sp} + \sum_{i=1}^{n} \phi_{ii}^{(st)}(a)E_{ii}E_{sp} - \phi_{st}^{(st)}(a)E_{sp}E_{st} - \sum_{i=1}^{n} \phi_{ii}^{(st)}(a)E_{sp}E_{ii} \\ &= (\phi_{ss}^{(st)}(a) - \phi_{pp}^{(st)}(a))E_{sp}, \end{aligned}$$

$$\begin{aligned} &[\psi(E_{sp}), aE_{st}] := \psi(E_{sp})aE_{st} - aE_{st}\psi(E_{sp}) \\ &= a\phi_{sp}^{(sp)}(1)E_{sp}E_{st} + \sum_{i=1}^{n} a\phi_{ii}^{(sp)}(1)E_{ii}E_{st} - a\phi_{sp}^{(sp)}(1)E_{st}E_{sp} - \sum_{i=1}^{n} a\phi_{ii}^{(sp)}(1)E_{st}E_{ii} \\ &= a(\phi_{ss}^{(sp)}(1) - \phi_{tt}^{(sp)}(1))E_{st}. \end{aligned}$$

Note that $[\psi(X), X] = 0$ for $X \in \{aE_{st} + E_{sp}, aE_{st}, E_{sp}\}$ yields $[\psi(aE_{st}), E_{sp}] + [\psi(E_{sp}), aE_{st}] = 0$. Since E_{st}, E_{sp} are linearly independent, we obtain

$$\phi_{ss}^{(st)}(a) = \phi_{pp}^{(st)}(a)$$

for all $a \in \mathbb{F}$. Then $\phi_{pp}^{(st)} = \phi_{ss}^{(st)}$ for every $p \in \{1, \ldots, n\} \backslash \{s, t\}$. Together with (3.12), we conclude that $\phi_{ii}^{(st)} = \varphi_{st}$ for $i = 1, \ldots, n$, where $\varphi_{st} : \mathbb{F} \to \mathbb{F}$ is an additive map. Hence claim (3.22) is proved.

Let $1 \leqslant s \leqslant n$ be an integer. We claim that

$$\phi_{ii}^{(ss)} = \phi_{jj}^{(ss)} \ \text{ for all } i, j \in \{1, \ldots, n\} \backslash \{s\}. \tag{3.23}$$

If $n = 2$, then there is nothing to show. Consider now $n \geqslant 3$. Let $p, q \in \{1, \ldots, n\} \backslash \{s\}$ be distinct integers and let $a \in \mathbb{F}$. Note that $[\psi(X), X] = 0$ for $X \in \{E_{pq} + aE_{ss}, E_{pq}, aE_{ss}\}$ leads to $[\psi(E_{pq}), aE_{ss}] + [\psi(aE_{ss}), E_{pq}] = 0$. It follows from (3.21) that

$$\begin{aligned}
[\psi(E_{pq}), aE_{ss}] &:= \psi(E_{pq})aE_{ss} - aE_{ss}\psi(E_{pq}) \\
&= \phi_{pq}^{(pq)}(1)aE_{pq}E_{ss} + \sum_{i=1}^{n} \phi_{ii}^{(pq)}(1)aE_{ii}E_{ss} \\
&\quad - \phi_{pq}^{(pq)}(1)aE_{ss}E_{pq} - \sum_{i=1}^{n} \phi_{ii}^{(pq)}(1)aE_{ss}E_{ii} \\
&= \phi_{ss}^{(pq)}(1)aE_{ss} - \phi_{ss}^{(pq)}(1)aE_{ss} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
[\psi(aE_{ss}), E_{pq}] &:= \psi(aE_{ss})E_{pq} - E_{pq}\psi(aE_{ss}) \\
&= \sum_{i=1}^{n} \phi_{ii}^{(ss)}(a)E_{ii}E_{pq} - \sum_{i=1}^{n} \phi_{ii}^{(ss)}(a)E_{pq}E_{ii} \\
&= \phi_{pp}^{(ss)}(a)E_{pq} - \phi_{qq}^{(ss)}(a)E_{pq}.
\end{aligned}$$

Consequently,

$$(\phi_{pp}^{(ss)}(a) - \phi_{qq}^{(ss)}(a))E_{pq} = 0.$$

Then $\phi_{pp}^{(ss)} = \phi_{qq}^{(ss)}$, and so $\phi_{ii}^{(ss)} = \phi_{jj}^{(ss)}$ for all $i, j \in \{1, \ldots, n\} \backslash \{s\}$, as claimed. By (3.23), we conclude that for each integer $1 \leqslant s \leqslant n$, there exists an additive map $\varphi_{ss} : \mathbb{F} \to \mathbb{F}$ such that

$$\phi_{ii}^{(ss)} = \varphi_{ss} \quad \text{for all } \ i \in \{1, \ldots, n\} \backslash \{s\}. \tag{3.24}$$

We now show that

$$\phi_{st}^{(st)} \text{ is linear } \quad \text{and} \quad \phi_{ss}^{(ss)} = \phi_{st}^{(st)} + \varphi_{ss} \tag{3.25}$$

for all distinct integers $1 \leqslant s, t \leqslant n$. Since $[\psi(X), X] = 0$ for all $X \in \{E_{ss} + aE_{st}, E_{ss}, aE_{st}\}$, we get $[\psi(E_{ss}), aE_{st}] + [\psi(aE_{st}), E_{ss}] = 0$ for all $a \in \mathbb{F}$. It follows from (3.21) and (3.24) that

$$
\begin{aligned}
[\psi(E_{ss}), aE_{st}] &:= \psi(E_{ss})aE_{st} - aE_{st}\psi(E_{ss}) \\
&= \sum_{i=1}^{n} \phi_{ii}^{(ss)}(1)aE_{ii}E_{st} - \sum_{i=1}^{n} \phi_{ii}^{(ss)}(1)aE_{st}E_{ii} \\
&= \phi_{ss}^{(ss)}(1)aE_{st} - \phi_{tt}^{(ss)}(1)aE_{st} \\
&= \phi_{ss}^{(ss)}(1)aE_{st} - \varphi_{ss}(1)aE_{st},
\end{aligned}
$$

$$
\begin{aligned}
[\psi(aE_{st}), E_{ss}] &:= \psi(aE_{st})E_{ss} - E_{ss}\psi(aE_{st}) \\
&= \phi_{st}^{(st)}(a)E_{st}E_{ss} + \sum_{i=1}^{n} \phi_{ii}^{(st)}(a)E_{ii}E_{ss} \\
&\quad - \phi_{st}^{(st)}(a)E_{ss}E_{st} - \sum_{i=1}^{n} \phi_{ii}^{(st)}(a)E_{ss}E_{ii} \\
&= \phi_{ss}^{(st)}(a)E_{ss} - \phi_{st}^{(st)}(a)E_{st} - \phi_{ss}^{(st)}(a)E_{ss} \\
&= -\phi_{st}^{(st)}(a)E_{st},
\end{aligned}
$$

so $(\phi_{ss}^{(ss)}(1)a - \varphi_{ss}(1)a - \phi_{st}^{(st)}(a))E_{st} = 0$, and thus $\phi_{st}^{(st)}(a) = (\phi_{ss}^{(ss)}(1) - \varphi_{ss}(1))a$ for all $a \in \mathbb{F}$. Then $\phi_{st}^{(st)}$ is linear for all distinct integers $1 \leqslant s, t \leqslant n$. Let $a \in \mathbb{F}$ and let $1 \leqslant s, t \leqslant n$ be distinct integers. Again, $[\psi(X), X] = 0$ for all $X \in \{aE_{ss} + E_{st}, aE_{ss}, E_{st}\}$ leads to

$$
\begin{aligned}
[\psi(aE_{ss}), E_{st}] &:= \psi(aE_{ss})E_{st} - E_{st}\psi(aE_{ss}) \\
&= \sum_{i=1}^{n} \phi_{ii}^{(ss)}(a)E_{ii}E_{st} - \sum_{i=1}^{n} \phi_{ii}^{(ss)}(a)E_{st}E_{ii} \\
&= \phi_{ss}^{(ss)}(a)E_{st} - \phi_{tt}^{(ss)}(a)E_{st} \\
&= \phi_{ss}^{(ss)}(a)E_{st} - \varphi_{ss}(a)E_{st},
\end{aligned}
$$

$$
\begin{aligned}
[\psi(E_{st}), aE_{ss}] &:= \psi(E_{st})aE_{ss} - aE_{ss}\psi(E_{st}) \\
&= \phi_{st}^{(st)}(1)aE_{st}E_{ss} + \sum_{i=1}^{n} \phi_{ii}^{(st)}(1)aE_{ii}E_{ss} \\
&\quad - \phi_{st}^{(st)}(1)aE_{ss}E_{st} - \sum_{i=1}^{n} \phi_{ii}^{(st)}(1)aE_{ss}E_{ii} \\
&= \phi_{ss}^{(st)}(1)aE_{ss} - \phi_{st}^{(st)}(1)aE_{st} - \phi_{ss}^{(st)}(1)aE_{ss} \\
&= -\phi_{st}^{(st)}(1)aE_{st},
\end{aligned}
$$

so $(\phi_{ss}^{(ss)}(a) - \varphi_{ss}(a) - \phi_{st}^{(st)}(1)a)E_{st} = 0$, and thus $\phi_{ss}^{(ss)}(a) = \varphi_{ss}(a) + \phi_{st}^{(st)}(1)a$. By the linearity of $\phi_{st}^{(st)}$, we have

$$\phi_{ss}^{(ss)}(a) = \varphi_{ss}(a) + \phi_{st}^{(st)}(1)a = \varphi_{ss}(a) + \phi_{st}^{(st)}(a) = (\varphi_{ss} + \phi_{st}^{(st)})(a)$$

for all $a \in \mathbb{F}$. Then $\phi_{ss}^{(ss)} = \varphi_{ss} + \phi_{st}^{(st)}$ for all distinct integers $1 \leqslant s, t \leqslant n$. Hence claim (3.25) is proved. Moreover, by (3.25), for each integer $1 \leqslant s \leqslant n$,

$$\phi_{si}^{(si)} = \phi_{sj}^{(sj)} \text{ for all } i, j \in \{1, \dots, n\} \backslash \{s\}. \tag{3.26}$$

In view of (3.21), (3.22), (3.24) and (3.25), we conclude that for any integers $1 \leqslant s, t \leqslant n$,

$$\psi(aE_{st}) = \begin{cases} \phi_{sp}^{(sp)}(a)E_{ss} + \varphi_{ss}(a)I_n & \text{when } s = t, \\ \phi_{st}^{(st)}(a)E_{st} + \varphi_{st}(a)I_n & \text{when } s \neq t \end{cases} \tag{3.27}$$

for all $a \in \mathbb{F}$, where p is any integer in $\{1, \dots, n\} \backslash \{s\}$. Finally, we claim that there exists a linear map $\varphi : \mathbb{F} \to \mathbb{F}$ such that

$$\phi_{st}^{(st)} = \varphi \tag{3.28}$$

for all distinct integers $s, t = 1, \dots, n$. Let $1 \leqslant s, t \leqslant n$ be distinct integers and let $a \in \mathbb{F}$. Since $[\psi(X), X] = 0$ for $X \in \{aE_{ss} + E_{ts}, aE_{ss}, E_{ts}\}$, it follows from (3.27) that

$$\begin{aligned} &[\psi(aE_{ss}), E_{ts}] := \psi(aE_{ss})E_{ts} - E_{ts}\psi(aE_{ss}) \\ &= \phi_{st}^{(st)}(a)E_{ss}E_{ts} + \varphi_{ss}(a)E_{ts} - \phi_{st}^{(st)}(a)E_{ts}E_{ss} - \varphi_{ss}(a)E_{ts} \\ &= -\phi_{st}^{(st)}(a)E_{ts}, \end{aligned}$$

$$\begin{aligned} &[\psi(E_{ts}), aE_{ss}] := \psi(E_{ts})aE_{ss} - aE_{ss}\psi(E_{ts}) \\ &= \phi_{ts}^{(ts)}(1)aE_{ts}E_{ss} + \varphi_{ts}(1)aE_{ss} - \phi_{ts}^{(ts)}(1)aE_{ss}E_{ts} - \varphi_{ts}(1)aE_{ss} \\ &= \phi_{ts}^{(ts)}(1)aE_{ts}, \end{aligned}$$

so $(\phi_{ts}^{(ts)}(1)a - \phi_{st}^{(st)}(a))E_{ts} = 0$ for all $a \in \mathbb{F}$. By the linearity of $\phi_{ts}^{(ts)}$,

$$\phi_{ts}^{(ts)} = \phi_{st}^{(st)} \tag{3.29}$$

for all distinct integers $1 \leqslant s, t \leqslant n$. If $n = 2$, then $\phi_{12}^{(12)} = \phi_{21}^{(21)}$ by (3.29) and claim

(3.28) is proved. Consider now $n \geqslant 3$. Putting (3.26) and (3.29) together, we obtain

$$\begin{aligned} \phi_{12}^{(12)} &= \phi_{13}^{(13)} = \cdots = \phi_{1n}^{(1n)} \\ &= \phi_{21}^{(21)} = \phi_{23}^{(23)} = \cdots = \phi_{2n}^{(2n)} \\ &\vdots \\ &= \phi_{n1}^{(n1)} = \phi_{n2}^{(n2)} = \cdots = \phi_{n,n-1}^{(n,n-1)}, \end{aligned} \tag{3.30}$$

so $\phi_{pq}^{(pq)} = \phi_{st}^{(st)}$ for all integers $1 \leqslant p, q, s, t \leqslant n$ with $p \neq q$ and $s \neq t$. Hence claim (3.28) is proved.

Let $\lambda = \varphi(1) \in \mathbb{F}$. Since φ is linear, $\varphi(a) = \lambda a$ for all $a \in \mathbb{F}$. By (3.27) and (3.28),

$$\psi(aE_{st}) = \lambda aE_{st} + \varphi_{st}(a)I_n$$

for all integers $1 \leqslant s, t \leqslant n$ and $a \in \mathbb{F}$. Let $A = \sum_{s,t=1}^{n} a_{st}E_{st} \in \mathcal{M}_n$. Then

$$\psi(A) = \sum_{s,t=1}^{n} \psi(a_{st}E_{st}) = \sum_{s,t=1}^{n} (\lambda a_{st}E_{st} + \varphi_{st}(a_{st})I_n) = \lambda A + \mu(A)I_n,$$

where $\mu : \mathcal{M}_n \to \mathbb{F}$ is the additive map defined by

$$\mu(A) = \sum_{s,t=1}^{n} \varphi_{st}(a_{st})$$

for all $A = \sum_{s,t=1}^{n} a_{st}E_{st} \in \mathcal{M}_n$. The proof is complete. $\square$

We continue with the study of additive maps satisfying condition (2.14) for $k \geqslant 2$.

Lemma 3.1.4. *Let $\mathbb{F}$ be a field and let $k, n_1, \ldots, n_k \geqslant 2$ be integers. Let $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ be an additive map. Then ψ is a commuting map on $\mathcal{S}_1^k$ if and only if there exist a commuting linear map $\varphi : \bigotimes_{i=2}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=2}^{k} \mathcal{M}_{n_i}$ on $\mathcal{S}_2^k$ and commuting additive maps $\varphi_{st} : \bigotimes_{i=2}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=2}^{k} \mathcal{M}_{n_i}$ on $\mathbb{F} \cdot \mathcal{S}_2^k$, $s, t = 1, \ldots, n_1$, such that*

$$\psi(\otimes_{i=1}^{k} A_i) = A_1 \otimes \varphi(\otimes_{i=2}^{k} A_i) + I_{n_1} \otimes \left(\sum_{s,t=1}^{n_1} \varphi_{st}(a_{st} \otimes_{i=2}^{k} A_i) \right)$$

for all $A_1 = (a_{st}) \in \mathcal{M}_{n_1}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$, where $\mathbb{F} \cdot \mathcal{S}_2^k = \{\alpha A : \alpha \in \mathbb{F},\ A \in \mathcal{S}_2^k\}$.

