CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction.

Let R be a ring. R is said to be *von Neumann regular* if for each element $x \in R$, there exists $y \in R$ such that $xyx = x$. Von Neumann regular ring is a part of noncommutative ring theory that was originally introduced by von Neumann in the mid-1930s to clarify certain aspects of operator algebras. Indeed, much of the impetus behind the development of regular rings is due to this and to some connections with functional analysis. As would be expected, von Neumann regular rings have also been extensively studied by ring theorists for their own sake. For convenience, we shall refer to von Neumann regular rings as just regular rings in the remainder of this thesis.

A ring R is said to be *strongly regular* if for each $x \in R$ there exists $y \in R$ such that $x^2y = x$. The study of strongly regular rings was pioneered by Arens and Kaplansky [AK]. They showed that a strongly regular ring is regular and that in a strongly regular ring, every one-sided ideal is two-sided. Forsythe and McCoy [FM] showed that a regular ring is strongly regular if and only if it has no nonzero nilpotent elements. Strongly regular rings have also been studied by Azumaya [Az], Goodearl [Go], Kennison [Ke], Yue [Y2], Li and Schein [LS], Zhang [Zh], Zhang and Lu [ZL] among others.

Strongly π-regular rings were introduced by Kaplansky [Ka] as a common generalization of algebraic algebras and artinian rings. A ring R is said to be *strongly π-regular* if for each $a \in R$ there exist a positive integer n and an element

$x \in R$ such that $a^n = a^{n+1}x$ and $ax = xa$. In the early years, strongly π-regular rings were studied extensively by Arens and Kaplansky [AK] and Azumaya [Az]. The theory of strongly π-regular rings has since developed into a variety of applications. For example, since algebraic algebras are strongly π-regular rings, the study of strongly π-regular rings has been used to solve some classical open questions on algebraic algebras. In more recent times, strongly π-regular rings have been studied by Dischinger [Di], Ara [Ar], Hirano [Hi], Shirley [Sh] and Badawi [B3] among others.

Topics in this dissertation are mainly concerned with several aspects of strongly π-regular and strongly regular rings. We shall also show how these rings are related to other types of "regular" rings. All rings considered in this dissertation are associative with identity unless stated otherwise and all modules are unitary. For any ring R, by an R-module M, we mean a right R-module and sometimes write M as M_R. In the remainder of this chapter, we shall fix notations and terminologies for later use.

1.2 Nil, nilpotent, idempotent and annihilator.

An element x of a ring R is *nilpotent* if there is a positive integer n such that $x^n = 0$. A right (left, or two-sided) ideal I of R is *nil* provided that every element of I is nilpotent. The ideal I is called *nilpotent* if $I^n = \{0\}$ for some positive integer n. Equivalently, I is nilpotent if for every choice of n elements $a_1, a_2, \ldots, a_n \in I$, the product $a_1 a_2 \ldots a_n = 0$. It is clear that nilpotent ideals are nil. A ring R is said to be *reduced* if it has no nonzero nilpotent elements. A nilpotent element $x \in R$ is said to have *index n* if n is the least positive integer such that $x^n = 0$. If the indices of all nilpotent elements of R are bounded, then R is said to have *bounded index (of nilpotency)*.

Let R be a ring. An element $e \in R$ is said to be an *idempotent* if $e^2 = e$. An element $f \in R$ is a *near idempotent* if f^n is an idempotent for some positive integer n. Clearly, every idempotent is a near idempotent. A ring R is said to be *indecomposable* if and only if 1 is the only nonzero central idempotent of R.

Proposition 1.2.1. *If the center of a ring R is a field, then R is indecomposable.*

Proof. Let $Z(R)$ denote the center of R and let e be a nonzero central idempotent of R. Since $e \in Z(R)$ and $Z(R)$ is a field, the inverse e^{-1} of e exists. It then follows from $e(e-1) = 0$ that $e = 1$. Hence, R is indecomposable. □

Let $a \in R$. Then

$$r_R(a) = \{x \in R \mid ax = 0\}$$

is called the *right annihilator* of a in R. Similarly,

$$l_R(a) = \{x \in R \mid xa = 0\}$$

is the *left annihilator* of a in R. It is straightforward to show that $r_R(a)$ ($l_R(a)$) is right (left) ideal of R.

