CHAPTER 2

AN OVERVIEW OF "REGULAR" RINGS

2.1 Introduction.

In this chapter, we shall study some relationships between various "regular" rings. We begin in Section 2 by giving some definitions and reviewing some basic results on regular, strongly regular and strongly π-regular rings, some of which will be needed in succeeding chapters. We shall also study how these rings are related to one another and to other types of "regular" rings.

Next in Section 3, we shall study some conditions under which the various "regular" conditions are equivalent. In particular, we shall see that if R is a commutative reduced ring, then all the different "regular" conditions on R as given in Section 2 are equivalent (see Theorem 2.3.6). Finally, in Section 4 we shall consider matrix rings over various "regular" rings.

2.2 Some relations between various "regular" rings.

Let R be a ring. R is said to be *regular* if for each element $x \in R$, there exists $y \in R$ such that $xyx = x$. For example, division rings are regular. It is not difficult to show that regular rings are closed under homomorphic images, direct limits and direct products. The following is a basic result on regular rings.

Theorem 2.2.1. *For a ring R, the following conditions are equivalent:*

(i) *R is regular;*

(ii) *Every principal right (left) ideal of R is generated by an idempotent;*

(iii) *Every finitely generated right (left) ideal of R is generated by an idempotent.*

Proof. See Theorem 1.1 of [Go]. $\square$

A regular ring R is said to be *abelian* if all its idempotents are central. It is clear that any commutative regular ring is abelian. An element x of a ring R is said to be *right* (or *left) regular* if there exists $y \in R$ such that $x^2y = x$ (or $yx^2 = x$). An element of R is *strongly regular* if it is both left and right regular. A ring R is called strongly regular if each element in R is strongly regular. The following theorem by Azumaya [Az] gives a characterization of strongly regular elements.

Theorem 2.2.2 (Azumaya, [Az]). *Let R be a ring and x a strongly regular element of R. Then there exists a unique element $y \in R$ such that $xy = yx$, $x^2y(= yx^2) = x$ and $xy^2(= y^2x) = y$. Moreover, y commutes with every element of R which is commutative with x.*

Proof. Let $a, b \in R$ such that $x^2a = x$ and $bx^2 = x$. Note that

$$xa = (bx^2)a = b(x^2a) = bx.$$

Therefore

$$xa^2 = (bx)a = b(xa) = b(bx) = b^2x.$$

Thus, we have that

$$xax = (bx)x = bx^2 = x = x^2a = xbx.$$

Let $y = xa^2$. Then

$$x^2y = x^2(xa^2) = x(x^2a^2) = x(xa) = x^2a = x.$$

We also have

$$yx = xa^2x = b(xax) = bx = xa = x^2a^2 = xy$$

and

$$xy^2 = x(xa^2)(xa^2) = x(bxa)(xa^2) = (xbx)axa^2 = (xax)a^2 = xa^2 = y.$$

Suppose that there exists an element $y' \in R$ which satisfies $xy' = y'x$, $x^2y' = x$ and $xy'^2 = y'$. Then

$$xy^2 = x^2y'y^2 = y'(x^2y)y = y'xy = y'(x^2y')y = y'^2x$$

and hence,

$$y = xy^2 = y'^2x = y'$$

which shows that y is unique. Finally, let $w \in R$ such that $xw = wx$. Then

$$yxw = ywx = ywx^2y = yx^2wy = xwy = wxy$$

and hence, w commutes with $yx = xy$. Therefore

$$yw = y^2xw = y(wyx) = ywxy = (yxw)y = (wyx)y = wy$$

and it follows that y commutes with every element which is commutative with x. $\square$

By Theorem 2.2.2 we see that a ring R is *strongly regular* if for each $x \in R$ there exists (a unique) $y \in R$ such that $x^2y = x$ and $xy = yx$. Another important characterization of strongly regular rings is given in Theorem 2.2.4 below (see also [Go, Theorem 3.5]). We first look at the following necessary conditions:

Proposition 2.2.3. *If R is a strongly regular ring, then R is semiprime and all idempotents in R are central.*

Proof. Let $x \in R$. Then $x = x^2 y$ for some $y \in R$ and it follows by induction that $x = x^{n+1}y^n$ for any positive integer n. We thus have from this that x is not nilpotent if $x \neq 0$. In other words, R has no nonzero nilpotent elements, that is, R is semiprime.

