CENTRALIZING ADDITIVE MAPS ON RANK R BLOCK TRIANGULAR MATRICES

MUHAMMAD HAZIM BIN ABDUL MUTALIB

FACULTY OF SCIENCE
UNIVERSITI MALAYA
KUALA LUMPUR

2021

CENTRALIZING ADDITIVE MAPS ON RANK R BLOCK TRIANGULAR MATRICES

MUHAMMAD HAZIM BIN ABDUL MUTALIB

DISSERTATION SUBMITTED IN FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

INSTITUTE OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
UNIVERSITI MALAYA
KUALA LUMPUR

2021

UNIVERSITI MALAYA

ORIGINAL LITERARY WORK DECLARATION

Name of Candidate: **MUHAMMAD HAZIM BIN ABDUL MUTALIB**

Registration/Matric No.: **17078603**

Name of Degree: **MASTER OF SCIENCE**

Title of Dissertation ("this Work"):

CENTRALIZING ADDITIVE MAPS ON RANK R BLOCK TRIANGULAR MATRICES

Field of Study: **PURE MATHEMATICS**

I do solemnly and sincerely declare that:

(1) I am the sole author/writer of this Work;
(2) This work is original;
(3) Any use of any work in which copyright exists was done by way of fair dealing and for permitted purposes and any excerpt or extract from, or reference to or reproduction of any copyright work has been disclosed expressly and sufficiently and the title of the Work and its authorship have been acknowledged in this Work;
(4) I do not have any actual knowledge nor do I ought reasonably to know that the making of this work constitutes an infringement of any copyright work;
(5) I hereby assign all and every rights in the copyright to this Work to the University of Malaya ("UM"), who henceforth shall be owner of the copyright in this Work and that any reproduction or use in any form or by any means whatsoever is prohibited without the written consent of UM having been first had and obtained;
(6) I am fully aware that if in the course of making this Work I have infringed any copyright whether intentionally or otherwise, I may be subject to legal action or any other action as may be determined by UM.

Candidate's Signature Date: 19 July 2021

Subscribed and solemnly declared before,

Witness's Signature Date: 20 July 2021

Name:
Designation:

CENTRALIZING ADDITIVE MAPS ON RANK R BLOCK TRIANGULAR MATRICES

ABSTRACT

In this dissertation, we study centralizing additive maps on block triangular matrix algebras. The main focus of this research is to classify centralizing additive maps on rank r block triangular matrices over an arbitrary field. Let $k, n_1, \ldots, n_k$ be positive integers with $n_1 + \cdots + n_k = n \geq 2$. Let $\mathcal{T}_{n_1,\ldots,n_k}$ be the $n_1, \ldots, n_k$ block triangular matrix algebra over a field $\mathbb{F}$ with center $Z(\mathcal{T}_{n_1,\ldots,n_k})$ and unity I_n. We first obtain a characterization of centralizing additive maps on $\mathcal{T}_{n_1,\ldots,n_k}$. Then, by using this preliminary result together with the classification of rank factorization of block triangular matrices, we characterize centralizing additive maps $\psi : \mathcal{T}_{n_1,\ldots,n_k} \to \mathcal{T}_{n_1,\ldots,n_k}$ on rank r block triangular matrices, i.e, additive maps ψ satisfying $A\psi(A) - \psi(A)A \in Z(\mathcal{T}_{n_1,\ldots,n_k})$ for all rank r matrices $A \in \mathcal{T}_{n_1,\ldots,n_k}$, where r is a fixed integer $1 < r \leq n$ such that $r \neq n$ when $\mathbb{F}$ is the Galois field of two elements, and we prove these additive maps ψ are of the form

$$\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n}$$

for all $A = (a_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$, where $\mu : \mathcal{T}_{n_1,\ldots,n_k} \to \mathbb{F}$ is an additive map, $\lambda, \alpha \in \mathbb{F}$ are scalars in which $\alpha \neq 0$ only if $r = n$, $n_1 = n_k = 1$ and $|\mathbb{F}| = 3$, and $E_{1n} \in \mathcal{T}_{n_1,\ldots,n_k}$ is the standard matrix unit whose $(1, n)$th entry is one and zero elsewhere. Using this result, together with the recent works on commuting additive maps on upper triangular matrices, we give a complete description of commuting additive maps on rank $r > 1$ upper triangular matrices.

Keywords: centralizing additive maps, commuting additive maps, block triangular matrices, upper triangular matrices, ranks.

PEMETAAN BERDAYA TAMBAH MEMUSAT PADA MATRIKS SEGITIGA BLOK BERPANGKAT R

ABSTRAK

Dalam disertasi ini, kami mengkaji pemetaan berdaya tambah memusat pada algebra matriks segitiga blok. Tumpuan utama dalam penyelidikan ini adalah mengelaskan pemetaan berdaya tambah memusat pada matriks segitiga blok berpangkat r terhadap medan sembarangan. Biar $k, n_1, \ldots, n_k$ merupakan integer dengan $n_1+\cdots+n_k = n \geq 2$. Biar $\mathcal{T}_{n_1,\ldots,n_k}$ menandakan $n_1, \ldots, n_k$ algebra matriks segitiga blok terhadap medan $\mathbb{F}$ dengan pusat $Z(\mathcal{T}_{n_1,\ldots,n_k})$ dan unit I_n. Kami mula dengan memperoleh suatu pencirian bagi pemetaan berdaya tambah memusat pada $\mathcal{T}_{n_1,\ldots,n_k}$. Dengan menggunakan hasil awal tersebut serta pencirian pemfaktoran pangkat bagi matriks segitiga blok, kami mencirikan pemetaan berdaya tambah ψ pada $\mathcal{T}_{n_1,\ldots,n_k}$ yang memenuhi $A\psi(A) - \psi(A)A \in Z(\mathcal{T}_{n_1,\ldots,n_k})$ bagi semua $A \in \mathcal{T}_{n_1,\ldots,n_k}$, di mana r adalah suatu integer tetap supaya $r \neq n$ bila $\mathbb{F}$ merupakan medan Galois berunsur dua, dan kami membuktikan pemetaan ψ berdaya tambah tersebut berstruktur $\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n}$ bagi semua $A = (a_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$, di mana $\mu : \mathcal{T}_{n_1,\ldots,n_k} \to \mathbb{F}$ merupakan pemetaan berdaya tambah, $\lambda, \alpha \in \mathbb{F}$ adalah skalar di mana $\alpha \neq 0$ hanya jika $r = n$, $n_1 = n_k = 1$ dan $|\mathbb{F}| = 3$, dan $E_{1n} \in \mathcal{T}_{n_1,\ldots,n_k}$ adalah adalah unit matriks piawai yang masukan $(1, n)$th satu dan sifar yang lain. Dengan ini, bersama dengan karya terbaru dalam pemetaan berdaya tambah kalis tukar tertib segitiga atas, kami memberikan suatu penyampaian lengkap bagi pemetaan berdaya tambah kalis tukar tertib pada matriks segitiga atas berpangkat $r > 1$.

Kata kunci: pemetaan berdaya tambah, pemetaan berdaya tambah kalis tukar tertib, matriks segitiga blok, matriks segitiga atas, pangkat.

ACKNOWLEDGEMENTS

First and foremost, I would like to take this opportunity to express my thanks to my supervisors Associate Professor Dr. Chooi Wai Leong and Dr. Kwa Kiam Heong for their guidance, patience and support throughout this research and the writing of this thesis. I could not have made it without them.

Next, I would like to thank all my lecturers who taught me during my undergraduate studies. I could not go down this road if it was not for them.

I would also like to thank my family for being supportive and understanding throughout my journey as a postgraduate student.

Lastly, I would like to take this opportunity to express my gratitude to Universiti Malaya for the Graduate Research Assistantship (No: GPF027B-2018) which has lessened my financial burden.

TABLE OF CONTENTS

LIST OF SYMBOLS

$\mathbb{F}$	: a field
$\lvert\mathbb{F}\rvert$	: the cardinality of the field $\mathbb{F}$
$\mathbb{F}_2$	: the Galois field of two elements
$[\,\cdot\,,\,\cdot\,]$	: the commutator
$[a,b]$	: the commutator of the elements a and b in a ring
$Z(R)$	: the center of the ring R
$M_{m,n}(\mathbb{F})$	: the linear space of $m\times n$ matrices over the field $\mathbb{F}$
$M_n(\mathbb{F})$	: the algebra of $n\times n$ matrices over the field $\mathbb{F}$
$T_n(\mathbb{F})$	: the algebra of $n\times n$ upper triangular matrices over the field $\mathbb{F}$
$\mathcal{T}_{n_1,\ldots,n_k}$	: the $n_1,\ldots,n_k$ block triangular matrix algebra over the field $\mathbb{F}$
E_{ij}	: the standard matrix unit whose (i,j)th entry is one and zeros elsewhere
I_n	: the $n\times n$ identity matrix over the field $\mathbb{F}$
tr(A)	: the trace of the square matrix A
A^t	: the transpose of the matrix A
J_n	: the $n\times n$ matrix with one on the anti-diagonal and zero elsewhere
A^+	: $J_nA^tJ_n$

CHAPTER 1: INTRODUCTION

1.1 Research objectives

The main objectives of this research are as follows:

1. To characterize centralizing additive maps on block triangular matrices over an arbitrary field.
2. To characterize centralizing additive maps on rank r block triangular matrices.
3. To study commuting additive maps on rank r upper triangular matrices over an arbitrary field.
4. To develop new mathematical tools in rank factorization for block triangular matrices in matrix theory.

Among the questions that we considered in this research are the following:

1. Is the structure of centralizing additive maps on block triangular matrix of the standard form?
2. What is the structure of centralizing additive maps on rank r block triangular matrices?
3. What is the complete description of commuting additive maps on rank $r > 1$ upper triangular matrices over an arbitrary field?

1.2 Significance of the study

The study of centralizing and commuting maps on matrices is an influential research topic in functional identity, linear preserver problems, ring theory and matrix theory. The study of this research topic was initiated when Posner (1957) studied centralizing derivation on prime rings. In recent years, many developments and new results in the study of centralizing and commuting additive maps on some sets of matrices can be observed

(Chooi et al., 2019, 2020; Franca, 2012, 2013; Liu, 2014a, 2014b; Liu et al., 2018; Liu & Yang, 2017; Wang, 2016; Xu & Yi, 2014). This research aims at facilitating the advancement of existing knowledge of linear algebra, matrix theory and functional identity. With the new mathematical techniques and tools on centralizing additive maps on rank r block triangular matrices developed from this study, the existing results on centralizing and commuting additive maps on some sets of matrices can be further extended and generalized. Some other relevant centralizing maps problems can also be alleviated and then solved by applying the techniques and results established from this study.

1.3 Organization of dissertation

In Chapter 2, we begin with some preliminaries on basic algebraic structures, prime rings, triangular rings and block triangular rings. We then proceed with the literature review of this research and followed by its methodology.

In Chapter 3, a characterization of centralizing additive maps on block triangular matrices over an arbitrary field is presented. The characterization is obtained by studying some relevant works in centralizing additive maps on triangular rings and some basic properties of block triangular matrices.

In Chapter 4, we present our main result of this dissertation. A characterization of centralizing additive maps on rank r block triangular matrices is obtained. This is achieved by the development of the rank factorization of block triangular matrices and several technical lemmas. Then we employ the characterization of centralizing additive maps obtained in Chapter 3 to achieve this result.

In Chapter 5, we study commuting additive maps on rank r upper triangular matrices over an arbitrary field. We employ the characterization of centralizing additive maps obtained in Chapter 4, together with some recent works in commuting additive maps, to give a complete description of commuting additive maps on rank $r > 1$ upper triangular matrices. We then continue to study commuting additive maps on rank one upper triangular matrices of orders two and three.

Lastly, in Chapter 6, we present the summary of our findings and list some open problems for future study.

CHAPTER 2: LITERATURE REVIEW AND SOME PRELIMINARIES

2.1 A brief overview

This chapter starts with some preliminaries on basic algebraic structures, prime rings, triangular rings and block triangular matrix algebras. These preliminaries consist of definitions, propositions and some theorems that will be used throughout this research. The following section is a literature review of some classical results and articles related to functional identities and centralizing maps on various rings and algebras. The last section provides us with a brief discussion of the methodology used in this research.

2.2 Some preliminaries

2.2.1 Some basic algebraic structures

Definition 2.2.1. A **group** $G = (G, \circ)$ is a set G together with a binary operation on G with the following properties:

(i) The operation $\circ$ is **associative**, i.e., $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b \in G$.

(ii) There exists an **identity element**, i.e., there is $e \in G$, $e \circ a = a \circ e = a$ for all $a \in G$.

(iii) Each element $a \in g$ has an **inverse**, i.e., for each $a \in G$, there is $b \in G$, $a \circ b = b \circ a = e$.

A group is **abelian** if its binary operation is commutative. We say that G is an **additive group** if the binary operation $\circ$ is referred as the additive notation, i.e., $a \circ b = a + b$ for $a, b \in G$.

Definition 2.2.2. A nonempty subset H of a group $(G, \circ)$ is a **subgroup** of G if $a, b \in H$ implies that $ab^{-1} \in H$. For additive group $(G, +)$, a subgroup H of G if and only if $H \neq \varnothing$ and $a - b \in H$ for every $a, b \in H$.

Definition 2.2.3. A **ring** $R = (R, +, \cdot)$ is a set R together with two binary operations on R, an addition "$+$" and a multiplication "$\cdot$" such that

(i) $(R, +)$ is an **abelian** group.

(ii) The multiplication is **associative**, i.e., $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in R$.

(iii) The multiplication is **distributive**, i.e., $(a+b)\cdot c = a\cdot c + b\cdot c$ and $a\cdot(b+c) = a\cdot b + a\cdot c$ for all $a, b, c \in R$.

An element $u \in R$ is called an **identity element** or a **unity** of R (for the multiplication) if $u \cdot a = a \cdot u = a$ for all $a \in R$. A ring with an identity element is called a **ring with unity** or a **unital** ring. The zero element of a ring is the identity element **0** for addition, i.e., $0 + a = a + 0 = a$ for all $a \in R$. A ring is **commutative** when its multiplication is commutative, i.e., $a \cdot b = b \cdot a$ for all $a, b \in R$. A **field** $\mathbb{F}$ is a commutative ring with unity $u \neq 0$ such that every nonzero element in $\mathbb{F}$ is invertible.

Definition 2.2.4. Let $(R, +, \cdot)$ be a ring. A nonempty subset S of R is a called a **subring** of R if $(S, +, \cdot)$ is itself a ring, or equivalently, S is a subring of R if and only if $a - b \in S$ and $ab \in S$ for all $a, b \in S$.

Definition 2.2.5. Let R be a ring. The **center** of R is the set

$$Z(R) = \{z \in R : zr = rz \text{ for all } r \in R\}.$$

Elements in $Z(R)$ are called **central elements** and $Z(R)$ is a subring of R.

Let n be a positive integer and let $\mathbb{F}$ be a field. We denote by $M_n(\mathbb{F})$ and $T_n(\mathbb{F})$ the set of all $n \times n$ matrices over $\mathbb{F}$ and the set of all $n \times n$ upper triangular matrices over $\mathbb{F}$, respectively. Under the usual matrix addition and matrix multiplication, $M_n(\mathbb{F})$ and $T_n(\mathbb{F})$ are noncommutative rings with unity I_n, the identity matrix. It can be verified that $T_n(\mathbb{F})$

is a subring of $M_n(\mathbb{F})$, and $Z(M_n(\mathbb{F})) = \mathbb{F} \cdot I_n = \{\lambda I_n : \lambda \in \mathbb{F}\}$ and $Z(T_n(\mathbb{F})) = \mathbb{F} \cdot I_n$. For a detailed proof, see for example (Chooi et al., 2020, Lemma 2.6).

Definition 2.2.6. Let R and R' be rings. A map $\varphi : R \to R'$ is said to be a **ring homomorphism** if

(i) $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in R$,

(ii) $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.

If a ring homomorphism φ is bijective, then φ is a **ring isomorphism**.

Definition 2.2.7. Let R be a ring. An element $e \in R$ is said to be an **idempotent** if $e^2 = e$. Moreover, if R is a unital ring, we say that $e \in R$ is a **nontrivial idempotent** if it is an idempotent different from 0 and 1 in R.

Let $E_{ij} \in M_n(\mathbb{F})$ be the matrix whose (i, j)th entry is one and zero elsewhere. If $n \geq 2$, then E_{ii} is a nontrivial idempotent element of $M_n(\mathbb{F})$.

Definition 2.2.8. Let $(R, +, \cdot)$ be a ring. A **left ideal** I (respectively, **right ideal**) of R is an additive subgroup of $(R, +)$ such that $a \in I$ and $r \in R$ implies that $ra \in I$ (respectively, $ar \in I$). We say that I is an **ideal** of R if it is two-sided ideal, i.e., I is is an additive subgroup of R such that $a \in I$ and $r \in R$ implies that $ra \in I$ and $ar \in I$. An ideal I of a ring R is a subring of R. Evidently, R and $\{0\}$ are two ideals of R and $\{0\}$ is called the **trivial ideal** of R. An ideal I of R is said to be **proper** if $I \neq R$.

Note that $T_n(\mathbb{F})$ is not an ideal of $M_n(\mathbb{F})$, because $E_{22} \in T_n(\mathbb{F})$ and $E_{21} \in M_n(\mathbb{F})$ but $E_{22}E_{21} \notin T_n(\mathbb{F})$. Also, $Z(M_n(\mathbb{F}))$ is not an ideal of $M_n(\mathbb{F})$. To see this, let $A \in Z(M_n(\mathbb{F}))$ be nonzero. Then $A = \alpha I_n$ for some nonzero $\alpha \in \mathbb{F}$. Take $B = E_{12} \in M_n(\mathbb{F})$. Then $AB = \alpha E_{12} \notin Z(M_n(\mathbb{F}))$, and so $Z(M_n(\mathbb{F}))$ is not an ideal of $M_n(\mathbb{F})$. Likewise, $Z(T_n(\mathbb{F}))$ is not an ideal of $T_n(\mathbb{F})$.

Definition 2.2.9. Let $\mathbb{F}$ be a field. An **algebra** over $\mathbb{F}$ is a nonempty set $\mathcal{A}$ together with two binary operations on $\mathcal{A}$: an addition "+" and a multiplication "·", and an external binary operation: scalar multiplication from $\mathbb{F} \times \mathcal{A}$ into $\mathcal{A}$, with the following properties:

(**A1**) $\mathcal{A}$ is a ring under addition and multiplication.

(**A2**) $\mathcal{A}$ is a linear space under addition and scalar multiplication.

(**A3**) $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{F}$.

Notice that an algebra $\mathcal{A}$ is a ring as well as a linear space which is endowed with an associative bilinear multiplication: $(\lambda a + b)c = \lambda(ac) + bc$ and $a(\lambda b + c) = \lambda(ab) + ac$ for all $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{F}$. A **subalgebra** is a subring and a subspace. A **unital** algebra is an algebra which is unital as a ring. The center of an algebra is just as the center of a ring.

We see that $M_n(\mathbb{F})$ and $T_n(\mathbb{F})$ are unital algebras over $\mathbb{F}$ with unity I_n and center $\mathbb{F} \cdot I_n$. Also, $T_n(\mathbb{F})$ is a subalgebra of $M_n(\mathbb{F})$.

2.2.2 Prime rings

Let R be a ring. Let $a, b \in R$ and let S be a subset of R. We define

$$aSb = \{asb : s \in S\}. \tag{2.1}$$

If S is a subring of R and $a, b \in S$, then aSb is a subring of R. To see this, we first note that $aSb \neq \varnothing$ since S is a subring of R. Let $x, y \in aSb$. Then $x = as_1b$ and $y = as_2b$ for some $s_1, s_2 \in S$. Note that

$$x - y = a(s_1 - s_2)b \in aSb \quad \text{and} \quad xy = a(s_1bas_2)b \in aSb$$

since $s_1 - s_2 \in S$, $ba \in S$ and $s_1bas_2 \in S$.

Note that if R is a ring with idempotent $e \in R$, then eRe is a unital ring with identity element e. Since eRe is a ring, it follows that $e = eee \in eRe$ is e which is the identity element of eRe. Let $x \in eRe$. Then $x = ere$ for some $r \in R$ and

$$ex = e(ere) = e^2re = ere = x \text{ and } xe = (ere)e = ere^2 = ere = x$$

for every $x \in eRe$. Hence eRe is a unital ring even if R is not.

Definition 2.2.10. A ring R is **prime** if for any $a, b \in R$, $aRb = \{0\}$ implies that $a = 0$ or $b = 0$.

Theorem 2.2.1. *Let n be a positive integer and let $\mathbb{F}$ be a field. Then $M_n(\mathbb{F})$ is a prime ring.*

Proof. The result is clear when $n = 1$. Consider $n \geq 2$. Let $A, B \in M_n(\mathbb{F})$ be such that $AM_n(\mathbb{F})B = \{0\}$. We claim that either $A = 0$ or $B = 0$. Suppose to the contrary that A and B are nonzero. It follows from the canonical rank factorization theorem that there exist invertible matrices $P_1, P_2, Q_1, Q_2 \in M_n(\mathbb{F})$ such that $A = P_1(E_{11} + \cdots + E_{pp})Q_1$ and $B = P_2(E_{11} + \cdots + E_{qq})Q_2$ for some integers $1 \leq p, q \leq n$. We thus obtain

$$(E_{11} + \cdots + E_{pp})M(E_{11} + \cdots + E_{qq}) = 0$$

for every $M \in M_n(\mathbb{F})$. Taking $M = E_{11}$, we have $(E_{11}+\cdots+E_{pp})E_{11}(E_{11}+\cdots+E_{qq}) = E_{11} \neq 0$, a contradiction. Hence $AM_n(\mathbb{F})B = 0$ implies either $A = 0$ or $B = 0$. □

When $n \geq 2$, we note that $T_n(\mathbb{F})$ is not a prime ring. Let $A = E_{nn}$ and $B = E_{11}$ be elements of $T_n(\mathbb{F})$. Note that $E_{nn}E_{ij}E_{11} = 0$ for every $1 \leq i \leq j \leq n$. Then $ATB = 0$ for every $T \in T_n(\mathbb{F})$.

2.2.3 Triangular rings

Definition 2.2.11. Let R be a ring. A **left module over** R, or a **left** R**-module** is an additive group $\mathcal{V}$ together with a map from $R \times \mathcal{V}$ into $\mathcal{V}$, $(r, u) \to ru$, such that for every $r, s \in R$ and $u, v \in \mathcal{V}$,

(L1) $(r + s)u = ru + su$,

(L2) $r(u + v) = ru + rv$,

(L3) $r(su) = (rs)u$.

If, additionally, R is a unital ring with unity 1 and

(L4) $1u = u$

for every $u \in \mathcal{V}$, then $\mathcal{V}$ is called a **unital left module over** R, or a **unital left** R**-module**. A **right module over** R, or a right R**-module** is an additive group $\mathcal{V}$ together with a map from $\mathcal{V} \times R$ into $\mathcal{V}$, $(u, r) \to ur$, such that for every $r, s \in R$ and $u, v \in \mathcal{V}$,

(R1) $u(r + s) = ur + us$,

(R2) $(u + v)r = ur + vr$,

(R3) $(us)r = u(sr)$.

If, additionally, R is a unital ring with unity 1 and

(R4) $u1 = u$

for every $u \in \mathcal{V}$, then $\mathcal{V}$ is called a **unital right module over** R, or a **unital right** R**-module**. If R is commutative, then every left R-module $\mathcal{V}$ becomes right R-module by defining $ur := ru$ for every $r \in R$ and $u \in \mathcal{V}$. Unless specified otherwise, we will henceforth adopt the following convention:

$$R - \text{module} := \text{left } R - \text{module}.$$

When it will be clear which ring R we have in mind, or when R will play just a formal role in our discussion, we will simply use the term "**module**".

Notice that a linear space over a field $\mathbb{F}$ is a unital $\mathbb{F}$-module.

Definition 2.2.12. Let R and S be rings. If $\mathcal{V}$ is both a left R-module and a right S-module satisfying

$$(ru)s = r(us)$$

for every $r \in R$, $u \in \mathcal{V}$ and $s \in S$, then $\mathcal{V}$ is called an (R, S)-**bimodule**. An (R, R)-**bimodule** is called an R-**bimodule**. In addition, we say that $\mathcal{V}$ is a unital (R, S)-**bimodule** if it is both a unital left R-module and a unital right S-module.

Definition 2.2.13. Let R be a ring. A left R-module $\mathcal{V}$ is said to be **faithfu**l if $r = 0$ is the only element in R satisfying $r\mathcal{V} = \{0\}$, or equivalently,

$$\{r \in R : r\mathcal{V} = \{0\}\} = \{0\},$$

where $r\mathcal{V} := \{rv : v \in \mathcal{V}\}$. Note that $r\mathcal{V} = \{0\}$ if and only if $rv = 0$ for all $v \in \mathcal{V}$. A right R-module $\mathcal{U}$ is said to be **faithful** if $r = 0$ is the only element in R satisfying $\mathcal{U}r = \{0\}$, or equivalently,

$$\{r \in R : \mathcal{U}r = \{0\}\} = \{0\},$$

where $\mathcal{U}r := \{ur : u \in \mathcal{U}\}$.

Definition 2.2.14. Let R and S be rings. A (R, S)-bimodule $\mathcal{M}$ is said to be **faithful** if $\mathcal{M}$ is faithful as a left R-module as well as a right S-module, i.e.,

$$\{r \in R : r\mathcal{M} = 0\} = \{0\} \text{ and } \{s \in S : \mathcal{M}s = 0\} = \{0\}.$$

Let R be a unital ring with unity 1. Two elements $r, s \in R$ are said to be **orthogonal** if $rs = sr = 0$. Notice that if $e \in R$ is a nontrivial idempotent, then $f := 1 - e$ is a nontrivial idempotent such that e and $1 - e$ are orthogonal. If R be a unital ring with nontrival idempotent e, then, by (2.1), it can be shown that

(i) eRe and fRf are unital rings with identity elements $e = eee \in eRe$ and $f = fff \in fRf$, respectively.

(ii) eRf is a left eRe-module and eRf is a right fRf-module, and so eRf is a unital (eRe, fRf)-bimodule. Note that $em = m$ and $mf = m$ for every $m \in eRf$.

Definition 2.2.15. Let R be a unital ring with unity 1. We say that R is a **triangular ring** if there exists a nontrivial idempotent $e \in R$ such that eRf is a faithful (eRe, fRf)-bimodule and $fRe = \{0\}$. Here, $f = 1 - e$ and eRe and fRf are called the **corner rings** corresponding to e.

We remark that when eRf is a faithful (eRe, fRf)-bimodule which means

$$\{r \in eRe : r(eRf) = \{0\}\} = \{0\} \text{ and } \{s \in fRf : (eRf)s = \{0\}\} = \{0\}.$$

Let $A \in M_{m,n}(\mathbb{F})$. Notice that $AM_{n,q}(\mathbb{F}) = \{0\}$ if and only if $AB = 0$ for every $B \in M_{n,q}(\mathbb{F})$. It is not difficult to verify that $AM_{n,q}(\mathbb{F}) = \{0\}$ if and only if $A = 0$. Likewise, $M_{p,m}(\mathbb{F})A = \{0\}$ if and only if $A = 0$.

Proposition 2.2.1. *Let R be a unital ring with unity and nontrivial idempotent $e \in R$. If eRf is a faithful (eRe, fRf)-bimodule, then $eRf \neq \{0\}$.*

Proof. Note that $0 \neq eee \in eRe$ and $(eee)eRf = eRf$. If $eRf = \{0\}$, then $(eee)eRf = \{0\}$ which contradicts to eRf is a faithful left eRe-module. □

Let $G_1, \ldots, G_n$ be additive subgroups of an additive group G. The sum $G_1 + \cdots + G_n = \{g_1 + \cdots + g_n : g_i \in G_i,\ i = 1, \ldots, n\}$. The sum is **direct**, denoted $G_1 \oplus \cdots \oplus G_n$, provided that

$$G_i \cap (G_1 + \cdots + G_{i-1} + G_{i+1} + \cdots + G_n) = \{0\}$$

for every $i = 1, \ldots, n$. Indeed, it can be shown that every element in $G_1 \oplus \cdots \oplus G_n$ can be written as $g_1 + \cdots + g_n$, where $g_i \in G_i, i = 1, \ldots, n$, in a unique way.

