ON STRONGLY π-REGULAR AND STRONGLY REGULAR RINGS

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ABSTRACT

Let R be an associative ring with identity $1 \neq 0$. An element $x \in R$ is said to be *right* (or *left*) *regular* if there exists y in R such that $x^2y = x$ (or $yx^2 = x$). If x is both left and right regular, then it is said to be *strongly regular.* The ring R is said to be *strongly regular* if every element of R is strongly regular. We say that x is a *left π-regular* element if there exist an integer $n > 0$ and an element $y \in R$ such that $yx^{n+1} = x^n$. A *right π-regular* element is defined analogously. An element of R is *strongly π-regular* if it is both left and right π-regular. R is *strongly π-regular* if every element of R is strongly π-regular. The main aim of this thesis is to study strongly regular and strongly π-regular rings. In particular, we shall be concerned with necessary and sufficient conditions for a ring to be strongly regular or strongly π-regular. We shall also study how these rings are related to other types of "regular" rings. A ring R is said to be an *(s,2)-ring* if every element of R is a sum of two units of R. If for any a, $b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible, then we say that R has *stable range one.* In a related thread, we shall study how various "regular" rings are related to (s,2)-rings and rings having stable range one.

ABSTRAK

Biar R suatu gelanggang kalis sekutuan dengan unsur identiti $1 \neq 0$. Suatu unsur $x \in R$ dikatakan *sekata kanan* (atau *kiri*) jika wujudnya $y \in R$ supaya $x^2y = x$ (atau $yx^2 = x$). Jika x adalah sekata kiri dan sekata kanan, maka x dikatakan *sekata kuat*. Gelanggang R dikatakan *sekata kuat* jika setiap unsur dalam R adalah sekata kuat. Unsur $x \in R$ dikatakan *π-sekata kiri* jika wujud suatu integer $n > 0$ dan suatu unsur $y \in R$ supaya $yx^{n+1} = x^n$. Unsur *π-sekata kanan* ditakrifkan secara beranalog. Suatu unsur dalam R dikatakan *π-sekata secara kuat* jika ianya π-sekata kiri dan π-sekata kanan. Gelanggang R dikatakan *π-sekata secara kuat* jika setiap unsur dalam R adalah π-sekata secara kuat. Tumpuan kajian disertasi ini adalah terhadap gelanggang-gelanggang sekata kuat dan π-sekata secara kuat. Khasnya, kita berminat kepada syarat-syarat perlu dan cukup bagi suatu gelanggang menjadi sekata kuat atau π-sekata secara kuat. Kita juga akan mengkaji bagaimana kedua-dua gelanggang ini berhubungkait dengan gelanggang-gelanggang "sekata" yang lain. Gelanggang R dikatakan *gelanggang-(s,2)* jika setiap unsur dalam R boleh ditulis sebagai hasil tambah dua unit dalam R. Jika bagi sebarang $a, b \in R$ yang memenuhi hubungan $aR+bR = R$, wujudnya $y \in R$ supaya $a + by$ adalah unsur bolehsongsang kanan, maka kita katakan R mempunyai *julat kestabilan satu*. Selaras dengan penyelidikan seterusnya, kita akan mengkaji bagaimana beberapa gelanggang "sekata" adalah berhubungkait dengan gelanggang-(s,2) dan gelanggang yang mempunyai julat kestabilan satu.

INTRODUCTION

In this thesis we shall investigate strongly π-regular and strongly regular rings as well as their relations with various other "regular" rings, (s,2)-rings and rings having stable range one.

In Chapter 1 we review some basic facts about rings (including group rings). We also use this chapter to set basic notations and give some definitions.

Chapter 2 deals with relationships between various "regular" rings such as strongly regular, strongly π-regular, (von Neumann) regular, unit regular, unit π-regular, π-regular, π'-regular and weakly π-regular rings. A diagram showing how these "regular" rings are related is given. We also consider matrix rings over "regular" rings in this chapter.

In Chapter 3 we focus our attention on strongly π-regular rings. We investigate properties of strongly π-regular rings and conditions which are equivalent to being strongly π-regular. In particular, one of the results which we obtain is that every element in a strongly π-regular ring can be written as the sum of two commuting elements, one of which is strongly regular and the other nilpotent. We also obtain some results on conditions which are necessary or sufficient for a group ring to be strongly π-regular. In the last section of this chapter, we look at Euler and exact-Euler rings which have been defined by Badawi [B3]. We shall provide different proofs for two of the results in [B3] relating Euler and exact-Euler rings with strongly π-regular rings.

Chapter 4 concerns strongly regular rings. We shall use arguments similar to those used in the proof of a result in [Sh] to show that every element in a strongly regular ring can be decomposed as the product of two commuting elements, one of which is an idempotent and the other an invertible element. We then demonstrate several applications of this result. In the last section of this chapter, we obtain conditions which are necessary or sufficient for a group ring to be strongly regular.

Finally, in Chapter 5 we study relations between various "regular" rings and (s,2)-rings as well as rings having stable range one. In particular, we shall give a different proof of the known result that a regular ring has stable range one if and only if it is unit regular.

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