Proof. By abuse of notation, we shall abbreviate $E_{ij}^{(n_1)}$ and $\bigotimes_{i=2}^k \mathcal{M}_{n_i}$ to E_{ij} and $\mathcal{M}$, respectively. We first prove the sufficiency. Let $\psi : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \bigotimes_{i=1}^k \mathcal{M}_{n_i}$ be the additive map defined by

$$\psi(\otimes_{i=1}^k A_i) = A_1 \otimes \varphi(\otimes_{i=2}^k A_i) + I_{n_1} \otimes \left(\sum_{s,t=1}^{n_1} \varphi_{st}(a_{st} \otimes_{i=2}^k A_i) \right) \tag{3.31}$$

for every $A_1 = (a_{st}) \in \mathcal{M}_{n_1}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$, where $\varphi : \mathcal{M} \to \mathcal{M}$ is a commuting linear map on $\mathcal{S}_2^k$ and $\varphi_{st} : \mathcal{M} \to \mathcal{M}$, $s, t = 1, \ldots, n_1$, are commuting additive maps on $\mathbb{F} \cdot \mathcal{S}_2^k$. Let $1 \leqslant s, t \leqslant n_1$ be integers. We claim that

$$[I_{n_1} \otimes \varphi_{st}(a \otimes_{i=2}^k A_i), \otimes_{i=1}^k A_i] = 0$$

for all $A_1 \in \mathcal{S}_{1,n_1}, \ldots, A_k \in \mathcal{S}_{k,n_k}$ and $a \in \mathbb{F}$. The result is clear when $a = 0$. Consider $a \neq 0$. Since φ_{st} is commuting on $\mathbb{F} \cdot \mathcal{S}_2^k$, we have

$$\begin{aligned}
(I_{n_1} \otimes \varphi_{st}(a \otimes_{i=2}^k A_i))(\otimes_{i=1}^k A_i) &= (I_{n_1} \otimes \varphi_{st}(a \otimes_{i=2}^k A_i))(a^{-1} A_1 \otimes (a \otimes_{i=2}^k A_i)) \\
&= a^{-1} A_1 \otimes \varphi_{st}(a \otimes_{i=2}^k A_i)(a \otimes_{i=2}^k A_i) \\
&= a^{-1} A_1 \otimes (a \otimes_{i=2}^k A_i)\, \varphi_{st}(a \otimes_{i=2}^k A_i) \\
&= (a^{-1} A_1 \otimes (a \otimes_{i=2}^k A_i))(I_{n_1} \otimes \varphi_{st}(a \otimes_{i=2}^k A_i)) \\
&= (\otimes_{i=1}^k A_i)(I_{n_1} \otimes \varphi_{st}(a \otimes_{i=2}^k A_i)).
\end{aligned}$$

Since φ is commuting on $\mathcal{S}_2^k$, $(A_1 \otimes \varphi(\otimes_{i=2}^k A_i))(\otimes_{i=1}^k A_i) = (\otimes_{i=1}^k A_i)(A_1 \otimes \varphi(\otimes_{i=2}^k A_i))$ for every $A_1 \in \mathcal{S}_{1,n_1}, \ldots, A_k \in \mathcal{S}_{k,n_k}$. It follows from (3.31) that ψ is commuting on $\mathcal{S}_1^k$.

We proceed to the necessity. Notice that $\{E_{ij} : i, j = 1, \ldots, n_1\}$ is a basis of $\mathcal{M}_{n_1}$. In view of (3.3), for each pair of integers $1 \leqslant s, t \leqslant n_1$, there exist additive maps $\phi_{ij}^{(st)} : \mathcal{M} \to \mathcal{M}$, $i, j = 1, \ldots, n_1$, such that

$$\varphi(E_{st} \otimes A) = \sum_{i,j=1}^{n_1} E_{ij} \otimes \phi_{ij}^{(st)}(A) \tag{3.32}$$

for every decomposable tensor $A \in D(\mathcal{M})$. Let $1 \leqslant s, t \leqslant n_1$ be integers. By (3.32), we obtain

$$\psi(E_{st}\otimes A)(E_{st}\otimes A)=\sum_{i,j=1}^{n_1}E_{ij}E_{st}\otimes\phi_{ij}^{(st)}(A)A=\sum_{i=1}^{n_1}E_{it}\otimes\phi_{is}^{(st)}(A)A,$$

$$(E_{st}\otimes A)\psi(E_{st}\otimes A)=\sum_{i,j=1}^{n_1}E_{st}E_{ij}\otimes A\phi_{ij}^{(st)}(A)=\sum_{j=1}^{n_1}E_{sj}\otimes A\phi_{tj}^{(st)}(A)$$

for all $A\in D(\mathcal{M})$. By virtue of $[\psi(E_{st}\otimes A),E_{st}\otimes A]=0$ for all $A\in\mathcal{S}_2^k$, we get $\sum_{i=1}^{n_1}E_{it}\otimes\phi_{is}^{(st)}(A)A+\sum_{j=1}^{n_1}E_{sj}\otimes A\phi_{tj}^{(st)}(A)=0$, so

$$E_{st}\otimes(\phi_{ss}^{(st)}(A)A-A\phi_{tt}^{(st)}(A))+\sum_{i=1,i\neq s}^{n_1}E_{it}\otimes\phi_{is}^{(st)}(A)A-\sum_{j=1,j\neq t}^{n_1}E_{sj}\otimes A\phi_{tj}^{(st)}(A)=0$$

for all $A\in\mathcal{S}_2^k$. Since $\{E_{ij}:i,j=1,\dots,n_1\}$ is a linearly independent set, it follows from Proposition 2.1.6 (a) that for each pair of integers $1\leqslant s,t\leqslant n_1$,

$$\phi_{ss}^{(st)}(A)A=A\,\phi_{tt}^{(st)}(A)\tag{3.33}$$

for all $A\in\mathcal{S}_2^k$, and

$$\phi_{is}^{(st)}(A)A=0\quad\text{for all }\ i\in\{1,\dots,n_1\}\backslash\{s\},$$

$$A\phi_{tj}^{(st)}(A)=0\quad\text{for all }\ j\in\{1,\dots,n_1\}\backslash\{t\}$$

for all $A\in\mathcal{S}_2^k$. By Theorem 3.1.2, we conclude that

$$\phi_{is}^{(st)}=0\quad\text{for all }\ i\in\{1,\dots,n_1\}\backslash\{s\},\tag{3.34}$$

$$\phi_{tj}^{(st)}=0\quad\text{for all }\ j\in\{1,\dots,n_1\}\backslash\{t\}\tag{3.35}$$

for all integers $1\leqslant s,t\leqslant n_1$. When $s=t$, by (3.33)–(3.35), for each integer $1\leqslant s\leqslant n_1$, we have

$$\phi_{is}^{(ss)}=0=\phi_{si}^{(ss)}\quad\text{for all }\ i\in\{1,\dots,n_1\}\backslash\{s\},\tag{3.36}$$

$$\phi_{ss}^{(ss)}\ \text{ is a commuting additive map on }\mathcal{S}_2^k.\tag{3.37}$$

Let $1\leqslant s,t\leqslant n_1$ be integers, $p\in\{1,\dots,n_1\}\backslash\{s,t\}$ and $A\in\mathcal{S}_2^k$. Since $[\psi(X),X]=0$ for all $X\in\{(E_{st}+E_{pp})\otimes A,E_{st}\otimes A,E_{pp}\otimes A\}$, we get $[\psi(E_{st}\otimes A),E_{pp}\otimes A]$

$+[\psi(E_{pp}\otimes A), E_{st}\otimes A]=0$. By virtue of (3.32), we obtain

$$\begin{aligned}
&[\psi(E_{st}\otimes A), E_{pp}\otimes A] := \psi(E_{st}\otimes A)(E_{pp}\otimes A)-(E_{pp}\otimes A)\psi(E_{st}\otimes A)\\
&=\sum_{i,j=1}^{n_1} E_{ij}E_{pp}\otimes \phi_{ij}^{(st)}(A)A-\sum_{i,j=1}^{n_1} E_{pp}E_{ij}\otimes A\phi_{ij}^{(st)}(A)\\
&=\sum_{i=1}^{n_1} E_{ip}\otimes \phi_{ip}^{(st)}(A)A-\sum_{j=1}^{n_1} E_{pj}\otimes A\phi_{pj}^{(st)}(A),
\end{aligned}$$

$$\begin{aligned}
&[\psi(E_{pp}\otimes A), E_{st}\otimes A] := \psi(E_{pp}\otimes A)(E_{st}\otimes A)-(E_{st}\otimes A)\psi(E_{pp}\otimes A)\\
&=\sum_{i,j=1}^{n_1} E_{ij}E_{st}\otimes \phi_{ij}^{(pp)}(A)A-\sum_{i,j=1}^{n_1} E_{st}E_{ij}\otimes A\phi_{ij}^{(pp)}(A)\\
&=\sum_{i=1}^{n_1} E_{it}\otimes \phi_{is}^{(pp)}(A)A-\sum_{j=1}^{n_1} E_{sj}\otimes A\phi_{tj}^{(pp)}(A).
\end{aligned}$$

Consequently,

$$\sum_{i=1}^{n_1} E_{ip}\otimes \phi_{ip}^{(st)}(A)A+\sum_{i=1}^{n_1} E_{it}\otimes \phi_{is}^{(pp)}(A)A=\sum_{j=1}^{n_1} E_{pj}\otimes A\phi_{pj}^{(st)}(A)+\sum_{j=1}^{n_1} E_{sj}\otimes A\phi_{tj}^{(pp)}(A) \tag{3.38}$$

for all $A\in\mathcal{S}_2^k$. Since $p\neq s,t$, it follows from (3.38) that

$$\begin{aligned}
E_{pp}\otimes(\phi_{pp}^{(st)}(A)A-A\phi_{pp}^{(st)}(A)) &+ \sum_{i=1,\, i\neq p,s}^{n_1} E_{ip}\otimes \phi_{ip}^{(st)}(A)A\\
&+ E_{pt}\otimes(\phi_{ps}^{(pp)}(A)A-A\phi_{pt}^{(st)}(A)) - \sum_{j=1,\, j\neq t,p}^{n_1} E_{pj}\otimes A\phi_{pj}^{(st)}(A)\\
&+ E_{sp}\otimes(\phi_{sp}^{(st)}(A)A-A\phi_{tp}^{(pp)}(A)) - \sum_{j=1,\, j\neq p,t}^{n_1} E_{sj}\otimes A\phi_{tj}^{(pp)}(A)\\
&+ E_{st}\otimes(\phi_{ss}^{(pp)}(A)A-A\phi_{tt}^{(pp)}(A)) + \sum_{i=1,\, i\neq s,p}^{n_1} E_{it}\otimes \phi_{is}^{(pp)}(A)A=0
\end{aligned}$$

for all $A\in\mathcal{S}_2^k$. It follows from Proposition 2.1.6 (a) that

(A) for each $i\in\{1,\ldots,n_1\}\backslash\{s,t\}$, $\phi_{ii}^{(st)}(A)A=A\phi_{ii}^{(st)}(A)$ for all $A\in\mathcal{S}_2^k$,

(B) for each $j\in\{1,\ldots,n_1\}\backslash\{s,t\}$, $i\in\{1,\ldots,n_1\}\backslash\{j,s\}$, $\phi_{ij}^{(st)}(A)A=0$ for all $A\in\mathcal{S}_2^k$,

(C) for each $i\in\{1,\ldots,n_1\}\backslash\{s,t\}$, $A\phi_{it}^{(st)}(A)=\phi_{is}^{(ii)}(A)A$ for all $A\in\mathcal{S}_2^k$,

(D) for each $j\in\{1,\ldots,n_1\}\backslash\{s,t\}$, $\phi_{sj}^{(st)}(A)A=A\phi_{tj}^{(jj)}(A)$ for all $A\in\mathcal{S}_2^k$.

In view of (A), we see that

$$\phi_{ii}^{(st)},\ i=1,\ldots,n_1,\ i\neq s,t,\ \text{are commuting additive maps on } \mathcal{S}_2^k. \tag{3.39}$$

By (B) and Theorem 3.1.2, we get

$$\phi_{ij}^{(st)}=0 \quad \text{for all } j\in\{1,\ldots,n_1\}\backslash\{s,t\} \text{ and } i\in\{1,\ldots,n_1\}\backslash\{j,s\}. \tag{3.40}$$

By virtue of (3.36), it follows from (C) that for each $i\in\{1,\ldots,n_1\}\backslash\{s,t\}$,

$$A\phi_{it}^{(st)}(A)=0 \tag{3.41}$$

for all $A\in\mathcal{S}_2^k$. By (3.41) and Theorem 3.1.2, we have

$$\phi_{it}^{(st)}=0 \quad \text{for all } i\in\{1,\ldots,n_1\}\backslash\{s,t\}. \tag{3.42}$$

Likewise, by (D), together with (3.36) and Theorem 3.1.2, we thus obtain

$$\phi_{sj}^{(st)}=0 \quad \text{for all } j\in\{1,\ldots,n_1\}\backslash\{s,t\}. \tag{3.43}$$

In view of (3.39) and (3.40), when $s=t$, we see that

$$\phi_{ii}^{(ss)},\ i=1,\ldots,n_1,\ i\neq s,\ \text{are commuting additive maps on } \mathcal{S}_2^k, \tag{3.44}$$

$$\phi_{ij}^{(ss)}=0 \quad \text{for all distinct } i,j\in\{1,\ldots,n_1\}\backslash\{s\}. \tag{3.45}$$

When $s=t$, in view of (3.32), (3.36), (3.37), (3.44) and (3.45), we have

$$\psi(E_{ss}\otimes A)=\sum_{i=1}^{n_1}E_{ii}\otimes\phi_{ii}^{(ss)}(A) \tag{3.46}$$

for all $A\in D(\mathcal{M})$, where $\phi_{ii}^{(ss)}$, $i=1,\ldots,n_1$, are commuting additive maps on $\mathcal{S}_2^k$.

When $s \neq t$, in view of (3.32), (3.34), (3.39), (3.40), (3.42) and (3.43), we obtain

$$\psi(E_{st} \otimes A) = E_{st} \otimes \phi_{st}^{(st)}(A) + \sum_{i=1}^{n_1} E_{ii} \otimes \phi_{ii}^{(st)}(A) \tag{3.47}$$

for all $A \in D(\mathcal{M})$, where $\phi_{ii}^{(st)}$, $i = 1, \ldots, n_1$, $i \neq s, t$, are commuting additive maps on $\mathcal{S}_2^k$. Putting (3.46) and (3.47) together, we conclude that for each pair of integers $1 \leqslant s, t \leqslant n_1$,

$$\psi(E_{st} \otimes A) = \begin{cases} \sum_{i=1}^{n_1} E_{ii} \otimes \phi_{ii}^{(ss)}(A) & \text{when } s = t, \\ E_{st} \otimes \phi_{st}^{(st)}(A) + \sum_{i=1}^{n_1} E_{ii} \otimes \phi_{ii}^{(st)}(A) & \text{when } s \neq t \end{cases} \tag{3.48}$$

for all $A \in D(\mathcal{M})$, where $\phi_{ii}^{(ss)}$, $i = 1, \ldots, n_1$, and $\phi_{jj}^{(st)}$, $j = 1, \ldots, n_1$, $j \neq s, t$, are commuting additive maps on $\mathcal{S}_2^k$.