Proposition 2.2.2. *Let R be a unital ring. If R is a triangular ring, then there exists a nontrivial idempotent $e \in R$ such that R can be represented by the **Peirce decomposition** with respect to*

$$R = eRe \oplus eRf \oplus fRf,$$

where $f = 1 - e$ and 1 is the unity of R. The Peirce decomposition of R with respect to a nontrivial idempotent $e \in R$ states that for each $r \in R$, there is a unique representation

$$r = ere + erf + frf.$$

Proof. We first show that $R = eRe + eRf + fRf$. Clearly, $eRe + eRf + fRf \subseteq R$. Let $r \in R$. Since $e+f = 1$, it follows that $r = 1r1 = (e+f)r(e+f) = ere+erf+frf$ since

$fre = 0$. Hence $R = eRe + eRf + fRf$. Suppose that $r = er_1e + er_2f + fr_3f$ for some $r_1, r_2, r_3 \in R$. Then $e(r-r_1)e+e(r-r_2)f+f(r-r_3)f = 0$. Since $ef = fe = 1$ and $e^2 = e$, it follows that $0 = e0e = e(e(r-r_1)e+e(r-r_2)f+f(r-r_3)f)e = r-r_1$. This implies that $r_1 = r$. Likewise, $r_2 = r$ and $r_3 = r$. Consequently, $R = eRe \oplus eRf \oplus fRf$. □

Definition 2.2.16. Let R and S be rings and let $\mathcal{M}$ be a (R, S)-bimodule. Let

$$\text{Tri}(R, \mathcal{M}, S) = \left\{ \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} : r \in R,\ s \in S,\ m \in \mathcal{M} \right\}. \tag{2.2}$$

We define the usual matrix-like addition and matrix-like multiplication on Tri$(R, \mathcal{M}, S)$ as follows:

$$\begin{bmatrix} r_1 & m_1 \\ 0 & s_1 \end{bmatrix} + \begin{bmatrix} r_2 & m_2 \\ 0 & s_2 \end{bmatrix} = \begin{bmatrix} r_1 + r_2 & m_1 + m_2 \\ 0 & s_1 + s_2 \end{bmatrix},$$

$$\begin{bmatrix} r_1 & m_1 \\ 0 & s_1 \end{bmatrix} \cdot \begin{bmatrix} r_2 & m_2 \\ 0 & s_2 \end{bmatrix} = \begin{bmatrix} r_1r_2 & r_1m_1 + m_2s_2 \\ 0 & s_1s_2 \end{bmatrix}$$

for every $r_1, r_2 \in R$, $s_1, s_2 \in S$ and $m_1, m_2 \in \mathcal{M}$. Then Tri$(R, \mathcal{M}, S)$ forms a ring under the usual matrix operations.

Let R be a unital ring. By Proposition 2.2.2, if R is a triangular ring, then there exists a nontrivial idempotent $e \in R$ such that eRf is a faithful (eRe, fRf)-bimodule and $fRe = \{0\}$, where $f = 1 - e$. By the Peirce decomposition of R with respect to e, we define the isomorphism $\Psi : R \to$ Tri(eRe, eRf, fRf) by

$$\Psi(r) = \begin{bmatrix} ere & erf \\ 0 & frf \end{bmatrix}$$

for every $r \in R$. Consequently, we summarize the following observation.

Theorem 2.2.2. *Let R be a unital ring. Then R is a triangular ring if and only if there exist unital rings $\mathcal{A}, \mathcal{B}$ and a unital faithful $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ such that R is isomorphic to the ring*

$$\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a \in \mathcal{A},\ b \in \mathcal{B},\ m \in \mathcal{M} \right\}$$

under the usual matrix addition and matrix multiplication.

Proof. If R is a triangular ring, then there exists a nontrivial idempotent $e \in R$ such that eRf is a faithful (eRe, fRf)-bimodule and $fRe = \{0\}$, where $f = 1 - e$ and 1 is the unity of R. Note that eRe and fRf are unital ring and eRf is a unital faithful (eRe, fRf)-bimodule. By the Peirce decomposition of R, we have $R = eRe \oplus eRf \oplus fRf$. Let $\Psi : R \to \text{Tri}(eRe, eRf, fRf)$ be the isomorphism defined by

$$\Psi(r) = \begin{bmatrix} ere & erf \\ 0 & frf \end{bmatrix}$$

for every $r \in R$. Hence R is isomorphic to $\text{Tri}(eRe, eRf, fRf)$.

Conversely, if R is isomorphic to the ring $\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are unital rings, and $\mathcal{M}$ is a unital faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then there exists a ring isomorphism $\Psi : \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) \to R$ such that

$$R = \Psi(\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})) = S \oplus \mathcal{N} \oplus T,$$

where $S = \Psi(\text{Tri}(\mathcal{A}, 0, 0))$, $T = \Psi(\text{Tri}(0, 0, \mathcal{B}))$ and $\mathcal{N} = \Psi(\text{Tri}(0, \mathcal{M}, 0))$. By virtue of Φ is a ring isomorphism, $\mathcal{A}$ and $\mathcal{B}$ are unital rings, and $\mathcal{M}$ is a unital faithful $(\mathcal{A}, \mathcal{B})$-bimodule, it follows that that S and T are unital subrings of R, and $\mathcal{N}$ is a unital faithful (S, T)-bimodule.

Let $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$ be the identity elements of the rings $\mathcal{A}$ and $\mathcal{B}$, respectively. Let

$$E_{\mathcal{A}} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix} \in \text{Tri}(\mathcal{A}, 0, 0) \quad \text{and} \quad E_{\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 0 & e_{\mathcal{B}} \end{bmatrix} \in \text{Tri}(0, 0, \mathcal{B}).$$

Since $E_{\mathcal{A}}^2 = E_{\mathcal{A}}$ and $E_{\mathcal{B}}^2 = E_{\mathcal{B}}$, we obtain $e = \Phi(E_{\mathcal{A}})$ and $f = \Phi(E_{\mathcal{B}})$ are nontrivial idempotents of R such that $e + f = 1$, and $eRe = \Phi(E_{\mathcal{A}})\Psi(\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}))\Phi(E_{\mathcal{A}}) = \Psi(\text{Tri}(\mathcal{A}, 0, 0)) = S$, $fRf = T$, $eRf = \mathcal{N}$ and $fRe = \{0\}$. Then R is a triangular ring. □

Proposition 2.2.3. *Let R be a triangular ring with with identity 1 and nontrivial idempotent e. Then*

$$Z(R) = \{a + b \in Z(eRe) \oplus Z(fRf) : am = mb \text{ for every } m \in eRf\},$$

where $f = 1 - e$. Moreover, if eRf is a faithful (eRe, fRf)-bimodule, then

$$Z(R) = \{a + b \in eRe \oplus fRf : am = mb \text{ for every } m \in eRf\}.$$

Proof. Denote $H = \{a + b \in Z(eRe) \oplus Z(fRf) : am = mb \text{ for every } m \in eRf\}$. Let $h \in H$. Then $h = a + b$ for some $a \in Z(eRe)$ and $b \in Z(fRf)$ such that $am = mb$ for all $m \in eRf$. Let $r \in R$. Since $R = eRe \oplus eRf \oplus fRf$, we have $r = a' + m + b'$ for some $a' \in eRe$, $m \in eRf$ and $b' \in fRf$. Note that

$$rh = (a'+m+b')(a+b) = a'a+mb+b'b = aa'+am+bb' = (a+b)(a'+m+b') = hr$$

since $a'b = ma = b'a = ab' = ba' = bm = 0$. Then $rh = hr$ for all $r \in R$. Hence $h \in Z(R)$, so $H \subseteq Z(R)$.

Let $x \in Z(R)$. Since $Z(R) \subseteq R = eRe \oplus eRf \oplus fRf$, it follows that $x = a_1 + m_1 + b_1$ for some $a_1 \in eRe$, $m_1 \in eRf$ and $b_1 \in fRf$. Note that

$$a_1 + m_1 = e(a_1 + m_1 + b_1) = (a_1 + m_1 + b_1)e = a_1 e = a_1$$

since $eb_1 = m_1 e = b_1 e = 0$. Then $m_1 = 0$, and so $x = a_1 + b_1$. Consequently, each element in $Z(R)$ is of the form $a + b$ for some $a \in eRe$ and $b \in fRf$.

We now claim $a_1 n = nb_1$ for all $n \in eRf$. Let $n \in eRf$. Then $a_1 n = (a_1 + b_1)n$ since $b_1 n = 0$. Since $a_1 + b_1 \in Z(R)$, we have $(a_1 + b_1)n = n(a_1 + b_1)$. So $a_1 n = n(a_1 + b_1) = nb_1$ since $na_1 = 0$.

We next claim that $a_1 \in Z(eRe)$ and $b_1 \in Z(fRf)$. Let $p \in eRe$ and $q \in fRf$. Note that $a_1 p + b_1 q = (a_1 + b_1)(p + q) = (p + q)(a_1 + b_1) = pa_1 + qb_1$. This implies that $(a_1 p - pa_1) + (b_1 q - qb_1) = 0$. Note that $a_1 p - pa_1 \in eRe$ and $b_1 q - qb_1 \in fRf$. Since $eRe \cap fRf = \{0\}$, we must have $a_1 p - pa_1 = 0$ and $b_1 q - qb_1 = 0$ due to unique representation. Hence $a_1 p = pa_1$ for all $p \in eRe$, and $b_1 q = qb_1$ for all $q \in fRf$. Hence $a_1 \in Z(eRe)$ and $b_1 \in Z(fRf)$, so $x \in H$. Consequently, $Z(R) = H$ as desired.

Consider now eRf is faithful. Let $K = \{a+b \in eRe \oplus fRf : am = mb \text{ for every } m \in eRf\}$. By virtue of $Z(R) = H$, we see that $Z(R) \subseteq K$. Let $k \in K$. Then $k = a + b$ for some $a \in eRe$ and $b \in fRf$ such that $am = mb$ for all $m \in eRf$. We claim that $a \in Z(eRe)$. Let $x \in eRe$. Note that $(ax - xa)m = (ax)m - (xa)m = a(xm) - x(am) = (xm)b - x(mb)$ since $xm \in eRf$. Hence $(ax - xa)m = xmb - xmb = 0$. Hence $(ax - xa)m = 0$ for all $m \in eRf$. Since eRf is a faithful (eRe, fRf)-bimodule, we must have $ax - xa = 0$, and so $a \in Z(eRe)$. Likewise, $b \in Z(fRf)$. Hence $k \in H = Z(R)$. The proof is complete. □

2.2.4 Block triangular matrix algebras

Let m, n be positive integers and let $\mathbb{F}$ be a field. Let $M_{m,n}(\mathbb{F})$ be the linear space of $m \times n$ matrices over $\mathbb{F}$. When $m = n$, we recall that $M_n(\mathbb{F})$ constitutes an algebra over $\mathbb{F}$ under the usual matrix multiplication. Let $k, n_1, \ldots, n_k$ be positive integers such that $n_1 + \cdots + n_k = n$. By $\mathcal{T}_{n_1,\ldots,n_k}$ we designate the subalgebra of $M_n(\mathbb{F})$ consisting of all upper triangular block matrices of the form

$$\begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1k} \\ 0 & A_{22} & \ldots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{kk} \end{bmatrix},$$

where $A_{ij} \in M_{n_i,n_j}(\mathbb{F})$ for all $1 \leq i \leq j \leq k$. We shall call $\mathcal{T}_{n_1,\ldots,n_k}$ the $n_1, \ldots, n_k$ **block triangular matrix algebra over** $\mathbb{F}$, or simply a **block triangular matrix algebra**. When $k = 1$, we have $\mathcal{T}_{n_1,\ldots,n_k} = M_n(\mathbb{F})$ and when $k = n$, we obtain $\mathcal{T}_{n_1,\ldots,n_k} = \mathcal{T}_n(\mathbb{F})$, the algebra of all $n \times n$ upper triangular matrices over $\mathbb{F}$.

Theorem 2.2.3. *Let $k, n_1, \ldots, n_k$ be positive integers such that $k \geq 2$. Then $\mathcal{T}_{n_1,\ldots,n_k}$ is a triangular ring.*

Proof. Let $n_1 + \cdots + n_k = n$. Note that $\mathcal{T}_{n_1,\ldots,n_k}$ is an associative ring with unity I_n. Let $E = E_{11} + \cdots + E_{n_1,n_1}$. Then E is a nontrivial idempotent of $\mathcal{T}_{n_1,\ldots,n_k}$ such that $E\mathcal{T}_{n_1,\ldots,n_k}E = \{A \oplus 0_{n-n_1} : A \in M_{n_1}(\mathbb{F})\}$, $F\mathcal{T}_{n_1,\ldots,n_k}F = \{0_{n_1} \oplus B : B \in \mathcal{T}_{n_2,\ldots,n_k}\}$, $F\mathcal{T}_{n_1,\ldots,n_k}E = \{0_n\}$ and

$$E\mathcal{T}_{n_1,\ldots,n_k}F = \left\{ \begin{bmatrix} 0_{n_1} & M \\ 0 & 0_{n-n_1} \end{bmatrix} : M \in M_{n_1,n-n_1}(\mathbb{F}) \right\},$$

where $F = I_n - E$. We note that $\mathcal{T}_{n_1,\dots,n_k}$ has the Peirce decomposition

$$\mathcal{T}_{n_1,\dots,n_k} = E\mathcal{T}_{n_1,\dots,n_k}E \oplus E\mathcal{T}_{n_1,\dots,n_k}F \oplus F\mathcal{T}_{n_1,\dots,n_k}F$$

with respective to E. Since $M_{n_1}(\mathbb{F})$, $\mathcal{T}_{n_2,\dots,n_k}$ are unital rings and $M_{n_1,n-n_1}(\mathbb{F})$ a unital faithful $(M_{n_1}(\mathbb{F}), \mathcal{T}_{n_2,\dots,n_k})$-bimodule, $\mathcal{T}_{n_1,\dots,n_k}$ is a unital triangular ring by Definition 2.2.15. □

Theorem 2.2.4. *Let $k, n_1, \dots, n_k$ be positive integers with $n_1 + \cdots + n_k = n$. Then $Z(\mathcal{T}_{n_1,\dots,n_k}) = \mathbb{F} \cdot I_n$.*

Proof. We argue by induction on k. When $k = 1$, $\mathcal{T}_{n_1} = M_{n_1}(\mathbb{F})$. Thus $Z(T_{n_1}) = \mathbb{F} \cdot I_{n_1}$. This validates the base step. Suppose that $Z(\mathcal{T}_{n_2,\dots,n_k}) = \mathbb{F} \cdot I_{n-n_1}$. Let $T \in Z(\mathcal{T}_{n_1,\dots,n_k})$. By Proposition 2.2.3, $T = A \oplus B$ for some $A \in M_{n_1}(\mathbb{F})$, $B \in \mathcal{T}_{n_2,\dots,n_k}$ such that $AM = MB$ for all $M \in M_{n_1,n-n_1}(\mathbb{F})$. Since $T \in Z(\mathcal{T}_{n_1,\dots,n_k})$, it follows that $A \in Z(M_{n_1}(\mathbb{F}))$ and $B \in Z(T_{n_2,\dots,n_k})$. So $A = \alpha I_{n_1}$ and $B \in \beta I_{n-n_1}$ for some $\alpha, \beta \in \mathbb{F}$. Note that $(\alpha - \beta)M = (\alpha I_{n_1})M - M(\beta I_{n-n_1}) = AM - MB = 0$ for every $M \in M_{n_1,n-n_1}(\mathbb{F})$. Since $M_{n_1,n-n_1}(\mathbb{F})$ is faithful, we obtain $\alpha = \beta$. So $Z(\mathcal{T}_{n_1,\dots,n_k}) = \mathbb{F} \cdot I_n$ as desired. □

2.3 Literature review

Let $\mathcal{R}$ be a ring with center $Z(\mathcal{R})$. We say that a map $\psi : \mathcal{R} \to \mathcal{R}$ is **centralizing** on a nonempty subset $\mathcal{S}$ of $\mathcal{R}$ if $[\psi(a), a] \in Z(\mathcal{R})$ for all $a \in \mathcal{S}$, and that is **commuting** on $\mathcal{S}$ if $[\psi(a), a] = 0$ for all $a \in \mathcal{S}$, where $[a, b] = ab - ba$ for $a, b \in \mathcal{R}$. The study of centralizing maps was originated by the classical result of Posner (1957) which states that a prime ring admitting a nonzero centralizing derivation must be commutative. Later, Mayne (1976) obtained an analogous result of centralizing automorphisms on prime rings. Brešar (1993) then gave a structural result of centralizing additive maps $\psi : \mathcal{R} \to \mathcal{R}$ on a prime ring $\mathcal{R}$

of characteristic not two and showed that ψ is of the form

$$\psi(a) = Za + \mu(a) \tag{2.3}$$

for every $a \in \mathcal{R}$, where Z is an element in the extended centroid $\mathcal{C}$ of $\mathcal{R}$ and $\mu : \mathcal{R} \to C$ is an additive map. This result has been extremely influential and stimulated considerable interest in centralizing maps and commuting maps on various rings and algebras (Ara & Mathieu, 1993; Beidar, 1998; Brešar et al., 1993; Cheung, 2001; Du & Wang, 2012; P. H. Lee & Lee, 1997; P. H. Lee & Wang, 2009; Wang, 2016). More importantly, together with the works by Beidar (1998) and Chebotar (1998), their efforts have activated the development of the theory of functional identities which can be informally described as the study of equations in which functions appear as unknowns. The main goal of this study is to determine the general forms and the classifications of all solutions for each functional identity. For an extensive survey of the subject, see the book "Functional Identities" by Brešar et al. (2007).

Lately, inspired by the study of linear preserver problems on matrices (Li & Pierce, 2001; Pierce, 1992), Franca (2012) studied commuting additive maps on invertible (respectively, singular) matrices over a field $\mathbb{F}$. He showed that if $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is an additive map satisfying $[\psi(A), A] = 0$ for all invertible (respectively, singular) matrices $A \in M_n(\mathbb{F})$, then there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : M_n(\mathbb{F}) \to \mathbb{F}$ such that

$$\psi(A) = \lambda A + \mu(A) I_n$$

for all $A \in M_n(\mathbb{F})$. Here, I_n is the identity matrix of $M_n(\mathbb{F})$.

Franca (2012) has advanced the study of functional identities to the set that are not closed under addition. Later, Liu (2014a) extended Franca's result and characterized centralizing additive maps on invertible (respectively, singular) matrices over division rings. This new line of research in functional identities has been continued in commuting additive maps on rank k matrices (Franca, 2013; Xu & Yi, 2014), power commuting additive maps on rank k matrices (Chou & Liu, 2019), commuting traces maps on invertible and singular matrices (Franca, 2015), strong commutativity preserving maps on rank k matrices (Liu et al., 2018) and additivity preserving maps on rank k matrices (Chooi & Kwa, 2019, 2020; Xu & Liu, 2017).

Let $\mathbb{F}$ be an arbitrary field and let $k, n_1, \ldots, n_k$ be positive integers such that $n_1 + \cdots + n_k = n$. Motivated by the recent development of functional identities on rank k matrices, in this dissertation, we classify centralizing additive maps $\psi : \mathcal{T}_{n_1,\ldots,n_k} \to \mathcal{T}_{n_1,\ldots,n_k}$ on block triangular matrices over $\mathbb{F}$. Next, we deduce from the obtained result a complete characterization of centralizing additive maps $\psi : \mathcal{T}_{n_1,\ldots,n_k} \to \mathcal{T}_{n_1,\ldots,n_k}$ on rank r block triangular matrices with $1 < r \leq n$ being a fixed integer such that $r \neq n$ when $\mathbb{F}$ is the Galois field of two elements. Together with the results in (Chooi et al., 2019, Theorems 2.8, 2.9, 2.10) and (Chooi et al., 2020, Theorem 1.1), we give a complete structural description of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank r upper triangular matrices over $\mathbb{F}$, where $1 < r \leq n$ is a fixed integer.

2.4 Methodology

The methodology that has been employed in this research consists of three main components. The first component involves a proper literature review of some classical and the latest articles on centralizing additive maps on algebras and rings. This is followed by a preliminary background study on triangular rings and matrix rings that will be employed in this research. Various techniques from research papers will be studied for possible application in our study. The second component concerns the classification of centralizing additive maps on block triangular matrices and on rank r block triangular matrices. During these studies, various useful techniques by other researchers will be noted and new mathematical tools will be developed. The third component is to apply the obtained result of centralizing additive maps on block triangular matrices in the study of commuting additive maps on rank r upper triangular matrices. This component will highlight the significance of our study in this research.

CHAPTER 3: CENTRALIZING ADDITIVE MAPS ON BLOCK TRIANGULAR MATRICES

3.1 A brief overview

This chapter describes the characterization of centralizing additive maps on block triangular matrices over an arbitrary field. Three lemmas are also presented to arrive at our main outcome of this chapter.

3.2 A characterization of centralizing additive maps on block triangular matrices

Throughout this dissertation, we keep in mind that $\mathcal{T}_{n_1,\ldots,n_k}$ is the $n_1,\ldots,n_k$ block triangular matrix algebra over a field $\mathbb{F}$ with $n_1+\cdots+n_k=n$, unity I_n and center $Z(\mathcal{T}_{n_1,\ldots,n_k})$. We refer to the commutator as $[\,\cdot\,,\,\cdot\,]$ and we write $E_{ij}\in M_n(\mathbb{F})$ for the standard matrix unit whose (i,j)th entry is one and zeros elsewhere.

We begin our discussion with a lemma that will be used to prove a result on rank factorization of block triangular matrices.

Lemma 3.2.1. *Let $r,k,n_1,\ldots,n_k$ be positive integers such that $n_1+\cdots+n_k=n\geq 2$ and $r\leq n$. If $A=E_{h_1,k_1}+\cdots+E_{h_r,k_r}$ is such that $E_{h_i,k_i}\in\mathcal{T}_{n_1,\ldots,n_k}$ for $i=1,\ldots,r$, and $h_i\neq h_j$ and $k_i\neq k_j$ whenever $1\leq i\neq j\leq r$, then there exists an invertible matrix $P\in\mathcal{T}_{n_1,\ldots,n_k}$ such that*

$$PA=E_{s_1,t_1}+\cdots+E_{s_r,t_r}$$

for some integers $1\leq s_i\leq t_i\leq n$ for $i=1,\ldots,r$ such that $s_1<\cdots<s_r$ and $t_i\neq t_j$ whenever $1\leq i\neq j\leq r$.

Proof. If $h_i\leq k_i$ for each $i=1,\ldots,r$, then the conclusion follows by taking $P=I_n$ and an appropriate rearrangement on E_{h_i,k_i} in which $PA=E_{s_1,t_1}+\cdots+E_{s_r,t_r}$ with

$s_1 < \cdots < s_r$ and $\{(s_i, t_i) : i = 1, \ldots, r\} = \{(h_i, k_i) : i = 1, \ldots, r\}$.

Suppose that there exists an integer $1 \leq p \leq r$ such that $h_p > k_p$. Since $E_{h_p,k_p} \in \mathcal{T}_{n_1,\ldots,n_k}$, there exists an integer $1 \leq l \leq k$ such that $\delta_{l-1} < k_p < h_p \leq \delta_l$, where $\delta_i = n_1 + \cdots + n_i$ for $i = 1, \ldots k$ and $\delta_0 = 0$. We first observe that for each pair of integers $\delta_{l-1} < i \leq \delta_l$ and $\delta_{l-1} < j \leq n$, $E_{ij} \in \mathcal{T}_{n_1,\ldots,n_k}$ and

$$M_{s,i}E_{ij} = E_{sj} \in \mathcal{T}_{n_1,\ldots,n_k} \tag{3.1}$$

for every integer $\delta_{l-1} < s \leq \delta_l$, where $M_{s,i}$ is the elementary matrix performed on I_n by interchanging rows s and i. We argue in the following three cases:

Case I: When $r = 1$, we have $A = E_{h_p,k_p}$. By (3.1), we obtain $M_{k_p,h_p}A = M_{k_p,h_p}E_{h_p,k_p} = E_{k_p,k_p}$.

Case II: When $r \geq 2$ and $k_p \neq h_j$ for every $j \in \{1, \ldots, r\}\backslash\{p\}$, we have $A = E_{h_p,k_p} + \sum_{i=1,i\neq p}^{r} E_{h_i,k_i}$. By (3.1), we obtain

$$\begin{aligned} M_{k_p,h_p}A &= M_{k_p,h_p}(E_{h_p,k_p} + \sum_{i=1,i\neq p}^{r} E_{h_i,k_i}) \\ &= M_{k_p,h_p}E_{h_p,k_p} + M_{h_p,k_p}\sum_{i=1,i\neq p}^{r} E_{h_i,k_i} \\ &= E_{k_p,k_p} + \sum_{i=1,i\neq p}^{r} E_{h_i,k_i}. \end{aligned}$$

Case III: When $r \geq 2$ and $k_p = h_q$ for some $q \in \{1, \ldots, r\}\backslash\{p\}$, we have $A = E_{h_p,k_p} + E_{h_q,k_q} + \sum_{i=1,i\neq p,q}^{r} E_{h_i,k_i}$. By (3.1), we obtain

$$
\begin{aligned}
M_{k_p,h_p}A &= M_{k_p,h_p}(E_{h_p,k_p} + E_{h_q,k_q} + \sum_{i=1,i\neq p,q}^{r} E_{h_i,k_i}) \\
&= M_{k_p,h_p}E_{h_p,k_p} + M_{k_p,h_p}E_{h_q,k_q} + M_{h_p,k_p}\sum_{i=1,i\neq p,q}^{r} E_{h_i,k_i} \\
&= E_{k_p,k_p} + E_{h_p,k_q} + \sum_{i=1,i\neq p,q}^{r} E_{h_i,k_i}.
\end{aligned}
$$

Hence $M_{k_p,h_p} \in \mathcal{T}_{n_1,\ldots,n_k}$ is invertible and

$$
M_{k_p,h_p}A = E_{k_p,k_p} + B, \tag{3.2}
$$

where

$$
B = \begin{cases} 0 & \text{if } r = 1, \\ \sum_{i=1,i\neq p}^{r} E_{h_i,k_i} & \text{if } r \geq 2,\ k_p \neq h_j \text{ for every } j \in \{1,\ldots,r\}\backslash\{p\}, \\ E_{h_p,k_q} + \sum_{i=1,i\neq p,q}^{r} E_{h_i,k_i} & \text{if } r \geq 2,\ k_p = h_q \text{ for some } q \in \{1,\ldots,r\}\backslash\{p\}. \end{cases}
$$

It follows from (3.1) and (3.2) that $B \in \mathcal{T}_{n_1,\ldots,n_k}$, and particularly $E_{h_p,k_q} \in \mathcal{T}_{n_1,\ldots,n_k}$ when $r \geq 2$ and $k_p = h_q$. If B is an upper triangular matrix, then the result holds by taking $P = M_{k_p,h_p}$. Otherwise, since $h_i \neq h_j$ and $k_i \neq k_j$ whenever $i \neq j$, and $M_{i,j}E_{k_p,k_p} = E_{k_p,k_p}$ for every $i, j \neq k_p$, we continue with a similar argument on B and deduce that there exists an invertible matrix $P \in \mathcal{T}_{n_1,\ldots,n_k}$ such that

$$
PA = E_{s_1,t_1} + \cdots + E_{s_r,t_r}
$$

for some integers $1 \leq s_i \leq t_i \leq n$ for $i = 1,\ldots,r$ such that $s_1 < \cdots < s_r$ and $t_i \neq t_j$ whenever $1 \leq i \neq j \leq r$. The proof is complete. $\square$

We are now ready to prove a result on rank factorization of block triangular matrices.

Lemma 3.2.2. *Let $k, n_1, \dots, n_k$ be positive integers such that $n_1 + \cdots + n_k = n \geq 1$. Then $A \in \mathcal{T}_{n_1,\dots,n_k}$ is of rank r if and only if there exist invertible matrices $P, Q \in \mathcal{T}_{n_1,\dots,n_k}$ such that*

$$A = P\Big(\sum_{i=1}^{r} E_{s_i,t_i}\Big)Q$$

for some integers $1 \leq s_i \leq t_i \leq n$ for $i = 1, \dots, r$ such that $s_1 < \cdots < s_r$ and $t_i \neq t_j$ whenever $1 \leq i \neq j \leq r$.

Proof. The sufficiency is clear. We now prove the necessity.