Now let e be an idempotent in R. Since $((1-e)Re)^2 = \{0\}$ and R is semiprime, so $(1-e)Re = \{0\}$. Similarly, $eR(1-e) = \{0\}$. Therefore, given any $r \in R$ we have $re = ere = er$. Thus, e is central, as required. □

Theorem 2.2.4 (Goodearl, [Go]). *A ring R is strongly regular if and only if it is abelian regular.*

Proof. Suppose that R is abelian regular. Then for any $x \in R$, there exists $y \in R$ such that $xyx = x$. Observe that

$$(xy)^2 = (xyx)y = xy,$$

that is, $xy \in Id(R)$. Since R is abelian regular, xy is central in R. It follows that

$$x = (xy)x = x(xy) = x^2 y.$$

Similarly, it can be shown that $yx \in Id(R)$ and $x = yx^2$. Thus x is strongly regular.

Conversely, suppose that R is strongly regular. By Proposition 2.2.3 it follows readily that all idempotents in R are central. It remains to show that R is regular. Let $x \in R$. Since R is strongly regular, it follows from Theorem 2.2.2 that there exists an element $y \in R$ such that $x = x^2 y$ and $xy = yx$. Thus $x = x^2 y = xyx$. Since x is arbitrary, it follows that R is regular. □

A regular ring is in general not necessarily strongly regular. In [LS], L. Li and B. M. Schein showed that a regular ring is not strongly regular if and only if it contains an isomorphic copy of $M_2(\mathbb{Q})$ or of $M_2(\mathbb{F}_p)$ for p a prime. (Here $\mathbb{Q}$ is the field of rationals, $\mathbb{F}_p$ is the field of integers modulo p and $M_2(R)$ is the ring of 2×2 matrices over R.) It follows from this that the smallest example of a regular but not strongly regular ring is $M_2(\mathbb{F}_2)$ which has order $2^4 = 16$.

Let R be a ring and let $x \in R$ be given. We say that x is a *left π-regular* element if there exist an integer $n > 0$ and an element $y \in R$ such that $x^n = yx^{n+1}$. A *right π-regular* element is defined analogously. If every element of R is left (right) π-regular, then R is said to be *left (right) π-regular.* An element of R is *strongly π-regular* if it is both left and right π-regular. R is *strongly π-regular* if every element of R is strongly π-regular.

The following useful characterization of strongly π-regular elements has been obtained by Azumaya [Az].

Theorem 2.2.5. *If an element x in a ring R is strongly π-regular, then there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$.*

Remark 2.2.1. We note by Theorem 2.2.5 that if x is a strongly π-regular element in a ring R, there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. We have from this that

$$\begin{aligned} x^n &= x(x^{n+1}y)y = x^{n+2}y^2 = x^2(x^{n+1}y)y^2 = x^{n+3}y^3 = \cdots = x^{2n}y^n \\ &= x^n y^n x^n. \end{aligned}$$

A ring R is *π-regular* if for every $a \in R$ there exist a positive integer n and an element $x \in R$ such that $a^n = a^n x a^n$. Clearly, regular rings are π-regular.

By Remark 2.2.1 it is also clear that strongly π-regular rings are π-regular. The converse of this is obviously true for commutative rings.

Theorem 2.2.6. *Let R be a ring with bounded index. The following conditions are equivalent.*

(i) *R is π-regular;*

(ii) *R is right π-regular;*

(iii) *R is left π-regular;*

(iv) *R is strongly π-regular.*

Proof. See Theorem 5 of [Az]. □

Very little was known about connections between right π-regular, left π-regular and strongly π-regular rings until a 1976 paper by Dischinger [Di] where he proved the following:

Theorem 2.2.7 (Dischinger, [Di]). *Every right π-regular ring is left π-regular.*

Proof. See Theorem 1 of [Di]. □

By using arguments analogous to the proof of Theorem 1 in [Di], it can be shown that left π-regular rings are also right π-regular. Hence right π-regular, left π-regular and strongly π-regular are all equivalent conditions on a ring.