We show that for each pair of distinct integers $1 \leqslant s, t \leqslant n_1$, there exists a commuting additive map $\varphi_{st} : \mathcal{M} \to \mathcal{M}$ on $\mathcal{S}_2^k$ such that

$$\phi_{ii}^{(st)} = \varphi_{st} \tag{3.49}$$

for all $i = 1, \ldots, n_1$. Let $1 \leqslant s, t \leqslant n_1$ be distinct integers and let $A \in \mathcal{S}_2^k$. Since $[\psi(X), X] = 0$ for $X \in \{(E_{ss} + E_{st}) \otimes A,\ E_{ss} \otimes A,\ E_{st} \otimes A\}$, we get $[\psi(E_{ss} \otimes A), E_{st} \otimes A] + [\psi(E_{st} \otimes A), E_{ss} \otimes A] = 0$. By (3.48),

$$\begin{aligned}
&[\psi(E_{ss} \otimes A), E_{st} \otimes A] := \psi(E_{ss} \otimes A)(E_{st} \otimes A) - (E_{st} \otimes A)\psi(E_{ss} \otimes A) \\
&= \sum_{i=1}^{n_1} E_{ii}E_{st} \otimes \phi_{ii}^{(ss)}(A)A - \sum_{i=1}^{n_1} E_{st}E_{ii} \otimes A\phi_{ii}^{(ss)}(A) \\
&= E_{st} \otimes (\phi_{ss}^{(ss)}(A)A - A\,\phi_{tt}^{(ss)}(A)),
\end{aligned}$$

$$\begin{aligned}
&[\psi(E_{st} \otimes A), E_{ss} \otimes A] := \psi(E_{st} \otimes A)(E_{ss} \otimes A) - (E_{ss} \otimes A)\psi(E_{st} \otimes A) \\
&= E_{st}E_{ss} \otimes \phi_{st}^{(st)}(A)A + \sum_{i=1}^{n_1} E_{ii}E_{ss} \otimes \phi_{ii}^{(st)}(A)A \\
&\quad - E_{ss}E_{st} \otimes A\phi_{st}^{(st)}(A) - \sum_{i=1}^{n_1} E_{ss}E_{ii} \otimes A\phi_{ii}^{(st)}(A) \\
&= E_{st} \otimes (-A\,\phi_{st}^{(st)}(A)) + E_{ss} \otimes (\phi_{ss}^{(st)}(A)A - A\,\phi_{ss}^{(st)}(A)).
\end{aligned}$$

Since E_{ss}, E_{st} are linearly independent, by Proposition 2.1.6 (a), $\phi_{ss}^{(st)}(A)A = A\,\phi_{ss}^{(st)}(A)$ for all $A \in \mathcal{S}_2^k$. Therefore $\phi_{ss}^{(st)}$ is a commuting additive map on $\mathcal{S}_2^k$. Moreover, by (3.33), we have

$$A\phi_{ss}^{(st)}(A) = \phi_{ss}^{(st)}(A)A = A\phi_{tt}^{(st)}(A)$$

for all $A \in \mathcal{S}_2^k$. It follows from Lemma 3.1.1 that

$$\phi_{tt}^{(st)} = \phi_{ss}^{(st)}. \tag{3.50}$$

Hence $\phi_{tt}^{(st)}$ is a commuting additive map on $\mathcal{S}_2^k$. Together with (3.39), we conclude that

$$\phi_{ii}^{(st)},\ i = 1, \ldots, n_1,\ \text{are commuting additive maps on } \mathcal{S}_2^k. \tag{3.51}$$

In view of (3.50), claim (3.49) is proved when $n_1 = 2$. Consider $n_1 \geqslant 3$. Let $p \in \{1, \ldots, n_1\} \backslash \{s, t\}$ and let $A \in \mathcal{S}_2^k$. Since $[\psi(X), X] = 0$ for $X \in \{(E_{st} + E_{sp}) \otimes A, E_{st} \otimes A, E_{sp} \otimes A\}$, it follows that $[\psi(E_{st} \otimes A), E_{sp} \otimes A] + [\psi(E_{sp} \otimes A), E_{st} \otimes A] = 0$. By (3.48), we have

$$\begin{aligned}
&[\psi(E_{st} \otimes A), E_{sp} \otimes A] := \psi(E_{st} \otimes A)(E_{sp} \otimes A) - (E_{sp} \otimes A)\psi(E_{st} \otimes A) \\
&= E_{st}E_{sp} \otimes \phi_{st}^{(st)}(A)A + \sum_{i=1}^{n_1} E_{ii}E_{sp} \otimes \phi_{ii}^{(st)}(A)A \\
&\quad - E_{sp}E_{st} \otimes A\phi_{st}^{(st)}(A) - \sum_{i=1}^{n_1} E_{sp}E_{ii} \otimes A\phi_{ii}^{(st)}(A) \\
&= E_{sp} \otimes (\phi_{ss}^{(st)}(A)A - A\,\phi_{pp}^{(st)}(A)).
\end{aligned}$$

Then by interchanging p and t, we get

$$[\psi(E_{sp} \otimes A), E_{st} \otimes A] = E_{st} \otimes (\phi_{ss}^{(sp)}(A)A - A\,\phi_{tt}^{(sp)}(A)).$$

Therefore,

$$E_{sp} \otimes (\phi_{ss}^{(st)}(A)A - A\,\phi_{pp}^{(st)}(A)) + E_{st} \otimes (\phi_{ss}^{(sp)}(A)A - A\,\phi_{tt}^{(sp)}(A)) = 0$$

for all $A \in \mathcal{S}_2^k$. Since E_{st}, E_{sp} are linearly independent, it follows from Proposition 2.1.6 (a) that

$$\phi_{ss}^{(st)}(A)A = A\,\phi_{pp}^{(st)}(A)$$

for all $A \in \mathcal{S}_2^k$. By (3.51), we obtain

$$\phi_{ss}^{(st)}(A)\,A = \phi_{pp}^{(st)}(A)A$$

for all $A \in \mathcal{S}_2^k$. By Lemma 3.1.1, $\phi_{pp}^{(st)} = \phi_{ss}^{(st)}$ for every $p \in \{1, \ldots, n_1\} \backslash \{s, t\}$. Together with (3.50) and (3.51), we conclude that $\phi_{ii}^{(st)} = \varphi_{st}$ for $i = 1, \ldots, n_1$, where $\varphi_{st} = \phi_{ss}^{(st)}$ is a commuting additive map on $\mathcal{S}_2^k$. Hence claim (3.49) is proved.

Next, we claim that for each integer $1 \leqslant s \leqslant n_1$, there exists a commuting additive map $\varphi_{ss} : \mathcal{M} \to \mathcal{M}$ on $\mathcal{S}_2^k$ such that

$$\phi_{ii}^{(ss)} = \varphi_{ss} \quad \text{for all } \ i \in \{1, \ldots, n_1\} \backslash \{s\}. \tag{3.52}$$

Let $1 \leqslant s \leqslant n_1$ be an integer. We show that

$$\phi_{pp}^{(ss)} = \phi_{qq}^{(ss)}$$

for all integers $p, q \in \{1, \ldots, n_1\} \backslash \{s\}$. If $n_1 = 2$, then there is nothing to show. Consider now $n_1 \geqslant 3$. Let $p, q \in \{1, \ldots, n_1\} \backslash \{s\}$ be distinct and $A \in \mathcal{S}_2^k$. Note that $[\psi(X), X] = 0$ for all $X \in \{(E_{pq} + E_{ss}) \otimes A,\ E_{pq} \otimes A,\ E_{ss} \otimes A\}$ leads to $[\psi(E_{pq} \otimes A), E_{ss} \otimes A] + [\psi(E_{ss} \otimes A), E_{pq} \otimes A] = 0$. It follows from (3.48) and (3.49) that

$$\begin{aligned}
&[\psi(E_{pq} \otimes A), E_{ss} \otimes A] := \psi(E_{pq} \otimes A)(E_{ss} \otimes A) - (E_{ss} \otimes A)\psi(E_{pq} \otimes A) \\
&= E_{pq}E_{ss} \otimes \phi_{pq}^{(pq)}(A)A + \sum_{i=1}^{n_1} E_{ii}E_{ss} \otimes \phi_{ii}^{(pq)}(A)A \\
&\quad - E_{ss}E_{pq} \otimes A\phi_{pq}^{(pq)}(A) - \sum_{i=1}^{n_1} E_{ss}E_{ii} \otimes A\phi_{ii}^{(pq)}(A) \\
&= E_{ss} \otimes (\phi_{ss}^{(pq)}(A)A - A\,\phi_{ss}^{(pq)}(A)) \\
&= E_{ss} \otimes (\varphi_{pq}(A)A - A\,\varphi_{pq}(A)),
\end{aligned}$$

$$\begin{aligned}
&[\psi(E_{ss} \otimes A), E_{pq} \otimes A] := \psi(E_{ss} \otimes A)(E_{pq} \otimes A) - (E_{pq} \otimes A)\psi(E_{ss} \otimes A) \\
&= \sum_{i=1}^{n_1} E_{ii}E_{pq} \otimes \phi_{ii}^{(ss)}(A)A - \sum_{i=1}^{n_1} E_{pq}E_{ii} \otimes A\phi_{ii}^{(ss)}(A) \\
&= E_{pq} \otimes (\phi_{pp}^{(ss)}(A)A - A\,\phi_{qq}^{(ss)}(A)).
\end{aligned}$$

Consequently,

$$E_{ss} \otimes (\varphi_{pq}(A)A - A\,\varphi_{pq}(A)) + E_{pq} \otimes (\phi_{pp}^{(ss)}(A)A - A\,\phi_{qq}^{(ss)}(A)) = 0$$

for all $A \in \mathcal{S}_2^k$. Since E_{ss}, E_{pq} are linearly independent, by Proposition 2.1.6 (a), we get $\phi_{pp}^{(ss)}(A)A = A\,\phi_{qq}^{(ss)}(A)$ for all $A \in \mathcal{S}_2^k$. It follows from (3.44) that $\phi_{pp}^{(ss)}(A)A = \phi_{qq}^{(ss)}(A)A$ for all $A \in \mathcal{S}_2^k$. Thus $\phi_{pp}^{(ss)} = \phi_{qq}^{(ss)}$ by Lemma 3.1.1. Hence, together with (3.44), claim (3.52) is proved.

We now show that for each integer $1 \leqslant s \leqslant n_1$,

$$\phi_{ss}^{(ss)} = \varphi_{ss} + \phi_{sj}^{(sj)} \quad \text{for all } j \in \{1, \ldots, n_1\} \backslash \{s\}, \tag{3.53}$$

$$\phi_{sj}^{(sj)},\ j = 1, \ldots, n_1,\ j \neq s, \text{ are commuting additive maps on } \mathcal{S}_2^k. \tag{3.54}$$

Let $1 \leqslant s, j \leqslant n_1$ be distinct integers and let $A \in \mathcal{S}_2^k$. Since $[\psi(X), X] = 0$ for all $X \in \{E_{ss} \otimes A, E_{sj} \otimes A, (E_{ss} + E_{sj}) \otimes A\}$, we get $[\psi(E_{ss} \otimes A), E_{sj} \otimes A] + [\psi(E_{sj} \otimes A), E_{ss} \otimes A] = 0$. It follows from (3.48), (3.49) and (3.52) that

$$\begin{aligned}
&[\psi(E_{ss} \otimes A), E_{sj} \otimes A] := \psi(E_{ss} \otimes A)(E_{sj} \otimes A) - (E_{sj} \otimes A)\psi(E_{ss} \otimes A) \\
&= \sum_{i=1}^{n_1} E_{ii}E_{sj} \otimes \phi_{ii}^{(ss)}(A)A - \sum_{i=1}^{n_1} E_{sj}E_{ii} \otimes A\phi_{ii}^{(ss)}(A) \\
&= E_{sj} \otimes (\phi_{ss}^{(ss)}(A)A - A\,\phi_{jj}^{(ss)}(A)) \\
&= E_{sj} \otimes (\phi_{ss}^{(ss)}(A)A - A\,\varphi_{ss}(A)),
\end{aligned}$$

$$\begin{aligned}
&[\psi(E_{sj} \otimes A), E_{ss} \otimes A] := \psi(E_{sj} \otimes A)(E_{ss} \otimes A) - (E_{ss} \otimes A)\psi(E_{sj} \otimes A) \\
&= E_{sj}E_{ss} \otimes \phi_{sj}^{(sj)}(A)A + \sum_{i=1}^{n_1} E_{ii}E_{ss} \otimes \phi_{ii}^{(sj)}(A)A \\
&\quad - E_{ss}E_{sj} \otimes A\phi_{sj}^{(sj)}(A) - \sum_{i=1}^{n_1} E_{ss}E_{ii} \otimes A\phi_{ii}^{(sj)}(A) \\
&= E_{ss} \otimes (\phi_{ss}^{(sj)}(A)A - A\phi_{ss}^{(sj)}(A)) - E_{sj} \otimes A\phi_{sj}^{(sj)}(A) \\
&= E_{ss} \otimes (\varphi_{sj}(A)A - A\varphi_{sj}(A)) + E_{sj} \otimes (-A\phi_{sj}^{(sj)}(A)) \\
&= E_{sj} \otimes (-A\phi_{sj}^{(sj)}(A)).
\end{aligned}$$

Therefore,

$$E_{sj} \otimes (\phi_{ss}^{(ss)}(A)A - A\,\varphi_{ss}(A) - A\,\phi_{sj}^{(sj)}(A)) = 0$$

for all $A \in \mathcal{S}_2^k$. Since $\phi_{ss}^{(ss)}$ is commuting on $\mathcal{S}_2^k$ by (3.37), using Proposition 2.1.6 (a), we have

$$A\,\phi_{ss}^{(ss)}(A) = A\,(\varphi_{ss} + \phi_{sj}^{(sj)})(A)$$

for all $A \in \mathcal{S}_2^k$. By Lemma 3.1.1, we obtain $\phi_{ss}^{(ss)} = \varphi_{ss} + \phi_{sj}^{(sj)}$, and hence, assertion (3.53) is proved. Notice that $\phi_{sj}^{(sj)} = \phi_{ss}^{(ss)} - \varphi_{ss}$. By (3.37) and (3.52), we see that $\phi_{sj}^{(sj)}$, $j = 1, \ldots, n_1$, $j \neq s$, are commuting additive maps on $\mathcal{S}_2^k$. Thus assertion (3.54) is claimed. Moreover, in view of (3.53), for each integer $1 \leqslant s \leqslant n$, we have

$$\phi_{si}^{(si)} = \phi_{sj}^{(sj)} \quad \text{for all } i, j \in \{1, \ldots, n_1\} \backslash \{s\}. \tag{3.55}$$