Let $A = (a_{ij}) \in \mathcal{T}_{n_1,\dots,n_k}$ be of rank r. The result is clear when $r = 0$ or $n = 1$. Consider now $r \geq 1$ and $n \geq 2$. We first claim that there exist invertible matrices $H, K \in \mathcal{T}_{n_1,\dots,n_k}$ such that

$$HAK = E_{p_1,q_1} + \cdots + E_{p_r,q_r} \tag{3.3}$$

for some integers $1 \leq p_1, q_1, \dots, p_r, q_r \leq n$ such that $E_{p_i,q_i} \in \mathcal{T}_{n_1,\dots,n_k}$ for $i = 1, \dots, r$, and $p_j < p_i$ and $q_i \neq q_j$ whenever $1 \leq i < j \leq r$. Let R_i and C_i denote the i-th row and the i-th column of A, respectively. Since $A \neq 0$, we let R_{p_1} be the nonzero row of A such that $R_i = 0$ for $i = p_1 + 1, \dots, n$, and let a_{p_1,q_1} be the first nonzero entry in row R_{p_1}. We may assume without loss of generality that $a_{p_1,q_1} = 1$. For each $1 \leq i \leq p_1 - 1$ and $q_1 + 1 \leq j \leq n$, we apply the following elementary row and column operations on A:

$$R_i \to R_i - a_{i,q_1}R_{p_1} \quad \text{and} \quad C_j \to C_j - a_{p_1,j}C_{q_1}. \tag{3.4}$$

Then there exist invertible matrices $H_1, K_1 \in \mathcal{T}_{n_1,\dots,n_k}$ such that

$$H_1AK_1 = E_{p_1,q_1} + B, \tag{3.5}$$

where $E_{p_1,q_1}, B \in \mathcal{T}_{n_1,\ldots,n_k}$. If $B = 0$, then claim (3.3) is proved. Consider now $0 \neq B = (b_{ij})$. In view of the operations performed in (3.4) on A, we see that $b_{i,q_1} = 0$ for every $1 \leq i \leq n$, and $b_{ij} = 0$ for every $p_1 \leq i \leq n$ and $1 \leq j \leq n$. By repeating a similar process on B, there exist integers $1 \leq p_2, q_2 \leq n$, with $p_2 < p_1$ and $q_2 \neq q_1$, and invertible matrices $H_2, K_2 \in \mathcal{T}_{n_1,\ldots,n_k}$, with $H_2E_{p_1,q_1}K_2 = E_{p_1,q_1}$, such that

$$H_2BK_2 = E_{p_2,q_2} + C$$

for some $E_{p_2,q_2}, C \in \mathcal{T}_{n_1,\ldots,n_k}$. Together with (3.5), we have

$$\begin{aligned}(H_2H_1)A(K_1K_2) &= H_2(H_1AK_1)K_2\\ &= H_2(E_{p_1,q_1} + B)K_2\\ &= H_2E_{p_1,q_1}K_2 + H_2BK_2\\ &= E_{p_1,q_1} + E_{p_2,q_2} + C.\end{aligned}$$

Continuing this process, since A is of rank r, we finally reach the desired result (3.3).

By Lemma 3.2.1, there is an invertible matrix $U \in \mathcal{T}_{n_1,\ldots,n_k}$ such that $U(E_{p_1,q_1} + \cdots + E_{p_r,q_r}) = E_{s_1,t_1} + \cdots + E_{s_r,t_r}$ for some integers $1 \leq s_i \leq t_i \leq n$ for $i = 1, \ldots, r$ such that $s_1 < \cdots < s_r$ and $t_i \neq t_j$ whenever $1 \leq i \neq j \leq r$. Together with (3.3) and

$$U(HAK) = U(E_{p_1,q_1} + \cdots + E_{p_r,q_r}) = E_{s_1,t_1} + \cdots + E_{s_r,t_r},$$

we thus obtain

$$A = (UH)^{-1}(E_{s_1,t_1} + \cdots + E_{s_r,t_r})K^{-}1$$
$$= P(E_{s_1,t_1} + \cdots + E_{s_r,t_r})Q,$$

where $P = (UH)^{-1}$ and $Q = K^{-1}$ are invertible matrices in $\mathcal{T}_{n_1,\dots,n_k}$ as desired. □

Given a nonempty subset $\mathcal{S}$ of $\mathcal{T}_{n_1,\dots,n_k}$, we let

$$Z_2(\mathcal{S}) = \{A \in \mathcal{S} : [[A, X], X] = 0 \text{ for all } X \in \mathcal{S}\}.$$

Note also that when $k \geq 2$, $\mathcal{T}_{n_1,\dots,n_k}$ is a triangular algebra because it can be represented as

$$\mathcal{T}_{n_1,\dots,n_k} = \left\{ \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} : A \in M_{n_1}(\mathbb{F}),\ B \in \mathcal{T}_{n_2,\dots,n_k},\ M \in M_{n_1,n-n_1}(\mathbb{F}) \right\} \quad (3.6)$$

with $M_{n_1,n-n_1}(\mathbb{F})$ being a faithful $(M_{n_1}(\mathbb{F}), \mathcal{T}_{n_2,\dots,n_k})$-bimodule.

We are now ready to prove the following lemma.

Lemma 3.2.3. *Let $k, n_1, \dots, n_k$ be positive integers such that $n_1 + \cdots + n_k = n \geq 1$. Then $Z_2(\mathcal{T}_{n_1,\dots,n_k}) = Z(\mathcal{T}_{n_1,\dots,n_k}) = \mathbb{F} \cdot I_n$.*

Proof. It follows from Theorem 2.2.4 that $Z(\mathcal{T}_{n_1,\dots,n_k}) = \mathbb{F} \cdot I_n$.

We now claim that $Z_2(\mathcal{T}_{n_1,\dots,n_k}) = \mathbb{F} \cdot I_n$. When $k \geq 2$, since $\mathcal{T}_{n_1,\dots,n_k}$ is a triangular algebra, the result follows immediately from (Wang, 2016, Lemma 2.2). Consider $k = 1$. Then $\mathcal{T}_{n_1,\dots,n_k} = M_n(\mathbb{F})$. Clearly, $\mathbb{F} \cdot I_n \subseteq Z_2(M_n(\mathbb{F}))$. Let $H = (h_{ij}) \in Z_2(M_n(\mathbb{F}))$. Then

$$[[H, X], X] = 0 \quad (3.7)$$

for all $X \in M_n(\mathbb{F})$. Let $1 \le i \le n$ be an integer. Taking $X = E_{ii}$ in (3.7), we obtain

$$
\begin{aligned}
0 &= [[H, E_{ii}], E_{ii}] \\
&= (HE_{ii} - E_{ii}H)E_{ii} - E_{ii}(HE_{ii} - E_{ii}H) \\
&= (I_n - E_{ii})HE_{ii} - E_{ii}H(E_{ii} - I_n)
\end{aligned}
$$

and since $(I_n - E_{ii})HE_{ii} = (\sum_{l=1,l\neq i}^{n} E_{ll})HE_{ii} = 0$, therefore we obtain $E_{ii}H = E_{ii}HE_{ii}$. Then $\sum_{l=1}^{n} h_{il}E_{il} = h_{ii}E_{ii}$ for $i = 1, \ldots, n$. Hence $H = \text{diag}(h_{11}, \ldots, h_{nn})$ is diagonal. Let $1 \le i, j \le n$ be distinct integers. We set $X = E_{ij} + E_{jj}$ in (3.7). Since $[[H, E_{jj}], E_{ij}] = 0$ and together with the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and (3.7), we obtain

$$
\begin{aligned}
0 &= [[H, E_{ij} + E_{jj}], E_{ij} + E_{jj}] \\
&= [[H, E_{ij}], E_{jj}] \\
&= [h_{ii}E_{ij} - h_{jj}E_{ij}, E_{jj}] \\
&= (h_{ii} - h_{jj})E_{ij}.
\end{aligned}
$$

Thus $(h_{ii} - h_{jj})E_{ij} = 0$, and so $h_{ii} = h_{jj}$ for every $1 \le i \neq j \le n$. Then $H \in \mathbb{F} \cdot I_n$, and hence $Z_2(\mathcal{T}_{n_1,\ldots,n_k}) = \mathbb{F} \cdot I_n$. □

We are now ready to characterize centralizing additive maps on block triangular matrices over an arbitrary field.

Theorem 3.2.1. *Let $k, n_1, \ldots, n_k$ be positive integers such that $n_1 + \cdots + n_k = n \ge 1$. Then $\psi : \mathcal{T}_{n_1,\ldots,n_k} \to \mathcal{T}_{n_1,\ldots,n_k}$ is an additive map satisfying $[\psi(A), A] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$ for all $A \in \mathcal{T}_{n_1,\ldots,n_k}$ if and only if there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : \mathcal{T}_{n_1,\ldots,n_k} \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A) I_n$$

for all $A \in \mathcal{T}_{n_1,\ldots,n_k}$.

Proof. For the sufficiency part, let $A \in \mathcal{T}_{n_1,\ldots,n_k}$, we see that

$$[\psi(A), A] = (\lambda A + \mu(A)I_n)A - A(\lambda A + \mu(A)I_n) = 0 \in Z(\mathcal{T}_{n_1,\ldots,n_k}).$$

Hence the additive map ψ is centralizing on $\mathcal{T}_{n_1,\ldots,n_k}$

For the necessity, if $k = 1$, then $\mathcal{T}_{n_1,\ldots,n_k} = M_n(\mathbb{F})$ is prime. The result follows immediately from (T. K. Lee, 1997, Theorem 2) or (Liu, 2014a, Lemma 2.2). Consider now $k > 1$. We divide the proof into two cases.

Case I: $\mathcal{T}_{n_1,\ldots,n_k} \neq T_2(\mathbb{F})$. Let $E = E_{11} + \cdots + E_{n_1,n_1}$ and $F = I_n - E$. Notice that

$$E\mathcal{T}_{n_1,\ldots,n_k}E = \{ETE : T \in \mathcal{T}_{n_1,\ldots,n_k}\} = \left\{ \begin{bmatrix} X & 0 \\ 0 & 0_{n-n_1} \end{bmatrix} : X \in M_{n_1}(\mathbb{F}) \right\},$$

$$F\mathcal{T}_{n_1,\ldots,n_k}F = \{FTF : T \in \mathcal{T}_{n_1,\ldots,n_k}\} = \left\{ \begin{bmatrix} 0_{n_1} & 0 \\ 0 & Y \end{bmatrix} : Y \in \mathcal{T}_{n_1,\ldots,n_k} \right\}.$$

By Lemma 3.2.3, $Z(E\mathcal{T}_{n_1,\ldots,n_k}E) = \mathbb{F}\cdot E = Z(\mathcal{T}_{n_1,\ldots,n_k})E$ and $Z(F\mathcal{T}_{n_1,\ldots,n_k}F) = \mathbb{F}\cdot F = Z(\mathcal{T}_{n_1,\ldots,n_k})F$. If $n_1 \geq 2$, then $E\mathcal{T}_{n_1,\ldots,n_k}E$ is isomorphic to $M_{n_1}(\mathbb{F})$, which does not contain nonzero central ideals. Moreover, $Z_2(E\mathcal{T}_{n_1,\ldots,n_k}E) = \mathbb{F}\cdot E = Z(E\mathcal{T}_{n_1,\ldots,n_k}E)$ by Lemma 3.2.3. If $n_1 = 1$, then $n - n_1 \geq 2$. Hence $F\mathcal{T}_{n_1,\ldots,n_k}F$ is isomorphic to $\mathcal{T}_{n_2,\ldots,n_k}$, which is either a triangular algebra or $M_{n-n_1}(\mathbb{F})$. So $F\mathcal{T}_{n_1,\ldots,n_k}F$ does not contain nonzero central ideals and $Z_2(F\mathcal{T}_{n_1,\ldots,n_k}F) = \mathbb{F}F = Z(F\mathcal{T}_{n_1,\ldots,n_k}F)$ by Lemma 3.2.3. Then the result follows from (Wang, 2016, Theorem 3.1).

Case II: $\mathcal{T}_{n_1,\ldots,n_k} = T_2(\mathbb{F})$. For each pair of integers $1 \le i \le j \le 2$, let

$$\psi(aE_{ij}) = \begin{bmatrix} f_{ij}(a) & h_{ij}(a) \\ 0 & g_{ij}(a) \end{bmatrix}$$

for every $a \in \mathbb{F}$, where $f_{ij}, g_{ij}, h_{ij} : \mathbb{F} \to \mathbb{F}$ are additive maps. Since $[\psi(aE_{ij}), aE_{ij}] \in Z(T_2(\mathbb{F})) = \mathbb{F} \cdot I_2$ for all $a \in \mathbb{F}$ and $1 \le i \le j \le 2$, it follows that

$$[\psi(aE_{11}), aE_{11}] = \begin{bmatrix} af_{11}(a) & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} af_{11}(a) & ah_{11}(a) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{11} \end{bmatrix},$$

$$[\psi(aE_{22}), aE_{22}] = \begin{bmatrix} 0 & ah_{22}(a) \\ 0 & ag_{22}(a) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & ag_{22}(a) \end{bmatrix} = \begin{bmatrix} \alpha_{22} & 0 \\ 0 & \alpha_{22} \end{bmatrix},$$

$$[\psi(aE_{12}), aE_{12}] = \begin{bmatrix} 0 & af_{12}(a) \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & ag_{12}(a) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_{12} & 0 \\ 0 & \alpha_{12} \end{bmatrix}$$

for some $\alpha_{11}, \alpha_{22}, \alpha_{12} \in \mathbb{F}$. Therefore

$$[\psi(aE_{ij}), aE_{ij}] = 0 \tag{3.8}$$

for all $a \in \mathbb{F}$ and $1 \le i \le j \le 2$. We also obtain

$$h_{11} = h_{22} = 0 \quad \text{and} \quad g_{12} = f_{12}. \tag{3.9}$$

We next see that $[\psi(aE_{11} + bE_{12}), aE_{11} + bE_{12}] \in \mathbb{F} \cdot I_2$ for all $a, b \in \mathbb{F}$. By the additivity of ψ, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and together with (3.8), we obtain

$$\begin{aligned} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} &= [\psi(aE_{11} + bE_{12}), aE_{11} + bE_{12}] \\ &= [\psi(aE_{11}), bE_{12}] + [\psi(bE_{12}), aE_{11}] \\ &= \begin{bmatrix} 0 & bf_{11}(a) \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & bg_{11}(a) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} af_{12}(b) & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} af_{12}(b) & ah_{12}(b) \\ 0 & 0 \end{bmatrix} \end{aligned}$$

for all $a, b \in \mathbb{F}$ and for some $\alpha \in \mathbb{F}$. We thus obtain $\alpha = 0$ and

$$ah_{12}(b) + b(g_{11}(a) - f_{11}(a)) = 0 \tag{3.10}$$

for all $a, b \in \mathbb{F}$. Taking $a = 1$ in (3.10), we obtain

$$h_{12}(b) = \lambda b \tag{3.11}$$

for all $b \in \mathbb{F}$, where $\lambda = f_{11}(1) - g_{11}(1)$. Setting $b = 1$ in (3.10), we obtain

$$f_{11}(a) = g_{11}(a) + \lambda a \tag{3.12}$$

for all $a \in \mathbb{F}$. Likewise, considering $[\psi(bE_{12} + aE_{22}), bE_{12} + aE_{22}] \in \mathbb{F} \cdot I_2$ for all $a, b \in \mathbb{F}$, together with the additivity of ψ, the bilinearity of $[\,\cdot\,,\,\cdot\,]$, (3.8) and (3.11), we obtain

$$\begin{aligned}
\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} &= [\psi(bE_{12} + aE_{22}), bE_{12} + aE_{22}] \\
&= [\psi(bE_{12}), aE_{22}] + [\psi(aE_{22}), bE_{12}] \\
&= \begin{bmatrix} 0 & \lambda ba \\ 0 & ag_{12}(b) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & ag_{12}(b) \end{bmatrix} + \begin{bmatrix} 0 & bf_{22}(a) \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & bg_{22}(a) \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

for all $a, b \in \mathbb{F}$ and for some $\alpha \in \mathbb{F}$. Hence we obtain $\alpha = 0$ and

$$\lambda ba + b(f_{22}(a) - g_{22}(a)) = 0 \tag{3.13}$$

for all $a, b \in \mathbb{F}$. Taking $b = 1$ in (3.13), we obtain

$$g_{22}(a) = f_{22}(a) + \lambda a \tag{3.14}$$

for all $a \in \mathbb{F}$. Let $\mu : T_2(\mathbb{F}) \to \mathbb{F}$ be the additive map defined by

$$\mu(A) = g_{11}(a_{11}) + g_{12}(a_{12}) + f_{22}(a_{22}) \tag{3.15}$$

for all $A = (a_{ij}) \in T_2(\mathbb{F})$. By virtue of (3.9), (3.11), (3.12), (3.14) and (3.15), we obtain

$$\begin{aligned}
&\psi(A) \\
&= \begin{bmatrix} f_{11}(a_{11}) & 0 \\ 0 & g_{11}(a_{11}) \end{bmatrix} + \begin{bmatrix} f_{12}(a_{12}) & h_{12}(a_{12}) \\ 0 & g_{12}(a_{12}) \end{bmatrix} + \begin{bmatrix} f_{22}(a_{22}) & 0 \\ 0 & g_{22}(a_{22}) \end{bmatrix} \\
&= \begin{bmatrix} g_{11}(a_{11}) + \lambda a_{11} & 0 \\ 0 & g_{11}(a_{11}) \end{bmatrix} + \begin{bmatrix} g_{12}(a_{12}) & \lambda a_{12} \\ 0 & g_{12}(a_{12}) \end{bmatrix} \\
&\quad + \begin{bmatrix} f_{22}(a_{22}) & 0 \\ 0 & f_{22}(a_{22}) + \lambda a_{22} \end{bmatrix} \\
&= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} \\ 0 & \lambda a_{22} \end{bmatrix} + (g_{11}(a_{11}) + g_{12}(a_{12}) + f_{22}(a_{22}))I_2 \\
&= \lambda A + \mu(A) I_2
\end{aligned}$$

for every $A = (a_{ij}) \in T_2(\mathbb{F})$ as desired. □

CHAPTER 4: CENTRALIZING ADDITIVE MAPS ON RANK R BLOCK TRIANGULAR MATRICES

4.1 A brief overview

This chapter presents the main result of this dissertation. We give a characterization of centralizing additive maps on rank r block triangular matrices over an arbitrary field. To accomplish this, several lemmas are developed in Section 4.2 and will then be used to prove our main result in section 4.3.

4.2 Preliminary results

We recall that $\mathcal{T}_{n_1,\ldots,n_k}$ is the $n_1, \ldots, n_k$ block triangular matrix algebra over a field $\mathbb{F}$ with $n_1 + \cdots + n_k = n$. Given $A \in \mathcal{T}_{n_1,\ldots,n_k}$, we denote by $A^+ = J_n A^t J_n$, where A^t is the transpose of A and J_n is the square matrix of order n with ones on the anti-diagonal and zeros elsewhere.

We begin our discussion with a technical lemma.

Lemma 4.2.1. *Let $r, k, n_1, \ldots, n_k$ be positive integers such that $n_1 + \cdots + n_k = n \geq 3$ and $1 < r < n$, and $(k, n) \neq (3, 3)$ when $|\mathbb{F}| = 2$. Then each rank one or rank two matrix $A \in \mathcal{T}_{n_1,\ldots,n_k}$ can be represented as a sum of three rank r matrices in $\mathcal{T}_{n_1,\ldots,n_k}$ among which the sum of any two is of rank r.*

Proof. For convenience, we denote $\mathcal{T} = \mathcal{T}_{n_1,\ldots,n_k}$. We first consider $A \in \mathcal{T}$ is of rank one. Without loss of generality that

$$A = E_{pq}$$

for some integers $1 \leq p \leq q \leq n$. Then $A \in T_n(\mathbb{F})$ and the result holds by (Chooi et al., 2020, Lemma 2.2) when $|\mathbb{F}| \geq 3$. Consider now $|\mathbb{F}| = 2$. By the hypothesis of

$(k,n) \neq (3,3)$, we see that $\mathcal{T} \neq T_3(\mathbb{F})$. We argue in the following two cases:

Case 1: $n = 3$. Note that $r = 2$ and $1 \leq k \leq 2$. We consider only the case $2 \leq n_1 \leq 3$, i.e., $\mathcal{T} \in \{\mathcal{T}_{2,1}, M_3(\mathbb{F})\}$ as the case $\mathcal{T} = \mathcal{T}_{1,2}$ can be treated similarly. We consider two subcases:

Subcase 1.1: $p = q$. When $A = E_{11}$, we set

$$X_1 = E_{11} + E_{12} + E_{23}, \quad Y_1 = E_{13} + E_{22} + E_{23} \quad \text{and} \quad Z_1 = E_{12} + E_{13} + E_{22}.$$

We thus obtain $X_1 + Y_1 + Z_1 = E_{11} = A$, where

$$X_1 + Y_1 = E_{11} + E_{12} + E_{13} + E_{22}, \quad X_1 + Z_1 = E_{11} + E_{13} + E_{22} + E_{23}$$

and

$$Y_1 + Z_1 = E_{12} + E_{23}$$

are of rank two. When $A = E_{33}$, since $E_{33} = E_{11}^{+}$, we have $A = X_1^{+} + Y_1^{+} + Z_1^{+}$ with $X_1^{+}, Y_1^{+}, Z_1^{+} \in \mathcal{T}$ as required. Consider now $A = E_{22}$. Since $\mathcal{T} \in \{\mathcal{T}_{2,1}, M_3(\mathbb{F})\}$, we see that $E_{21} \in \mathcal{T}$. We set

$$X_2 = E_{12} + E_{21} + E_{22}, \quad Y_2 = E_{21} + E_{33} \quad \text{and} \quad Z_2 = E_{12} + E_{33}.$$

We thus obtain $X_2 + Y_2 + Z_2 = E_{22} = A$, where

$$X_2 + Y_2 = E_{12} + E_{22} + E_{33},$$

$$X_2 + Z_2 = E_{21} + E_{22} + E_{33} \quad \text{and} \quad Y_2 + Z_2 = E_{12} + E_{21}$$

are of rank two.

Subcase 1.2: $p < q$. When $A = E_{12}$, we set

$$X_3 = E_{11} + E_{12} + E_{33}, \quad Y_3 = E_{22} + E_{33}, \quad \text{and} \quad Z_3 = E_{11} + E_{22}.$$

We thus obtain $X_3 + Y_3 + Z_3 = E_{12} = A$, where

$$X_3 + Y_3 = E_{11} + E_{12} + E_{22},$$

$$X_3 + Z_3 = E_{12} + E_{22} + E_{33} \quad \text{and} \quad Y_3 + Z_3 = E_{11} + E_{33}$$

are of rank two. When $A = E_{23}$, since $E_{23} = E_{12}^{+}$, we have $A = X_3^{+} + Y_3^{+} + Z_3^{+}$ with $X_3^{+}, Y_3^{+}, Z_3^{+} \in \mathcal{T}$ as required. When $A = E_{13}$, we set

$$X_4 = E_{12} + E_{13} + E_{23}, \quad Y_4 = E_{11} + E_{33} \quad \text{and} \quad Z_4 = E_{11} + E_{12} + E_{23} + E_{33}.$$

We thus obtain $X_4 + Y_4 + Z_4 = E_{13} = A$, where

$$X_4 + Y_4 = E_{11} + E_{12} + E_{13} + E_{23} + E_{33},$$

$$X_4 + Z_4 = E_{11} + E_{13} + E_{33} \quad \text{and} \quad Y_4 + Z_4 = E_{12} + E_{23}$$

are of rank two.

Case 2 : $n \geq 4$. Note that $2 \leq r \leq n-1$. We consider two subcases:

Subcase 2.1: $p = q$. Consider $A = E_{pp}$ with $1 \leq p \leq n-2$. We select $r-1$ distinct integers $h_1, \ldots, h_{r-1} \in \{1, \ldots, n-1\} \backslash \{p\}$ and set

$$X_{pp} = E_{pp} + E_{pn} + \sum_{i=1}^{r-1} E_{h_i,h_i}, \quad Y_{pp} = E_{p,p+1} + E_{pn} + \sum_{i=1}^{r-1} E_{h_i,h_i+1}$$

and

$$Z_{pp} = E_{p,p+1} + \sum_{i=1}^{r-1} E_{h_i,h_i} + \sum_{i=1}^{r-1} E_{h_i,h_i+1}.$$

We thus obtain $X_{pp} + Y_{pp} + Z_{pp} = E_{pp} = A$, where

$$X_{pp} + Y_{pp} = E_{pp} + E_{p,p+1} + \sum_{i=1}^{r-1} E_{h_i,h_i} + \sum_{i=1}^{r-1} E_{h_i,h_i+1},$$

$$X_{pp} + Z_{pp} = E_{pp} + E_{p,p+1} + E_{pn} + \sum_{i=1}^{r-1} E_{h_i,h_i+1}$$

and

$$Y_{pp} + Z_{pp} = E_{pn} + \sum_{i=1}^{r-1} E_{h_i,h_i}$$

are of rank r. Consider $A = E_{n+1-p,n+1-p}$ with $1 \leq p \leq 2$. Since $E_{n+1-p,n+1-p} = E_{pp}^{+}$, we have $A = X_{pp}^{+} + Y_{pp}^{+} + Z_{pp}^{+}$ with $X_{pp}^{+}, Y_{pp}^{+}, Z_{pp}^{+} \in \mathcal{T}$ as required.

Subcase 2.2: $p < q$. Consider $A = E_{pq}$ with $p + 2 \leq q$. We select $r - 1$ distinct integers $h_1, \ldots, h_{r-1} \in \{1, \ldots, n-1\} \backslash \{p\}$ and set

$$X_{pq} = E_{pq} + E_{pp} + E_{p,p+1} + \sum_{i=1}^{r-1} E_{h_i,h_i} + \sum_{i=1}^{r-1} E_{h_i,h_i+1},$$

$$Y_{pq} = E_{pp} + \sum_{i=1}^{r-1} E_{h_i,h_i} \quad \text{and} \quad Z_{pq} = E_{p,p+1} + \sum_{i=1}^{r-1} E_{h_i,h_i+1}.$$

We thus obtain $X_{pq} + Y_{pq} + Z_{pq} = E_{pq} = A$, where

$$X_{pq} + Y_{pq} = E_{pq} + E_{p,p+1} + \sum_{i=1}^{r-1} E_{h_i,h_i+1},$$

$$X_{pq} + Z_{pq} = E_{pq} + E_{pp} + \sum_{i=1}^{r-1} E_{h_i,h_i}$$

and

$$Y_{pq} + Z_{pq} = E_{pp} + E_{p,p+1} + \sum_{i=1}^{r-1} E_{h_i,h_i} + \sum_{i=1}^{r-1} E_{h_i,h_i+1}$$

are of rank r. Consider $A = E_{p,p+1}$ with $1 \leq p \leq n-2$. We select $r-1$ distinct integers $h_1, \ldots, h_{r-1} \in \{1, \ldots, n-1\} \backslash \{p\}$. Let $\mathcal{I}_p = \{i : 1 \leq h_i < p\}$ and $\mathcal{J}_p = \{i : p < h_i < n\}$. Note that $\mathcal{I}_p \cup \mathcal{J}_p = \{1, \ldots, r-1\}$, $\mathcal{I}_p = \varnothing$ when $p = 1$ and $n \notin \mathcal{J}_p \neq \varnothing$. We set

$$X_{p,p+1} = E_{pp} + E_{p,p+1} + E_{pn} + \sum_{i=1}^{r-1} E_{h_i,h_i} + \sum_{i=1}^{r-1} E_{h_i,h_i+1},$$

$$Y_{p,p+1} = E_{pp} + \sum_{i \in \mathcal{I}_p} E_{h_i,h_i} + \sum_{i \in \mathcal{J}_p} E_{h_i,h_i+1}$$

and

$$Z_{p,p+1} = E_{pn} + \sum_{i \in \mathcal{J}_p} E_{h_i,h_i} + \sum_{i \in \mathcal{I}_p} E_{h_i,h_i+1}.$$

We thus obtain $X_{p,p+1} + Y_{p,p+1} + Z_{p,p+1} = E_{p,p+1} = A$, where

$$X_{p,p+1} + Y_{p,p+1} = E_{p,p+1} + E_{pn} + \sum_{i \in \mathcal{J}_p} E_{h_i,h_i} + \sum_{i \in \mathcal{I}_p} E_{h_i,h_i+1},$$

$$X_{p,p+1} + Z_{p,p+1} = E_{pp} + E_{p,p+1} + \sum_{i \in \mathcal{I}_p} E_{h_i,h_i} + \sum_{i \in \mathcal{J}_p} E_{h_i,h_i+1}$$

and

$$Y_{p,p+1} + Z_{p,p+1} = E_{pp} + E_{pn} + \sum_{i=1}^{r-1} E_{h_i,h_i} + \sum_{i=1}^{r-1} E_{h_i,h_i+1}$$

are of rank r. Consider $A = E_{n-1,n}$. Since $E_{n-1,n} = E_{12}^{+}$, we have $A = X_{12}^{+} + Y_{12}^{+} + Z_{12}^{+}$ with $X_{12}^{+}, Y_{12}^{+}, Z_{12}^{+} \in \mathcal{T}$ as required. This completes the proof for A is of rank one.