As mentioned in Chapter 1, strongly π-regular rings generalize artinian rings. Indeed, suppose that R is a ring which is not strongly π-regular. Then there exists an element $x \in R$ such that given any $y \in R$, $x^n \neq x^{n+1}y$ for any positive integer n. Thus, the descending chain

$$xR \supseteq x^2R \supseteq \cdots \supseteq x^nR \supseteq x^{n+1}R \supseteq \dots$$

of right ideals of R does not terminate which implies that R is not artinian.

It is clear by definition that strongly regular rings are strongly π-regular. Then since strongly π-regular rings are π-regular, so are strongly regular rings. As for regular rings, a π-regular ring R is said to be *abelian* provided that all idempotents in R are central.

Proposition 2.2.8. *Let R be an abelian π-regular ring. Then R is strongly π-regular.*

Proof. Let $x \in R$. Since R is π-regular, there exist $y \in R$ and a positive integer n such that $x^n = x^n y x^n$. Let $e = x^n y$. Note that

$$e^2 = (x^n y x^n) y = x^n y = e.$$

Thus $e \in Id(R)$ and hence,

$$x^n = e x^n = x^n e = x^n (x^n y) = x^{n+1}(x^{n-1} y).$$

It follows that R is right π-regular and therefore strongly π-regular by Theorem 2.2.7. □

We next look at unit regular rings which were introduced by Ehrlich [Eh]. An element x in a ring R is said to be *unit regular* if there exists an invertible element $u \in R$ such that $xux = x$. A ring R is *unit regular* provided that every element in R is unit regular. Unit regular rings are nice because of various cancellation properties associated with them, one of which is as follows:

Let R be a unit regular ring. Then $A \oplus B \cong A \oplus C$ implies that $B \cong C$ for all finitely generated projective right R-modules A, B and C.

The proof of this fact is given in Theorem 4.5 of [Go]. It is clear that every unit regular ring is regular. A ring R is *directly finite* if $xy = 1$ implies $yx = 1$ for all x, $y \in R$. The following proposition by R. Yue [Y1] shows that a unit regular element can be factored into the product of an idempotent and an invertible element.

Proposition 2.2.9. *Let R be a ring. The following conditions are equivalent:*

(i) *R is unit regular;*

(ii) *Every element of R is the product of an idempotent and an invertible element;*

(iii) *R is a directly finite ring such that for any $0 \neq a \in R$, either a is right invertible or there exist a nontrivial idempotent $e \in R$ and a left regular element $d \in R$ such that $a = ed$.*

Proof. See Remark 8 of [Y1]. $\square$

A ring R is *unit π-regular* provided that, for each $x \in R$, there is an invertible element $u \in R$ and a positive integer n such that $x^n = x^n u x^n$. It is obvious that every unit π-regular ring is π-regular. It will be shown in Chapter 3 (see Proposition 3.3.3) that a strongly π-regular ring is unit π-regular.

Proposition 2.2.10. *Every abelian regular ring is unit regular.*

Proof. See Corollary 4.2 of [Go]. $\square$

Since strongly regular rings are abelian regular, it follows readily from Proposition 2.2.10 that every strongly regular ring is unit regular. A different proof of this result will be given in Chapter 4 (see Corollary 4.3.5). A strongly π-regular ring is not necessarily unit regular as shown in the following example. However, a strongly π-regular regular ring is unit regular as has been shown by Goodearl and Menal [GM] (see also [Sh]).

Example 2.2.1: Let R be the ring of 3×3 upper triangular matrices over the field $\mathbb{F}_2$. Since R is a finite dimensional algebra, so it is strongly π-regular. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in R.$$

Note that $AXA = 0$ for all $X \in R$. It follows that A is not regular and hence, not unit regular.