Next, we show that there exists a commuting additive map $\varphi : \mathcal{M} \to \mathcal{M}$ on $\mathcal{S}_2^k$ such that

$$\phi_{st}^{(st)} = \varphi \tag{3.56}$$

for all distinct integers $1 \leqslant s, t \leqslant n_1$. Let $1 \leqslant s \leqslant n_1$ be an integer, $i \in \{1, \ldots, n_1\} \backslash \{s\}$ and $A \in \mathcal{S}_2^k$. Note that $[\psi(X), X] = 0$ for all $X \in \{(E_{ss} + E_{is}) \otimes A, E_{ss} \otimes A, E_{is} \otimes A\}$ leads to $[\psi(E_{ss} \otimes A), E_{is} \otimes A] + [\psi(E_{is} \otimes A), E_{ss} \otimes A] = 0$. It follows from (3.48), (3.49) and (3.52) that

$$\begin{aligned}
&[\psi(E_{ss} \otimes A), E_{is} \otimes A] := \psi(E_{ss} \otimes A)(E_{is} \otimes A) - (E_{is} \otimes A)\psi(E_{ss} \otimes A) \\
&= \sum_{j=1}^{n_1} E_{jj}E_{is} \otimes \phi_{jj}^{(ss)}(A)A - \sum_{j=1}^{n_1} E_{is}E_{jj} \otimes A\phi_{jj}^{(ss)}(A) \\
&= E_{is} \otimes (\phi_{ii}^{(ss)}(A)A - A\phi_{ss}^{(ss)}(A)) \\
&= E_{is} \otimes (\varphi_{ss}(A)A - A\phi_{ss}^{(ss)}(A)),
\end{aligned}$$

$$\begin{aligned}
&[\psi(E_{is} \otimes A), E_{ss} \otimes A] := \psi(E_{is} \otimes A)(E_{ss} \otimes A) - (E_{ss} \otimes A)\psi(E_{is} \otimes A) \\
&= E_{is}E_{ss} \otimes \phi_{is}^{(is)}(A)A + \sum_{j=1}^{n_1} E_{jj}E_{ss} \otimes \phi_{jj}^{(is)}(A)A \\
&\quad - E_{ss}E_{is} \otimes A\phi_{is}^{(is)}(A) - \sum_{j=1}^{n_1} E_{ss}E_{jj} \otimes A\phi_{jj}^{(is)}(A)
\end{aligned}$$

$$= E_{is} \otimes \phi_{is}^{(is)}(A)A + E_{ss} \otimes (\phi_{ss}^{(is)}(A)A - A\phi_{ss}^{(is)}(A))$$
$$= E_{is} \otimes \phi_{is}^{(is)}(A)A + E_{ss} \otimes (\varphi_{is}(A)A - A\varphi_{is}(A))$$
$$= E_{is} \otimes \phi_{is}^{(is)}(A)A,$$

so $E_{is} \otimes (\varphi_{ss}(A)A - A\phi_{ss}^{(ss)}(A) + \phi_{is}^{(is)}(A)A) = 0$ for all $A \in \mathcal{S}_2^k$. Since $\phi_{ss}^{(ss)}$ is commuting on $\mathcal{S}_2^k$ by (3.37), using Proposition 2.1.6 (a), we have $\phi_{ss}^{(ss)}(A)A = (\varphi_{ss} + \phi_{is}^{(is)})(A)A$ for all $A \in \mathcal{S}_2^k$. By Lemma 3.1.1, we obtain

$$\phi_{ss}^{(ss)} = \varphi_{ss} + \phi_{is}^{(is)} \quad \text{for all } i \in \{1, \ldots, n_1\} \backslash \{s\}. \tag{3.57}$$

By virtue of (3.53) and (3.57), for each integer $1 \leqslant s \leqslant n_1$,

$$\phi_{sj}^{(sj)} = \phi_{is}^{(is)} \quad \text{for all } i, j \in \{1, \ldots, n_1\} \backslash \{s\}. \tag{3.58}$$

Putting (3.55) together with (3.58) and using the same argument as in (3.30), we obtain $\phi_{st}^{(st)} = \phi_{pq}^{(pq)}$ for all integers $1 \leqslant s, t, p, q \leqslant n_1$ with $s \neq t$ and $p \neq q$. Hence claim (3.56) is proved.

Summarizing from (3.48), (3.49), (3.52) and (3.56), we conclude that

$$\psi(E_{st} \otimes A) = E_{st} \otimes \varphi(A) + I_{n_1} \otimes \varphi_{st}(A) \tag{3.59}$$

for all integers $1 \leqslant s, t \leqslant n_1$ and $A \in D(\mathcal{M})$. We further show that φ is a linear map and φ_{st}, $s, t = 1, \ldots, n_1$, are commuting additive maps on $\mathbb{F} \cdot \mathcal{S}_2^k$. To see φ is linear, let $A \in \mathcal{S}_2^k$ and $\lambda \in \mathbb{F}$. By virtue of $[\psi(X), X] = 0$ for all $X \in \{(E_{21} + \lambda E_{12}) \otimes A, E_{21} \otimes A, \lambda E_{12} \otimes A\}$, we obtain $[\psi(E_{12} \otimes \lambda A), E_{21} \otimes A] + [\psi(E_{21} \otimes A), E_{12} \otimes \lambda A] = 0$. It follows from (3.59) that

$$[\psi(E_{12} \otimes \lambda A), E_{21} \otimes A] := \psi(E_{12} \otimes \lambda A)(E_{21} \otimes A) - (E_{21} \otimes A)\psi(E_{12} \otimes \lambda A)$$
$$= E_{12}E_{21} \otimes \varphi(\lambda A)A + E_{21} \otimes \varphi_{12}(\lambda A)A - E_{21}E_{12} \otimes A\varphi(\lambda A) - E_{21} \otimes A\varphi_{12}(\lambda A)$$
$$= E_{11} \otimes \varphi(\lambda A)A + E_{21} \otimes (\varphi_{12}(\lambda A)A - A\varphi_{12}(\lambda A)) - E_{22} \otimes A\varphi(\lambda A),$$

$$
\begin{aligned}
&[\psi(E_{21}\otimes A), E_{12}\otimes \lambda A] := \psi(E_{21}\otimes A)(E_{12}\otimes \lambda A) - (E_{12}\otimes \lambda A)\psi(E_{21}\otimes A)\\
&= E_{21}E_{12}\otimes \varphi(A)(\lambda A) + E_{12}\otimes \lambda\varphi_{12}(A)A - E_{12}E_{21}\otimes \lambda A\varphi(A) - E_{12}\otimes \lambda A\varphi_{12}(A)\\
&= -E_{11}\otimes \lambda A\varphi(A) + E_{12}\otimes (\lambda\varphi_{12}(A)A - \lambda A\varphi_{12}(A)) + E_{22}\otimes \lambda\varphi(A)A\\
&= E_{11}\otimes (-\lambda A\varphi(A)) + E_{22}\otimes \lambda\varphi(A)A.
\end{aligned}
$$

Since $E_{11}, E_{12}, E_{21}, E_{22}$ are linearly independent, by Proposition 2.1.6 (a), we get

$$\varphi(\lambda A)\, A = \lambda A\varphi(A) \tag{3.60}$$

for all $\lambda \in \mathbb{F}$ and $A \in \mathcal{S}_2^k$. Let $\alpha \in \mathbb{F}$ and let $\zeta_\alpha : \mathcal{M} \to \mathcal{M}$ be the additive map defined by

$$\zeta_\alpha(X) = \varphi(\alpha X) - \alpha\varphi(X)$$

for all $X \in \mathcal{M}$. Let $A \in \mathcal{S}_2^k$. Note that

$$\zeta_\alpha(A)A = \varphi(\alpha A)A - \alpha\varphi(A)A = \varphi(\alpha A)A - \alpha A\varphi(A)$$

since $\varphi(A)A = A\varphi(A)$. It follows from (3.60) that $\zeta_\alpha(A)A = 0$ for all $A \in \mathcal{S}_2^k$. By Theorem 3.1.2, $\zeta_\alpha = 0$ for all $\alpha \in \mathbb{F}$. Then $\varphi(\alpha X) = \alpha\varphi(X)$ for all $\alpha \in \mathbb{F}$ and $X \in \mathcal{M}$. Thus φ is linear.

We next verify that φ_{st}, $s, t = 1, \ldots, n_1$, are commuting additive maps on $\mathbb{F}\cdot\mathcal{S}_2^k$. Let $1 \leqslant s, t \leqslant n_1$ be integers, $\alpha \in \mathbb{F}$ and $A \in \mathcal{S}_2^k$. By (3.59) and the linearity of φ, we have

$$\psi(\alpha E_{st}\otimes A) = \psi(E_{st}\otimes \alpha A) = \alpha E_{st}\otimes \varphi(A) + I_{n_1}\otimes \varphi_{st}(\alpha A).$$

Since $[\psi(\alpha E_{st}\otimes A), \alpha E_{st}\otimes A] = 0$ and

$$
\begin{aligned}
&[\alpha E_{st}\otimes \varphi(A), \alpha E_{st}\otimes A] := (\alpha E_{st}\otimes \varphi(A))(\alpha E_{st}\otimes A) - (\alpha E_{st}\otimes A)(\alpha E_{st}\otimes \varphi(A))\\
&= (\alpha\alpha E_{st}E_{st})\otimes \varphi(A)A \; - \; (\alpha\alpha E_{st}E_{st})\otimes A\varphi(A)\\
&= (\alpha\alpha E_{st}E_{st})\otimes \varphi(A)A \; - \; (\alpha\alpha E_{st}E_{st})\otimes \varphi(A)A\\
&= 0,
\end{aligned}
$$

it follows from Proposition 2.1.6 (a) that

$$\begin{aligned}
&[I_{n_1} \otimes \varphi_{st}(\alpha A), E_{st} \otimes \alpha A] = 0 \\
&\implies (I_{n_1} \otimes \varphi_{st}(\alpha A))(E_{st} \otimes \alpha A) - (E_{st} \otimes \alpha A)(I_{n_1} \otimes \varphi_{st}(\alpha A)) = 0 \\
&\implies E_{st} \otimes (\varphi_{st}(\alpha A)(\alpha A) - (\alpha A)\varphi_{st}(\alpha A)) = 0 \\
&\implies \varphi_{st}(\alpha A)(\alpha A) = (\alpha A)\varphi_{st}(\alpha A)
\end{aligned}$$

for all $\alpha \in \mathbb{F}$ and $A \in \mathcal{S}_2^k$. Hence φ_{st}, $s, t = 1, \ldots, n_1$, are commuting additive maps on $\mathbb{F} \cdot \mathcal{S}_2^k$.

Let $A_1 = \sum_{s,t=1}^{n_1} a_{st} E_{st} \in \mathcal{M}_{n_1}$, $a_{st} \in \mathbb{F}$ and let $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$. By (3.59),

$$\begin{aligned}
\psi(\otimes_{i=1}^k A_i) &= \psi\left(\sum_{s,t=1}^{n_1} (E_{st} \otimes a_{st}(\otimes_{i=2}^k A_i))\right) \\
&= \sum_{s,t=1}^{n_1} \psi(E_{st} \otimes a_{st}(\otimes_{i=2}^k A_i)) \\
&= \sum_{s,t=1}^{n_1} (a_{st} E_{st} \otimes \varphi(\otimes_{i=2}^k A_i) + I_{n_1} \otimes \varphi_{st}(a_{st} \otimes_{i=2}^k A_i)) \\
&= A_1 \otimes \varphi(\otimes_{i=2}^k A_i) + I_{n_1} \otimes \left(\sum_{s,t=1}^{n_1} \varphi_{st}(a_{st} \otimes_{i=2}^k A_i)\right).
\end{aligned}$$

This completes the proof. □

The following lemma will be needed to prove the results in the next section.

Lemma 3.1.5. (Chooi et al., 2019, Lemma 2.7) *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Then $\tau : \mathcal{M}_n \to \mathbb{F}$ is a linear functional if and only if there exists a matrix $H \in \mathcal{M}_n$ such that*

$$\tau(A) = \operatorname{tr}(H^t A)$$

for all $A \in \mathcal{M}_n$.

3.2 Main Results

We are now ready to prove the main results.

Theorem 3.2.1. *Let $\mathbb{F}$ be a field and let $m, n \geqslant 2$ be integers. Let $\psi : \mathcal{M}_m \bigotimes \mathcal{M}_n \to \mathcal{M}_m \bigotimes \mathcal{M}_n$ be an additive map. Then ψ is a commuting map on $\mathcal{S}_1^2$ if and only if there exist a scalar $\lambda \in \mathbb{F}$, an additive map $\mu : \mathcal{M}_m \bigotimes \mathcal{M}_n \to \mathbb{F}$ and matrices $K \in \mathcal{M}_m$, $H \in \mathcal{M}_n$ such that*

$$\psi(A \otimes B) = \lambda(A \otimes B) + \mu(A \otimes B)I_{mn} + \text{tr}\,(H^t B)(A \otimes I_n) + \text{tr}\,(K^t A)(I_m \otimes B) \quad (3.61)$$

for all $A \in \mathcal{M}_m$ and $B \in \mathcal{M}_n$.

Proof. We first prove the sufficiency. Let $\psi : \mathcal{M}_m \bigotimes \mathcal{M}_n \to \mathcal{M}_m \bigotimes \mathcal{M}_n$ be the additive map defined by

$$\psi(A \otimes B) = \lambda(A \otimes B) + \mu(A \otimes B)I_{mn} + \text{tr}\,(H^t B)(A \otimes I_n) + \text{tr}\,(K^t A)(I_m \otimes B)$$

for all $A \in \mathcal{M}_m$ and $B \in \mathcal{M}_n$, where $\lambda \in \mathbb{F}$, $K \in \mathcal{M}_m$, $H \in \mathcal{M}_n$ and $\mu : \mathcal{M}_m \bigotimes \mathcal{M}_n \to \mathbb{F}$ is an additive map. Let $A \in \mathcal{S}_{1,m}$ and $B \in \mathcal{S}_{2,n}$. Note that

$$\begin{aligned}
(\text{tr}\,(H^t B)(A \otimes I_n))(A \otimes B) &= \text{tr}\,(H^t B)(AA \otimes I_n B) \\
&= \text{tr}\,(H^t B)(AA \otimes B I_n) \\
&= \text{tr}\,(H^t B)((A \otimes B)(A \otimes I_n)) \\
&= (A \otimes B)(\text{tr}\,(H^t B)(A \otimes I_n)),
\end{aligned}$$

$$\begin{aligned}
(\text{tr}\,(K^t A)(I_m \otimes B))(A \otimes B) &= \text{tr}\,(K^t A)(I_m A \otimes BB) \\
&= \text{tr}\,(K^t A)(A I_m \otimes BB) \\
&= \text{tr}\,(K^t A)((A \otimes B)(I_m \otimes B)) \\
&= (A \otimes B)(\text{tr}\,(K^t A)(I_m \otimes B)).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\psi(A \otimes B)(A \otimes B) = {}& (\lambda(A \otimes B))(A \otimes B) + (\mu(A \otimes B)I_{mn})(A \otimes B) \\
&+ (\text{tr}\,(H^t B)(A \otimes I_n))(A \otimes B) + (\text{tr}\,(K^t A)(I_m \otimes B))(A \otimes B)
\end{aligned}$$

$$
\begin{aligned}
&= (A\otimes B)(\lambda(A\otimes B)) + (A\otimes B)(\mu(A\otimes B)I_{mn})\\
&\quad + (A\otimes B)(\mathrm{tr}\,(H^tB)(A\otimes I_n)) + (A\otimes B)(\mathrm{tr}\,(K^tA)(I_m\otimes B))\\
&= (A\otimes B)\psi(A\otimes B)
\end{aligned}
$$

for all $A\in\mathcal{S}_{1,m}$ and $B\in\mathcal{S}_{2,n}$. Hence ψ is commuting on $\mathcal{S}_1^2$.