Next, we consider A is of rank two. Invoking Lemma 3.2.2, we may assume without loss of generality that

$$A = E_{pq} + E_{st}$$

for some integers $1 \leq p \leq q \leq n$ and $1 \leq s \leq t \leq n$ with $p < s$ and $q \neq t$. Then $A \in T_n(\mathbb{F})$, and the result holds by (Chooi et al., 2020, Lemma 2.2) when $|\mathbb{F}| \geq 3$. Consider now $|\mathbb{F}| = 2$. Recall that $\mathcal{T} \neq T_3(\mathbb{F})$. We distinguish the following two cases:

Case A: $n = 3$. Note that $r = 2$. Again, we consider $\mathcal{T} \in \{\mathcal{T}_{2,1}, M_3(\mathbb{F})\}$ as the case $\mathcal{T} = \mathcal{T}_{1,2}$ can be treated similarly. We consider three subcases:

Subcase A.1: $p = q$ and $s = t$. Consider $A = E_{11} + E_{22}$. We set

$$X_1 = E_{11} + E_{12} + E_{33}, \quad Y_1 = E_{22} + E_{23} + E_{33} \quad \text{and} \quad Z_1 = E_{12} + E_{23}.$$

We thus obtain $X_1 + Y_1 + Z_1 = E_{11} + E_{22} = A$, where

$$X_1 + Y_1 = E_{11} + E_{12} + E_{22} + E_{23}, \quad X_1 + Z_1 = E_{11} + E_{23} + E_{33}$$

and

$$Y_1 + Z_1 = E_{12} + E_{22} + E_{33}$$

are of rank two. Consider $A = E_{11} + E_{33}$. We set

$$X_2 = E_{11} + E_{12} + E_{22}, \quad Y_2 = E_{22} + E_{23} + E_{33} \quad \text{and} \quad Z_2 = E_{12} + E_{23}.$$

We thus obtain $X_2 + Y_2 + Z_2 = E_{11} + E_{33} = A$, where

$$X_2 + Y_2 = E_{11} + E_{12} + E_{23} + E_{33}, \quad X_2 + Z_2 = E_{11} + E_{22} + E_{23}$$

and

$$Y_2 + Z_2 = E_{12} + E_{22} + E_{33}$$

are of rank two. Consider $A = E_{22} + E_{33}$. Since $E_{22} + E_{33} = (E_{11} + E_{22})^+$, we have $A = X_1^+ + Y_1^+ + Z_1^+$ with $X_1^+, Y_1^+, Z_1^+ \in \mathcal{T}$ as required.

Subcase A.2: $p = q$ or $s = t$. Consider $A = E_{11} + E_{23}$. We set

$$X_3 = E_{11} + E_{12} + E_{22},$$

$$Y_3 = E_{13} + E_{21} + E_{22} + E_{23}, \quad \text{and} \quad Z_3 = E_{12} + E_{13} + E_{21}.$$

We thus obtain $X_3 + Y_3 + Z_3 = E_{11} + E_{23} = A$, where

$$X_3 + Y_3 = E_{11} + E_{12} + E_{13} + E_{21} + E_{23},$$

$$X_3 + Z_3 = E_{11} + E_{13} + E_{21} + E_{22} \quad \text{and} \quad Y_3 + Z_3 = E_{12} + E_{22} + E_{23}$$

are of rank two. Consider $A = E_{13} + E_{22}$. We set

$$X_4 = E_{12} + E_{13} + E_{21} + E_{22} + E_{23},$$

$$Y_4 = E_{21} + E_{33} \quad \text{and} \quad Z_4 = E_{12} + E_{23} + E_{33}.$$

We thus obtain $X_4 + Y_4 + Z_4 = E_{13} + E_{22} = A$, where

$$X_4 + Y_4 = E_{12} + E_{22} + E_{13} + E_{23} + E_{33},$$

$$X_4 + Z_4 = E_{21} + E_{22} + E_{13} + E_{33} \quad \text{and} \quad Y_4 + Z_4 = E_{12} + E_{21} + E_{23}$$

are of rank two. Consider $A = E_{12} + E_{33}$. We set

$$X_5 = E_{11} + E_{12} + E_{21}, \quad Y_5 = E_{21} + E_{23} + E_{33} \quad \text{and} \quad Z_5 = E_{11} + E_{23}.$$

We thus obtain $X_5 + Y_5 + Z_5 = E_{12} + E_{33} = A$, where

$$X_5 + Y_5 = E_{11} + E_{12} + E_{23} + E_{33},$$

$$X_5 + Z_5 = E_{12} + E_{21} + E_{23} \quad \text{and} \quad Y_5 + Z_5 = E_{11} + E_{21} + E_{33}$$

are of rank two.

Subcase A.3: $p < q$ and $s < t$. Then $A = E_{12} + E_{23}$. We set

$$X_6 = E_{11} + E_{12} + E_{33}, \quad Y_6 = E_{11} + E_{22} + E_{23} \quad \text{and} \quad Z_6 = E_{22} + E_{33}.$$

We thus obtain $X_6 + Y_6 + Z_6 = E_{12} + E_{23} = A$, where

$$X_6 + Y_6 = E_{12} + E_{22} + E_{23} + E_{33}, \quad X_6 + Z_6 = E_{11} + E_{12} + E_{22}$$

and

$$Y_6 + Z_6 = E_{11} + E_{23} + E_{33}$$

are of rank two.

Case B : $n \geq 4$. Recall that $2 \leq r \leq n-1$. We consider the following four subcases:

Subcase B.1: $p = q$ and $s = t$. Then $1 \leq p < s \leq n$. Firstly, we consider $A = E_{pp} + E_{ss}$ with $1 \leq p < s \leq n-2$. We select $r-2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots, n-1\} \backslash \{p, s\}$ and set

$$X_{ps} = E_{pp} + E_{p,p+1} + E_{pn} + E_{s,s+1} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$Y_{ps} = E_{ss} + E_{pn} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i}$$

and

$$Z_{ps} = E_{p,p+1} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}.$$

We thus obtain $X_{p,p+1} + Y_{p,p+1} + Z_{p,p+1} = E_{pp} + E_{ss} = A$, where

$$X_{ps} + Y_{ps} = E_{pp} + E_{p,p+1} + E_{ss} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$X_{ps} + Z_{ps} = E_{pp} + E_{pn} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i}$$

and

$$Y_{ps} + Z_{ps} = E_{p,p+1} + E_{pn} + E_{ss} + E_{s,s+1} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

are of rank r. Consider $A = E_{pp} + E_{n-1,n-1}$ with $1 \leq p \leq n-3$. When $r = 2$, we set

$$X_{p,n-1} = E_{pp} + E_{p,n-1} + E_{pn} + E_{n-1,n} + E_{nn},$$

$$Y_{p,n-1} = E_{n-1,n-1} + E_{pn} + E_{nn} \quad \text{and} \quad Z_{p,n-1} = E_{p,n-1} + E_{n-1,n}.$$

We thus obtain $X_{p,n-1} + Y_{p,n-1} + Z_{p,n-1} = E_{pp} + E_{n-1,n-1} = A$, where

$$X_{p,n-1} + Y_{p,n-1} = E_{pp} + E_{p,n-1} + E_{n-1,n-1} + E_{n-1,n},$$

$$X_{p,n-1} + Z_{p,n-1} = E_{pp} + E_{pn} + E_{nn}$$

and

$$Y_{p,n-1} + Z_{p,n-1} = E_{p,n-1} + E_{n-1,n-1} + E_{pn} + E_{n-1,n} + E_{nn}$$

are of rank two. When $3 \leq r \leq n-1$, we select $r-3$ distinct integers $h_1, \ldots, h_{r-3} \in \{1, \ldots, n-2\} \backslash \{p, p+1\}$ and set

$$X_{p,n-1} = E_{pp} + E_{p,p+1} + E_{pn} + E_{p+1,p+1} + E_{n-1,n} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i},$$

$$Y_{p,n-1} = E_{pn} + E_{nn} + E_{p+1,p+1} + E_{p+1,p+2} + E_{n-1,n-1} + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}$$

and

$$Z_{p,n-1} = E_{p,p+1} + E_{p+1,p+2} + E_{n-1,n} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}.$$

We thus obtain $X_{p,n-1} + Y_{p,n-1} + Z_{p,n-1} = E_{pp} + E_{n-1,n-1} = A$, where

$$X_{p,n-1} + Y_{p,n-1} = E_{pp} + E_{p,p+1} + E_{p+1,p+2} + E_{n-1,n-1} + E_{n-1,n} + \sum_{i=1}^{r-3} E_{h_i,h_i+1},$$

$$X_{p,n-1} + Z_{p,n-1} = E_{pp} + E_{pn} + E_{p+1,p+1} + E_{p+1,p+2} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}$$

and

$$Y_{p,n-1} + Z_{p,n-1} = E_{p,p+1} + E_{p+1,p+1} + E_{n-1,n-1} + E_{pn} + E_{n-1,n} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i}$$

are of rank r. Consider now $A = E_{n-2,n-2} + E_{n-1,n-1}$. We select $r-2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots, n-3\}$ and set

$$X_{n-2,n-1} = E_{n-2,n-2} + E_{n-2,n-1} + E_{n-2,n} + E_{n-1,n} + E_{nn} + \sum_{i=1}^{r-2} E_{h_i,h_i},$$

$$Y_{n-2,n-1} = E_{n-1,n-1} + E_{n-2,n} + E_{nn} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

and

$$Z_{n-2,n-1} = E_{n-2,n-1} + E_{n-1,n} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}.$$

We thus obtain $X_{n-2,n-1}+Y_{n-2,n-1}+Z_{n-2,n-1}=E_{n-2,n-2}+E_{n-1,n-1}=A$, where

$$X_{n-2,n-1}+Y_{n-2,n-1}=E_{n-2,n-2}+E_{n-2,n-1}+E_{n-1,n-1}+E_{n-1,n} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$X_{n-2,n-1}+Z_{n-2,n-1}=E_{n-2,n-2}+E_{n-2,n}+E_{nn}+\sum_{i=1}^{r-2} E_{h_i,h_i}+\sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

and

$$Y_{n-2,n-1}+Z_{n-2,n-1}=E_{n-2,n-1}+E_{n-1,n-1}+E_{n-2,n}+E_{n-1,n}+E_{nn} + \sum_{i=1}^{r-2} E_{h_i,h_i}$$

are of rank r. Consider $A=E_{11}+E_{nn}$. When $r=2$, we set

$$X_{1n}=E_{11}+E_{12}+E_{13}+E_{n-2,n}+E_{n-1,n}+E_{nn},$$

$$Y_{1n}=E_{12}+E_{n-1,n} \quad \text{and} \quad Z_{1n}=E_{13}+E_{n-2,n}.$$

We thus obtain $X_{1n}+Y_{1n}+Z_{1n}=E_{11}+E_{n}=A$, where

$$X_{1n}+Y_{1n}=E_{11}+E_{13}+E_{n-2,n}+E_{nn},$$

$$X_{1n}+Z_{1n}=E_{11}+E_{12}+E_{n-1,n}+E_{nn}$$

and

$$Y_{1n}+Z_{1n}=E_{12}+E_{13}+E_{n-2,n}+E_{n-1,n}$$

are of rank two. When $3 \le r \le n-1$, we select $r-3$ distinct integers $h_1, \ldots, h_{r-3} \in \{1, \ldots, n-3\}$ and set

$$X_{1n} = E_{11} + E_{1,n-1} + E_{n-2,n-2} + E_{n-2,n-1} + E_{n-1,n-1} + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1},$$

$$Y_{1n} = E_{12} + E_{1,n-1} + E_{n-2,n-2} + E_{n-1,n} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i}$$

and

$$Z_{1n} = E_{12} + E_{n-2,n-1} + E_{n-1,n-1} + E_{n-1,n} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}.$$

We thus obtain $X_{1n} + Y_{1n} + Z_{1n} = E_{11} + E_{nn} = A$, where

$$X_{1n} + Y_{1n} = E_{11} + E_{12} + E_{n-2,n-1} + E_{n-1,n-1} + E_{n-1,n} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i+1},$$

$$X_{1n} + Z_{1n} = E_{11} + E_{12} + E_{1,n-1} + E_{n-2,n-2} + E_{n-1,n} + \sum_{i=1}^{r-3} E_{h_i,h_i}$$

and

$$Y_{1n} + Z_{1n} = E_{1,n-1} + E_{n-1,n-1} + E_{n-2,n-2} + E_{n-2,n-1} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}$$

are of rank r. Consider now $A = E_{pp} + E_{nn}$ with $2 \le p \le n-1$. Since $E_{pp} + E_{nn} = (E_{11} + E_{n+1-p,n+1-p})^+$, we have $A = X^+_{1,n+1-p} + Y^+_{1,n+1-p} + Z^+_{1,n+1-p}$ with $X^+_{1,n+1-p}, Y^+_{1,n+1-p}, Z^+_{1,n+1-p} \in \mathcal{T}$ as desired.

Subcase B.2: $p = q$ and $s < t$. Then $1 \le p < s < t \le n$. So $s \le n-1$ and $s+1 \le t$. Consider $A = E_{pp} + E_{st}$ with $1 \le p < s < t < n$. We select $r-2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots, n-1\} \backslash \{p, s\}$. Let $\mathcal{I}_s = \{i : 1 \le h_i < s\}$ and $\mathcal{J}_s = \{i : s < h_i < n\}$. Note that $\mathcal{I}_s \cup \mathcal{J}_s = \{1, \ldots, r-2\}$, $\mathcal{I}_s = \mathcal{J}_s = \varnothing$ when $r = 2$, and $\mathcal{I}_s = \varnothing$ and $p = 1$ when $s = 2$. We set

$$X_{pst} = E_{pp} + E_{p,p+1} + E_{pn} + E_{st} + E_{sn} + \sum_{i \in \mathcal{I}_s} E_{h_i,h_i+1} + \sum_{i \in \mathcal{J}_s} E_{h_i,h_i},$$

$$Y_{pst} = \begin{cases} E_{p,p+1} + E_{ss} + E_{st} + E_{sn} + \sum_{i \in \mathcal{I}_s} E_{h_i,h_i} \\ + \sum_{i=1}^{r-2} E_{h_i,h_i+1} & \text{if } t = s+1, \\ E_{p,p+1} + E_{ss} + E_{s,s+1} + E_{st} + E_{sn} + \sum_{i \in \mathcal{I}_s} E_{h_i,h_i} \\ + \sum_{i=1}^{r-2} E_{h_i,h_i+1} & \text{if } t > s+1, \end{cases}$$

and

$$Z_{pst} = \begin{cases} E_{pn} + E_{ss} + E_{st} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i \in \mathcal{J}_s} E_{h_i,h_i+1} & \text{if } t = s+1, \\ E_{pn} + E_{ss} + E_{s,s+1} + E_{st} + \sum_{i=1}^{r-2} E_{h_i,h_i} \\ + \sum_{i \in \mathcal{J}_s} E_{h_i,h_i+1} & \text{if } t > s+1. \end{cases}.$$

We thus obtain $X_{pst} + Y_{pst} + Z_{pst} = E_{pp} + E_{st} = A$, where

$$X_{pst} + Y_{pst} = \begin{cases} E_{pp} + E_{pn} + E_{ss} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i \in \mathcal{J}_s} E_{h_i,h_i+1} & \text{if } t = s+1, \\ E_{pp} + E_{pn} + E_{ss} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i} \\ + \sum_{i \in \mathcal{J}_s} E_{h_i,h_i+1} & \text{if } t > s+1, \end{cases}$$

$$X_{pst} + Z_{pst} = \begin{cases} E_{pp} + E_{p,p+1} + E_{ss} + E_{sn} + \sum_{i \in \mathcal{I}_s} E_{h_i,h_i} \\ + \sum_{i=1}^{r-2} E_{h_i,h_i+1} & \text{if } t = s+1, \\ E_{pp} + E_{p,p+1} + E_{ss} + E_{s,s+1} + E_{sn} + \sum_{i \in \mathcal{I}_s} E_{h_i,h_i} \\ + \sum_{i=1}^{r-2} E_{h_i,h_i+1} & \text{if } t > s+1, \end{cases}$$

and

$$Y_{pst} + Z_{pst} = E_{p,p+1} + E_{pn} + E_{sn} + \sum_{i \in \mathcal{I}_s} E_{h_i,h_i+1} + \sum_{i \in \mathcal{J}_s} E_{h_i,h_i}$$

are of rank r. Consider now $A = E_{pp} + E_{sn}$ for $1 \leq p < s < n-1$. We select $r-2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots n-1\} \backslash \{p, s\}$. Let $\mathcal{I}_s = \{i : 1 \leq h_i < s\}$ and $\mathcal{J}_s = \{i : s < h_i < n\}$. We set

$$X_{psn} = E_{pp} + E_{p,p+1} + E_{pn} + E_{s,s+1} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$Y_{psn} = E_{p,p+1} + E_{ss} + E_{s,s+1} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i \in \mathcal{I}_s} E_{h_i,h_i+1}$$

and

$$Z_{psn} = E_{pn} + E_{ss} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i \in \mathcal{J}_s} E_{h_i,h_i+1}.$$

We thus obtain $X_{psn} + Y_{psn} + Z_{psn} = E_{pp} + E_{sn} = A$, where

$$X_{psn} + Y_{psn} = E_{pp} + E_{pn} + E_{ss} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i \in \mathcal{J}_s} E_{h_i,h_i+1},$$

$$X_{psn} + Z_{psn} = E_{pp} + E_{p,p+1} + E_{ss} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i \in \mathcal{I}_s} E_{h_i,h_i+1}$$

and

$$Y_{psn} + Z_{psn} = E_{p,p+1} + E_{pn} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

are of rank r. Note that $t = n$ when $s = n-1$. We consider $A = E_{pp} + E_{n-1,n}$ with $1 \le p < n-2$. When $r = 2$, we set

$$X_{p,n-1,n} = E_{pp} + E_{p,p+1} + E_{p,p+2} + E_{n-2,n} + E_{n-1,n} + E_{nn},$$

$$Y_{p,n-1,n} = E_{p,p+1} + E_{n-2,n} \quad \text{and} \quad Z_{p,n-1,n} = E_{p,p+2} + E_{nn}.$$

We thus obtain $X_{p,n-1} + Y_{p,n-1} + Z_{p,n-1} = E_{pp} + E_{n-1,n} = A$, where

$$X_{p,n-1,n} + Y_{p,n-1,n} = E_{pp} + E_{p,p+2} + E_{n-1,n} + E_{nn},$$

$$X_{p,n-1,n} + Z_{p,n-1,n} = E_{pp} + E_{p,p+1} + E_{n-2,n} + E_{n-1,n}$$

and

$$Y_{p,n-1,n} + Z_{p,n-1,n} = E_{p,p+1} + E_{p,p+2} + E_{n-2,n} + E_{nn}$$

are of rank two. When $3 \le r \le n-1$, we select $r-3$ distinct integers $h_1, \ldots, h_{r-3} \in \{1, \ldots, n-3\} \backslash \{p\}$ and set

$$X_{p,n-1,n} = E_{pp} + E_{pn} + E_{n-2,n-2} + E_{n-1,n-1} + \sum_{i=1}^{r-3} E_{h_i,h_i},$$

$$Y_{p,n-1,n} = E_{p,p+1} + E_{n-2,n-2} + E_{n-2,n-1} + E_{n-1,n} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}$$

and

$$Z_{p,n-1,n} = E_{p,p+1} + E_{pn} + E_{n-2,n-1} + E_{n-1,n-1} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}.$$

We thus obtain $X_{p,n-1,n} + Y_{p,n-1,n} + Z_{p,n-1,n} = E_{pp} + E_{n-1,n} = A$, where

$$\begin{aligned} X_{p,n-1,n} + Y_{p,n-1,n} = E_{pp} + E_{p,p+1} + E_{pn} + E_{n-2,n-1} + E_{n-1,n-1} + E_{n-1,n} \\ + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}, \end{aligned}$$

$$\begin{aligned} X_{p,n-1,n} + Z_{p,n-1,n} = E_{pp} + E_{p,p+1} + E_{n-2,n-2} + E_{n-2,n-1} + E_{nn} + \\ + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1} \end{aligned}$$

and

$$Y_{p,n-1,n} + Z_{p,n-1,n} = E_{pn} + E_{n-2,n-2} + E_{n-1,n-1} + E_{n-1,n} + \sum_{i=1}^{r-3} E_{h_i,h_i}$$

are of rank r. Consider now $A = E_{n-2,n-2} + E_{n-1,n}$. When $r = 2$, we set

$$X_{n-2,n-1,n} = E_{n-3,n-3} + E_{n-3,n} + E_{n-2,n-2},$$

$$Y_{n-2,n-1,n} = E_{n-3,n-3} + E_{n-3,n-2} + E_{n-3,n} + E_{n-2,n} + E_{n-1,n}$$

and

$$Z_{n-2,n-1,n} = E_{n-3,n-2} + E_{n-2,n}.$$

We thus obtain $X_{n-2,n-1,n} + Y_{n-2,n-1,n} + Z_{n-2,n-1,n} = E_{n-2,n-2} + E_{n-1,n} = A$, where

$$X_{n-2,n-1,n} + Y_{n-2,n-1,n} = E_{n-3,n-2} + E_{n-2,n-2} + E_{n-2,n} + E_{n-1,n},$$

$$X_{n-2,n-1,n} + Z_{n-2,n-1,n} = E_{n-3,n-3} + E_{n-3,n-2} + E_{n-3,n} + E_{n-2,n-2} + E_{n-2,n}$$

and

$$Y_{n-2,n-1,n} + Z_{n-2,n-1,n} = E_{n-3,n-3} + E_{n-3,n} + E_{n-1,n}$$

are of rank two. When $3 \leq r \leq n-1$, we select $r-3$ distinct integers $h_1, \ldots, h_{r-3} \in \{1, \ldots, n-4\}$ and set

$$X_{n-2,n-1,n} = E_{n-3,n-3} + E_{n-2,n-2} + E_{n-2,n} + E_{n-1,n-1} + \sum_{i=1}^{r-3} E_{h_i,h_i},$$

$$Y_{n-2,n-1,n} = E_{n-3,n-3} + E_{n-3,n-2} + E_{n-2,n-1} + E_{n-1,n} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}$$

and

$$Z_{n-2,n-1,n} = E_{n-3,n-2} + E_{n-2,n-1} + E_{n-1,n-1} + E_{n-2,n} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}.$$

We thus obtain $X_{n-2,n-1,n} + Y_{n-2,n-1,n} + Z_{n-2,n-1,n} = E_{n-2,n-2} + E_{n-1,n} = A$, where

$$X_{n-2,n-1,n} + Y_{n-2,n-1,n} = E_{n-3,n-2} + E_{n-2,n-2} + E_{n-2,n-1} + E_{n-1,n-1} + E_{n-1,n} + E_{n-2,n} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i+1},$$

$$X_{n-2,n-1,n} + Z_{n-2,n-1,n} = E_{n-3,n-3} + E_{n-3,n-2} + E_{n-2,n-2} + E_{n-2,n-1} + E_{nn} + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}$$

and

$$Y_{n-2,n-1,n} + Z_{n-2,n-1,n} = E_{n-3,n-3} + E_{n-2,n} + E_{n-1,n-1} + E_{n-1,n} + \sum_{i=1}^{r-3} E_{h_i,h_i}$$

are of rank r.

Subcase B.3: $p < q$ and $s = t$. We consider the following two subcases:

Subcase B.3.1: $q < s$. We thus have $1 \leq p < q < s \leq n$. Consider $A = E_{pq} + E_{ss}$ for $1 \leq p < q < s \leq n$. Since $E_{pq} + E_{ss} = (E_{ll} + E_{uv})^+$, where $l = n+1-s$, $u = n+1-q$ and $v = n+1-p$ are integers such that $1 \leq l < u < v \leq n$, it follows from Subcase B.2 that $A = (E_{ll} + E_{uv})^+ = X^+_{luv} + Y^+_{luv} + Z^+_{luv}$ is the sum of three rank r matrices $X^+_{luv}, Y^+_{luv}, Z^+_{luv}$ in $\mathcal{T}$ among which the sum of any two is of rank r as claimed.

Subcase B.3.2: $s < q$. We thus obtain $1 \leq p < s < q \leq n$. Consider $A = E_{pq} + E_{ss}$ for $1 \leq p < s < q \leq n$ and $s \leq n-2$. We select $r-2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots, n-1\backslash\{p, s\}$ and set

$$X_{psq} = E_{p,p+1} + E_{pq} + E_{s,s+1} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$Y_{psq} = E_{pp} + E_{ss} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i}$$

and

$$Z_{psq} = E_{pp} + E_{p,p+1} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}.$$

We thus obtain $X_{psq} + Y_{psq} + Z_{psq} = E_{pq} + E_{ss} = A$, where

$$X_{psq} + Y_{psq} = E_{pp} + E_{p,p+1} + E_{pq} + E_{ss} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$X_{psq} + Z_{psq} = E_{pp} + E_{pq} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i}$$

and

$$Y_{psq} + Z_{psq} = E_{p,p+1} + E_{ss} + E_{s,s+1} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

are of rank r. Note that $q = n$ when $s = n-1$. Consider $A = E_{pn} + E_{n-1,n-1}$. Since $E_{pn} + E_{n-1,n-1} = (E_{uv} + E_{ll})^+$, where $u = 1$, $v = n+1-p$ and $l = 2$ are integers satisfying $1 \leq u < l < v \leq n$ and $l \leq n-2$, it follows that $A = (E_{uv} + E_{ll})^+ = X_{ulv}^+ + Y_{ulv}^+ + Z_{ulv}^+$ is the sum of three rank r matrices $X_{ulv}^+, X_{ulv}^+, X_{ulv}^+ \in \mathcal{T}$ among which the sum of any two is of rank r as desired.

Subcase B.4: $p < q$ and $s < t$. Then $p \leq q-1$ and $s \leq t-1$ with $1 \leq p < s < n$ and $1 < q \neq t \leq n$. We consider the following four subcases:

Subcase B.4.1: $p < q-1$ and $s < t-1$. Consider $A = E_{pq} + E_{st}$ with $q \geq p+2$ and $t \geq s+2$. We select $r-2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots, n-1\} \backslash \{p, s\}$ and set

$$X = E_{pp} + E_{p,p+1} + E_{pq} + E_{ss} + E_{s,s+1} + E_{st} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$Y = E_{pp} + E_{ss} + \sum_{i=1}^{r-2} E_{h_i,h_i} \quad \text{and} \quad Z = E_{p,p+1} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}.$$

We thus obtain $X + Y + Z = E_{pq} + E_{st} = A$, where

$$X + Y = E_{p,p+1} + E_{pq} + E_{s,s+1} + E_{st} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$X + Z = E_{pp} + E_{pq} + E_{ss} + E_{st} + \sum_{i=1}^{r-2} E_{h_i,h_i}$$

and

$$Y + Z = E_{pp} + E_{p,p+1} + E_{ss} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

are of rank r.