Given a ring R, we use the notation $\mathrm{Nil}(R)$ to denote the set of all nilpotent elements of R. We also use $Id(R)$ and $U(R)$ to denote the set of all idempotent and invertible elements of R, respectively.

1.3 Artinian rings.

A ring R is said to be *right (left) artinian* if for any descending chain

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \ldots$$

of right (left) ideals of R, there exists an integer n such that $I_m = I_n$ for all $m \geq n$. As mentioned earlier, strongly π-regular rings generalize artinian rings. We shall provide details of this statement in the following chapter.

1.4 Prime and Jacobson radicals.

Let R be a ring. A proper ideal P of R is said to be *prime* if for any elements $a, b \in R$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$. The *prime radical* of a ring R, denoted by $P(R)$, is the intersection of all the prime ideals of R. It is known that $P(R)$ contains all the nilpotent ideals of R and that $P(R)$ is a nil ideal. A ring R is said to be *semiprime* if $P(R) = \{0\}$.

A proper right (left) ideal I of R is said to be *maximal* if whenever J is a right (left) ideal of R with $I \subset J \subseteq R$, then $J = R$. The *Jacobson radical* of R, denoted by $J(R)$, is the intersection of all the maximal right ideals of R. It can be shown that $J(R)$ is also the intersection of all the maximal left ideals of R. If $J(R) = \{0\}$, then R is said to be *semiprimitive*. It is known that $J(R)$ consists of all $x \in R$ such that for all $r \in R$, $1 - xr$ has a right inverse. In addition, $J(R)$ is an ideal of R which contains all the nil ideals of R; hence, $P(R) \subseteq J(R)$.

1.5 Some basic results on group rings.

Let R be a ring and G a multiplicative group. The group ring RG consists of all formal sums of the form $\sum_{g \in G} r_g \cdot g$ with $r_g \in R$, where only finitely many $r_g \neq 0$. The operations of addition, scalar multiplication and multiplication in RG are defined as follows:

$$\sum_{g \in G} r_g \cdot g + \sum_{g \in G} s_g \cdot g = \sum_{g \in G} (r_g + s_g) \cdot g;$$

$$\left(\sum_{g\in G} r_g \cdot g\right) a = \sum_{g\in G} (r_g a) \cdot g, \qquad a \in R;$$

$$a \left(\sum_{g\in G} r_g \cdot g\right) = \sum_{g\in G} (a r_g) \cdot g, \qquad a \in R;$$

$$\left(\sum_{h\in G} r_h \cdot h\right) \left(\sum_{k\in G} s_k \cdot k\right) = \sum_{g\in G} \left(\sum_{hk=g} r_h s_k\right) \cdot g.$$

If 1_R and 1_G are the identity elements of R and G respectively, then RG has identity element $1 = 1_R \cdot 1_G$. If we identify $x \in G$ with $1_R \cdot x \in RG$, then we have $G \subseteq RG$, and the elements of G form a basis of RG over R. Similarly, we may also identify $r \in R$ with $r \cdot 1_G \in RG$. With these identifications, $r_g \cdot g$ can be written as $r_g g$.

Let $r = \sum_{g\in G} r_g g \in RG$. The *support* of r is defined to be

$$Supp(r) = \{g \in G \mid r_g \neq 0\}.$$

Since $r_g \neq 0$ for only finitely many $g \in G$, so $Supp(r)$ is a finite subset of G. Furthermore, $Supp(r) = \emptyset$ if and only if $r = 0$. The *norm* of r, $\delta(r)$, is defined as

$$\delta(r) = \sum_{g\in G} r_g.$$

A mapping ω from the lattice of subgroups of G to the lattice of right ideals of the group ring RG is defined as follows: If H is a subgroup of G, then ωH is the right ideal of RG generated by $\{1 - h \mid h \in H\}$. If $H = G$, then $\omega G = \Delta$ is called the *augmentation ideal* of RG.