For the necessity, it follows from Lemma 3.1.4 that there exist a commuting linear map $\varphi:\mathcal{M}_n\to\mathcal{M}_n$ on $\mathcal{S}_{2,n}=\{\,E_{st}^{(n)}+\alpha E_{pq}^{(n)}\in\mathcal{M}_n:\alpha\in\mathbb{F}$ and $1\leqslant p,q,s,t\leqslant n_i$ are not all distinct$\}$ and commuting additive maps $\varphi_{ij}:\mathcal{M}_n\to\mathcal{M}_n$ on $\mathbb{F}\cdot\mathcal{S}_{2,n}$, $i,j=1,\ldots,m$, such that

$$
\psi(A\otimes B) = A\otimes\varphi(B) +\, I_m\otimes\left(\sum_{i,j=1}^{m}\varphi_{ij}(a_{ij}B)\right)
$$

for all $A=(a_{ij})\in\mathcal{M}_m$ and $B\in\mathcal{M}_n$, where $\mathbb{F}\cdot\mathcal{S}_{2,n}=\{\alpha X:\alpha\in\mathbb{F},\,X\in\mathcal{S}_{2,n}\}$. By Theorem 3.1.3, there exist scalars $\lambda,\lambda_{ij}\in\mathbb{F}$, $i,j=1,\ldots,m$, a linear functional $\tau:\mathcal{M}_n\to\mathbb{F}$ and additive maps $\tau_{ij}:\mathcal{M}_n\to\mathbb{F}$, $i,j=1,\ldots,m$, such that

$$
\varphi(B)=\lambda B+\tau(B)I_n \quad\text{and}\quad \varphi_{ij}(B)=\lambda_{ij}B+\tau_{ij}(B)I_n
$$

for all $B\in\mathcal{M}_n$ and $i,j=1,\ldots,m$. Then by (2.8),

$$
\begin{aligned}
\psi(A\otimes B) &= A\otimes(\lambda B+\tau(B)I_n) + I_m\otimes\left(\sum_{i,j=1}^{m}\lambda_{ij}a_{ij}B+\sum_{i,j=1}^{m}\tau_{ij}(a_{ij}B)I_n\right)\\
&= \lambda(A\otimes B)+\tau(B)(A\otimes I_n)+\left(\sum_{i,j=1}^{m}\lambda_{ij}a_{ij}\right)(I_m\otimes B)+\left(\sum_{i,j=1}^{m}\tau_{ij}(a_{ij}B)\right)I_{mn}
\end{aligned}
$$

for all $A=(a_{ij})\in\mathcal{M}_m$ and $B\in\mathcal{M}_n$. Let $\mu:\mathcal{M}_m\bigotimes\mathcal{M}_n\to\mathbb{F}$ be the additive map defined by

$$
\mu(A\otimes B)=\sum_{i,j=1}^{m}\tau_{ij}(a_{ij}B)
$$

for all $A=(a_{ij})\in\mathcal{M}_m$ and $B\in\mathcal{M}_n$. We first check that μ is well defined in two cases.

Case I: $A\otimes B=0$. Since μ is additive, $\mu(0)=0$. On the other hand, if $A\otimes B=0$, then $A=0$ or $B=0$ by Proposition 2.1.5. Therefore, $\mu(A\otimes B)=\sum_{i,j=1}^{m}\tau_{ij}(0)=0$ as τ_{ij}, $i,j=1,\ldots,m$, are additive.

Case II: $A \otimes B \neq 0$. Let $A' \in \mathcal{M}_m$ and $B' \in \mathcal{M}_n$ such that $A' \otimes B' = A \otimes B$. By Proposition 2.1.5, $A' = \alpha A$ and $B' = \alpha^{-1} B$ for some nonzero $\alpha \in \mathbb{F}$. Then $\mu(A' \otimes B') = \sum_{i,j=1}^{m} \tau_{ij}\left((\alpha a_{ij})(\alpha^{-1}B)\right) = \sum_{i,j=1}^{m} \tau_{ij}(a_{ij}B) = \mu(A \otimes B)$.

Hence, μ is well defined. Since $\tau : \mathcal{M}_n \to \mathbb{F}$ is linear, it follows from Lemma 3.1.5 that there exists $H \in \mathcal{M}_n$ such that $\tau(B) = \mathrm{tr}\,(H^t B)$ for all $B \in \mathcal{M}_n$. Consequently,

$$\psi(A \otimes B) = \lambda(A \otimes B) + \mu(A \otimes B) I_{mn} + \mathrm{tr}\,(H^t B)(A \otimes I_n) + \mathrm{tr}\,(K^t A)(I_m \otimes B)$$

for all $A = (a_{ij}) \in \mathcal{M}_m$ and $B \in \mathcal{M}_n$, where $K = (\lambda_{ij}) \in \mathcal{M}_m$. We are done. $\square$

Let $k, n_1, \ldots, n_k \geqslant 2$ be integers. For each integer $1 \leqslant h \leqslant k$, we denote by $Q_{h,k}$ the totality of strictly increasing sequences $\alpha = (\alpha_i)$ of h integers $\alpha_1 < \cdots < \alpha_h$ chosen from $1, \ldots, k$. Let $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$ and let $\alpha = (\alpha_i) \in Q_{h,k}$. We designate

$$(\otimes_{i=1}^{k} A_i)_\alpha = \otimes_{i=1}^{k} B_i \in \mathcal{M}_{n_1 \cdots n_k}, \tag{3.62}$$

where

$$B_i = \begin{cases} I_{n_i} & \text{if } i \in \{\alpha_1, \ldots, \alpha_h\}, \\ A_i & \text{if } i \notin \{\alpha_1, \ldots, \alpha_h\} \end{cases}$$

for $i = 1, \ldots, k$. Evidently, $[(\otimes_{i=1}^{k} A_i)_\alpha, \otimes_{i=1}^{k} A_i] = 0$ for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$. This is because $(\otimes_{i=1}^{k} A_i)_\alpha (\otimes_{i=1}^{k} A_i) = (\otimes_{i=1}^{k} B_i)(\otimes_{i=1}^{k} A_i) = \otimes_{i=1}^{k} B_i A_i = \otimes_{i=1}^{k} A_i B_i = (\otimes_{i=1}^{k} A_i)(\otimes_{i=1}^{k} B_i) = (\otimes_{i=1}^{k} A_i)(\otimes_{i=1}^{k} A_i)_\alpha$ for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$.

Theorem 3.2.2. *Let $\mathbb{F}$ be a field and let $k, n_1, \ldots, n_k \geqslant 2$ be integers. Let $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ be an additive map. Then ψ is a commuting map on $\mathcal{S}_1^k$ if and only if there exist a scalar $\lambda \in \mathbb{F}$, an additive map $\mu : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \mathbb{F}$ and matrices $H_\alpha \in \mathcal{M}_{n_{\alpha_1} \cdots n_{\alpha_h}}$ for each $\alpha = (\alpha_i) \in Q_{h,k}$, $h = 1, \ldots, k-1$, such that*

$$\psi(\otimes_{i=1}^{k} A_i) = \lambda(\otimes_{i=1}^{k} A_i) + \mu(\otimes_{i=1}^{k} A_i) I_{n_1 \cdots n_k} + \sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i) \in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t (\otimes_{i=1}^{h} A_{\alpha_i}))(\otimes_{i=1}^{k} A_i)_\alpha$$

for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$.

Proof. We first prove the sufficiency. Let $\psi : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \bigotimes_{i=1}^k \mathcal{M}_{n_i}$ be the additive map defined by

$$\psi(\otimes_{i=1}^k A_i) = \lambda(\otimes_{i=1}^k A_i) + \mu(\otimes_{i=1}^k A_i) I_{n_1\cdots n_k} + \sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=1}^k A_i)_\alpha$$

for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$, where $\lambda \in \mathbb{F}$, $H_\alpha \in \mathcal{M}_{n_{\alpha_1}\cdots n_{\alpha_h}}$ for each $\alpha = (\alpha_i) \in Q_{h,k}$, $h = 1, \ldots, k-1$, and $\mu : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \mathbb{F}$ is an additive map. Let $A_i \in \mathcal{S}_{i,n_i}$ for $i = 1, \ldots, k$. Since $[(\otimes_{i=1}^k A_i)_\alpha, \otimes_{i=1}^k A_i] = 0$, it follows that

$$\begin{aligned}
&\left(\sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=1}^k A_i)_\alpha\right)(\otimes_{i=1}^k A_i)\\
&= \sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))((\otimes_{i=1}^k A_i)_\alpha(\otimes_{i=1}^k A_i))\\
&= \sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))((\otimes_{i=1}^k A_i)(\otimes_{i=1}^k A_i)_\alpha)\\
&= (\otimes_{i=1}^k A_i)\left(\sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=1}^k A_i)_\alpha\right).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\psi(\otimes_{i=1}^k A_i)(\otimes_{i=1}^k A_i) &= (\lambda(\otimes_{i=1}^k A_i))(\otimes_{i=1}^k A_i) + (\mu(\otimes_{i=1}^k A_i) I_{n_1\cdots n_k})(\otimes_{i=1}^k A_i)\\
&\quad + \left(\sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=1}^k A_i)_\alpha\right)(\otimes_{i=1}^k A_i)\\
&= (\otimes_{i=1}^k A_i)(\lambda(\otimes_{i=1}^k A_i)) + (\otimes_{i=1}^k A_i)(\mu(\otimes_{i=1}^k A_i) I_{n_1\cdots n_k})\\
&\quad + (\otimes_{i=1}^k A_i)\left(\sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=1}^k A_i)_\alpha\right)\\
&= (\otimes_{i=1}^k A_i)\psi(\otimes_{i=1}^k A_i)
\end{aligned}$$

for all $A_i \in \mathcal{S}_{i,n_i}$, $i = 1, \ldots, k$. Hence ψ is commuting on $\mathcal{S}_1^k$.

For the necessity, we argue by induction on the order k. The base case $k = 2$ is true by Theorem 3.2.1. Suppose that $k \geqslant 3$ and that the result holds for $k-1$. In view of Lemma 3.1.4, since $\mathcal{S}_2^k \subseteq \mathbb{F} \cdot \mathcal{S}_2^k$, there exist a commuting linear map $\varphi : \bigotimes_{i=2}^k \mathcal{M}_{n_i} \to$

$\bigotimes_{i=2}^{k} \mathcal{M}_{n_i}$ on $\mathcal{S}_2^k$ and commuting additive maps $\varphi_{pq} : \bigotimes_{i=2}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=2}^{k} \mathcal{M}_{n_i}$ on $\mathcal{S}_2^k$, $p, q = 1, \ldots, n_1$, such that

$$\psi(\otimes_{i=1}^{k} A_i) = A_1 \otimes \varphi(\otimes_{i=2}^{k} A_i) + I_{n_1} \otimes \left(\sum_{p,q=1}^{n_1} \varphi_{pq}(a_{pq} \otimes_{i=2}^{k} A_i) \right) \tag{3.63}$$

for all $A_1 = (a_{pq}) \in \mathcal{M}_{n_1}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$. For each integer $1 \leqslant h \leqslant k-2$, we denote $Q_{h,k}^* = \{(\alpha_1, \ldots, \alpha_h) \in Q_{h,k} : \alpha_1 \geqslant 2\}$. By the induction hypothesis, there exist a scalar $\lambda \in \mathbb{F}$, a linear functional $\eta_1 : \bigotimes_{i=2}^{k} \mathcal{M}_{n_i} \to \mathbb{F}$ and matrices $H_\alpha \in \mathcal{M}_{n_{\alpha_1} \cdots n_{\alpha_h}}$ for all $\alpha = (\alpha_i) \in Q_{h,k}^*$, $h = 1, \ldots, k-2$, such that

$$\varphi(\otimes_{i=2}^{k} A_i) = \lambda(\otimes_{i=2}^{k} A_i) + \eta_1(\otimes_{i=2}^{k} A_i) I_N + \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i) \in Q_{h,k}^*} \operatorname{tr}\left(H_\alpha^t (\otimes_{i=1}^{h} A_{\alpha_i})\right) (\otimes_{i=2}^{k} A_i)_\alpha \tag{3.64}$$

for every $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$, where $N = n_2 \cdots n_k$ and $(\otimes_{i=2}^{k} A_i)_\alpha$ is as defined in (3.62). Moreover, for each pair of integers $1 \leqslant p, q \leqslant n_1$, since φ_{pq} is a commuting additive map on $\mathcal{S}_2^k$, it follows from the induction hypothesis that there exist a scalar $\lambda_{pq} \in \mathbb{F}$, an additive map $\eta_{pq} : \bigotimes_{i=2}^{k} \mathcal{M}_{n_i} \to \mathbb{F}$ and matrices $X_\alpha^{p,q} \in \mathcal{M}_{n_{\alpha_1} \cdots n_{\alpha_h}}$ for all $\alpha = (\alpha_i) \in Q_{h,k}^*$, $h = 1, \ldots, k-2$, such that

$$\begin{aligned} &\varphi_{pq}(a_{pq} \otimes_{i=2}^{k} A_i) = \lambda_{pq}(a_{pq} \otimes_{i=2}^{k} A_i) + \eta_{pq}(a_{pq} \otimes_{i=2}^{k} A_i) I_N + \\ &\sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i) \in Q_{h,k}^*} \operatorname{tr}\left((X_\alpha^{p,q})^t \left((a_{pq} A_{\alpha_1}) \otimes (\otimes_{i=2}^{h} A_{\alpha_i})\right)\right) (A_2 \otimes \cdots \otimes (a_{pq} A_{\alpha_1}) \otimes \cdots \otimes A_k)_\alpha \end{aligned}$$

for all $a_{pq} \in \mathbb{F}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$. Consequently,

$$\begin{aligned} \varphi_{pq}(a_{pq} \otimes_{i=2}^{k} A_i) &= \lambda_{pq}(a_{pq} \otimes_{i=2}^{k} A_i) + \eta_{pq}(a_{pq} \otimes_{i=2}^{k} A_i) I_N \\ &\quad + \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i) \in Q_{h,k}^*} \operatorname{tr}\left((X_\alpha^{p,q})^t (a_{pq} \otimes_{i=1}^{h} A_{\alpha_i})\right) (\otimes_{i=2}^{k} A_i)_\alpha \end{aligned} \tag{3.65}$$

for all $a_{pq} \in \mathbb{F}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$.

Note first that $\eta_1 : \bigotimes_{i=2}^{k} \mathcal{M}_{n_i} \to \mathbb{F}$ is a linear functional. Since $\bigotimes_{i=2}^{k} \mathcal{M}_{n_i} \cong \mathcal{M}_{n_2 \cdots n_k}$ by (2.11), it follows from Lemma 3.1.5 that there exists a matrix $H_{(2,\ldots,k)} \in \mathcal{M}_{n_2 \cdots n_k}$ such that $\eta_1(\otimes_{i=2}^k A_i) = \mathrm{tr}\,(H^t_{(2,\ldots,k)}(\otimes_{i=2}^k A_i))$ for all $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$. We thus have

$$A_1 \otimes\, \eta_1(\otimes_{i=2}^k A_i) I_N = \eta_1(\otimes_{i=2}^k A_i)(A_1 \otimes I_N) = \mathrm{tr}\,(H^t_{(2,\ldots,k)}(\otimes_{i=2}^k A_i))(\otimes_{i=1}^k A_i)_{(2,\ldots,k)} \tag{3.66}$$

for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$, where $(2, \ldots, k) \in Q^*_{k-1,k}$.