Subcase B.4.2: $p = q - 1$ and $s = t - 1$. Then $1 < s \leq n - 1$ and $1 \leq p < s$. When $s < n-1$ and $p < s$, we consider $A = E_{p,p+1} + E_{s,s+1}$ with $1 \leq p < s < n-1$. We select $r - 2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots, n-1\} \backslash \{p, s\}$. Let $\mathcal{I}_{p,s} = \{i : p < h_i < s\}$ and $\mathcal{J}_{p,s} = \{i : 1 \leq h_i < p\} \cup \{i : s < h_i < n\}$. Note that $\mathcal{I}_{p,s} \cup \mathcal{J}_{p,s} = \{1, \ldots, r-2\}$, $\mathcal{I}_{p,s} = \mathcal{J}_{p,s} = \varnothing$ when $r = 2$, and $\mathcal{I}_{p,s} = \varnothing$ and $h_i \neq p, p+1$ for $i = 1, \ldots r - 2$ when $s = p + 1$. We set

$$X = E_{pp} + E_{p,p+1} + E_{p,s+1} + E_{ss} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$Y = E_{p,s+1} + E_{ss} + E_{s,s+1} + \sum_{i \in \mathcal{I}_{p,s}} E_{h_i,h_i} + \sum_{i \in \mathcal{J}_{p,s}} E_{h_i,h_i+1}$$

and

$$Z = E_{pp} + E_{sn} + \sum_{i \in \mathcal{J}_{ps}} E_{h_i,h_i} + \sum_{i \in \mathcal{I}_{p,s}} E_{h_i,h_i+1}.$$

We thus obtain $X+Y+Z = E_{p,p+1} + E_{s,s+1} = A$, where

$$X+Y = E_{pp} + E_{p,p+1} + E_{s,s+1} + E_{sn} + \sum_{i\in\mathcal{J}_{p,s}} E_{h_i,h_i} + \sum_{i\in\mathcal{I}_{p,s}} E_{h_i,h_i+1},$$

$$X+Z = E_{p,p+1} + E_{p,s+1} + E_{ss} + \sum_{i\in\mathcal{I}_{p,s}} E_{h_i,h_i} + \sum_{i\in\mathcal{J}_{p,s}} E_{h_i,h_i+1}$$

and

$$Y+Z = E_{pp} + E_{p,s+1} + E_{ss} + E_{s,s+1} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

are of rank r. When $s = n-1$ and $p < s-1$, we consider $A = E_{p,p+1} + E_{n-1,n}$ with $1 \leq p \leq n-3$. If $r = 2$, then we set

$$X_r = E_{pp} + E_{p,p+1} + E_{p,p+2} + E_{n-2,n} + E_{n-1,n} + E_{nn},$$

$$Y_r = E_{pp} + E_{n-2,n} \quad \text{and} \quad Z_r = E_{p,p+2} + E_{nn}.$$

We thus obtain $X_r + Y_r + Z_r = E_{p,p+1} + E_{n-1,n}$, where

$$X_r + Y_r = E_{p,p+1} + E_{p,p+2} + E_{n-1,n} + E_{nn},$$

$$X_r + Z_r = E_{pp} + E_{p,p+1} + E_{n-2,n} + E_{n-1,n}$$

and

$$Y_r + Z_r = E_{pp} + E_{p,p+2} + E_{n-2,n} + E_{nn}$$

are of rank two. If $3 \leq r \leq n-1$, then we select $r-3$ distinct integers $h_1, \ldots, h_{r-3} \in$

$\{1,\ldots,n-2\}\backslash\{p,p+1\}$. Let $\mathcal{I}_p = \{i : 1 \leq h_i < p\}$ and $\mathcal{K}_p = \{i : p+1 < h_i \leq n-2\}$. Note that $\mathcal{I}_p \cup \mathcal{K}_p = \{1,\ldots,r-3\}$, $\mathcal{I}_p = \mathcal{K}_p = \varnothing$ when $r=3$, $\mathcal{I}_p = \varnothing$ when $p=1$, and $\mathcal{K}_p = \varnothing$ when $p = n-3$. We set

$$X_r = E_{pp} + E_{p,p+1} + E_{p+1,p+2} + E_{nn} + \sum_{i\in\mathcal{I}_p} E_{h_i,h_i} + \sum_{i\in\mathcal{K}_p} E_{h_i,h_i+1},$$

$$Y_r = E_{pn} + E_{nn} + E_{p+1,p+1} + E_{n-1,n-1} + E_{n-1,n} + \sum_{i\in\mathcal{K}_p} E_{h_i,h_i} + \sum_{i\in\mathcal{I}_p} E_{h_i,h_i+1}$$

and

$$Z_r = E_{pp} + E_{pn} + E_{p+1,p+1} + E_{p+1,p+2} + E_{n-1,n-1} + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}.$$

We thus obtain $X_r + Y_r + Z_r = E_{p,p+1} + E_{n-1,n} = A$, where

$$\begin{aligned} X_r + Y_r = {} & E_{pp} + E_{p,p+1} + E_{pn} + E_{p+1,p+1} + E_{p+1,p+2} + E_{n-1,n-1} + E_{n-1,n} \\ & + \sum_{i=1}^{r-3} E_{h_i,h_i} + \sum_{i=1}^{r-3} E_{h_i,h_i+1}, \end{aligned}$$

$$\begin{aligned} X_r + Z_r = {} & E_{pn} + E_{nn} + E_{p,p+1} + E_{p+1,p+1} + E_{n-1,n-1} + \sum_{i\in\mathcal{K}_p} E_{h_i,h_i} \\ & + \sum_{i\in\mathcal{I}_p} E_{h_i,h_i+1} \end{aligned}$$

and

$$Y_r + Z_r = E_{pp} + E_{p+1,p+2} + E_{n-1,n} + E_{nn} + \sum_{i\in\mathcal{I}_p} E_{h_i,h_i} + \sum_{i\in\mathcal{K}_p} E_{h_i,h_i+1}$$

are of rank r. When $s = n-1$ and $p = s-1$, we thus obtain $p = n-2$ and $A =$

$E_{n-2,n-1} + E_{n-1,n}$. We select $r-2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots, n-3\}$ and set

$$X = E_{n-2,n-2} + E_{n-2,n-1} + E_{nn} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$Y = E_{n-1,n-1} + E_{n-2,n} + E_{n-1,n} + E_{nn} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

and

$$Z = E_{n-2,n-2} + E_{n-2,n} + E_{n-1,n-1} + \sum_{i=1}^{r-2} E_{h_i,h_i}.$$

We thus obtain $X + Y + Z = E_{n-2,n-1} + E_{n-1,n} = A$, where

$$X + Y = E_{n-2,n-2} + E_{n-2,n-1} + E_{n-2,n} + E_{n-1,n-1} + E_{n-1,n} + \sum_{i=1}^{r-2} E_{h_i,h_i},$$

$$X + Z = E_{n-2,n-1} + E_{n-1,n-1} + E_{n-2,n} + E_{nn} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

and

$$Y + Z = E_{n-2,n-2} + E_{n-1,n} + E_{nn} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

are of rank r.

Subcase B.4.3: $p = q-1$ and $s < t-1$. Then $q = p+1$ and $1 \leq p < s < t-1 \leq n-1$. Consider $A = E_{p,p+1} + E_{st}$ with $1 \leq p < s < t-1 \leq n-1$. We select $r-2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots, n-1\} \backslash \{p, s\}$. Let $\mathcal{I}_p = \{i : 1 \leq h_i < p\}$ and $\mathcal{J}_p = \{i : p < h_i \leq n-1\}$. Note that $\mathcal{I}_p \cup \mathcal{J}_p = \{1, \ldots, r-2\}$, $\mathcal{I}_p = \mathcal{J}_p = \varnothing$ when $r = 2$, and $\mathcal{I}_p = \varnothing$ when $p = 1$. Since $p < n-2$, we set

$$X_{p,p+1,s,t} = E_{pp} + E_{p,p+1} + E_{pn} + E_{ss} + E_{s,s+1} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$Y_{p,p+1,s,t} = E_{pn} + E_{ss} + E_{st} + \sum_{i\in\mathcal{J}_p} E_{h_i,h_i} + \sum_{i\in\mathcal{I}_p} E_{h_i,h_i+1}$$

and

$$Z_{p,p+1,s,t} = E_{pp} + E_{s,s+1} + \sum_{i\in\mathcal{I}_p} E_{h_i,h_i} + \sum_{i\in\mathcal{J}_p} E_{h_i,h_i+1}.$$

We thus obtain $X_{p,p+1,s,t} + Y_{p,p+1,s,t} + Z_{p,p+1,s,t} = E_{p,p+1} + E_{st} = A$, where

$$X_{p,p+1,s,t} + Y_{p,p+1,s,t} = E_{pp} + E_{p,p+1} + E_{s,s+1} + E_{st} + \sum_{i\in\mathcal{I}_p} E_{h_i,h_i} + \sum_{i\in\mathcal{J}_p} E_{h_i,h_i+1},$$

$$X_{p,p+1,s,t} + Z_{p,p+1,s,t} = E_{p,p+1} + E_{pn} + E_{ss} + \sum_{i\in\mathcal{J}_p} E_{h_i,h_i} + \sum_{i\in\mathcal{I}_p} E_{h_i,h_i+1}$$

and

$$Y_{p,p+1,s,t} + Z_{p,p+1,s,t} = E_{pp} + E_{pn} + E_{ss} + E_{s,s+1} + E_{st} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

are of rank r.

Subcase B.4.4: $p < q-1$ and $s = t-1$. If $q \leq s$, then we obtain $t = s+1$ and $1 \leq p < q-1 < q \leq s < n$. Consider $A = E_{pq} + E_{s,s+1}$ with $1 \leq p < q-1 < s < n$. Since $E_{pq} + E_{s,s+1} = (E_{l,l+1} + E_{uv})^+$, where $l = n-s$, $u = n+1-q$ and $v = n+1-p$ are integers satisfying $1 \leq l < u < v-1 \leq n-1$. It follows from Subcase B.4.3 that $A = (E_{l,l+1} + E_{uv})^+ = X^+_{l,l+1,u,v} + Y^+_{l,l+1,u,v} + Z^+_{l,l+1,u,v}$ is the sum of three rank r matrices $X^+_{l,l+1,u,v}, Y^+_{l,l+1,u,v}, Z^+_{l,l+1,u,v} \in \mathcal{T}$ among which the sum of any two is of rank r. If $s < q$, then $1 \leq p < s < s+1 < q \leq n$ because $t = s+1$ and $q \neq t$.

Consider $A = E_{pq} + E_{s,s+1}$ with $1 \le p < s < s+1 < q \le n$. We select $r-2$ distinct integers $h_1, \ldots, h_{r-2} \in \{1, \ldots, n-1\} \backslash \{p, s\}$. Let $I_s = \{i : 1 \le h_i < s\}$ and $J_s = \{i : s < h_i \le n-1\}$. Note that $I_s \cup J_s = \{1, \ldots, r-2\}$, $I_s = J_s = \varnothing$ when $r = 2$, and $I_s = \varnothing$ when $s = 2$. Since $2 \le s \le n-2$ and $p < n-2$, we set

$$X = E_{pp} + E_{p,p+1} + E_{pq} + E_{ss} + E_{s,s+1} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1},$$

$$Y = E_{pp} + E_{ss} + \sum_{i \in I_s} E_{h_i,h_i} + \sum_{i \in J_s} E_{h_i,h_i+1}$$

and

$$Z = E_{p,p+1} + E_{sn} + \sum_{i \in J_s} E_{h_i,h_i} + \sum_{i \in I_s} E_{h_i,h_i+1}.$$

We thus obtain $X + Y + Z = E_{pq} + E_{s,s+1} = A$, where

$$X + Y = E_{p,p+1} + E_{pq} + E_{s,s+1} + E_{sn} + \sum_{i \in J_s} E_{h_i,h_i} + \sum_{i \in I_s} E_{h_i,h_i+1},$$

$$X + Z = E_{pp} + E_{pq} + E_{ss} + E_{s,s+1} + \sum_{i \in I_s} E_{h_i,h_i} + \sum_{i \in J_s} E_{h_i,h_i+1}$$

and

$$Y + Z = E_{pp} + E_{p,p+1} + E_{ss} + E_{sn} + \sum_{i=1}^{r-2} E_{h_i,h_i} + \sum_{i=1}^{r-2} E_{h_i,h_i+1}$$

are of rank r. This completes the proof. $\square$

We now prove the following.

Lemma 4.2.2. *Let $\mathbb{F}$ be a field and $n \ge 2$ be an integer. Suppose that $\mathcal{S}$ is a subset of $M_n(\mathbb{F})$ such that $\mathcal{S}$ is closed under addition and that $\mathcal{Z} = \{A \in \mathcal{S} : [A, X] = 0$ for all*

$X \in \mathcal{S}\} \neq \varnothing$. Let $\psi : \mathcal{S} \to \mathcal{S}$ be an additive map satisfying $[\psi(A), A] \in \mathcal{Z}$ for every rank r matrix $A \in \mathcal{S}$, where $1 \leq r \leq n$ is a fixed integer. If $H \in \mathcal{S}$ is a sum of three rank r matrices in $\mathcal{S}$ among which the sum of any two is of rank r, then $[\psi(H), H] \in \mathcal{Z}$.

Proof. Note first that $\mathcal{Z}$ is closed under addition. Let $A, B \in \mathcal{Z}$. Then $(A + B)X = AX + BX = XA + XB = X(A + B)$ for any $X \in \mathcal{S}$ implies that $A + B \in \mathcal{S}$. Let $H = X_1 + X_2 + X_3$ be a sum of three rank r matrices $X_1, X_2, X_3 \in \mathcal{S}$ among which the sum of any two is of rank r. Let $1 \leq i < j \leq 3$ be a pair of distinct integers. Since $[\psi(X_i + X_j), X_i + X_j], [\psi(X_i), X_i]$ and $[\psi(X_j), X_j]$ are in $\mathcal{Z}$, it follows that

$$\begin{aligned}
&[\psi(X_i), X_j] + [\psi(X_j), X_i] \\
&= [\psi(X_i + X_j), X_i + X_j] - [\psi(X_i), X_i] - [\psi(X_j), X_j] \\
&\in \mathcal{Z}.
\end{aligned}$$

Then

$$[\psi(H), H] = \sum_{i=1}^{3} [\psi(X_i), X_i] + \sum_{1 \leq i < j \leq 3} ([\psi(X_i), X_j] + [\psi(X_j), X_i])$$

as desired. □

We continue our discussion by proving a result related to centralizing additive maps on rank n block triangular matrix algebra over a field $\mathbb{F}$ with $|\mathbb{F}| \geq 3$.

Lemma 4.2.3. *Let $k, n_1, \ldots, n_k$ be positive integers such that $n_1 + \cdots + n_k = n \geq 2$ and let $\mathcal{T}_{n_1,\ldots,n_k}$ be a block triangular matrix algebra over a field $\mathbb{F}$ with $|\mathbb{F}| \geq 3$. If $\psi : \mathcal{T}_{n_1,\ldots,n_k} \to \mathcal{T}_{n_1,\ldots,n_k}$ is a centralizing additive map on rank n matrices, then there* *exists a unique additive map* $\tau : \mathbb{F} \to \mathbb{F}$ *such that*

$$\psi(\lambda I_n) + \tau(\lambda) E_{1n} \in Z(\mathcal{T}_{n_1,\ldots,n_k})$$

for every $\lambda \in \mathbb{F}$*, where* $\tau = 0$ *when either* $(n_1, n_k) \neq (1,1)$ *or* $|\mathbb{F}| > 3$*.*

Proof. Let $\lambda \in \mathbb{F}$. The result is clear when $\lambda = 0$. Consider $\lambda \neq 0$. We first claim that

$$[\psi(\lambda I_n), E_{st}] = 0 \tag{4.1}$$

for every $(s,t) \in \triangle$ with $s \neq t$. Here, $\triangle = \{(i,j) : E_{ij} \in \mathcal{T}_{n_1,\ldots,n_k}\}$. We denote $\psi(\lambda I_n) = (a_{ij})$. By virtue of $|\mathbb{F}| \geq 3$, there exists a nonzero scalar $\alpha \in \mathbb{F}$ such that $\alpha \neq \lambda$. Setting $B = E_{st} - \alpha I_n$ yields B and $\lambda I_n + B$ are of rank n. Then $[\psi(\lambda I_n + B), \lambda I_n + B]$ and $[\psi(B), B]$ are in $Z(\mathcal{T}_{n_1,\ldots,n_k})$. Since $[\psi(\lambda I_n + B), \lambda I_n] = 0$, it follows from the additivity of ψ and the bilinearity of $[\,\cdot\,,\,\cdot\,]$ that

$$[\psi(\lambda I_n), B] = [\psi(\lambda I_n + B), \lambda I_n + B] - [\psi(B), B] \in Z(\mathcal{T}_{n_1,\ldots,n_k}).$$

Since $[\psi(\lambda I_n), -\alpha I_n] = 0$, we thus have $[\psi(\lambda I_n), E_{st}] = [\psi(\lambda I_n), B] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$. Consequently, for each $(s,t) \in \triangle$ with $s \neq t$, there exists $\xi_{st} \in \mathbb{F}$ such that $\psi(\lambda I_n)E_{st} - E_{st}\psi(\lambda I_n) = \xi_{st} I_n$. When $n \geq 3$, there exists an integer $1 \leq p \leq n$ such that $p \neq s, t$. Then

$$E_{pp}(\psi(\lambda I_n)E_{st} - E_{st}\psi(\lambda I_n)) = E_{pp}(\xi_{st} I_n) \implies a_{ps}E_{pt} = \xi_{st}E_{pp}.$$

Hence $\xi_{st} = 0$. Consider now $n = 2$. When $\mathcal{T}_{n_1,\ldots,n_k} = T_2(\mathbb{F})$, we have $(s,t) = (1,2)$.

Then

$$E_{22}(\psi(\lambda I_2)E_{12} - E_{12}\psi(\lambda I_2)) = E_{22}(\xi_{12}I_2) \implies a_{21}E_{22} = \xi_{12}E_{22}.$$

Since $a_{21} = 0$, we obtain $\xi_{12} = 0$. If $\mathcal{T}_{n_1,\ldots,n_k} = M_2(\mathbb{F})$, then either $(s,t) = (1,2)$ or $(s,t) = (2,1)$. When $(s,t) = (1,2)$, $\psi(\lambda I_2)E_{12} - E_{12}\psi(\lambda I_2) = \xi_{12}I_2$ yields $-a_{21}E_{11} + (a_{11} - a_{22})E_{12} + a_{21}E_{22} = \xi_{12}I_2$. Then $\xi_{12} = a_{21} = -a_{21}$, and so $\xi_{12} = 0$. Likewise, when $(s,t) = (2,1)$, we see that $\psi(\lambda I_2)E_{21} - E_{21}\psi(\lambda I_2) = \xi_{21}I_2$ implies that $a_{12}E_{11} + (a_{22} - a_{11})E_{21} - a_{12}E_{22} = \xi_{21}I_2$. Then $\xi_{21} = a_{12} = -a_{12}$, and so $\xi_{21} = 0$. Hence claim (4.1) is proved.

We now distinguish our argument in the following two cases:

Case 1: $|\mathbb{F}| > 3$. We claim that

$$[\psi(\lambda I_n), E_{ss}] = 0 \tag{4.2}$$

for every $1 \leq s \leq n$. Since $|\mathbb{F}| > 3$, there exists a nonzero scalar $\beta \in \mathbb{F}$ such that $\beta \neq \alpha, \alpha - \lambda$. Setting $C = \beta E_{ss} - \alpha I_n$ yields C and $\lambda I_n + C$ are of rank n. Then $[\psi(\lambda I_n + C), \lambda I_n + C]$ and $[\psi(C), C]$ are in $Z(\mathcal{T}_{n_1,\ldots,n_k})$. Since $[\psi(\lambda In + C), \lambda I_n] = 0$, we have $[\psi(\lambda I_n), C] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$. By $[\psi(\lambda I_n), -\alpha I_n] = 0$, we thus obtain $[\psi(\lambda I_n), E_{ss}] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$. Therefore for each $(s,s) \in \triangle$, there exists $\xi_{ss} \in \mathbb{F}$ such that $\psi(\lambda I_n)E_{ss} - E_{ss}\psi(\lambda I_n) = \xi_{ss}I_n$. We take an integer $1 \leq p \leq n$ such that $p \neq s$. Then

$$E_{pp}(\psi(\lambda I_n)E_{ss} - E_{ss}\psi(\lambda I_n)) = E_{pp}(\xi_{ss}I_n) \implies a_{ps}E_{ps} = \xi_{ss}E_{pp}.$$

Hence $\xi_{ss} = 0$, and thus claim (4.2) is proved. In view of (4.1) and (4.2), we conclude that $[\psi(\lambda I_n), E_{st}] = 0$ for every $(s,t) \in \triangle$. Let $X = (x_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$. Then

$[\psi(\lambda I_n), X] = \sum_{(i,j)\in\triangle} x_{ij}[\psi(\lambda I_n), E_{ij}] = 0$ implies that $\psi(\lambda I_n) \in Z(\mathcal{T}_{n_1,\ldots,n_k})$ for each $\lambda \in \mathbb{F}$. Consequently, the result holds with τ the zero map on $\mathbb{F}$.

Case 2: $|\mathbb{F}| = 3$. Let $1 \leq h \leq n$ be an integer and let $(s,t) \in \triangle$ with $s \neq t$. From (4.1),

$$E_{hh}\psi(\lambda I_n)E_{st} = E_{hh}E_{st}\psi(\lambda I_n) \implies a_{hs}E_{ht} = \delta_{hs}E_{ht}\psi(\lambda I_n) \tag{4.3}$$

where δ_{ij} is the Kronecker delta. Note that $a_{hs} = 0$ for all $(h,s),(s,t) \in \triangle$ with $h \neq s$ and $s \neq t$. In particular, taking $t = n$ yields $a_{hs} = 0$ for each $1 \leq s < n$ and $(h,s) \in \triangle$ with $h \neq s$. So

$$\psi(\lambda I_n) = \sum_{i=1}^{n} a_{ii}E_{ii} + \sum_{i=1}^{n-1} a_{in}E_{in}. \tag{4.4}$$

On the other hand, when $h = s = 1$, for each integer $1 < t \leq n$, we note from (4.3) and (4.4) that

$$\begin{aligned} a_{11}E_{1t} &= E_{1t}\psi(\lambda I_n) \\ &= E_{1t}\left(\sum_{i=1}^{n} a_{ii}E_{ii} + \sum_{i=1}^{n-1} a_{in}E_{in}\right) \\ &= \begin{cases} a_{tt}E_{1t} + a_{tn}E_{1n} & \text{if } 1 < t < n, \\ a_{nn}E_{1n} & \text{if } t = n. \end{cases} \end{aligned}$$

Then $a_{in} = 0$ for each $1 < i < n$, and $a_{ii} = a_{11}$ for $1 < i \leq n$. Hence $\psi(\lambda I_n) = a_{11}I_n + a_{1n}E_{1n}$. When $(n_1, n_k) = (1,1)$, there exist maps $\tau, \eta : \mathbb{F} \to \mathbb{F}$ such that

$$\psi(\lambda I_n) + \tau(\lambda)E_{1n} = \eta(\lambda)I_n \in Z(\mathcal{T}_{n_1,\ldots,n_k})$$

for every $\lambda \in \mathbb{F}$. By the additivity of ψ and the linear independence of E_{1n} and I_n, it can be shown that τ and η are additive maps uniquely determined by ψ as required.

Consider now $(n_1, n_k) \neq (1, 1)$. When $n_1 \geq 2$, we have $E_{21} \in \mathcal{T}_{n_1,\ldots,n_k}$. By virtue of (4.1), we have $[\psi(\lambda In), E_{21}] = 0$ yields $a_{1n}E_{2n} = 0$ when $n \geq 3$, and $a_{12}E_{11} - a_{12}E_{22} = 0$ when $n = 2$. In both cases, we obtain $a_{1n} = 0$. Likewise, when $n_k \geq 2$, we have $[\psi(\lambda I_n), E_{n,n-1}] = 0$ leads to $a_{1n} = 0$. Then $\psi(\lambda I_n) \in Z(\mathcal{T}_{n_1,\ldots,n_k})$ for each $\lambda \in \mathbb{F}$. Consequently, the result holds with τ the zero map on $\mathbb{F}$. $\square$

We now prove a result related to centralizing additive maps on rank two upper triangular matrices of order three over the Galois field of two elements.

Lemma 4.2.4. *Let $\mathbb{F}$ be the field with $|\mathbb{F}| = 2$. If $\psi : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ is a centralizing additive map on rank two matrices and $D = E_{12} + E_{23} \in T_3(\mathbb{F})$, then the following hold.*

(i) $[\psi(I_3), D + E_{ii}] \in Z(T_3(\mathbb{F}))$ *for* $i = 1, 2, 3$.

(ii) $[\psi(I_3), D] + [\psi(E_{ii}), E_{ii}] \in Z(T_3(\mathbb{F}))$ *for* $i = 1, 2, 3$.

(iii) $[\psi(E_{st} + I_3 + E_{ii}), E_{st}] + [\psi(E_{st}), E_{ii}] \in Z(T_3(\mathbb{F}))$ *for all integers* $1 \leq s < t \leq 3$ *and* $i = 1, 2, 3$.

(iv) $[\psi(D), E_{ij}] + [\psi(E_{ij}), D] \in Z(T_3(\mathbb{F}))$ *for all integers* $1 \leq i < j \leq 3$.

Proof. (i) Let $1 \leq i \leq 3$ be an integer and let $F_{ii} = I_3 + E_{ii}$. Since $E_{ii} + D$ and $F_{ii} + D$ are of rank two, it follows that $[\psi(E_{ii} + D), E_{ii} + D], [\psi(F_{ii} + D), F_{ii} + D] \in Z(T_3(\mathbb{F}))$. Note that

$$
\begin{aligned}
&[\psi(I_3), E_{ii} + D] + [\psi(E_{ii} + D), E_{ii} + D] \\
&= [\psi(F_{ii} + D), E_{ii} + D] \\
&= [\psi(F_{ii} + D), F_{ii} + D]
\end{aligned}
$$

as $[\psi(F_{ii}+D), I_3] = 0$. Hence $[\psi(I_3), D+E_{ii}] \in Z(T_3(\mathbb{F}))$ for $i = 1, 2, 3$.

(ii) Let $1 \leq i \leq 3$ be an integer. Note that $[\psi(F_{ii}), E_{ii}] = [\psi(F_{ii}), F_{ii}] \in Z(T_3(\mathbb{F}))$ since $[\psi(F_{ii}), I_3] = 0$ and F_{ii} is of rank two. By (i), we obtain

$$[\psi(I_3), D] + [\psi(E_{ii}), E_{ii}] + [\psi(F_{ii}), E_{ii}] = [\psi(I_3), D+E_{ii}] \in Z(T_3(\mathbb{F})).$$

Thus $[\psi(I_3), D] + [\psi(E_{ii}), E_{ii}] \in Z(T_3(\mathbb{F}))$ for $i = 1, 2, 3$.

(iii) Let $1 \leq i \leq 3$ and $1 \leq s < t \leq 3$ be integers. Since $[\psi(F_{ii}+E_{st}), I_3] = 0$, we see that

$$\begin{aligned}[\psi(F_{ii}+E_{st}), F_{ii}+E_{st}] &= [\psi(F_{ii}+E_{st}), E_{ii}+E_{st}] \\ &= [\psi(F_{ii}+E_{st}), E_{st}] + [\psi(F_{ii}), E_{ii}] + [\psi(E_{st}), E_{ii}].\end{aligned}$$

Since $F_{ii}+E_{st}$ and F_{ii} are of rank two, we have $[\psi(F_{ii}+E_{st}), F_{ii}+E_{st}], [\psi(F_{ii}), E_{ii}] \in Z(T_3(\mathbb{F}))$. We thus obtain $[\psi(F_{ii}+E_{st}), E_{st}] + [\psi(E_{st}), E_{ii}] \in Z(T_3(\mathbb{F}))$ for all integers $1 \leq s < t \leq 3$ and $i = 1, 2, 3$.

(iv) Let $1 \leq i < j \leq 3$ be integers. Note that

$$\begin{aligned}&[\psi(F_{11}+D+E_{ij}), F_{11}+D+E_{ij}] \\ &= [\psi(F_{11}+D), F_{11}+D] + [\psi(F_{11}+D), E_{ij}] + [\psi(E_{ij}), F_{11}+D+E_{ij}].\end{aligned}$$

Since $F_{11}+D+E_{ij}$ and $F_{11}+D$ are of rank two, it follows that $[\psi(F_{11}+D+E_{ij}), F_{11}+D+E_{ij}], [\psi(F_{11}+D), F_{11}+D] \in Z(T_3(\mathbb{F}))$. Then

$$[\psi(F_{11}+D), E_{ij}] + [\psi(E_{ij}), F_{11}+D+E_{ij}] \in Z(T_3(\mathbb{F}))$$

$$\implies [\psi(F_{11}), E_{ij}] + [\psi(D), E_{ij}] + [\psi(E_{ij}), E_{ij}] + [\psi(E_{ij}), F_{11}+D] \in Z(T_3(\mathbb{F}))$$

$$\implies [\psi(F_{11}+E_{ij}), E_{ij}] + [\psi(D), E_{ij}] + [\psi(E_{ij}), E_{11}] + [\psi(E_{ij}), D] \in Z(T_3(\mathbb{F}))$$

because $[\psi(E_{ij}), I_3] = 0$. By (iii), we see that $[\psi(F_{11}+E_{ij}), E_{ij}] + [\psi(E_{ij}), E_{11}] \in Z(T_3(\mathbb{F}))$. It follows that $[\psi(D), E_{ij}] + [\psi(E_{ij}), D] \in Z(T_3(\mathbb{F}))$ for all integers $1 \leq i < j \leq 3$. $\square$

The following result is a continuation of the previous lemma.