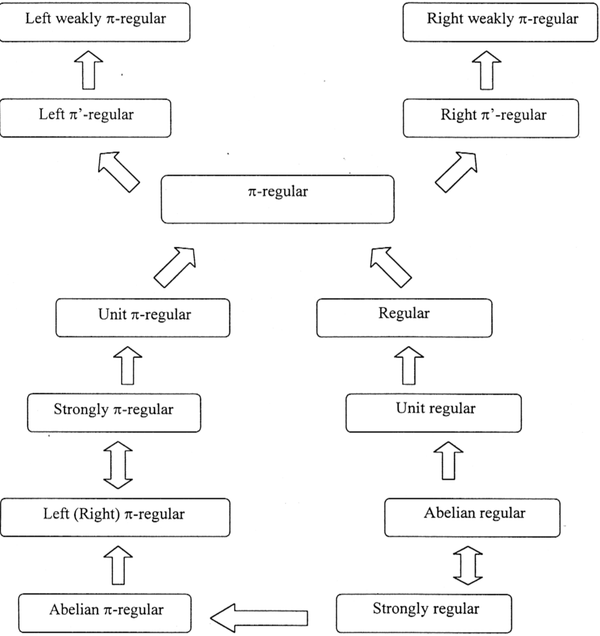
Next we consider weakly π-regular rings. A ring R is called *right (left) weakly π-regular* if for every $x \in R$ there exists a positive integer $n = n(x)$, depending on x, such that $x^n \in (x^n R)^2$ ($x^n \in (Rx^n)^2$). R is *weakly π-regular* if it is both left and right weakly π-regular. These definitions can be found in [Gu] and [T2].

In [Hi], a ring R is defined to be *right (left) π'-regular* if for each $x \in R$ there exists a positive integer n such that $x^n = x^n y x^n z$ ($x^n = z x^n y x^n$) for some $y, z \in R$. It is obvious that every π-regular ring is right and left π'-regular.

Proposition 2.2.11. *Every right (left) π'-regular ring is right (left) weakly π-regular.*

Proof. Let R be a right π'-regular ring. Then for any $x \in R$, there exist a positive integer n and elements $y, z \in R$ such that $x^n = x^n y x^n z \in (x^n R)^2$. Thus, R is right weakly π-regular. The proof for the case R is left π'-regular follows analogously. $\square$

We close this section by illustrating how different types of "regular" rings are related to one another in Figure 1.



Left weakly π-regular
Right weakly π-regular
Left π'-regular
Right π'-regular
π-regular
Unit π-regular
Regular
Strongly π-regular
Unit regular
Left (Right) π-regular
Abelian regular
Abelian π-regular
Strongly regular

FIGURE 1

2.3 Some equivalent conditions.

In this section we shall look at situations where various "regular" rings are equivalent. By Theorem 2.2.4, Proposition 2.2.10 and the fact that unit regular rings are regular, we obtain the following:

Proposition 2.3.1. *Let R be a ring such that every idempotent in R is central. Then the following conditions are equivalent:*

(i) *R is strongly regular;*

(ii) *R is unit regular;*

(iii) *R is regular.*

By definition and using some of the results given in the previous section, we have

Proposition 2.3.2. *Let R be a commutative ring. Then the following conditions are equivalent:*

(i) *R is strongly π-regular;*

(ii) *R is unit π-regular;*

(iii) *R is π-regular;*

(iv) *R is right (left) π'-regular;*

(v) *R is right (left) weakly π-regular.*

A ring R is said to be *right (left) duo* if every right (left) ideal of R is two-sided. R is said to be *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided. If R is both left and right quasi-duo, then we say that R is *quasi-duo*. It is known and will be shown in Chapter 4 (see Proposition 4.3.9) that a strongly regular ring is left and right duo. For quasi-duo rings, we have the following:

Theorem 2.3.3. *Let R be a left quasi-duo ring. Then the following statements are equivalent:*