Next, since $A_1 \otimes (\otimes_{i=2}^k A_i)_\alpha = (\otimes_{i=1}^k A_i)_\alpha$ for any $\alpha \in Q^*_{h,k}$, it follows that

$$A_1 \otimes \left(\sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i)\in Q^*_{h,k}} \mathrm{tr}\,(H^t_\alpha(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=2}^k A_i)_\alpha \right) = \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i)\in Q^*_{h,k}} \mathrm{tr}\,(H^t_\alpha(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=1}^k A_i)_\alpha \tag{3.67}$$

for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$.

We denote $H_{(1)} = (\lambda_{ij}) \in \mathcal{M}_{n_1}$. Then

$$\mathrm{tr}\,(H^t_{(1)} A_1)(\otimes_{i=1}^k A_i)_{(1)} = \left(\sum_{p,q=1}^{n_1} \lambda_{pq} a_{pq} \right) (I_{n_1} \otimes (\otimes_{i=2}^k A_i)) = I_{n_1} \otimes \left(\sum_{p,q=1}^{n_1} \lambda_{pq} (a_{pq} \otimes_{i=2}^k A_i) \right) \tag{3.68}$$

for all $A_1 = (a_{pq}) \in \mathcal{M}_{n_1}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$, where $(1) \in Q_{1,k}$.

Let $\mu : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \mathbb{F}$ be the additive map defined by

$$\mu(\otimes_{i=1}^k A_i) = \sum_{p,q=1}^{n_1} \eta_{pq}(a_{pq} \otimes_{i=2}^k A_i)$$

for all $A_1 = (a_{pq}) \in \mathcal{M}_{n_1}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$. We first check that μ is well defined in two cases.

Case I: $\otimes_{i=1}^k A_i = 0$. Since μ is additive, $\mu(0) = 0$. On the other hand, if $\otimes_{i=1}^k A_i = 0$, then $A_i = 0$ for some integer $1 \leqslant i \leqslant k$ by Proposition 2.1.5. Therefore, $\mu(\otimes_{i=1}^k A_i) = \sum_{p,q=1}^{n_1} \eta_{pq}(0) = 0$ as η_{pq}, $p, q = 1, \ldots, n_1$, are additive.

Case II: $\otimes_{i=1}^k A_i \neq 0$. Let $A_1' \in \mathcal{M}_{n_1}, \ldots, A_k' \in \mathcal{M}_{n_k}$ such that $\otimes_{i=1}^k A_i' = \otimes_{i=1}^k A_i$. By Proposition 2.1.5, $A_1' = \alpha A_1$ and $\otimes_{i=2}^k A_i' = \alpha^{-1} \otimes_{i=2}^k A_i$ for some nonzero $\alpha \in \mathbb{F}$. Then $\mu(\otimes_{i=1}^k A_i') = \sum_{p,q=1}^{n_1} \eta_{pq}((\alpha a_{pq})(\alpha^{-1} \otimes_{i=2}^k A_i)) = \sum_{p,q=1}^{n_1} \eta_{pq}(a_{pq} \otimes_{i=2}^k A_i) = \mu(\otimes_{i=1}^k A_i)$.

Hence, μ is well defined. It follows that

$$I_{n_1} \otimes \left(\sum_{p,q=1}^{n_1} \eta_{pq}(a_{pq} \otimes_{i=2}^k A_i) I_N \right) = \left(\sum_{p,q=1}^{n_1} \eta_{pq}(a_{pq} \otimes_{i=2}^k A_i) \right) I_{n_1 \cdots n_k} = \mu(\otimes_{i=1}^k A_i) I_{n_1 \cdots n_k} \tag{3.69}$$

for all $A_1 = (a_{pq}) \in \mathcal{M}_{n_1}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$.

For each $\alpha = (\alpha_i) \in Q^*_{h,k}$ with $1 \leqslant h \leqslant k-2$, we let $\tau_\alpha : \mathcal{M}_{n_1 \cdot n_{\alpha_1} \cdots n_{\alpha_h}} \to \mathbb{F}$ be the linear functional defined by

$$\tau_\alpha(A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h}) = \sum_{p,q=1}^{n_1} \operatorname{tr}\left((X_\alpha^{p,q})^t \left(a_{pq} \otimes_{i=1}^h A_{\alpha_i}\right)\right) \tag{3.70}$$

for all $A_1 = (a_{pq}) \in \mathcal{M}_{n_1}$ and $A_{\alpha_1} \in \mathcal{M}_{n_{\alpha_1}}, \ldots, A_{\alpha_h} \in \mathcal{M}_{n_{\alpha_h}}$. We first check that τ_α is well defined in two cases.

Case I: $A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h} = 0$. Since τ_α is linear, $\tau_\alpha(0) = 0$. On the other hand, if $A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h} = 0$, then $A_1 = 0$ or $A_{\alpha_i} = 0$ for some integer $1 \leqslant i \leqslant h$ by Proposition 2.1.5. Therefore, $\tau_\alpha(A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h}) = \sum_{p,q=1}^{n_1} \operatorname{tr}\left((X_\alpha^{p,q})^t (0)\right) = 0$.

Case II: $A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h} \neq 0$. Let $A_1' \in \mathcal{M}_{n_1}$ and $A_{\alpha_1}' \in \mathcal{M}_{n_{\alpha_1}}, \ldots, A_{\alpha_h}' \in \mathcal{M}_{n_{\alpha_h}}$ such that $A_1' \otimes A_{\alpha_1}' \otimes \cdots \otimes A_{\alpha_h}' = A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h}$. By Proposition 2.1.5, we obtain $A_1' = \beta A_1$ and $\otimes_{i=1}^h A_{\alpha_i}' = \beta^{-1} \otimes_{i=1}^h A_{\alpha_i}$ for some nonzero $\beta \in \mathbb{F}$. Consequently, $\tau_\alpha(A_1' \otimes A_{\alpha_1}' \otimes \cdots \otimes A_{\alpha_h}') = \sum_{p,q=1}^{n_1} \operatorname{tr}\left((X_\alpha^{p,q})^t \left((\beta a_{pq})(\beta^{-1} \otimes_{i=1}^h A_{\alpha_i})\right)\right) = \sum_{p,q=1}^{n_1} \operatorname{tr}\left((X_\alpha^{p,q})^t \left(a_{pq} \otimes_{i=1}^h A_{\alpha_i}\right)\right) = \tau_\alpha(A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h})$.

Hence, τ_α is well defined. By virtue of Lemma 3.1.5, there exists a matrix $Y_\alpha \in \mathcal{M}_{n_1 \cdot n_{\alpha_1} \cdots n_{\alpha_h}}$ such that

$$\tau_\alpha(A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h}) = \operatorname{tr}\left(Y_\alpha^t (A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h})\right) \tag{3.71}$$

for all $A_1 \in \mathcal{M}_{n_1}$ and $A_{\alpha_1} \in \mathcal{M}_{n_{\alpha_1}}, \ldots, A_{\alpha_h} \in \mathcal{M}_{n_{\alpha_h}}$. It follows from (3.70)–(3.71) that

$$\begin{aligned}
& I_{n_1} \otimes \left(\sum_{p,q=1}^{n_1} \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i) \in Q^*_{h,k}} \operatorname{tr}\left((X_\alpha^{p,q})^t \left(a_{pq} \otimes_{i=1}^h A_{\alpha_i}\right)\right) (\otimes_{i=2}^k A_i)_\alpha \right) \\
&= \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i) \in Q^*_{h,k}} \left(\sum_{p,q=1}^{n_1} \operatorname{tr}\left((X_\alpha^{p,q})^t \left(a_{pq} \otimes_{i=1}^h A_{\alpha_i}\right)\right) \right) (I_{n_1} \otimes (\otimes_{i=2}^k A_i)_\alpha) \\
&= \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i) \in Q^*_{h,k}} \tau_\alpha(A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h}) (I_{n_1} \otimes (\otimes_{i=2}^k A_i)_\alpha) \\
&= \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i) \in Q^*_{h,k}} \operatorname{tr}\left(Y_\alpha^t (A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h})\right) (I_{n_1} \otimes (\otimes_{i=2}^k A_i)_\alpha)
\end{aligned}$$

for all $A_1 = (a_{pq}) \in \mathcal{M}_{n_1}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$. For each $\alpha = (\alpha_1, \ldots, \alpha_h) \in Q^*_{h,k}$ with $1 \leqslant h \leqslant k-2$, we denote $(1, \alpha) = (1, \alpha_1, \ldots, \alpha_h) \in Q_{h+1,k}$ and $H_{(1,\alpha)} = Y_\alpha \in \mathcal{M}_{n_1 \cdot n_{\alpha_1} \cdots n_{\alpha_h}}$. Then

$$\begin{aligned} I_{n_1} \otimes & \left(\sum_{p,q=1}^{n_1} \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i)\in Q^*_{h,k}} \operatorname{tr}\left((X^{p,q}_\alpha)^t \left(a_{pq} \otimes_{i=1}^h A_{\alpha_i}\right)\right)(\otimes_{i=2}^k A_i)_\alpha \right) \\ &= \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i)\in Q^*_{h,k}} \operatorname{tr}\left(H^t_{(1,\alpha)}(A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h})\right)(\otimes_{i=1}^k A_i)_{(1,\alpha)} \end{aligned} \tag{3.72}$$

for all $A_1 = (a_{pq}) \in \mathcal{M}_{n_1}$ and $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$.

We are now ready to combine (3.64) and (3.65) into (3.63). Let $A_1 = (a_{pq}) \in \mathcal{M}_{n_1}$ and let $A_2 \in \mathcal{M}_{n_2}, \ldots, A_k \in \mathcal{M}_{n_k}$. In view of (3.64) and (3.65), we see that

$$\begin{aligned} \psi(\otimes_{i=1}^k A_i) &= A_1 \otimes \varphi(\otimes_{i=2}^k A_i) + I_{n_1} \otimes \left(\sum_{p,q=1}^{n_1} \varphi_{pq}(a_{pq} \otimes_{i=2}^k A_i) \right) \\ &= A_1 \otimes \left(\lambda(\otimes_{i=2}^k A_i) + \eta_1(\otimes_{i=2}^k A_i) I_N + \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i)\in Q^*_{h,k}} \operatorname{tr}\left(H^t_\alpha(\otimes_{i=1}^h A_{\alpha_i})\right)(\otimes_{i=2}^k A_i)_\alpha \right) \\ &+ I_{n_1} \otimes \left(\sum_{p,q=1}^{n_1} \lambda_{pq}(a_{pq} \otimes_{i=2}^k A_i) \right) + I_{n_1} \otimes \left(\sum_{p,q=1}^{n_1} \eta_{pq}(a_{pq} \otimes_{i=2}^k A_i) I_N \right) \\ &+ I_{n_1} \otimes \left(\sum_{p,q=1}^{n_1} \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i)\in Q^*_{h,k}} \operatorname{tr}\left((X^{p,q}_\alpha)^t \left(a_{pq} \otimes_{i=1}^h A_{\alpha_i}\right)\right)(\otimes_{i=2}^k A_i)_\alpha \right). \end{aligned}$$

It follows from (3.66)–(3.69) and (3.72) that

$$\begin{aligned} \psi(\otimes_{i=1}^k A_i) &= \lambda(\otimes_{i=1}^k A_i) + \operatorname{tr}\left(H^t_{(2,\ldots,k)}(\otimes_{i=2}^k A_i)\right)(\otimes_{i=1}^k A_i)_{(2,\ldots,k)} \\ &\quad + \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i)\in Q^*_{h,k}} \operatorname{tr}\left(H^t_\alpha(\otimes_{i=1}^h A_{\alpha_i})\right)(\otimes_{i=1}^k A_i)_\alpha \\ &\quad + \operatorname{tr}\left(H^t_{(1)} A_1\right)(\otimes_{i=1}^k A_i)_{(1)} + \mu(\otimes_{i=1}^k A_i) I_{n_1 \cdots n_k} \\ &\quad + \sum_{h=1}^{k-2} \sum_{\alpha=(\alpha_i)\in Q^*_{h,k}} \operatorname{tr}\left(H^t_{(1,\alpha)}(A_1 \otimes A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_h})\right)(\otimes_{i=1}^k A_i)_{(1,\alpha)} \\ &= \lambda(\otimes_{i=1}^k A_i) + \mu(\otimes_{i=1}^k A_i) I_{n_1 \cdots n_k} + \sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \operatorname{tr}\left(H^t_\alpha(\otimes_{i=1}^h A_{\alpha_i})\right)(\otimes_{i=1}^k A_i)_\alpha \end{aligned}$$

for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$. This completes the proof. $\square$

3.3 Remarks

First, one may ask why the set

$$\mathcal{S}_{1,n} = \Big\{E_{st} + \alpha E_{pq} : \alpha \in \mathbb{F} \text{ and } 1 \leqslant p, q, s, t \leqslant n \text{ are not all distinct integers}\Big\}$$

is considered in Theorem 3.1.3. The reason is, inspired by the set

$$Y = \Big\{\alpha E_{pq} + \beta E_{ps} \ \text{ or } \ \alpha E_{pq} + \beta E_{sq} : \alpha, \beta \in \mathbb{F} \text{ and } 1 \leqslant p, q, s \leqslant n \text{ are integers}\Big\}$$

in Franca (2017), we want to choose a set which looks alike such that commuting additive maps on the particular set are of the standard form. It should be noted that commuting additive maps on Y are not necessarily of the standard form, see (Franca, 2017, Theorem 15) and (Franca, 2013, Example 1). The condition $1 \leqslant p, q, s, t \leqslant n$ are not all distinct in $\mathcal{S}_{1,n}$ implies that there are at most three distinct integers, which is also the case for $1 \leqslant p, q, s \leqslant n$ in Y.

We end our discussion with a review of the structural form as illustrated in (3.61). Let $m, n \geqslant 2$ be integers and let $\psi : \mathcal{M}_m \bigotimes \mathcal{M}_n \to \mathcal{M}_m \bigotimes \mathcal{M}_n$ be the additive map defined by

$$\psi(A \otimes B) = \varphi_1(A) \otimes \varphi_2(B) \quad \text{for all } A \in \mathcal{M}_m, B \in \mathcal{M}_n, \tag{3.73}$$

where $\varphi_1 : \mathcal{M}_m \to \mathcal{M}_m$ is a commuting additive map on $\mathcal{M}_m$ and $\varphi_2 : \mathcal{M}_n \to \mathcal{M}_n$ is a commuting additive map on $\mathcal{M}_n$. Clearly, ψ is commuting on $D(\mathcal{M}_m \bigotimes \mathcal{M}_n)$, so ψ is commuting on $\mathcal{S}_1^2$. Then it can be represented in form (3.61). One may ask whether commuting additive maps on $D(\mathcal{M}_m \bigotimes \mathcal{M}_n)$ of form (3.61) can always be represented as in form (3.73). Unfortunately, the answer is negative in general. For example, we consider additive maps $\psi_1, \psi_2 : \mathcal{M}_m \bigotimes \mathcal{M}_n \to \mathcal{M}_m \bigotimes \mathcal{M}_n$ defined by

$$\psi_1(A \otimes B) = \operatorname{tr}(E_{11}^{(m)} A)(I_m \otimes B) + \operatorname{tr}(E_{22}^{(n)} B)(A \otimes I_n),$$

$$\psi_2(A \otimes B) = A \otimes B + \mu(A \otimes B) I_{mn}$$

for all $A \in \mathcal{M}_m$, $B \in \mathcal{M}_n$, where $\mu : \mathcal{M}_m \bigotimes \mathcal{M}_n \to \mathbb{F}$ is the linear functional defined by

$$\mu(E_{ij}^{(m)} \otimes E_{st}^{(n)}) = \begin{cases} 1 & \text{if } (i,j) = (1,1),\ (s,t) = (2,2), \\ \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that both maps ψ_1 and ψ_2 are commuting on $D(\mathcal{M}_m \bigotimes \mathcal{M}_n)$. Note that both

$$\psi_1(E_{11}^{(m)} \otimes E_{22}^{(n)}) = E_{11}^{(m)} \otimes I_n + I_m \otimes E_{22}^{(n)} \quad \text{and} \quad \psi_2(E_{11}^{(m)} \otimes E_{22}^{(n)}) = E_{11}^{(m)} \otimes E_{22}^{(n)} + I_m \otimes I_n$$

are not expressible as decomposable tensors. If, however, ψ_i, $i = 1, 2$ can be represented as in form (3.73), then $\psi_i(E_{11}^{(m)} \otimes E_{22}^{(n)}) = \varphi_1(E_{11}^{(m)}) \otimes \varphi_2(E_{22}^{(n)})$ yields a decomposable tensor independent of choices of φ_1 and φ_2, a contradiction.