Lemma 4.2.5. *Let $\mathbb{F}$ be the field with $|\mathbb{F}| = 2$. Then $\psi : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ is a centralizing additive map on rank two matrices if and only if there exist a scalar $\lambda \in \mathbb{F}$ and a matrix $H \in T_3(\mathbb{F})$ such that*

$$\psi(A) = \lambda A + \text{tr}(H^t A) I_3$$

for every $A \in T_3(\mathbb{F})$. Here, $\text{tr}(A)$ *denotes the trace of A.*

Proof. Let $\psi(I_3) = (a_{ij}) \in T_3(\mathbb{F})$ and let $D = E_{12} + E_{23} \in T_3(\mathbb{F})$. By Lemma 4.2.4 (i), we have $[\psi(I_3), D + E_{11}] \in Z(T_3(\mathbb{F}))$. So

$$a_{11} = a_{12} + a_{22}, \quad a_{12} = a_{13} + a_{23} \quad \text{and} \quad a_{22} = a_{33}. \tag{4.5}$$

By $[\psi(I_3), D + E_{22}] \in Z(T_3(\mathbb{F}))$, we see that

$$a_{12} = a_{23}, \quad a_{22} = a_{11} + a_{12} \quad \text{and} \quad a_{22} = a_{23} + a_{33}. \tag{4.6}$$

Solving the equations in (4.5) and (4.6) yields $a_{12} = a_{13} = a_{23} = 0$ and $a_{11} = a_{22} = a_{33}$.

Then

$$\psi(I_3) = \alpha I_3 \tag{4.7}$$

for some $\alpha \in \mathbb{F}$.

For each integer $1 \leq s \leq t \leq 3$, let $\psi(E_{st}) = (b_{ij}^{(st)}) \in T_3(\mathbb{F})$ and let $\psi(D) = (c_{ij}) \in T_3(\mathbb{F})$. Since $[\psi(I_3), D] = [\alpha I_3, D] = 0$ by (4.7), it follows from Lemma 4.2.4 (ii) that $[\psi(E_{ii}), E_{ii}] \in Z(T_3(\mathbb{F}))$ for $i = 1, 2, 3$. Then

$$b_{12}^{(11)} = b_{13}^{(11)} = 0, \tag{4.8}$$

$$b_{12}^{(22)} = b_{23}^{(22)} = 0, \tag{4.9}$$

$$b_{13}^{(33)} = b_{23}^{(33)} = 0, \tag{4.10}$$

Since $\psi(E_{11}) + \psi(E_{22}) + \psi(E_{33}) = \psi(I_3) = \lambda I_3$, together with (4.8)-(4.10), we obtain

$$b_{23}^{(11)} = b_{13}^{(22)} = b_{12}^{(33)} = 0, \tag{4.11}$$

$$b_{ii}^{(11)} = b_{ii}^{(22)} = b_{ii}^{(33)} = \alpha, \tag{4.12}$$

for $i = 1, 2, 3$. In view of (4.8)-(4.11), we conclude that

$$\psi(E_{ii}) = b_{11}^{(ii)} E_{11} + b_{22}^{(ii)} E_{22} + b_{33}^{(ii)} E_{33} \tag{4.13}$$

for $i = 1, 2, 3$. By virtue of $[\psi(D), D] \in Z(T_3(\mathbb{F}))$, we see that

$$c_{11} = c_{22} = c_{33} \quad \text{and} \quad c_{23} = c_{12}. \tag{4.14}$$

Let $\lambda = c_{12}$. By Lemma 4.2.4 (iv), $[\psi(D), E_{12}] + [\psi(E_{12}), D] \in Z(T_3(\mathbb{F}))$, together with (4.14), we obtain

$$b_{11}^{(12)} = b_{22}^{(12)} = b_{33}^{(12)} \quad \text{and} \quad b_{23}^{(12)} = b_{12}^{(12)} + \lambda. \tag{4.15}$$

By Lemma 4.2.4 (iii), $[\psi(E_{12} + I_3 + E_{11}), E_{12}] + [\psi(E_{12}), E_{11}] \in Z(T_3(\mathbb{F}))$, together with (4.7), (4.13) and (4.15), we have

$$b_{12}^{(12)} = b_{11}^{(11)} + b_{22}^{(11)} \quad \text{and} \quad b_{13}^{(12)} = b_{23}^{(12)}. \tag{4.16}$$

By $[\psi(E_{12} + I_3 + E_{22}), E_{12}] + [\psi(E_{12}), E_{22}] \in Z(T_3(\mathbb{F}))$, (4.7), (4.13) and (4.15), we obtain

$$b_{12}^{(12)} = b_{11}^{(22)} + b_{22}^{(22)} \quad \text{and} \quad b_{23}^{(12)} = 0. \tag{4.17}$$

It follows from (4.15) and (4.16) that $b_{12}^{(12)} = \lambda$ and $b_{13}^{(12)} = 0$. Then

$$\psi(E_{12}) = \alpha_{12} I_3 + \lambda E_{12}. \tag{4.18}$$

where $\alpha_{12} = b_{11}^{(12)}$. Likewise, by $[\psi(D), E_{23}] + [\psi(E_{23}), D] \in Z(T_3(\mathbb{F}))$ and (4.14), we obtain

$$b_{11}^{(23)} = b_{22}^{(23)} = b_{33}^{(23)} \quad \text{and} \quad b_{23}^{(23)} = b_{12}^{(23)} + \lambda. \tag{4.19}$$

By $[\psi(E_{23} + I_3 + E_{11}), E_{23}] + [\psi(E_{23}), E_{11}] \in Z(T_3(\mathbb{F}))$, (4.7), (4.13) and (4.19), we obtain $b_{12}^{(23)} = b_{13}^{(23)} = 0$ and $b_{33}^{(11)} = b_{22}^{(11)}$, and so $b_{23}^{(23)} = \lambda$. Then

$$\psi(E_{23}) = \alpha_{23} I_3 + \lambda E_{23}, \tag{4.20}$$

where $\alpha_{23} = b_{11}^{(23)}$. Note that $b_{12}^{(12)} = b_{11}^{(11)} + b_{22}^{(11)}$ by (4.16). It follows from (4.13), $b_{33}^{(11)} = b_{22}^{(11)}$ and $\lambda = b_{12}^{(12)}$ that

$$\psi(E_{11}) = \alpha_{11} I_3 + \lambda E_{11}, \tag{4.21}$$

where $\alpha_{11} = b_{22}^{(11)}$. Similarly, by $[\psi(D), E_{13}] + [\psi(E_{13}, D] \in Z(T_3(\mathbb{F}))$ and (4.14), we obtain

$$b_{11}^{(13)} = b_{22}^{(13)} = b_{33}^{(13)} \quad \text{and} \quad b_{23}^{(13)} = b_{12}^{(13)}. \tag{4.22}$$

By $[\psi(E_{13} + I_3 + E_{11}), E_{13}] + [\psi(E_{13}), E_{11}] \in Z(T_3(\mathbb{F}))$, (4.7), (4.21) and (4.22), we have $b_{12}^{(13)} = 0$ and $b_{13}^{(13)} = \lambda$. Consequently,

$$\psi(E_{13}) = \alpha_{13} I_3 + \lambda E_{13}, \tag{4.23}$$

where $\alpha_{13} = b_{11}^{(13)}$. Next, consider $[\psi(E_{13} + I_3 + E_{22}), E_{13}] + [\psi(E_{13}), E_{22}] \in Z(T_3(\mathbb{F}))$. Together with (4.7), (4.13) and (4.23), we obtain $b_{11}^{(22)} = b_{33}^{(22)}$. By (4.17), we have $\lambda = b_{11}^{(22)} + b_{22}^{(22)}$. Then

$$\psi(E_{22}) = \alpha_{22} I_3 + \lambda E_{22}, \tag{4.24}$$

where $\alpha_{22} = b_{11}^{(22)}$. In view of (4.12), we see that $b_{11}^{(11)} + b_{11}^{(22)} + b_{11}^{(33)} = b_{22}^{(11)} + b_{22}^{(22)} + b_{22}^{(33)}$.

Since $b_{11}^{(11)} + b_{22}^{(11)} = b_{11}^{(22)} + b_{22}^{(22)}$ by (4.16) and (4.17), we obtain $b_{11}^{(33)} = b_{22}^{(33)}$. By (4.12), $b_{11}^{(11)}+b_{11}^{(22)}+b_{11}^{(33)} = b_{33}^{(11)}+b_{33}^{(22)}+b_{33}^{(33)}$ and $b_{11}^{(22)} = b_{33}^{(22)}$ imply that $b_{11}^{(33)}+b_{33}^{(33)} = b_{11}^{(11)}+b_{33}^{(11)}$. Since $b_{33}^{(11)} = b_{22}^{(11)}$, we obtain $b_{11}^{(33)} + b_{33}^{(33)} = \lambda$. It follows from (4.13) that

$$\psi(E_{33}) = \alpha_{33}I_3 + \lambda E_{33}, \tag{4.25}$$

where $\alpha_{33} = b_{11}^{(33)}$. Let $H = (\alpha_{ij}) \in T_3(\mathbb{F})$. By virtue of (4.18), (4.20), (4.21), (4.23)-(4.25), we obtain

$$\begin{aligned}\psi(A) &= \sum_{1\le i\le j\le 3} \psi(a_{ij}E_{ij}) \\ &= \Big(\sum_{1\le i\le j\le 3} \alpha_{ij}a_{ij}\Big)I_3 + \lambda A \\ &= \text{tr}(H^tA)I_3 + \lambda A\end{aligned}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$. □

4.3 A characterization of centralizing additive maps on rank r block triangular matrices

With the several technical lemmas developed in the previous section, we are now ready to prove the main result of this dissertation.

Theorem 4.3.1. *Let $k, n_1, \ldots, n_k$ be positive integers such that $n_1 + \cdots + n_k = n \geq 2$ and let $\mathcal{T}_{n_1,\ldots,n_k}$ be a block triangular matrix algebra over a field $\mathbb{F}$. Let $1 < r \leq n$ be a fixed integer such that $r \neq n$ when $|\mathbb{F}| = 2$. Then $\psi : \mathcal{T}_{n_1,\ldots,n_k} \to \mathcal{T}_{n_1,\ldots,n_k}$ is a centralizing additive map on rank r matrices if and only if there exist scalars $\lambda, \alpha \in \mathbb{F}$ and an additive map $\mu : \mathcal{T}_{n_1,\ldots,n_k} \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n}$$

for every $A = (a_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$, *where* $\alpha \neq 0$ *only if* $r = n$, $n_1 = n_k = 1$ *and* $|\mathbb{F}| = 3$.

Proof. For sufficiency, we note that the additivity of ψ is obvious. We show that ψ is centralizing on rank r matrices. When $\alpha = 0$, we have $\psi(A) = \lambda A + \mu(A)I_n$ for $A \in \mathcal{T}_{n_1,\ldots,n_k}$. Then the result follows because $[\psi(A), A] = 0$ for all $A \in \mathcal{T}_{n_1,\ldots,n_k}$. Consider now $\alpha \neq 0$. Then $r = n$, $n_1 = n_k = 1$ and $|\mathbb{F}| = 3$. Let $A = (a_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$ be of rank n. Since $n_1 = n_k = 1$, we must have $a_{11}, a_{nn} \neq 0$ and $E_{1n}A - AE_{1n} = (a_{nn} - a_{11})E_{1n}$. Thus

$$[\psi(A), A] = \alpha(a_{11} + a_{nn})(E_{1n}A - AE_{1n}) = \alpha(a_{11} + a_{nn})(a_{nn} - a_{11})E_{1n}.$$

Since $|\mathbb{F}| = 3$, it follows that $a_{11} + a_{nn} = 0$ when $a_{11} \neq a_{nn}$. Then $[\psi(A), A] = 0 \in Z(\mathcal{T}_{n_1,\ldots,n_k})$ for all rank n matrices $A \in \mathcal{T}_{n_1,\ldots,n_k}$ as claimed.

To show necessity, we consider the following two cases:

Case I: $|\mathbb{F}| = 2$ and $(k, n) = (3, 3)$. Thus $r = 2$ and $\mathcal{T}_{n_1,\ldots,n_k} = T_3(\mathbb{F})$. The result follows immediately from Lemma 4.2.5. Then there exist a scalar $\lambda \in \mathbb{F}$ and a matrix $H \in T_3(\mathbb{F})$ such that

$$\psi(A) = \lambda A + \text{tr}(H^t A)I_3$$

for every $A \in T_3(\mathbb{F})$ as required.

Case II: $|\mathbb{F}| \geq 2$ and $(k, n) \neq (3, 3)$ when $|\mathbb{F}| = 2$. Recall that $\triangle = \{(i, j) : E_{ij} \in \mathcal{T}_{n_1,\ldots,n_k}\}$. Let $A = (a_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$. By the bilinearity of $[\,\cdot\,,\,\cdot\,]$, we obtain

$$[\psi(A), A] = \sum_{(i,j)\in\triangle} [\psi(a_{ij}E_{ij}), a_{ij}E_{ij}] + \sum_{(i,j)\neq(s,t)\in\triangle} [\psi(a_{ij}E_{ij}), a_{st}E_{st}] \qquad (4.26)$$

To prove $[\psi(A), A] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$, we only need to show that

$$[\psi(a_{ij}E_{ij}), a_{ij}E_{ij}] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$$

and

$$[\psi(a_{ij}E_{ij}), a_{st}E_{st}] + [\psi(a_{st}E_{st}), a_{ij}E_{ij}] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$$

for every pair of distinct indexes $(i, j), (s, t) \in \triangle$. We distinguish two subcases:

Subcase II-1: $r = n$. Then $|\mathbb{F}| \geq 3$. Let $\varphi : \mathcal{T}_{n_1,\ldots,n_k} \to \mathcal{T}_{n_1,\ldots,n_k}$ be the map defined by

$$\varphi(A) = \begin{cases} \psi(A) - \tau(a_{11} + a_{nn})E_{1n} & \text{when} \quad n_1 = n_k = 1 \quad \text{and} \quad |\mathbb{F}| = 3, \\ \psi(A) & \text{otherwise} \end{cases} \tag{4.27}$$

for every $A = (a_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$, where $\tau : \mathbb{F} \to \mathbb{F}$ is the additive map uniquely determined by ψ as described in Lemma 4.2.3. We claim that φ is a centralizing additive map on rank n matrices such that $\varphi(Z(\mathcal{T}_{n_1,\ldots,n_k})) \subseteq Z(\mathcal{T}_{n_1,\ldots,n_k})$. The result is clear when $(n_1, n_k) \neq (1, 1)$ or $|\mathbb{F}| > 3$ by Lemma 4.2.3. Consider $n_1 = n_k = 1$ and $|\mathbb{F}| = 3$. Let $A = (a_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$ be of rank n. Then

$$\begin{aligned} [\varphi(A), A] &= [\psi(A), A] + A\tau(a_{11} + a_{nn})E_{1n} - \tau(a_{11} + a_{nn})E_{1n}A \\ &= [\psi(A), A] + (a_{11} - a_{nn})\tau(a_{11} + a_{nn})E_{1n}. \end{aligned}$$

Since A is invertible and $n_1 = n_k = 1$, we have $a_{11}, a_{nn} \neq 0$. If $a_{11} \neq a_{nn}$, then $a_{11} + a_{nn} = 0$ by $|\mathbb{F}| = 3$. Consequently, $(a_{11} - a_{nn})\tau(a_{11} + a_{nn}) = 0$. Since $[\psi(A), A] \in$

$Z(\mathcal{T}_{n_1,\dots,n_k})$, we have $[\varphi(A), A] \in Z(\mathcal{T}_{n_1,\dots,n_k})$ for every rank n matrices $A \in \mathcal{T}_{n_1,\dots,n_k}$. We next show that $\varphi(Z(\mathcal{T}_{n_1,\dots,n_k})) \subseteq Z(\mathcal{T}_{n_1,\dots,n_k})$. Let $X \in Z(\mathcal{T}_{n_1,\dots,n_k})$. It follows from Lemma 3.2.3 that $X = \lambda I_n$ for some $\lambda \in \mathbb{F}$. Since $\lambda + \lambda = -\lambda$, we have $\varphi(X) = \psi(\lambda I_n) - \tau(\lambda+\lambda)E_{1n} = \psi(\lambda I_n) + \tau(\lambda)E_{1n} \in Z(\mathcal{T}_{n_1,\dots,n_k})$ by Lemma 4.2.3. Hence φ is a centralizing additive map on rank n matrices and $\varphi(Z(\mathcal{T}_{n_1,\dots,n_k})) \subseteq Z(\mathcal{T}_{n_1,\dots,n_k})$.

Let $A = (a_{ij}) \in \mathcal{T}_{n_1,\dots,n_k}$. We first show that for each $(i,j) \in \triangle$,

$$[\varphi(a_{ij}E_{ij}), a_{ij}E_{ij}] \in Z(\mathcal{T}_{n_1,\dots,n_k}). \tag{4.28}$$

Let $(i,j) \in \triangle$. The result is clear if $a_{ij} = 0$. Consider now $a_{ij} \neq 0$. Since $|\mathbb{F}| \geq 3$, there exists a nonzero $a \in \mathbb{F}$ such that $a_{ij}E_{ij} + aI_n$ is of rank n. By virtue of $\varphi(aI_n) \in Z(\mathcal{T}_{n_1,\dots,n_k})$ and $[\varphi(a_{ij}E_{ij} + aI_n), a_{ij}E_{ij} + aI_n], [\varphi(a_{ij}E_{ij} + aI_n), aI_n] \in Z(\mathcal{T}_{n_1,\dots,n_k})$, claim (4.28) is proved. We next claim that for each $(i,j), (s,t) \in \triangle$ with $(i,j) \neq (s,t)$,

$$[\varphi(a_{ij}E_{ij)}, a_{st}E_{st}] + [\varphi(a_{st}E_{st}), a_{ij}E_{ij}] \in Z(\mathcal{T}_{n_1,\dots,n_k}). \tag{4.29}$$

When $i \neq j$ or $s \neq t$ or $|\mathbb{F}| > 3$, there exists a nonzero $b \in \mathbb{F}$ such that $a_{ij}E_{ij} + a_{st}E_{st} + bI_n$ is of rank n. By $[\varphi(a_{ij}E_{ij} + a_{st}E_{st} + bI_n), a_{ij}E_{ij} + a_{st}E_{st} + bI_n], [\varphi(a_{ij}E_{ij} + a_{st}E_{st} + bI_n), bI_n] \in Z(\mathcal{T}_{n_1,\dots,n_k})$, $\varphi(bI_n) \in Z(\mathcal{T}_{n_1,\dots,n_k})$ and (4.28), claim (4.29) is proved. Consider now $i = j$, $s = t$ and $|\mathbb{F}| = 3$. Then there exists a nonzero $c \in \mathbb{F}$ such that $E_{ii} + E_{ss} + cI_n$ is of rank n. Using the facts that $[\varphi(E_{ii} + E_{ss} + cI_n), E_{ii} + E_{ss} + cI_n]$, $[\varphi(E_{ii} + E_{ss} + cI_n), cI_n] \in Z(\mathcal{T}_{n_1,\dots,n_k})$, $\varphi(cI_n) \in Z(\mathcal{T}_{n_1,\dots,n_k})$ and (4.28), we obtain

$$[\varphi(E_{ii}), E_{ss}] + [\varphi(E_{ss}), E_{ii}] \in Z(\mathcal{T}_{n_1,\dots,n_k}). \tag{4.30}$$

Moreover, since φ is linear when $|\mathbb{F}| = 3$, it follows from (4.30) and the bilinearity of

$[\,\cdot\,,\,\cdot\,]$ that

$$[\varphi(a_{ii}E_{ii}), a_{ss}E_{ss}] + [\varphi(a_{ss}E_{ss}), a_{ii}E_{ii}] = a_{ii}a_{ss}([\varphi(E_{ii}), E_{ss}] + [\varphi(E_{ss}), E_{ii}])$$
$$\in Z(\mathcal{T}_{n_1,\ldots,n_k}).$$

Hence claim (4.29) is proved. By (4.28) and (4.29), together with the observation in (4.26), we conclude that $[\varphi(A), A] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$ for every $A \in \mathcal{T}_{n_1,\ldots,n_k}$.

Subcase II-2: $1 < r < n$. Then $n \geq 3$ and $(k, n) \neq (3, 3)$ when $|\mathbb{F}| = 2$. Let $A = (a_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$ and let $(i, j), (s, t) \in \triangle$ be such that $(i, j) \neq (s, t)$. Note that $a_{ij}E_{ij} + a_{st}E_{st} \in \mathcal{T}_{n_1,\ldots,n_k}$ is of at most rank two. By Lemma 4.2.1, if $a_{ij}E_{ij} + a_{st}E_{st}$ is nonzero, then it can be represented as a sum of three rank r matrices in $\mathcal{T}_{n_1,\ldots,n_k}$ among which the sum of any two is of rank r. Then $[\psi(a_{ij}E_{ij} + a_{st}E_{st}), a_{ij}E_{ij} + a_{st}E_{st}] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$ by Lemma 4.2.2. By a similar argument on $a_{ij}E_{ij}$ and $a_{st}E_{st}$, we have $[\psi(a_{ij}E_{ij}).a_{ij}E_{ij}], [\psi(a_{st}E_{st}), a_{st}E_{st}] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$. Then $[\psi(a_{ij}E_{ij}), a_{st}E_{st}] + [\psi(a_{st}E_{st}), a_{ij}E_{ij}] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$. It follows from (4.26) that $[\psi(A), A] \in Z(\mathcal{T}_{n_1,\ldots,n_k})$ for every $A \in \mathcal{T}_{n_1,\ldots,n_k}$.

In view of (4.27) and Theorem 3.2.1, we see that there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : \mathcal{T}_{n_1,\ldots,n_k} \to \mathbb{F}$ such that

$$\psi(A) = \lambda A + \mu(A)I_n + \tau(a_{11} + a_{nn})E_{1n}$$

for every $A = (a_{ij}) \in \mathcal{T}_{n_1,\ldots,n_k}$, where $\tau = 0$ when $1 < r < n$ or $(n_1, n_k) \neq (1, 1)$ or $|\mathbb{F}| > 3$. Moreover, when $|\mathbb{F}| = 3$, $n_1 = n_k = 1$ and $r = n$, the additivity of τ yields τ is linear. Then either $\tau = 0$ or τ is bijective. When $\tau \neq 0$, we have either τ is the identity, or $\tau(0) = 0$, $\tau(1) = -1$ and $\tau(-1) = 1$. We thus conclude that there exists a scalar $\alpha \in \mathbb{F}$ such that $\tau(x) = \alpha x$ for every $x \in \mathbb{F}$. This completes the proof. $\square$

As an immediate consequence of Theorem 4.3.1, we obtain a classification of centralizing additive map $\psi : \mathcal{M} \to \mathcal{M}$ on rank r matrices, where $\mathcal{M} \in \{M_n(\mathbb{F}), T_n(\mathbb{F})\}$ and $1 < r \leq n$ is a fixed integer with $r \neq n$ when $|\mathbb{F}| = 2$.

When $\mathcal{M} = M_n(\mathbb{F})$, we have the following result.

Theorem 4.3.2. *Let $\mathbb{F}$ be a field and let $n \geq 2$ be an integer. Let $1 < r \leq n$ be a fixed integer such that $r \neq n$ when $|\mathbb{F}| = 2$. Then $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a centralizing additive map on rank r matrices if and only if there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : M_n(\mathbb{F}) \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A) I_n$$

for every $A \in M_n(\mathbb{F})$.

When $\mathcal{M} = T_n(\mathbb{F})$, we obtain the following result.

Theorem 4.3.3. *Let $\mathbb{F}$ be a field and let $n \geq 2$ be an integer. Let $1 < r \leq n$ be a fixed integer such that $r \neq n$ when $|\mathbb{F}| = 2$. Then $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is a centralizing additive map on rank r matrices if and only if there exist scalars $\lambda, \alpha \in \mathbb{F}$ and an additive map $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A) I_n + \alpha(a_{11} + a_{nn}) E_{1n}$$

for every $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\alpha \neq 0$ only if $r = n$ and $|\mathbb{F}| = 3$.

CHAPTER 5: COMMUTING ADDITIVE MAPS ON RANK R UPPER TRIANGULAR MATRICES

5.1 A brief overview

In this chapter, we will apply the characterization of centralizing additive maps on rank r block triangular matrices and employ the most recent results (Chooi et al., 2019, 2020) to give a complete description of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank $r \geq 2$ upper triangular matrices over an arbitrary field. In Section 5.3 we continue our study of commuting additive maps on rank one upper triangular matrices of orders two and three.

5.2 A complete description of commuting additive maps on rank $r \geq 2$ upper triangular matrices

We begin with the illustration of the most recent results by Chooi et al. (2019, 2020) in the study of commuting additive maps on rank r upper triangular matrices

We will state without proof the following results related to commuting additive maps on rank r upper triangular matrices.

Lemma 5.2.1. (Chooi et al., 2020, Theorem 1.1) *Let $2 \leq r \leq n$ be fixed integers and let $\mathbb{F}$ be a field with $|\mathbb{F}| \geq 3$. Let $T_n(\mathbb{F})$ be the ring of $n \times n$ upper triangular matrices over $\mathbb{F}$ with center $Z(T_n(\mathbb{F}))$. Then $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is an additive map satisfying $A\psi(A) = \psi(A)A$ for all rank r matrices $A \in T_n(\mathbb{F})$ if and only if there exist an additive map $\mu : T_n(\mathbb{F}) \to Z(T_n(\mathbb{F}))$, $Z \in Z(T_n(\mathbb{F}))$ and $\alpha \in \mathbb{F}$ in which $\alpha = 0$ when $|\mathbb{F}| > 3$ or $r < n$ such that*

$$\psi(A) = ZA + \mu(A) + \alpha(a_{11} + a_{nn})E_{1n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$.

Lemma 5.2.2. (Chooi et al., 2019, Theorem 2.8) *Let $n \geq 4$ be an integer. Then $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is a commuting additive map on invertible matrices if and only if there exist scalars $\lambda, \alpha, \beta_1, \beta_2 \in \mathbb{F}_2$ and matrices $H, K \in T_n(\mathbb{F}_2)$ and $X_1, \ldots, X_n \in T_n(\mathbb{F}_2)$ satisfying $X_1 + \cdots + X_n = 0$ such that*

$$\psi(A) = \lambda A + \text{tr}(H^t A) I_n + \text{tr}(K^t A) E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A) + \sum_{i=1}^{n} a_{ii} X_i$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$, where $\Psi_{\alpha,\beta_1,\beta_2} : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the additive map defined by

$$\Psi_{\alpha,\beta_1,\beta_2}(A) = (\alpha a_{12} + \beta_1(a_{n-1,n} + a_{nn}))E_{1,n-1} + (\alpha a_{n-1,n} + \beta_2(a_{11} + a_{12}))E_{2n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$.

Lemma 5.2.3. (Chooi et al., 2019, Theorem 2.9) *$\psi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ is a commuting additive map on invertible matrices if and only if there exist scalars $\lambda, \alpha, \beta, \gamma \in \mathbb{F}_2$ and matrices $H, K \in T_3(\mathbb{F}_2)$ and $X_1, X_2, X_3 \in T_3(\mathbb{F}_2)$ satisfying $X_1 + X_2 + X_3 = 0$ such that*

$$\psi(A) = \lambda A + \text{tr}(H^t A) I_3 + \text{tr}(K^t A) E_{13} + \Psi_{\alpha,\beta}(A) + \Phi_\gamma(A) + \sum_{i=1}^{3} a_{ii} X_i$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$, where $\Psi_{\alpha,\beta} : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ and $\Phi_\gamma : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ are the additive maps defined by

$$\Psi_{\alpha,\beta}(A) = (\alpha(a_{23} + a_{33}))E_{12} + (\beta(a_{11} + a_{12}))E_{23},$$

$$\Phi_\gamma(A) = \gamma((a_{12} + a_{22})E_{22} + (a_{11} + a_{12} + a_{23} + a_{33})E_{33} + a_{13}(E_{12} + E_{23}))$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$.

Lemma 5.2.4. (Chooi et al., 2019, Theorem 2.10) $\psi : T_2(\mathbb{F}_2) \to T_2(\mathbb{F}_2)$ *is a commuting additive map on invertible matrices if and only if there exist some scalars* $\lambda_1, \lambda_2 \in \mathbb{F}_2$ *and matrices* $X_1, X_2 \in T_2(\mathbb{F}_2)$ *such that*

$$\psi(A) = (a_{11} + a_{12})X_1 + (a_{22} + a_{12})X_2 + \lambda_1 a_{12} I_2 + \lambda_2 a_{12} E_{12}$$

for all $A = (a_{ij}) \in T_2(\mathbb{F})$.