Proposition 1.5.1 (Connell, [Co]). *Let R be a ring, G a group and H a subgroup of G.*

(i) *If H is generated by $\{h_i\}$, then ωH is generated by $\{1 - h_i\}$.*

$$\left(\sum_{g\in G} r_g \cdot g\right) a = \sum_{g\in G} (r_g a)\cdot g, \qquad a \in R;$$

$$a\left(\sum_{g\in G} r_g \cdot g\right) = \sum_{g\in G} (a r_g)\cdot g, \qquad a \in R;$$

$$\left(\sum_{h\in G} r_h \cdot h\right)\left(\sum_{k\in G} s_k \cdot k\right) = \sum_{g\in G}\left(\sum_{hk=g} r_h s_k\right)\cdot g.$$

If 1_R and 1_G are the identity elements of R and G respectively, then RG has identity element $1 = 1_R \cdot 1_G$. If we identify $x \in G$ with $1_R \cdot x \in RG$, then we have $G \subseteq RG$, and the elements of G form a basis of RG over R. Similarly, we may also identify $r \in R$ with $r \cdot 1_G \in RG$. With these identifications, $r_g \cdot g$ can be written as $r_g g$.

Let $r = \sum_{g\in G} r_g g \in RG$. The *support* of r is defined to be

$$Supp(r) = \{g \in G \mid r_g \neq 0\}.$$

Since $r_g \neq 0$ for only finitely many $g \in G$, so $Supp(r)$ is a finite subset of G. Furthermore, $Supp(r) = \emptyset$ if and only if $r = 0$. The *norm* of r, $\delta(r)$, is defined as

$$\delta(r) = \sum_{g\in G} r_g.$$

A mapping ω from the lattice of subgroups of G to the lattice of right ideals of the group ring RG is defined as follows: If H is a subgroup of G, then ωH is the right ideal of RG generated by $\{1 - h \mid h \in H\}$. If $H = G$, then $\omega G = \triangle$ is called the *augmentation ideal* of RG.

Proposition 1.5.1 (Connell, [Co]). *Let R be a ring, G a group and H a subgroup of G.*

(i) *If H is generated by $\{h_i\}$, then ωH is generated by $\{1 - h_i\}$.*

(ii) *ωH is a two-sided ideal of RG if and only if H is a normal subgroup of G. Moreover, $R(G/H) \cong RG/\omega H$ if H is a normal subgroup of G. In particular, $R \cong RG/\triangle$.*

(iii) *If J is a right ideal of R, then JG is a right ideal of RG. Conversely, if J' is a right ideal of RG, then $J' \cap R$ is a right ideal of R.*

(iv) *If J is an ideal of R, then JG is an ideal of RG and $(R/J)G \cong RG/JG$.*

(v) *For any element $r \in RG$, $r \in \triangle$ if and only if $\delta(r) = 0$.*

Proposition 1.5.2. *Let R be a ring and G a group. If RG has property (P) and homomorphic images of RG also have property (P), then $(R/I)G$ has property (P) for every ideal I of R.*

Proof. This is straightforward since $(R/I)G \cong RG/IG$. □

A group G is said to be *locally finite* if every finitely generated subgroup of G is finite. Necessary and sufficient conditions for a group ring to be regular are known and are as follows:

Theorem 1.5.3 (Auslander, [Au]; Connell, [Co]; Villamayor, [Vi]). *Let R be a ring and G a group. Then RG is regular if and only if*

(i) *R is regular,*

(ii) *G is locally finite, and*

(iii) *the order of every finite subgroup of G is a unit in R.*

The following result is also well-known:

Theorem 1.5.4 (Connell, [Co]; Passman, [Pa]). *Let R be a ring and G a group. Then RG is artinian if and only if R is artinian and G is finite.*