CHAPTER 4: COMMUTING ADDITIVE MAPS ON TENSOR PRODUCTS OF FIXED-RANK MATRICES

As an application of Theorems 3.1.3 and 3.2.2, we now turn our attention to study commuting additive maps $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ on tensor products of k fixed-rank matrices. A matrix $A \in \mathcal{M}_n$, $n \geqslant 2$, is said to be of **bounded rank two** if A is of rank at most two. We look at some preliminary results before we prove the main results.

4.1 Preliminary Results

Lemma 4.1.1. *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Let $1 < r \leqslant n$ be a fixed integer such that $r \neq n$ when $|\mathbb{F}| = 2$. Then each nonzero bounded rank two matrix in $\mathcal{M}_n$ can be represented by a sum of three rank r matrices in $\mathcal{M}_n$ among which the sum of any two is of rank r.*

Proof. Let $A \in \mathcal{M}_n$. We argue in the following two cases.

Case I: A is of rank one. Since two matrices are equivalent if and only if they have the same rank, we assume without loss of generality that $A = E_{11}$. Consider first $r = n$. We let $\alpha \in \mathbb{F} \backslash \{0, 1\}$ and select

$$X_1 = E_{11} + \sum_{i=1}^{n} E_{i,n+1-i}, \quad Y_1 = -\sum_{i=1}^{n} \alpha E_{i,n+1-i}, \quad Z_1 = \sum_{i=1}^{n} (\alpha - 1) E_{i,n+1-i}.$$

Then X_1, Y_1, Z_1 are of rank n such that $A = X_1 + Y_1 + Z_1$ and among which the sum of any two is of rank n. Consider now $1 < r < n$. We select

$$X_2 = -E_{1,r+1} + \sum_{i=1}^{r} E_{ii}, \quad Y_2 = E_{1,r+1} + \sum_{i=1}^{r} E_{i,i+1}, \quad Z_2 = -\sum_{i=2}^{r} E_{ii} - \sum_{i=1}^{r} E_{i,i+1}.$$

Then X_2, Y_2, Z_2 are of rank r such that $A = X_2 + Y_2 + Z_2$ and among which the sum of any two is of rank r.

Case II: A is of rank two. We may assume $A = E_{11} + E_{22}$. Consider $r = n$. We let $\alpha \in \mathbb{F} \backslash \{0, 1\}$ and select

$$X_3 = E_{11} + E_{12} + E_{21} + \alpha E_{22} - \sum_{i=3}^{n} E_{ii}, \quad Y_3 = \alpha E_{11} + \sum_{i=2}^{n} (1 - \alpha) E_{ii},$$

$$Z_3 = -\alpha E_{11} - E_{12} - E_{21} + \sum_{i=3}^{n} \alpha E_{ii}.$$

Then X_3, Y_3, Z_3 are of rank n such that $A = X_3 + Y_3 + Z_3$ and among which the sum of any two is of rank n. Consider now $1 < r < n$. We take

$$X_4 = -E_{1,r+1} + E_{21} + \sum_{i=1}^{r} E_{ii}, \quad Y_4 = E_{1,r+1} + \sum_{i=1}^{r} E_{i,i+1},$$

$$Z_4 = -E_{21} - \sum_{i=3}^{r} E_{ii} - \sum_{i=1}^{r} E_{i,i+1}.$$

Then X_4, Y_4, Z_4 are of rank r such that $A = X_4 + Y_4 + Z_4$ and among which the sum of any two is of rank r as required. $\square$

It should be noted that the idea of the proof of Lemma 4.1.1 has been mentioned in Xu and Yi (2014). Our proof is a different approach. The following lemma is due to Lemmas 2.3 and 2.4 in Xu and Yi (2014).

Lemma 4.1.2. (Xu & Yi, 2014, Lemmas 2.3 and 2.4) *Let $\mathbb{F}$ be the field of two elements and let $n \geqslant 3$ be an integer. Then each rank two matrix in $\mathcal{M}_n$ can be represented by a sum of three rank n matrices in $\mathcal{M}_n$ among which the sum of any two is of rank n.*

Lemma 4.1.3. *Let $k \geqslant 1$ and $n_1, \ldots, n_k \geqslant 2$ be integers. Let $A_{i1}, A_{i2}, A_{i3} \in \mathcal{M}_{n_i}$ for $i = 1, \ldots, k$. If $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ is an additive map such that $[\psi(\otimes_{i=1}^{k} Z_i), \otimes_{i=1}^{k} Z_i] = 0$ for all $Z_i \in \{A_{i1},\ A_{i2},\ A_{i3},\ A_{i1} + A_{i2},\ A_{i1} + A_{i3},\ A_{i2} + A_{i3}\}$, $i = 1, \ldots, k$, then*

$$[\psi(\otimes_{i=1}^{k}(A_{i1} + A_{i2} + A_{i3})), \otimes_{i=1}^{k}(A_{i1} + A_{i2} + A_{i3})] = 0.$$

Proof. We denote $\mathcal{H}_i = \{A_{i1},\ A_{i2},\ A_{i3},\ A_{i1} + A_{i2},\ A_{i1} + A_{i3},\ A_{i2} + A_{i3}\}$ and $X_i = A_{i1} + A_{i2} + A_{i3}$ for $i = 1, \ldots, k$. We claim that for each integer $1 \leqslant h \leqslant k$,

$$[\psi((\otimes_{i=1}^{h} X_i) \otimes (\otimes_{i=h+1}^{k} Z_i)),\ (\otimes_{i=1}^{h} X_i) \otimes (\otimes_{i=h+1}^{k} Z_i)] = 0 \tag{4.1}$$

for all $Z_i \in \mathcal{H}_i$, $i = h+1, \ldots, k$. We argue by induction on h. Consider $h = 1$. Let $1 \leqslant s, t \leqslant 3$ be distinct integers and let $Z_i \in \mathcal{H}_i$ for $i = 2, \ldots, k$. For simplicity, we denote $Z = \otimes_{i=2}^{k} Z_i$. Since $[\psi(Y \otimes Z), Y \otimes Z] = 0$ for all $Y \in \{A_{1s} + A_{1t},\ A_{1s},\ A_{1t}\}$, we get $[\psi(A_{1s} \otimes Z), A_{1t} \otimes Z] + [\psi(A_{1t} \otimes Z), A_{1s} \otimes Z] = 0$ for all distinct integers $1 \leqslant s, t \leqslant 3$. Thus

$$\begin{aligned}
[\psi(X_1 \otimes Z), X_1 \otimes Z] &= \sum_{i=1}^{3} [\psi(A_{1i} \otimes Z), A_{1i} \otimes Z] \\
&\quad + \sum_{1 \leqslant i < j \leqslant 3} [\psi(A_{1i} \otimes Z), A_{1j} \otimes Z] + [\psi(A_{1j} \otimes Z), A_{1i} \otimes Z] \\
&= 0
\end{aligned}$$

for all $Z = \otimes_{i=2}^{k} Z_i$ and $Z_i \in \mathcal{H}_i$, $i = 2, \ldots, k$. This validates the base case $h = 1$. If $k = 1$, then the proof is complete. Consider now $k \geqslant 2$. Suppose that $h \geqslant 2$ and that assertion (4.1) holds for $h - 1$. Let $1 \leqslant s, t \leqslant 3$ be distinct integers and let $Z_i \in \mathcal{H}_i$ for $i = h + 1, \ldots, k$. For simplicity, we denote $X = \otimes_{i=1}^{h-1} X_i$ and $W = \otimes_{i=h+1}^{k} Z_i$. The induction hypothesis guarantees

$$[\psi(X \otimes Y \otimes W),\ X \otimes Y \otimes W] = 0$$

for all $Y \in \{A_{hs} + A_{ht},\ A_{hs},\ A_{ht}\}$. It follows that $[\psi(X \otimes A_{hs} \otimes W),\ X \otimes A_{ht} \otimes W] + [\psi(X \otimes A_{ht} \otimes W), X \otimes A_{hs} \otimes W] = 0$ for all distinct integers $1 \leqslant s, t \leqslant 3$. Consequently,

$$\begin{aligned}
&[\psi(X \otimes X_h \otimes W), X \otimes X_h \otimes W] \\
&\quad = \sum_{i=1}^{3} [\psi(X \otimes A_{hi} \otimes W), X \otimes A_{hi} \otimes W] \\
&\qquad + \sum_{1 \leqslant i < j \leqslant 3} [\psi(X \otimes A_{hi} \otimes W), X \otimes A_{hj} \otimes W] + [\psi(X \otimes A_{hj} \otimes W), X \otimes A_{hi} \otimes W] \\
&\quad = 0.
\end{aligned}$$

Hence the inductive step is completed. By induction, we conclude that assertion (4.1) is proved. When $h = k$, we get $[\psi(\otimes_{i=1}^{k}(A_{i1} + A_{i2} + A_{i3})), \otimes_{i=1}^{k}(A_{i1} + A_{i2} + A_{i3})] = 0$. $\square$

4.2 Main Results

We are now ready to prove the main results.

Theorem 4.2.1. *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Let $1 < r \leqslant n$ be a fixed integer such that $r \neq n$ when $n = 2$ and $|\mathbb{F}| = 2$. Then $\psi : \mathcal{M}_n \to \mathcal{M}_n$ is a commuting additive map on rank r matrices if and only if there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : \mathcal{M}_n \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A) I_n$$

for all $A \in \mathcal{M}_n$.

Proof. The sufficiency is clear. For the necessity, we first claim that $[\psi(A), A] = 0$ for every bounded rank two matrix $A \in \mathcal{M}_n$. Since it is obvious that $[\psi(A), A] = 0$ when $A = 0$, we show only $[\psi(A), A] = 0$ for every nonzero bounded rank two matrix $A \in \mathcal{M}_n$. We argue in two cases.

Case I: $1 < r \leqslant n$ and $r \neq n$ when $|\mathbb{F}| = 2$. Let $A \in \mathcal{M}_n$ be nonzero bounded rank two. By Lemma 4.1.1, A can be represented by a sum of three rank r matrices in $\mathcal{M}_n$ among which the sum of any two is of rank r. It follows from Lemma 4.1.3 that $[\psi(A), A] = 0$ as desired.

Case II: $r = n \geqslant 3$ and $|\mathbb{F}| = 2$. We show that

$$[\psi(X), X] = 0 \tag{4.2}$$

for all rank two matrices $X \in \mathcal{M}_n$. By Lemma 4.1.2, X can be represented by a sum of three rank n matrices in $\mathcal{M}_n$ among which the sum of any two is of rank n. By Lemma 4.1.3, we get $[\psi(X), X] = 0$. Next, we show that

$$[\psi(Y), Y] = 0 \tag{4.3}$$

for all rank one matrices $Y \in \mathcal{M}_n$. Since $n \geqslant 3$, by Lemma 4.1.1, Y can be represented by a sum of three rank two matrices in $\mathcal{M}_n$ among which the sum of any two is of rank two. It follows from (4.2) and Lemma 4.1.3 that $[\psi(Y), Y] = 0$. Putting (4.2) and (4.3) together, we have $[\psi(A), A] = 0$ for every nonzero bounded rank two matrix $A \in \mathcal{M}_n$.

Since $\mathcal{S}_{1,n}$ is contained in the set of all bounded rank two matrices in $\mathcal{M}_n$, it follows that $[\psi(A), A] = 0$ for all $A \in \mathcal{S}_{1,n}$, and so ψ is a commuting additive map on $\mathcal{S}_{1,n}$. The result follows immediately from Theorem 3.1.3. $\square$

Remark 4.2.2. It is noted that the assumption on r in Theorem 4.2.1 is indispensable. Theorem 4.2.1 does not hold for the case $r = 1$ when $n \geqslant 3$ and the case $r = n$ when $|\mathbb{F}| = 2 = n$. For example, the reader may refer to Example 1 in Franca (2013) for $r = 1$ when $n \geqslant 3$, and see Theorem 2.12 in Chooi et al. (2019) for $r = n$ when $|\mathbb{F}| = 2 = n$.

Let $n \geqslant 2$ and $1 \leqslant r \leqslant n$ be integers. We denote by $\mathcal{R}_r^n$ the set of all rank r matrices in $\mathcal{M}_n$. By Theorem 3.2.2, Lemmas 4.1.1, 4.1.2 and 4.1.3 and by using a similar argument as in the proof of Theorem 4.2.1, we obtain the following result.

Theorem 4.2.3. *Let $\mathbb{F}$ be a field and let $k, n_1, \ldots, n_k \geqslant 2$ be integers. Let $1 < r_i \leqslant n_i$ be a fixed integer such that $r_i \neq n_i$ when $|\mathbb{F}| = 2$ and $n_i = 2$ for $i = 1, \ldots, k$. If $\psi : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \bigotimes_{i=1}^k \mathcal{M}_{n_i}$ is an additive map, then ψ satisfies*

$$\psi(\otimes_{i=1}^k A_i)(\otimes_{i=1}^k A_i) = (\otimes_{i=1}^k A_i)\, \psi(\otimes_{i=1}^k A_i)$$

for all $A_1 \in \mathcal{R}_{r_1}^{n_1}, \ldots, A_k \in \mathcal{R}_{r_k}^{n_k}$ if and only if there exist a scalar $\lambda \in \mathbb{F}$, an additive map $\mu : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \mathbb{F}$ and matrices $H_\alpha \in \mathcal{M}_{n_{\alpha_1} \cdots n_{\alpha_h}}$ for each $\alpha = (\alpha_i) \in Q_{h,k}$, $h = 1, \ldots, k-1$, such that

$$\psi(\otimes_{i=1}^k A_i) = \lambda(\otimes_{i=1}^k A_i) + \mu(\otimes_{i=1}^k A_i) I_{n_1 \cdots n_k} + \sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=1}^k A_i)_\alpha$$

for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$.

Proof. The sufficiency is clear. For the necessity, we first claim that $[\psi(\otimes_{i=1}^k A_i), \otimes_{i=1}^k A_i] = 0$ for every bounded rank two matrix $A_i \in \mathcal{M}_{n_i}$, $i = 1, \ldots, k$. Since it is obvious that $[\psi(\otimes_{i=1}^k A_i), \otimes_{i=1}^k A_i] = 0$ when $A_i = 0$ for some integer $1 \leqslant i \leqslant k$, we show only $[\psi(\otimes_{i=1}^k A_i), \otimes_{i=1}^k A_i] = 0$ for every nonzero bounded rank two matrix $A_i \in \mathcal{M}_{n_i}$, $i = 1, \ldots, k$. We argue in two cases.