Using Lemmas 5.2.1, 5.2.2, 5.2.3, 5.2.4 and Theorem 4.3.3, we obtain a complete description of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank r matrices over an arbitrary field $\mathbb{F}$, where $1 < r \leq n$ is a fixed integer.

Theorem 5.2.1. *Let* $\mathbb{F}$ *be a field and let* $n \geq 2$ *be an integer. Let* $1 < r \leq n$ *be a fixed integer. Then* $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ *is a commuting additive map on rank r matrices if and only if*

- *when* $r < n$ *or* $|\mathbb{F}| \neq 2$, *there exist scalars* $\lambda, \alpha \in \mathbb{F}$ *and an additive map* $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ *such that*

$$\psi(A) = \lambda A + \mu(A) I_n + \alpha(a_{11} + a_{nn}) E_{1n}$$

  for all $A = (a_{ij}) \in T_n(\mathbb{F})$, *where* $\alpha \neq 0$ *only if* $r = n$ *and* $|\mathbb{F}| = 3$,

- *when* $r = n \geq 4$ *and* $|\mathbb{F}| = 2$, *there exist scalars* $\lambda, \alpha, \beta_1, \beta_2 \in \mathbb{F}$ *and matrices* $H, K \in T_n(\mathbb{F})$ *and* $X_1, \ldots X_n \in T_n(\mathbb{F})$ *satisfying* $X_1 + \cdots + X_n = 0$ *such that*

$$\psi(A) = \lambda A + \text{tr}(H^t A) I_n + \text{tr}(K^t A) E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A) + \sum_{i=1}^{n} a_{ii} X_i$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$*, where* $\Psi_{\alpha,\beta_1,\beta_2} : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ *is the additive map defined by*

$$\Psi_{\alpha,\beta_1,\beta_2}(A) = (\alpha a_{12} + \beta_1(a_{n-1,n} + a_{nn}))E_{1,n-1} + (\alpha a_{n-1,n} + \beta_2(a_{11} + a_{12}))E_{2n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$*,*

- *when* $r = n = 3$ *and* $|\mathbb{F}| = 2$*, there exist scalars* $\lambda, \alpha, \beta, \gamma \in \mathbb{F}$ *and matrices* $H, K \in T_3(\mathbb{F})$ *and* $X_1, X_2, X_3 \in T_3(\mathbb{F})$ *satisfying* $X_1 + X_2 + X_3 = 0$ *such that*

$$\psi(A) = \lambda A + \text{tr}(H^t A)I_3 + \text{tr}(K^t A)E_{13} + \Psi_{\alpha,\beta}(A) + \Phi_\gamma(A) + \sum_{i=1}^{3} a_{ii}X_i$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$*, where* $\Psi_{\alpha,\beta} : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ *and* $\Phi_\gamma : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ *are the additive maps defined by*

$$\Psi_{\alpha,\beta}(A) = (\alpha(a_{23} + a_{33}))E_{12} + (\beta(a_{11} + a_{12}))E_{23},$$

$$\Phi_\gamma(A) = \gamma((a_{12} + a_{22})E_{22} + (a_{11} + a_{12} + a_{23} + a_{33})E_{33} + a_{13}(E_{12} + E_{23}))$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$*, and*

- *when* $r = n = 2$ *and* $|\mathbb{F}| = 2$*, there exist scalars* $\lambda_1, \lambda_2 \in \mathbb{F}$ *and matrices* $X_1, X_2 \in T_2(\mathbb{F})$ *such that*

$$\psi(A) = (a_{11} + a_{12})X_1 + (a_{22} + a_{12})X_2 + \lambda_1 a_{12} I_2 + \lambda_2 a_{12} E_{12}$$

for all $A = (a_{ij}) \in T_2(\mathbb{F})$*.*

5.3 A study of commuting additive maps on rank one upper triangular matrices of orders two and three

In this section, we characterize commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank one upper triangular matrices for $n = 2$ and $n = 3$ respectively. As we will see in Theorem 5.3.2, the structure of commuting additive maps $\psi : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ is rather complicated and complex.

We begin with a result on commuting additive maps on rank one upper triangular matrices of order two.

Theorem 5.3.1. *Let $\mathbb{F}$ be a field. Then $\psi : T_2(\mathbb{F}) \to T_2(\mathbb{F})$ is a commuting additive map on rank one matrices if and only if there exist a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : T_2(\mathbb{F}) \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_2$$

for every $A \in T_2(\mathbb{F})$.

Proof. For the sufficiency part, let $A \in T_2(\mathbb{F})$, we see that

$$\psi(A)A = (\lambda A + \mu(A)I_2)A = \lambda A^2 + \mu(A)A = A(\lambda A + \mu(A)I_2) = A\psi(A).$$

Hence the additive map ψ is commuting on $T_2(\mathbb{F})$, and so ψ is a commuting additive map on rank one matrices in $T_2(\mathbb{F})$.

We now proceed to prove the necessity part. Since ψ is an additive map, it follows that for each pair of integers $1 \leq i \leq j \leq 2$, there exist additive maps $f_{ij}, g_{ij}, h_{ij} : \mathbb{F} \to \mathbb{F}$ such that

$$\psi(aE_{ij}) = \begin{bmatrix} f_{ij}(a) & h_{ij}(a) \\ 0 & g_{ij}(a) \end{bmatrix}$$

for all $a \in \mathbb{F}$. Since $0 = [\psi(aE_{ij}), aE_{ij}] = \psi(aE_{ij})aE_{ij} - aE_{ij}(aE_{ij})$ for all integers $1 \leq i \leq j \leq 2$ and $a \in \mathbb{F}$, it follows that

$$\begin{bmatrix} f_{11}(a)a & 0 \\ 0 & 0 \end{bmatrix} = \psi(aE_{11})aE_{11} = aE_{11}\psi(aE_{11}) = \begin{bmatrix} af_{11}(a) & ah_{11}(a) \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & f_{12}(a)a \\ 0 & 0 \end{bmatrix} = \psi(aE_{12})aE_{12} = aE_{12}\psi(aE_{12}) = \begin{bmatrix} 0 & ag_{12}(a) \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & h_{22}(a)a \\ 0 & g_{22}(a)a \end{bmatrix} = \psi(aE_{22})aE_{22} = aE_{22}\psi(aE_{22}) = \begin{bmatrix} 0 & 0 \\ 0 & ag_{22}(a) \end{bmatrix}$$

for all $a \in \mathbb{F}$. Therefore

$$g_{12} = f_{12} \quad \text{and} \quad h_{11} = h_{22} = 0. \tag{5.1}$$

We next see that $0 = [\psi(aE_{11} + bE_{12}), aE_{11} + bE_{12}] = \psi(aE_{11} + bE_{12})(aE_{11} + bE_{12}) - (aE_{11} + bE_{12})\psi(aE_{11} + bE_{12})$ for all $a, b \in \mathbb{F}$. By the additivity of ψ, together with (5.1), we obtain

$$\begin{bmatrix} f_{11}(a)a + f_{12}(b)a & af_{11}(a)b + g_{12}(b)b \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} af_{11}(a) + af_{12}(b) & ah_{12}(b) + bg_{11}(a) + bg_{12}(b) \\ 0 & 0 \end{bmatrix}$$

for all $a, b \in \mathbb{F}$. We thus obtain

$$ah_{12}(b) + b(g_{11}(a) - f_{11}(a)) = 0 \tag{5.2}$$

for all $a, b \in \mathbb{F}$. Taking $a = 1$ in (5.2), we obtain

$$h_{12}(b) = \lambda b \tag{5.3}$$

for all $b \in \mathbb{F}$, where $\lambda = f_{11}(1) - g_{11}(1)$. Setting $b = 1$ in (5.2), we obtain

$$f_{11}(a) = g_{11}(a) + \lambda a \tag{5.4}$$

for all $a \in \mathbb{F}$. Likewise, considering $[\psi(bE_{12} + aE_{22}), bE_{12} + aE_{22}] = 0$ for all $a, b \in \mathbb{F}$, together with the additivity of ψ, (5.1) and (5.3), we obtain

$$\begin{bmatrix} 0 & f_{12}(b)b + f_{22}(a)b + \lambda ba \\ 0 & af_{12}(b) + ag_{22}(a) \end{bmatrix} = \begin{bmatrix} 0 & bf_{12}(b) + bg_{22}(a) \\ 0 & f_{12}(b)a + g_{22}(a)a \end{bmatrix}$$

for all $a, b \in \mathbb{F}$. Then

$$g_{22}(a) = f_{22}(a) + \lambda a \tag{5.5}$$

for all $a \in \mathbb{F}$. Let $\mu : T_2(\mathbb{F}) \to \mathbb{F}$ be the additive map defined by

$$\mu(A) = g_{11}(a_{11}) + f_{12}(a_{12}) + f_{22}(a_{22}) \tag{5.6}$$

for all $A = (a_{ij}) \in T_2(\mathbb{F})$. By virtue of (5.1), (5.3), (5.4), (5.5), (5.6) and together with the additivity of ψ, we obtain

$$\begin{aligned} \psi(A) &= \psi(a_{11}E_{11}) + \psi(a_{12}E_{12}) + \psi(a_{22}E_{22}) \\ &= \begin{bmatrix} g_{11}(a_{11}) + \lambda a_{11} & 0 \\ 0 & g_{11}(a_{11}) \end{bmatrix} + \begin{bmatrix} f_{12}(a_{12}) & \lambda a_{12} \\ 0 & f_{12}(a_{12}) \end{bmatrix} + \\ &\quad \begin{bmatrix} f_{22}(a_{22}) & 0 \\ 0 & f_{22}(a_{22}) + \lambda a_{22} \end{bmatrix} \\ &= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} \\ 0 & \lambda a_{22} \end{bmatrix} + (g_{11}(a_{11}) + f_{12}(a_{12}) + f_{22}(a_{22}))I_2 \\ &= \lambda A + \mu(A)I_2 \end{aligned}$$

for every $A = (a_{ij}) \in T_2(\mathbb{F})$ as desired. $\square$

We now show the structure of commuting additive maps on rank one upper triangular matrices of order three.

Theorem 5.3.2. *Let $\mathbb{F}$ be a field. Then $\psi : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ is a commuting additive map on rank one matrices if and only if there exists scalar $\theta, \vartheta, \tau_{12}, \tau_{13}, \tau_{23} \in \mathbb{F}$ and additive maps $\mu : T_3(\mathbb{F}) \to \mathbb{F}$, $\phi : \mathbb{F} \to \mathbb{F}$ such that*

$$\psi(A) = \mu(A)I_3 + \psi_{\tau_{12},\tau_{13},\tau_{23}}(A) + \psi_{\theta,\vartheta}(A) + \psi_{\phi}(A)$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$. Here $\psi_{\tau_{12},\tau_{13},\tau_{23}}(A)$ is the linear map defined by

$$\begin{bmatrix} -a_{22}\tau_{12} - a_{33}\tau_{13} & a_{12}\tau_{12} & a_{13}\tau_{13} \\ 0 & -a_{33}\tau_{23} - a_{11}\tau_{12} & a_{23}\tau_{23} \\ 0 & 0 & -a_{11}\tau_{13} - a_{22}\tau_{23} \end{bmatrix}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$, and $\psi_{\theta,\vartheta}(A)$ is the linear map defined by

$$\begin{bmatrix} 0 & a_{33}\vartheta & -a_{23}\vartheta - a_{12}\theta \\ 0 & 0 & a_{11}\theta \\ 0 & 0 & 0 \end{bmatrix}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$, and $\psi_{\phi}(A)$ is the additive map defined by

$$\begin{bmatrix} 0 & 0 & \phi(a_{22}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$.

Proof. Firstly, we see that the additive map $A \to \mu(A)I_3$ is commuting on triangular matrices $A \in T_3(\mathbb{F})$. So it is a commuting additive map on rank one triangular matrices. We now proceed to prove $\psi_{\tau_{12},\tau_{13},\tau_{23}}$, $\psi_{\theta,\vartheta}$ and ψ_{ϕ} are, respectively, commuting additive maps on rank one triangular matrices $T_3(\mathbb{F})$. Note that

$$\psi_{\tau_{12},\tau_{13},\tau_{23}}(aE_{ij}) = \begin{cases} -a\tau_{12}E_{22} - a\tau_{13}E_{33} & \text{when } i = j = 1, \\ -a\tau_{12}E_{11} - a\tau_{23}E_{33} & \text{when } i = j = 2, \\ -a\tau_{13}E_{11} - a\tau_{23}E_{22} & \text{when } i = j = 3 \\ a\tau_{ij}E_{ij} & \text{when } 1 \leq i < j \leq 3 \end{cases} \tag{5.7}$$

for all $a \in \mathbb{F}$. By (5.7), we obtain

$$[\psi_{\tau_{12},\tau_{13},\tau_{23}}(aE_{ij}), aE_{ij}] = 0 \tag{5.8}$$

for all $a \in \mathbb{F}$ and integers $1 \leq i \leq j \leq 3$. Since $[\psi_{\tau_{12},\tau_{13},\tau_{23}}(aE_{ij}), bE_{ik}] = 0 = [\psi_{\tau_{12},\tau_{13},\tau_{23}}(bE_{ik}), aE_{ij}]$ for all $a, b \in \mathbb{F}$ and integers $1 \leq i \leq j, k \leq 3$ such that $j \neq k$, and together with the additivity of $\psi_{\tau_{12},\tau_{13},\tau_{23}}$, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and (5.8), it follows that

$$[\psi_{\tau_{12},\tau_{13},\tau_{23}}(aE_{ij} + bE_{ik}), aE_{ij} + bE_{ik}] = 0 \tag{5.9}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i \leq j, k \leq 3$ such that $j \neq k$. Also, since $[\psi_{\tau_{12},\tau_{13},\tau_{23}}(aE_{ij}), bE_{kj}] = 0 = [\psi_{\tau_{12},\tau_{13},\tau_{23}}(bE_{kj}), aE_{ij}]$ for all $a, b \in \mathbb{F}$ and integers $1 \leq i, k \leq j \leq 3$ such that $i \neq k$, and together with the additivity of $\psi_{\tau_{12},\tau_{13},\tau_{23}}$, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and (5.8), it follows that

$$[\psi_{\tau_{12},\tau_{13},\tau_{23}}(aE_{ij} + bE_{kj}), aE_{ij} + bE_{kj}] = 0 \tag{5.10}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i, k \leq j \leq 3$ such that $i \neq k$. By the additivity of $\psi_{\tau_{12},\tau_{13},\tau_{23}}$, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and together with (5.9), we obtain

$$[\psi_{\tau_{12},\tau_{13},\tau_{23}}(aE_{11}+bE_{12}+cE_{13}), aE_{11}+bE_{12}+cE_{13}] = 0 \tag{5.11}$$

for all $a, b, c \in \mathbb{F}$. By the additivity of $\psi_{\tau_{12},\tau_{13},\tau_{23}}$, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and together with (5.10), we obtain

$$[\psi_{\tau_{12},\tau_{13},\tau_{23}}(aE_{33}+bE_{23}+cE_{13}), aE_{33}+bE_{23}+cE_{13}] = 0 \tag{5.12}$$

for all $a, b, c \in \mathbb{F}$. By virtue of (5.8)-(5.12), it follows that $\psi_{\tau_{12},\tau_{13},\tau_{23}}$ is a commuting additive map on rank one triangular matrices $T_3(\mathbb{F})$. Next, we see that

$$\psi_{\theta,\vartheta}(aE_{ij}) = \begin{cases} a\vartheta E_{23} & \text{when } (i,j)=(1,1), \\ -a\vartheta E_{13} & \text{when } (i,j)=(1,2), \\ 0 & \text{when } (i,j)=(1,3), \\ 0 & \text{when } (i,j)=(2,2), \\ -a\theta E_{13} & \text{when } (i,j)=(2,3), \\ a\theta E_{12} & \text{when } (i,j)=(3,3), \end{cases} \tag{5.13}$$

for all $a \in \mathbb{F}$. By (5.13), we obtain

$$[\psi_{\theta,\vartheta}(aE_{ij}), aE_{ij}] = 0 \tag{5.14}$$

for all $a \in \mathbb{F}$ and integers $1 \leq i \leq j \leq 3$. Since $[\psi_{\theta,\vartheta}(aE_{ij}), bE_{ik}] = 0 = [\psi_{\theta,\vartheta}(bE_{ik}), aE_{ij}]$ for all $a, b \in \mathbb{F}$ and integers $1 \leq i \leq j, k \leq 3$ such that $j \neq k$, and together with the additivity of $\psi_{\theta,\vartheta}$, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and (5.14), it follows that

$$[\psi_{\theta,\vartheta}(aE_{ij} + bE_{ik}), aE_{ij} + bE_{ik}] = 0 \tag{5.15}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i \leq j, k \leq 3$ such that $j \neq k$. Also, since $[\psi_{\theta,\vartheta}(aE_{ij}), bE_{kj}] = 0 = [\psi_{\theta,\vartheta}(bE_{kj}), aE_{ij}]$ for all $a, b \in \mathbb{F}$ and integers $1 \leq i, k \leq j \leq 3$ such that $i \neq k$, and together with the additivity of $\psi_{\theta,\vartheta}$, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and (5.14), it follows that

$$[\psi_{\theta,\vartheta}(aE_{ij} + bE_{kj}), aE_{ij} + bE_{kj}] = 0 \tag{5.16}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i, k \leq j \leq 3$ such that $i \neq k$. By the additivity of $\psi_{\theta,\vartheta}$, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and together with (5.15), we obtain

$$[\psi_{\theta,\vartheta}(aE_{11} + bE_{12} + cE_{13}), aE_{11} + bE_{12} + cE_{13}] = 0 \tag{5.17}$$

for all $a, b, c \in \mathbb{F}$. By the additivity of $\psi_{\theta,\vartheta}$, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and together with (5.16), we obtain

$$[\psi_{\theta,\vartheta}(aE_{33} + bE_{23} + cE_{13}), aE_{33} + bE_{23} + cE_{13}] = 0 \tag{5.18}$$

for all $a, b, c \in \mathbb{F}$. By virtue of (5.14)-(5.18), it follows that $\psi_{\theta,\vartheta}$ is a commuting additive map on rank one triangular matrices $T_3(\mathbb{F})$. We also see that

$$\psi_\phi(aE_{ij}) = \begin{cases} \phi(a)E_{13} & \text{when } (i,j) = (2,2) \\ 0 & \text{when } (i,j) \neq (2,2) \end{cases} \tag{5.19}$$

for all $a \in \mathbb{F}$ and integers $1 \leq i \leq j \leq 3$. By (5.19), we obtain

$$[\psi_\phi(aE_{ij}), aE_{ij}] = 0 \tag{5.20}$$

for all $a \in \mathbb{F}$ and integers $1 \leq i \leq j \leq k$. Since $[\psi_\phi(aE_{ij}), bE_{ik}] = 0 = [\psi_\phi(bE_{ik}), aE_{ij}]$ for all $a, b \in \mathbb{F}$ and integers $1 \leq i \leq j, k \leq 3$ such that $j \neq k$, and together with the additivity of ψ_ϕ, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and (5.20), it follows that

$$[\psi_\phi(aE_{ij} + bE_{ik}), aE_{ij} + bE_{ik}] = 0 \tag{5.21}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i \leq j, k \leq 3$ such that $j \neq k$. Also, since $[\psi_\phi(aE_{ij}), bE_{kj}] = 0 = [\psi_\phi(bE_{kj}), aE_{ij}]$ for all $a, b \in \mathbb{F}$ and integers $1 \leq i, k \leq j \leq 3$ such that $i \neq k$, and together with the additivity of ψ_ϕ, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and (5.20), it follows that

$$[\psi_\phi(aE_{ij} + bE_{ik}), aE_{ij} + bE_{ik}] = 0 \tag{5.22}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i, k \leq j \leq 3$ such that $i \neq k$. By the additivity of ψ_ϕ, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and together with (5.21), we obtain

$$[\psi_\phi(aE_{11} + bE_{12} + cE_{13}), aE_{11} + bE_{12} + cE_{13}] = 0 \tag{5.23}$$

for all $a, b, c \in \mathbb{F}$. By the additivity of ψ_ϕ, the bilinearity of $[\,\cdot\,,\,\cdot\,]$ and together with (5.21), we obtain

$$[\psi_\phi(aE_{33} + bE_{23} + cE_{13}), aE_{33} + bE_{23} + cE_{13}] = 0 \tag{5.24}$$

for all $a, b, c \in \mathbb{F}$. By (5.20)-(5.24), it follows that ψ_ϕ is a commuting additive map on rank one triangular matrices $T_3(\mathbb{F})$. This proves the sufficiency part.

For the necessity part, since ψ is an additive map, it follows that for each pair of integers $1 \leq i \leq j \leq 3$ and integers $1 \leq s \leq t \leq 3$, there exist additive maps $f_{st}^{(ij)}(a)E_{st} : \mathbb{F} \to \mathbb{F}$ such that $\psi(aE_{ij}) = \sum_{1 \leq s \leq t \leq 3} f_{st}^{(ij)}(a)E_{st}$ for all $a \in \mathbb{F}$. We note that

$$\psi(aE_{ij})(aE_{ij}) = aE_{ij}\psi(aE_{ij}) \tag{5.25}$$

for all $a \in \mathbb{F}$ and integers $1 \leq i \leq j \leq 3$. By (5.25), we have

$$E_{1k} \sum_{1 \leq s \leq t \leq 3} f_{st}^{(ij)}(a)E_{st}(aE_{ij}) = E_{1k}(aE_{ij}) \sum_{1 \leq s \leq t \leq 3} f_{st}^{(ij)}(a)E_{st}$$

for all $a \in \mathbb{F}$ and integers $1 \leq i, j, k \leq 3$ with $i \leq j$. Therefore

$$af_{ki}^{(ij)}(a)E_{1j} = \begin{cases} a\sum_{t=j}^{3} f_{jt}^{(ij)}(a)E_{1t} & \text{when } k = i, \\ \\ 0 & \text{when } k \neq i \end{cases} \tag{5.26}$$

for all $a \in \mathbb{F}$ and integers $1 \leq k \leq i \leq j \leq 3$. Setting $(i, j) = (1, 1)$ in (5.26), we obtain $k = 1$ and so

$$af_{11}^{(11)}(a)E_{11} = a(f_{11}^{(11)}(a)E_{11} + f_{12}^{(11)}(a)E_{12} + f_{13}^{(11)}(a)E_{13})$$

for all $a \in \mathbb{F}$. Hence we obtain

$$f_{12}^{(11)} = f_{13}^{(11)} = 0. \tag{5.27}$$

Setting $(i, j) = (2, 2)$ in (5.26), we obtain $k = 1, 2$ and so

$$af_{12}^{(22)}(a)E_{12} = 0 \quad \text{and} \quad af_{22}^{(22)}(a)E_{12} = a(f_{22}^{(22)}(a)E_{12} + f_{23}^{(22)}(a)E_{13})$$

for all $a \in \mathbb{F}$. Hence we obtain

$$f_{12}^{(22)} = f_{23}^{(22)} = 0. \tag{5.28}$$

Setting $(i, j) = (3, 3)$ in (5.26), we obtain $k = 1, 2, 3$ and so

$$af_{13}^{(33)}(a)E_{13} = 0, \quad af_{23}^{(33)}(a)E_{13} = 0 \quad \text{and} \quad af_{33}^{(33)}(a)E_{13} = af_{33}^{(33)}(a)E_{13}$$

for all $a \in \mathbb{F}$. Hence we obtain

$$f_{13}^{(33)} = f_{23}^{(33)} = 0 \tag{5.29}$$

Setting $(i, j) = (1, 2)$ in (5.26), we obtain $k = 1$ and so

$$af_{11}^{(12)}(a)E_{12} = a(f_{22}^{(12)}(a)E_{12} + f_{23}^{(12)}(a)E_{13})$$

for all $a \in \mathbb{F}$. Hence we obtain

$$f_{11}^{(12)} = f_{22}^{(12)}, \tag{5.30}$$

$$f_{23}^{(12)} = 0. \tag{5.31}$$

Setting $(i, j) = (1, 3)$ in (5.26), we obtain $k = 1$ and so

$$af_{11}^{(13)}(a)E_{13} = af_{33}^{(13)}(a)E_{13}$$

for all $a \in \mathbb{F}$. Hence we obtain

$$f_{11}^{(13)} = f_{33}^{(13)}. \tag{5.32}$$

Setting $(i,j) = (2,3)$ in (5.26), we obtain $k = 1,2$ and so

$$af_{12}^{(23)}(a)E_{13} = 0 \quad \text{and} \quad af_{22}^{(23)}(a)E_{13} = af_{33}^{(23)}(a)E_{13}$$

for all $a \in \mathbb{F}$. Hence we obtain

$$f_{12}^{(23)} = 0, \tag{5.33}$$

$$f_{22}^{(23)} = f_{33}^{(23)}. \tag{5.34}$$

We also see that

$$\psi(aE_{ij} + bE_{il})(aE_{ij} + bE_{il}) = (aE_{ij} + bE_{il})\psi(aE_{ij} + bE_{il}) \tag{5.35}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i \leq j, l \leq 3$. By the additivity of ψ and (5.35), we have

$$\begin{aligned} &E_{1k}\bigg(\sum_{1\leq s\leq t\leq 3} f_{st}^{(ij)}(a)E_{st}(bE_{il}) + \sum_{1\leq s\leq t\leq 3} f_{st}^{(il)}(b)E_{st}(aE_{ij})\bigg) \\ &= E_{1k}\bigg((aE_{ij})\sum_{1\leq s\leq t\leq 3} f_{st}^{(il)}(b)E_{st} + (bE_{il})\sum_{1\leq s\leq t\leq 3} f_{st}^{(ij)}(a)E_{st}\bigg). \end{aligned}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i, j, l, k \leq 3$ with $i \leq j, l$. Therefore

$$bf_{ki}^{(ij)}(a)E_{1l} + af_{ki}^{(il)}(b)E_{1j}$$
$$= \begin{cases} a\sum_{t=j}^{3} f_{jt}^{(il)}(b)E_{1t} + b\sum_{t=l}^{3} f_{lt}^{(ij)}(a)E_{1t} & \text{when } k = i, \\ 0 & \text{when } k \neq i \end{cases} \tag{5.36}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq k \leq i \leq j, l \leq 3$. Setting $(i, j, l) = (1, 1, 2)$ in (5.36), we obtain $k = 1$ and so

$$bf_{11}^{(11)}(a)E_{12} + af_{11}^{(12)}(b)E_{11}$$
$$= a(f_{11}^{(12)}(b)E_{11} + f_{12}^{(12)}(b)E_{12} + f_{13}^{(12)}(b)E_{13}) + b(f_{22}^{(11)}(a)E_{12} + f_{23}^{(11)}(a)E_{13})$$

for all $a, b \in \mathbb{F}$. Hence we obtain

$$b(f_{11}^{(11)}(a) - f_{22}^{(11)}(a)) = af_{12}^{(12)}(b), \tag{5.37}$$

$$af_{13}^{(12)}(b) = -bf_{23}^{(11)}(a) \tag{5.38}$$

for all $a, b \in \mathbb{F}$. Taking $a = 1$ in (5.37) and (5.38), we obtain

$$f_{12}^{(12)}(b) = b\tau_{12}, \tag{5.39}$$

$$f_{13}^{(12)}(b) = -b\theta \tag{5.40}$$

for all $b \in \mathbb{F}$, where $\tau_{12} = f_{11}^{(11)}(1) - f_{22}^{(11)}(1)$ and $\theta = f_{23}^{(11)}(1)$. Taking $b = 1$ in (5.37) and (5.38), and together with (5.39) and (5.40) respectively, we obtain

$$f_{11}^{(11)}(a) = f_{22}^{(11)}(a) + a\tau_{12}, \tag{5.41}$$

$$f_{23}^{(11)}(a) = a\theta \tag{5.42}$$

for all $a \in \mathbb{F}$. Setting $(i, j, l) = (1, 1, 3)$ in (5.36), we obtain $k = 1$ and so

$$\begin{aligned} &bf_{11}^{(11)}(a)E_{13} + af_{11}^{(13)}(b)E_{11} \\ &= a(f_{11}^{(13)}(b)E_{11} + f_{12}^{(13)}(b)E_{12} + f_{13}^{(13)}(b)E_{13}) + bf_{33}^{(11)}(a)E_{13} \end{aligned}$$

for all $a, b \in \mathbb{F}$. Hence we obtain

$$b(f_{11}^{(11)}(a) - f_{33}^{(11)}(a)) = af_{13}^{(13)}(b) \tag{5.43}$$

for all $a, b \in \mathbb{F}$, and

$$f_{12}^{(13)} = 0. \tag{5.44}$$

Taking $a = 1$ in (5.43), we obtain

$$f_{13}^{(13)}(b) = b\tau_{13} \tag{5.45}$$

for all $b \in \mathbb{F}$, where $\tau_{13} = f_{11}^{(11)}(1) - f_{33}^{(11)}(1)$. Taking $b = 1$ in (5.43) and together with (5.45), we obtain

$$f_{11}^{(11)}(a) = f_{33}^{(11)}(a)) + a\tau_{13} \tag{5.46}$$

for all $a \in \mathbb{F}$. Setting $(i,j,l) = (1,2,3)$ in (5.36), we obtain $k = 1$ and so

$$bf_{11}^{(12)}(a)E_{13} + af_{11}^{(13)}(b)E_{12} = a(f_{22}^{(13)}(b)E_{12} + f_{23}^{(13)}(b)E_{13}) + bf_{33}^{(12)}(a)E_{13}$$

for all $a, b \in \mathbb{F}$. Hence we obtain

$$b(f_{11}^{(12)}(a) - f_{33}^{(12)}(a)) = af_{23}^{(13)}(b) \tag{5.47}$$

for all $a, b \in \mathbb{F}$, and

$$f_{11}^{(13)} = f_{22}^{(13)}. \tag{5.48}$$

Setting $(i,j,l) = (2,2,3)$ in (5.36), we obtain $k = 1, 2$ and so

$$bf_{12}^{(22)}(a)E_{13} + af_{12}^{(23)}(a)E_{12} = 0$$

and

$$bf_{22}^{(22)}(a)E_{13} + af_{22}^{(23)}(b)E_{12} = a(f_{22}^{(23)}(b)E_{12} + f_{23}^{(23)}(b)E_{13}) + bf_{33}^{(22)}(a)E_{13}$$

for all $a, b \in \mathbb{F}$. Hence we obtain

$$b(f_{22}^{(22)}(a) - f_{33}^{(22)}(a)) = af_{23}^{(23)}(b) \tag{5.49}$$

for all $a, b \in \mathbb{F}$, and

$$f_{12}^{(22)} = f_{12}^{(23)} = 0. \tag{5.50}$$

Taking $a = 1$ in (5.49), we obtain

$$f_{23}^{(23)}(b) = b\tau_{23} \tag{5.51}$$

for all $b \in \mathbb{F}$, where $\tau_{23} = f_{22}^{(22)}(1) - f_{33}^{(22)}(1)$. Taking $b = 1$ in (5.42) and together with (5.47), we obtain

$$f_{22}^{(22)}(a) = f_{33}^{(22)}(a) + a\tau_{23} \tag{5.52}$$

for all $a \in \mathbb{F}$. We see that

$$\psi(aE_{ij} + bE_{lj})(aE_{ij} + bE_{lj}) = (aE_{ij} + bE_{lj})\psi(aE_{ij} + bE_{lj}) \tag{5.53}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i, l \leq j \leq 3$. By the additivity of ψ and (5.44), we obtain

$$\begin{aligned}
&\Bigg(\sum_{1\leq s\leq t\leq 3} f_{st}^{(ij)}(a)E_{st}(bE_{lj}) + \sum_{1\leq s\leq t\leq 3} f_{st}^{(lj)}(b)E_{st}(aE_{ij})\Bigg)E_{k3} \\
&= \Bigg((aE_{ij})\sum_{1\leq s\leq t\leq 3} f_{st}^{(lj)}(b)E_{st} + (bE_{lj})\sum_{1\leq s\leq t\leq 3} f_{st}^{(ij)}(a)E_{st}\Bigg)E_{k3}.
\end{aligned}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i, l, j, k \leq 3$ with $i, l \leq j$. Therefore

$$\begin{aligned}
&af_{jk}^{(lj)}(b)E_{i3} + bf_{jk}^{(ij)}(a)E_{l3} \\
&= \begin{cases} b\sum_{s=1}^{l} f_{sl}^{(ij)}(a)E_{s3} + a\sum_{s=1}^{i} f_{si}^{(lj)}(b)E_{s3} & \text{when } k = j, \\ 0 & \text{when } k \neq j \end{cases}
\end{aligned} \tag{5.54}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leq i, l \leq j \leq k \leq 3$. Setting $(j, i, l) = (2, 1, 2)$ in (5.54), we obtain $k = 2, 3$ and so

$$af_{22}^{(22)}(b)E_{13} + bf_{22}^{(12)}(a)E_{23} = b(f_{12}^{(12)}(a)E_{13} + f_{22}^{(12)}(a)E_{23}) + af_{11}^{(22)}(b)E_{13}$$

and

$$af_{23}^{(22)}(b)E_{13} + bf_{23}^{(12)}(a) = 0$$

for all $a, b \in \mathbb{F}$. Hence we obtain

$$a(f_{22}^{(22)}(b) - f_{11}^{(22)}(b)) = bf_{12}^{(12)}(a) \tag{5.55}$$

for all $a, b \in \mathbb{F}$ and so

$$f_{23}^{(22)} = f_{23}^{(12)} = 0. \tag{5.56}$$

Taking $a = 1$ in (5.55) and by (5.39), we obtain

$$f_{22}^{(22)}(b) = f_{11}^{(22)}(b) + b\tau_{12} \tag{5.57}$$

for all $b \in \mathbb{F}$. Setting $(j, i, l) = (3, 1, 2)$ in (5.54), we obtain $k = 3$ and so

$$af_{33}^{(23)}(b)E_{13} + bf_{33}^{(13)}(a)E_{23} = b(f_{12}^{(13)}(a)E_{13} + f_{22}^{(13)}(a)E_{23}) + af_{11}^{(23)}(b)E_{13}$$

for all $ab \in \mathbb{F}$. Hence we obtain

$$a(f_{33}^{(23)}(b) - f_{11}^{(23)}(b)) = bf_{12}^{(13)}(a) \tag{5.58}$$

for all $a, b \in \mathbb{F}$ and so

$$f_{33}^{(13)} = f_{22}^{(13)}. \tag{5.59}$$

In view of (5.44) and (5.58), we obtain

$$f_{33}^{(23)} = f_{11}^{(23)} \tag{5.60}$$

Setting $(j, i, l) = (3, 1, 3)$ in (5.54), we obtain $k = 3$ and so

$$\begin{aligned} & af_{33}^{(33)}(b)E_{13} + bf_{33}^{(13)}(a)E_{33} \\ & = b(f_{13}^{(13)}(a)E_{13} + f_{23}^{(13)}(a)E_{23} + f_{33}^{(13)}(a)E_{33}) + af_{11}^{(33)}(b)E_{13} \end{aligned}$$

for all $a, b \in \mathbb{F}$. Hence we obtain

$$a(f_{33}^{(33)}(b) - f_{11}^{(33)}(b)) = bf_{13}^{(13)}(a) \tag{5.61}$$

for all $a, b \in \mathbb{F}$, and

$$f_{23}^{(13)} = 0. \tag{5.62}$$

In view of (5.47) and (5.62), we obtain

$$f_{11}^{(12)} = f_{33}^{(12)}. \tag{5.63}$$

Taking $a = 1$ in (5.61) and together with (5.45), we obtain

$$f_{33}^{(33)}(b) = f_{11}^{(33)}(b) + b\tau_{13} \tag{5.64}$$

for all $a \in \mathbb{F}$. Setting $(j,i,l) = (3,2,3)$ in (5.54), we obtain $k = 3$ and so

$$
\begin{aligned}
& af_{33}^{(33)}(b)E_{23} + bf_{33}^{(23)}(a)E_{33} \\
& = b(f_{13}^{(23)}(a)E_{13} + f_{23}^{(23)}(a)E_{23} + f_{33}^{(23)}(a)E_{33}) + a(f_{12}^{(33)}(b)E_{13} + f_{22}^{(33)}(b)E_{23})
\end{aligned}
$$

for all $a, b \in \mathbb{F}$. Hence we obtain

$$a(f_{33}^{(33)}(b) - f_{22}^{(33)}(b)) = bf_{23}^{(23)}(a), \tag{5.65}$$

$$bf_{13}^{(23)}(a) = -af_{12}^{(33)}(b) \tag{5.66}$$

for all $a, b \in \mathbb{F}$. Taking $b = 1$ in (5.66), we obtain

$$f_{13}^{(23)}(a) = -a\vartheta \tag{5.67}$$

for all $a \in \mathbb{F}$, where $\vartheta = f_{12}^{(33)}(1)$. Taking $a = 1$ in (5.66) and by (5.67), we obtain

$$f_{12}^{(33)}(b) = b\vartheta \tag{5.68}$$

for all $b \in \mathbb{F}$. Taking $a = 1$ in (5.66) and together with (5.51), we obtain

$$f_{33}^{(33)}(b) = f_{22}^{(33)}(b) + b\tau_{23} \tag{5.69}$$

for all $b \in \mathbb{F}$. We next see that

$$\psi(aE_{11}+bE_{12}+cE_{13})(aE_{11}+bE_{12}+cE_{13})$$
$$=(aE_{11}+bE_{12}+cE_{13})\psi(aE_{11}+bE_{12}+cE_{13}) \tag{5.70}$$

for all $a,b,c\in\mathbb{F}$. By the additivity of ψ and (5.70), we have

$$E_{11}\Bigg(\sum_{1\le s\le t\le 3} f_{st}^{(11)}(a)E_{st}(bE_{12})+\sum_{1\le s\le t\le 3} f_{st}^{(11)}(a)E_{st}(cE_{13})$$
$$+\sum_{1\le s\le t\le 3} f_{st}^{(12)}(b)E_{st}(aE_{11})+\sum_{1\le s\le t\le 3} f_{st}^{(12)}(b)E_{st}(cE_{13})$$
$$+\sum_{1\le s\le t\le 3} f_{st}^{(13)}(c)E_{st}(aE_{11})+\sum_{1\le s\le t\le 3} f_{st}^{(13)}(c)E_{st}(bE_{12})\Bigg)$$
$$=$$
$$E_{11}\Bigg((bE_{12})\sum_{1\le s\le t\le 3} f_{st}^{(13)}(c)E_{st}+(aE_{11})\sum_{1\le s\le t\le 3} f_{st}^{(13)}(c)E_{st}$$
$$+(cE_{13})\sum_{1\le s\le t\le 3} f_{st}^{(12)}(b)E_{st}+(aE_{11})\sum_{1\le s\le t\le 3} f_{st}^{(12)}(b)E_{st}$$
$$+(cE_{13})\sum_{1\le s\le t\le 3} f_{st}^{(11)}(a)E_{st}+(bE_{12})\sum_{1\le s\le t\le 3} f_{st}^{(11)}(a)E_{st}\Bigg)$$

for all $a,b,c\in\mathbb{F}$. Therefore

$$bf_{11}^{(11)}(a)E_{12}+cf_{11}^{(11)}(a)E_{13}+af_{11}^{(12)}(b)E_{11}$$
$$+cf_{11}^{(12)}(b)E_{13}+af_{11}^{(13)}(c)E_{11}+bf_{11}^{(13)}(c)E_{12}$$
$$=$$
$$b\sum_{t=2}^{3} f_{2t}^{(13)}(c)E_{1t}+a\sum_{t=1}^{3} f_{1t}^{(13)}(c)E_{1t}+c\sum_{t=3}^{3} f_{3t}^{(12)}(b)E_{1t}$$
$$+a\sum_{t=1}^{3} f_{1t}^{(12)}(b)E_{1t}+c\sum_{t=3}^{3} f_{3t}^{(11)}(a)E_{1t}+b\sum_{t=2}^{3} f_{2t}^{(11)}(a)E_{1t}$$

for all $a,b,c\in\mathbb{F}$, and so

$$c(f_{11}^{(11)}(a)+f_{11}^{(12)}(b)-f_{33}^{(12)}(b)-f_{33}^{(11)}(a))$$
$$=a(f_{13}^{(13)}(c)+f_{13}^{(12)}(b))+b(f_{23}^{(13)}(c)+f_{23}^{(11)}(a)) \tag{5.71}$$

for all $a,b,c\in\mathbb{F}$. By (5.40), (5.42), (5.45), (5.46), (5.62) and (5.71), we obtain

$$f_{11}^{(12)}=f_{33}^{(12)}. \tag{5.72}$$

We then see that

$$\psi(aE_{13}+bE_{23}+cE_{33})(aE_{13}+bE_{23}+cE_{33})$$
$$=(aE_{13}+bE_{23}+cE_{33})\psi(aE_{13}+bE_{23}+cE_{33}) \tag{5.73}$$

for all $a,b,c\in\mathbb{F}$. By the additivity of ψ and (5.73), we have

$$\Bigg(\sum_{1\le s\le t\le 3} f_{st}^{(13)}(a)E_{st}(bE_{23})+\sum_{1\le s\le t\le 3} f_{st}^{(13)}(a)E_{st}(cE_{33})$$
$$+\sum_{1\le s\le t\le 3} f_{st}^{(23)}(b)E_{st}(aE_{13})+\sum_{1\le s\le t\le 3} f_{st}^{(23)}(b)E_{st}(cE_{33})$$
$$+\sum_{1\le s\le t\le 3} f_{st}^{(33)}(c)E_{st}(aE_{13})+\sum_{1\le s\le t\le 3} f_{st}^{(33)}(c)E_{st}(bE_{23})\Bigg)E_{33}$$
$$=$$
$$\Bigg((bE_{23})\sum_{1\le s\le t\le 3} f_{st}^{(33)}(c)E_{st}+(aE_{13})\sum_{1\le s\le t\le 3} f_{st}^{(33)}(c)E_{st}$$
$$+(cE_{33})\sum_{1\le s\le t\le 3} f_{st}^{(23)}(b)E_{st}+(aE_{13})\sum_{1\le s\le t\le 3} f_{st}^{(23)}(b)E_{st}$$
$$+(cE_{33})\sum_{1\le s\le t\le 3} f_{st}^{(13)}(a)E_{st}+(bE_{23})\sum_{1\le s\le t\le 3} f_{st}^{(13)}(a)E_{st}\Bigg)E_{33}$$

for all $a,b,c\in\mathbb{F}$. Therefore

$$bf_{33}^{(13)}(a)E_{23} + cf_{33}^{(13)}(a)E_{33} + af_{33}^{(23)}(b)E_{13}$$

$$+ cf_{33}^{(23)}(b)E_{33} + af_{33}^{(33)}(c)E_{13} + bf_{33}^{(33)}(c)E_{23}$$

$$=$$

$$b\sum_{s=1}^{2} f_{s2}^{(13)}(a)E_{s3} + c\sum_{s=1}^{3} f_{s3}^{(13)}(a)E_{s3} + a\sum_{s=1}^{1} f_{s1}^{(23)}(b)E_{s3}$$
$$+ c\sum_{s=1}^{3} f_{s3}^{(23)}(b)E_{s3} + a\sum_{s=1}^{1} f_{s1}^{(33)}(c)E_{s3} + b\sum_{s=1}^{2} f_{s2}^{(33)}(c)E_{s3}$$

for all $a, b, c \in \mathbb{F}$, and so

$$a(f_{33}^{(23)}(b) + f_{33}^{(33)}(c) - f_{11}^{(23)}(b) - f_{11}^{(33)}(c))$$
$$= b(f_{12}^{(13)}(a) + f_{12}^{(33)}(c)) + c(f_{13}^{(13)}(a) + f_{13}^{(23)}(b)) \tag{5.74}$$

for all $a, b, c \in \mathbb{F}$. By (5.44), (5.45), (5.64), (5.67), (5.68) and (5.74), we obtain

$$f_{33}^{(23)} = f_{11}^{(23)}. \tag{5.75}$$

Solving equations (5.37), (5.39), (5.43) and (5.45), we obtain

$$a(\tau_{12} - \tau_{13}) = f_{33}^{(11)}(a) - f_{22}^{(11)}(a) \tag{5.76}$$

for all $a \in \mathbb{F}$. Solving equations (5.37), (5.39), (5.43) and (5.45), we obtain

$$b(\tau_{12} - \tau_{23}) = f_{33}^{(22)}(b) - f_{11}^{(22)}(b) \tag{5.77}$$

for all $b \in \mathbb{F}$. Solving equations (5.45), (5.51), (5.61) and (5.65), we obtain

$$c(\tau_{13} - \tau_{23}) = f_{22}^{(33)}(c) - f_{11}^{(33)}(c) \tag{5.78}$$

for all $c \in \mathbb{F}$. By (5.77), (5.78), together with (5.52) and (5.69), it follows that

$$f_{11}^{(22)}(b) + f_{11}^{(33)}(c) = -b\tau_{12} - c\tau_{13} + f_{22}^{(22)}(b) + f_{33}^{(33)}(c) \tag{5.79}$$

for all $b, c \in \mathbb{F}$. By (5.76), (5.78), together with (5.46) and (5.64), it follows that

$$f_{22}^{(11)}(a) + f_{22}^{(33)}(c) = -a\tau_{12} - c\tau_{23} + f_{11}^{(11)}(a) + f_{33}^{(33)}(c) \tag{5.80}$$

for all $a, c \in \mathbb{F}$. By (5.76), (5.77), together with (5.41) and (5.57), it follows that

$$f_{33}^{(11)}(a) + f_{33}^{(22)}(b) = -a\tau_{13} - b\tau_{23} + f_{11}^{(11)}(a) + f_{22}^{(22)}(b) \tag{5.81}$$

for all $a, b \in \mathbb{F}$. Let $\mu : T_3(\mathbb{F}) \to \mathbb{F}$ be the additive map defined by

$$\begin{aligned}\mu(A) = {} & f_{11}^{(11)}(a_{11}) + f_{11}^{(12)}(a_{12}) + f_{11}^{(13)}(a_{13}) + f_{22}^{(22)}(a_{22}) \\ & + f_{11}^{(23)}(a_{23}) + f_{33}^{(33)}(a_{33})\end{aligned} \tag{5.82}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$. Let $\psi_{\tau_{12},\tau_{13},\tau_{23}}, \psi_{\theta,\vartheta}(A) : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ be the linear maps defined by

$$\begin{aligned}&\psi_{\tau_{12},\tau_{13},\tau_{23}}(A) \\ &= \begin{bmatrix} -a_{22}\tau_{12} - a_{33}\tau_{13} & a_{12}\tau_{12} & a_{13}\tau_{13} \\ 0 & -a_{33}\tau_{23} - a_{11}\tau_{12} & a_{23}\tau_{23} \\ 0 & 0 & -a_{11}\tau_{13} - a_{22}\tau_{23} \end{bmatrix}\end{aligned} \tag{5.83}$$

and

$$\psi_{\theta,\vartheta}(A) = \begin{bmatrix} 0 & a_{33}\vartheta & -a_{23}\vartheta - a_{12}\theta \\ 0 & 0 & a_{11}\theta \\ 0 & 0 & 0 \end{bmatrix} \tag{5.84}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$ and for some scalars $\theta, \vartheta, \tau_{12}, \tau_{13}, \tau_{23} \in \mathbb{F}$. Let $\psi_\phi(A) : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ be the additive map defined by

$$\psi_\phi(A) = \begin{bmatrix} 0 & 0 & f_{13}^{(22)}(a_{22}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.85}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$. By virtue of (5.27)-(5.34), (5.39), (5.40), (5.42), (5.44), (5.45), (5.48), (5.45), (5.50), (5.51), (5.56), (5.59), (5.60), (5.62), (5.63), (5.67), (5.68), (5.72), (5.75), (5.79)-(5.85) and together with the additivity of ψ, we obtain

$$\begin{aligned}
\psi(A) &= \psi(a_{11}E_{11}) + \psi(a_{12}E_{12}) + \psi(a_{13}E_{13}) + \psi(a_{22}E_{22}) + \psi(a_{23}E_{23}) \\
&\quad + \psi(a_{33}E_{33}) \\
&= \begin{bmatrix} f_{11}^{(11)}(a_{11}) & 0 & 0 \\ 0 & f_{22}^{(22)}(a_{22}) & 0 \\ 0 & 0 & f_{33}^{(33)}(a_{33}) \end{bmatrix} + \\
&\quad \begin{bmatrix} f_{11}^{(22)}(a_{22}) + f_{11}^{(33)}(a_{33}) & f_{12}^{(12)}(a_{12}) & f_{13}^{(13)}(a_{13}) \\ 0 & f_{22}^{(11)}(a_{11}) + f_{22}^{(33)}(a_{33}) & f_{23}^{(23)}(a_{23}) \\ 0 & 0 & f_{33}^{(11)}(a_{11}) + f_{33}^{(22)}(a_{22}) \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & f_{12}^{(33)}(a_{33}) & f_{13}^{(12)}(a_{12}) + f_{13}^{(23)}(a_{23}) \\ 0 & 0 & f_{23}^{(11)}(a_{11}) \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & f_{13}^{(22)}(a_{22}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&\quad + (f_{11}^{(12)}(a_{12}) + f_{11}^{(13)}(a_{13}) + f_{11}^{(23)}(a_{23}))I_3 \\
&= \begin{bmatrix} -a_{22}\tau_{12} - a_{33}\tau_{13} & a_{12}\tau_{12} & a_{13}\tau_{13} \\ 0 & -a_{33}\tau_{23} - a_{11}\tau_{12} & a_{23}\tau_{23} \\ 0 & 0 & -a_{11}\tau_{13} - a_{22}\tau_{23} \end{bmatrix} + \\
&\quad \begin{bmatrix} 0 & a_{33}\vartheta & -a_{23}\vartheta - a_{12}\theta \\ 0 & 0 & a_{11}\theta \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & f_{13}^{(22)}(a_{22}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu(A)I_3 \\
&= \mu(A)I_3 + \psi_{\tau_{12},\tau_{13},\tau_{23}}(A) + \psi_{\theta,\vartheta}(A) + \psi_\phi(A)
\end{aligned}$$

for every $A = (a_{ij}) \in T_3(\mathbb{F})$ as desired. □

CHAPTER 6: CONCLUSION

6.1 A brief overview

This final chapter briefly discusses the main findings of this research. Some open problems related to the study of this dissertation are also listed for future study.

6.2 Summary

In summary, this research explored centralizing (commuting) additive maps on some matrix rings. We first studied centralizing additive maps on block triangular matrices over an arbitrary field and showed its characterization, see Theorem 3.2.1. We extended this result by characterizing centralizing additive maps on rank r block triangular matrices over an arbitrary field, see Theorem 4.3.1. Consequently, we also obtained a classification of centralizing additive maps on rank r square matrices and centralizing additive maps on rank r upper triangular matrices, see Theorem 4.3.2 and Theorem 4.3.3.

Next, we explored commuting additive maps on rank r upper triangular matrices over an arbitrary field. By (Chooi et al., 2020, Theorem 1.1), (Chooi et al., 2019, Theorems 2.8, 2.9, 2.10) and together with our main result, i.e. Theorem 4.3.1, we obtained a complete structural characterization of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank r matrices over an arbitrary field $\mathbb{F}$, where $2 \leq r \leq n$ is a fixed integer, see Theorem 5.2.1. At the end of this dissertation, we continued studying on commuting additive maps on rank one upper triangular matrices over an arbitrary field as it was not covered in Theorem 5.2.1. We managed to show the characterization on commuting additive maps on rank one upper triangular matrices of order two and three. We must highlight that the structure of commuting additive map on rank one upper triangular matrices is rather complex and complicated.

6.3 Some open problems

We end this chapter with some open problems related to the study of this dissertation.

1. Determine the structure of centralizing additive maps on invertible block triangular matrices over the Galois field of two elements.
2. Determine the structure of centralizing additive maps on rank one block triangular matrices over fields.
3. Determine the structure of centralizing additive maps on rank one block triangular matrices over division rings.

REFERENCES

Ara, P., & Mathieu, M. (1993). An application of local multipliers to centralizing mappings of c*-algebras. *The Quarterly Journal of Mathematics*, *44*(2), 129-138.

Beidar, K. I. (1998). On functional identities and commuting additive mappings. *Communications in Algebra*, *26*(6), 1819-1850.

Brešar, M. (1993). Centralizing mappings and derivations in prime rings. *Journal of Algebra*, *156*(2), 385-394.

Brešar, M., Chebotar, M. A., & Martindale, W. S. (2007). Functional identities. *Birkhäuser Verlag, Basel*.

Brešar, M., Martindale, W. S., & Miers, C. R. (1993). Centralizing maps in prime rings with involution. *Journal of Algebra.*, *161*(2), 342-357.

Chebotar, M. A. (1998). On generalized functional identities on prime rings. *Journal of Algebra*, *202*(2), 665-670.

Cheung, W. S. (2001). Commuting maps of triangular algebras. *Journal of the London Mathematical Society*, *63*(1), 117-127.

Chooi, W. L., & Kwa, K. H. (2019). Additive maps of rank r tensors and symmetric tensors. *Linear and Multilinear Algebra*, *67*(6), 1269-1293.

Chooi, W. L., & Kwa, K. H. (2020). Additive maps of rank k bivectors. *The Electronic Journal of Linear Algebra*, *36*(36), 847-856.

Chooi, W. L., Kwa, K. H., & Tan, L. Y. (2019). Commuting maps on invertible triangular matrices over $\mathbb{F}_2$. *Linear Algebra and its Applications*, *583*, 77-101.

Chooi, W. L., Kwa, K. H., & Tan, L. Y. (2020). Commuting maps on rank k triangular matrices. *Linear and Multilinear Algebra*, *68*(5), 1021-1030.

Chou, P. H., & Liu, C. K. (2019). Power commuting additive maps on rank-k linear transformations. *Linear and Multilinear Algebra*, 1-25.

Du, Y., & Wang, Y. (2012). k-commuting maps on triangular algebras. *Linear Algebra and its Applications*, *436*(5), 1367-1375.

Franca, W. (2012). Commuting maps on some subsets of matrices that are not closed under addition. *Linear Algebra and its Applications*, *437*(1), 388-391.

Franca, W. (2013). Commuting maps on rank-k matrices. *Linear Algebra and its Applications*, *438*(6), 2813-2815.

Franca, W. (2015). Commuting traces on invertible and singular operators. *Operators and Matrices*, *2*, 305-310.

Lee, P. H., & Lee, T. K. (1997). Linear identities and commuting maps in rings with involution. *Communications in Algebra*, *25*(9), 2881-2895.

Lee, P. H., & Wang, Y. (2009). Supercentralizing maps in prime superalgebras. *Communications in Algebra*, *37*(3), 840-854.

Lee, T. K. (1997). Derivations and centralizing mappings in prime rings. *Taiwanese Journal of Mathematics*, *1*(3), 333-342.

Li, C. K., & Pierce, S. (2001). Linear preserver problems. *The American Mathematical Monthly*, *108*(7), 591-605.

Liu, C. K. (2014a). Centralizing maps on invertible or singular matrices over division rings. *Linear Algebra and its Applications*, *440*, 318-324.

Liu, C. K. (2014b). Strong commutativity preserving maps on subsets of matrices that are not closed under addition. *Linear Algebra and its Applications*, *458*, 280-290.

Liu, C. K., Liau, P. K., & Tsai, Y. T. (2018). Nonadditive strong commutativity preserving maps on rank-k matrices over division rings. *Operators and Matrices*, *12*, 563–578.

Liu, C. K., & Yang, J. J. (2017). Power commuting additive maps on invertible or singular matrices. *Linear Algebra and its Applications*, *530*, 127-149.

Mayne, J. H. (1976). Centralizing automorphisms of prime rings. *Canadian Mathematical Bulletin*, *19*(1), 113-115.

Pierce, S. (1992). A survey of linear preserver problems contents. *Linear and Multilinear Algebra*, *33*, 1-129.

Posner, E. C. (1957). Derivations in prime rings. *Proceedings of the American Mathematical Society*, *8*(6), 1093-1100.

Wang, Y. (2016). On functional identities of degree 2 and centralizing maps in triangular rings. *Operators and Matrices*, *10*, 485-499.

Xu, X., & Liu, H. (2017). Additive maps on rank-s matrices. *Linear and Multilinear Algebra*, *65*(4), 806-812.

Xu, X., & Yi, X. (2014). Commuting maps on rank-k matrices. *The Electronic Journal of Linear Algebra*, *27*, 735-741.

LIST OF PUBLICATIONS

List of Publications

1. Chooi, W.L., **Mutalib, M.H.A.**, Tan, L.Y. (2021). Centralizing additive maps on rank r block triangular matrices. *Acta Scientiarum Mathematicarum (Szeged), 87*, 63-94.

2. Chooi, W.L., **Mutalib, M.H.A.**, Tan, L.Y. (2021). Commuting maps on rank one triangular matrices. *Linear Algebra and its Applications, 626*, 34-55.