Case I: $|\mathbb{F}| \geqslant 3$. Let $A_i \in \mathcal{M}_{n_i}$ be nonzero bounded rank two for $i = 1, \ldots, k$. By Lemma 4.1.1, each A_i can be represented by a sum of three rank r_i matrices in $\mathcal{M}_{n_i}$ among which the sum of any two is of rank r_i. It follows from Lemma 4.1.3 that $[\psi(\otimes_{i=1}^k A_i), \otimes_{i=1}^k A_i] = 0$ for every nonzero bounded rank two matrix $A_i \in \mathcal{M}_{n_i}$, $i = 1, \ldots, k$.

Case II: $|\mathbb{F}| = 2$ and $n_1, \ldots, n_k \geqslant 3$. We show that

$$[\psi(\otimes_{i=1}^k X_i), \otimes_{i=1}^k X_i] = 0 \tag{4.4}$$

for all rank two matrices $X_i \in \mathcal{M}_{n_i}$, $i = 1, \ldots, k$. For $1 < r_i < n_i$, by Lemma 4.1.1, each X_i can be represented by a sum of three rank r_i matrices in $\mathcal{M}_{n_i}$ among which the sum of any two is of rank r_i. For $r_i = n_i$, by Lemma 4.1.2, each X_i can be represented by a sum of three rank r_i matrices in $\mathcal{M}_{n_i}$ among which the sum of any two is of rank r_i. By Lemma 4.1.3, we get $[\psi(\otimes_{i=1}^k X_i), \otimes_{i=1}^k X_i] = 0$. Now we show that

$$[\psi(\otimes_{i=1}^k A_i), \otimes_{i=1}^k A_i] = 0$$

for all nonzero bounded rank two matrices $A_i \in \mathcal{M}_{n_i}$, $i = 1, \ldots, k$. Since $n_1, \ldots, n_k \geqslant 3$, by Lemma 4.1.1, each A_i can be represented by a sum of three rank two matrices in $\mathcal{M}_{n_i}$ among which the sum of any two is of rank two. It follows from (4.4) and Lemma 4.1.3 that $[\psi(\otimes_{i=1}^k A_i), \otimes_{i=1}^k A_i] = 0$ for every nonzero bounded rank two matrix $A_i \in \mathcal{M}_{n_i}$, $i = 1, \ldots, k$, as desired.

Since $\mathcal{S}_{i,n_i}$ is contained in the set of all bounded rank two matrices in $\mathcal{M}_{n_i}$ for $i = 1, \ldots, k$, it follows that $[\psi(\otimes_{i=1}^k A_i), \otimes_{i=1}^k A_i] = 0$ for all $A_i \in \mathcal{S}_{i,n_i}$, $i = 1, \ldots, k$, and so ψ is a commuting additive map on $\mathcal{S}_1^k$. The result follows immediately from Theorem 3.2.2. $\square$

CHAPTER 5: CONCLUSION

In this chapter, we summarize the main results in Chapters 3 and 4 for convenience. We also propose some potential open problems related to the study in this dissertation.

5.1 Main Results in Chapter 3

Theorem 5.1.1. *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Then $\psi : \mathcal{M}_n \to \mathcal{M}_n$ is a commuting additive map on* $\mathcal{S}_{1,n} = \{E_{st} + \alpha E_{pq} : \alpha \in \mathbb{F}$ and $1 \leqslant p, q, s, t \leqslant n$ are not all distinct integers$\}$ *if and only if there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : \mathcal{M}_n \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A) I_n$$

for all $A \in \mathcal{M}_n$.

Theorem 5.1.2. *Let $\mathbb{F}$ be a field and let $k, n_1, \ldots, n_k \geqslant 2$ be integers. Let $\psi : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \bigotimes_{i=1}^k \mathcal{M}_{n_i}$ be an additive map. Then ψ is a commuting map on $\mathcal{S}_1^k$ if and only if there exist a scalar $\lambda \in \mathbb{F}$, an additive map $\mu : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \mathbb{F}$ and matrices $H_\alpha \in \mathcal{M}_{n_{\alpha_1} \cdots n_{\alpha_h}}$ for each $\alpha = (\alpha_i) \in Q_{h,k}$, $h = 1, \ldots, k-1$, such that*

$$\psi(\otimes_{i=1}^k A_i) = \lambda(\otimes_{i=1}^k A_i) + \mu(\otimes_{i=1}^k A_i) I_{n_1 \cdots n_k} + \sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \mathrm{tr}\,(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=1}^k A_i)_\alpha$$

for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$.

In particular, when $k = 2$, we obtain:

Theorem 5.1.3. *Let $\mathbb{F}$ be a field and let $m, n \geqslant 2$ be integers. Let $\psi : \mathcal{M}_m \bigotimes \mathcal{M}_n \to \mathcal{M}_m \bigotimes \mathcal{M}_n$ be an additive map. Then ψ is a commuting map on $\mathcal{S}_1^2$ if and only if there exist a scalar $\lambda \in \mathbb{F}$, an additive map $\mu : \mathcal{M}_m \bigotimes \mathcal{M}_n \to \mathbb{F}$ and matrices $K \in \mathcal{M}_m$, $H \in \mathcal{M}_n$ such that*

$$\psi(A \otimes B) = \lambda(A \otimes B) + \mu(A \otimes B) I_{mn} + \mathrm{tr}\,(H^t B)(A \otimes I_n) + \mathrm{tr}\,(K^t A)(I_m \otimes B)$$

for all $A \in \mathcal{M}_m$ and $B \in \mathcal{M}_n$.

5.2 Main Results in Chapter 4

Theorem 5.2.1. *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Let $1 < r \leqslant n$ be a fixed integer such that $r \neq n$ when $n = 2$ and $|\mathbb{F}| = 2$. Then $\psi : \mathcal{M}_n \to \mathcal{M}_n$ is a commuting additive map on rank r matrices if and only if there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : \mathcal{M}_n \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A) I_n$$

for all $A \in \mathcal{M}_n$.

Theorem 5.2.2. *Let $\mathbb{F}$ be a field and let $k, n_1, \ldots, n_k \geqslant 2$ be integers. Let $1 < r_i \leqslant n_i$ be a fixed integer such that $r_i \neq n_i$ when $|\mathbb{F}| = 2$ and $n_i = 2$ for $i = 1, \ldots, k$. If $\psi : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \bigotimes_{i=1}^k \mathcal{M}_{n_i}$ is an additive map, then ψ satisfies*

$$\psi(\otimes_{i=1}^k A_i)(\otimes_{i=1}^k A_i) = (\otimes_{i=1}^k A_i)\, \psi(\otimes_{i=1}^k A_i)$$

for all $A_1 \in \mathcal{R}_{r_1}^{n_1}, \ldots, A_k \in \mathcal{R}_{r_k}^{n_k}$ if and only if there exist a scalar $\lambda \in \mathbb{F}$, an additive map $\mu : \bigotimes_{i=1}^k \mathcal{M}_{n_i} \to \mathbb{F}$ and matrices $H_\alpha \in \mathcal{M}_{n_{\alpha_1} \cdots n_{\alpha_h}}$ for each $\alpha = (\alpha_i) \in Q_{h,k}$, $h = 1, \ldots, k-1$, such that

$$\psi(\otimes_{i=1}^k A_i) = \lambda(\otimes_{i=1}^k A_i) + \mu(\otimes_{i=1}^k A_i) I_{n_1 \cdots n_k} + \sum_{h=1}^{k-1} \sum_{\alpha=(\alpha_i)\in Q_{h,k}} \operatorname{tr}(H_\alpha^t(\otimes_{i=1}^h A_{\alpha_i}))(\otimes_{i=1}^k A_i)_\alpha$$

for all $A_1 \in \mathcal{M}_{n_1}, \ldots, A_k \in \mathcal{M}_{n_k}$.

5.3 Some Open Problems

1. Determine the structure of commuting additive maps on tensor products of rank one matrices.
2. Determine the structure of centralizing additive maps on tensor products of matrices.
3. Determine the structure of power commuting additive maps on tensor products of matrices.
4. Determine the structure of strong commutativity additive maps on tensor products of matrices.

REFERENCES

Beidar, K. I. (1998). On functional identities and commuting additive mappings. *Communications in Algebra*, *26*, 1819–1850.

Beidar, K. I., Brešar, M., & Chebotar, M. A. (2000). Functional identities on upper triangular matrix algebras. *Journal of Mathematical Sciences*, *102*, 4557–4565.

Brešar, M. (1993). Centralizing mappings and derivations in prime rings. *Journal of Algebra*, *156*, 385–394.

Brešar, M. (2004). Commuting maps: a survey. *Taiwanese Journal of Mathematics*, *8*, 361–397.

Brešar, M. (2014). *Introduction to Noncommutative Algebra. Universitext*. Springer International Publishing.

Brešar, M. (2016a). Functional identities on tensor products of algebras. *Journal of Algebra*, *455*, 108–136.

Brešar, M. (2016b). Jordan $\{g, h\}$-derivations on tensor products of algebras. *Linear and Multilinear Algebra*, *64*, 2199–2207.

Brešar, M. (2017). Derivations of tensor products of nonassociative algebras. *Linear Algebra and its Applications*, *530*, 244–252.

Brešar, M., Chebotar, M. A., & Martindale III, W. S. (2007). *Functional Identities. Frontiers in Mathematics*. Birkhäuser Basel.

Cheung, W.-S. (2001). Commuting maps of triangular algebras. *Journal of the London Mathematical Society*, *63*, 117–127.

Chooi, W. L., Kwa, K. H., & Tan, L. Y. (2019). Commuting maps on invertible triangular matrices over $\mathbb{F}_2$. *Linear Algebra and its Applications*, *583*, 77–101.

Chou, P.-H., & Liu, C.-K. (2021). Power commuting additive maps on rank-k linear transformations. *Linear and Multilinear Algebra*, *69*, 403–427.

Conrad, K. (n.d.). Tensor products II. Retrieved from http://kconrad.math.uconn.edu/blurbs/linmultialg/tensorprod2.pdf

Eremita, D. (2018). Biderivations on tensor products of algebras. *Communications in Algebra*, *46*, 1722–1726.

Fošner, A., Huang, Z., Li, C.-K., & Sze, N.-S. (2013). Linear preservers and quantum information science. *Linear and Multilinear Algebra*, *61*, 1377–1390.

Franca, W. (2013). Commuting maps on rank-k matrices. *Linear Algebra and its Applications*, *438*, 2813–2815.

Franca, W. (2017). Weakly commuting maps on the set of rank-1 matrices. *Linear and Multilinear Algebra*, *65*, 475–495.

Friedland, S., Li, C.-K., Poon, Y.-T., & Sze, N.-S. (2011). The automorphism group of separable states in quantum information theory. *Journal of Mathematical Physics*, *52*, 042203.1–042203.8.

Frobenius, G. (1897). Über die darstellung der endlichen gruppen durch lineare substitutionen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, 994–1015.

Gallier, J., & Quaintance, J. (2020). *Differential Geometry and Lie Groups: A Second Course. Geometry and Computing. Volume 13*. Springer International Publishing.

Hou, J., & Qi, X. (2013). Linear maps preserving separability of pure states. *Linear Algebra and its Applications*, *439*, 1245–1257.

Lee, P.-H., & Lee, T.-K. (1997). Linear identities and commuting maps in rings with involution. *Communications in Algebra*, *25*, 2881–2895.

Lee, T.-K., & Lee, T.-C. (1996). Commuting additive mappings in semiprime rings. *Bulletin of the Institute of Mathematics Academia Sinica*, *24*, 259–268.

Li, C.-K., & Pierce, S. (2001). Linear preserver problems. *American Mathematical Monthly*, *108*, 591–605.

Lim, M. H. (2012). Additive preservers of tensor product of rank one Hermitian matrices. *Electronic Journal of Linear Algebra*, *23*, 356–374.

Molnár, L. (2007). *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces. Lecture Notes in Mathematics. Volume 1895*. Springer-Verlag Berlin Heidelberg.

Pierce, S., Lim, M. H., Loewy, R., Li, C.-K., Tsing, N.-K., McDonald, B. R., & Beasley, L. (1992). A survey of linear preserver problems. *Linear and Multilinear Algebra*, *33*, 1–129.

Posner, E. C. (1957). Derivations in prime rings. *Proceedings of the American Mathematical Society*, *8*, 1093–1100.

Xu, X., & Yi, X. (2014). Commuting maps on rank-k matrices. *Electronic Journal of Linear Algebra*, *27*, 735–741.

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Commuting additive maps on tensor products of matrices

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ABSTRACT

Let $k, n_1, \ldots, n_k$ be positive integers such that $n_i \geqslant 2$ for $i = 1, \ldots, k$ and let $\mathcal{M}_{n_i}$ denote the algebra of $n_i \times n_i$ matrices over a field $\mathbb{F}$ for $i = 1, \ldots, k$. Let $\bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ be the tensor product of $\mathcal{M}_{n_1}, \ldots, \mathcal{M}_{n_k}$. We obtain a structural characterization of additive maps $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ satisfying

$$\psi\left(\bigotimes_{i=1}^{k} A_i\right)\left(\bigotimes_{i=1}^{k} A_i\right) = \left(\bigotimes_{i=1}^{k} A_i\right)\psi\left(\bigotimes_{i=1}^{k} A_i\right)$$

for all $A_1 \in \mathcal{S}_{n_1}, \ldots, A_k \in \mathcal{S}_{n_k}$, where

$$\mathcal{S}_{n_i} = \Big\{E_{st}^{(n_i)} + \alpha E_{pq}^{(n_i)} : \alpha \in \mathbb{F},$$
$$1 \leqslant p, q, s, t \leqslant n_i \text{ are not all distinct integers}\Big\}$$

and $E_{st}^{(n_i)}$ is the standard matrix unit in $\mathcal{M}_{n_i}$ for $i = 1, \ldots, k$. In particular, we show that $\psi : \mathcal{M}_{n_1} \to \mathcal{M}_{n_1}$ is an additive map commuting on $\mathcal{S}_{n_1}$ if and only if there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : \mathcal{M}_{n_1} \to \mathbb{F}$ such that

$$\psi(A) = \lambda A + \mu(A) I_{n_1}$$

for all $A \in \mathcal{M}_{n_1}$. As an application, we classify additive maps $\psi : \bigotimes_{i=1}^{k} \mathcal{M}_{n_i} \to \bigotimes_{i=1}^{k} \mathcal{M}_{n_i}$ satisfying $\psi(\bigotimes_{i=1}^{k} A_i)(\bigotimes_{i=1}^{k} A_i) = (\bigotimes_{i=1}^{k} A_i)\psi(\bigotimes_{i=1}^{k} A_i)$ for all $A_1 \in \mathcal{R}_{r_1}^{n_1}, \ldots, A_k \in \mathcal{R}_{r_k}^{n_k}$. Here, $\mathcal{R}_{r_i}^{n_i}$ denotes the set of rank r_i matrices in $\mathcal{M}_{n_i}$ and each $1 < r_i \leqslant n_i$ is a fixed integer such that $r_i \neq n_i$ when $n_i = 2$ and $|\mathbb{F}| = 2$ for $i = 1, \ldots, k$.

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1. Introduction

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ and let $\mathcal{S}$ be a nonempty subset of $\mathcal{A}$. A map $\psi : \mathcal{A} \to \mathcal{A}$ is said to be *commuting* on $\mathcal{S}$ if $[\psi(a), a] = 0$ for all $a \in \mathcal{S}$, where $[a, b]$ denotes the commutator $ab - ba$ of elements $a, b \in \mathcal{A}$. The study of commuting maps was initiated by Posner in the 1950s. He proved that a prime ring admitting a nonzero commuting derivation must

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