(i) *R is left weakly π-regular;*

(ii) *R is strongly π-regular.*

Proof. It suffices to show that (i) $\Rightarrow$ (ii). Suppose that R is left weakly π-regular. Then for any $a \in R$ there exists a positive integer n such that $Ra^n = Ra^nRa^n$. We claim that $Ra + l_R(a^n) = R$. If this does not occur, then $Ra + l_R(a^n) \subseteq M$ for some maximal left ideal M of R. Then $Ra^n \subseteq Ra + l_R(a^n) \subseteq M$ and hence, $Ra^n = Ra^nRa^n \subseteq MRa^n \subseteq Ma^n$. The reverse inclusion $Ma^n \subseteq Ra^n$ is clear. Thus, $Ra^n = Ma^n$ and hence, $a^n = ma^n$ for some $m \in M$. It follows that $(1-m)a^n = 0$ and so $1-m \in l_R(a^n) \subseteq M$. But this implies that $1 \in M$ which is a contradiction. Thus $Ra + l_R(a^n) = R$ and so, a is left π-regular. Consequently, R is left π-regular and therefore strongly π-regular. $\square$

The analogue of Theorem 2.3.3 for right quasi-duo rings has in fact been shown in [HKKL, Theorem 7]. We thus have the following:

Theorem 2.3.4. *Let R be a quasi-duo ring. Then the following conditions are equivalent:*

(i) *R is strongly π-regular;*

(ii) *R is unit π-regular;*

(iii) *R is π-regular;*

(iv) *R is right (left) π'-regular;*

(v) *R is right (left) weakly π-regular.*

The proof of the following result will be given in Chapter 4 (see Theorem

4.3.6).

Theorem 2.3.5. *Let R be a ring. Then the following statements are equivalent:*

(i) *R is reduced and strongly π-regular;*

(ii) *R is strongly regular.*

In the next theorem we shall see that if R is a commutative reduced ring, then all the different "regular" conditions on R as given in Figure 1 are equivalent.

Theorem 2.3.6. *Let R be a commutative ring. If R is reduced, then the following conditions are equivalent:*

(i) *R is regular;*

(ii) *R is unit regular;*

(iii) *R is strongly regular;*

(iv) *R is strongly π-regular;*

(v) *R is right (left) π-regular;*

(vi) *R is unit π-regular;*

(vii) *R is π-regular;*

(viii) *R is right (left) π'-regular;*

(ix) *R is right (left) weakly π-regular.*

Proof. By Propositions 2.2.10, 2.3.2 and Theorems 2.2.4, 2.3.5. □

Let $K[z_1, \ldots, z_n]$ be the polynomial ring over the field K in the noncommuting variables $z_1, \ldots, z_n$. An algebra R over K is said to satisfy a polynomial identity (PI) if there exists $f(z_1, \ldots, z_n) \in K[z_1, \ldots, z_n]$, $f \neq 0$ with $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in R$.

Theorem 2.3.7. *If R is a PI-ring then the following conditions are equivalent:*

(i) *R is strongly π-regular;*

(ii) *R is π-regular;*

(iii) *$R/P(R)$ is π-regular;*

(iv) *$R/J(R)$ is π-regular and $J(R)$ is nil;*

(v) *R is weakly π-regular;*

(vi) *R is right (left) weakly π-regular;*

(vii) *$R/J(R)$ is right weakly π-regular and $J(R)$ is nil;*

(viii) *Every prime factor ring of R is right (left) weakly π-regular;*

(ix) *Every prime ideal of R is maximal;*

(x) *Every prime factor ring of R is simple artinian.*

Proof. See Corollary 18 of [HKKL]. □

2.4 On matrix rings over "regular" rings.

For a ring R, we use $M_n(R)$ to denote the ring of $n \times n$ matrices over R. It is known that the ring of all $n \times n$ matrices over a regular ring is regular (see [Go, Theorem 1.7]). However, the ring of $n \times n$ matrices over a strongly regular ring is not necessarily strongly regular as shown in the following example:

Example 2.4.1: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{F}_2)$. Clearly, $\mathbb{F}_2$ is a strongly regular ring. Since $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $A \neq A^2 X$ for any $X \in M_2(\mathbb{F}_2)$. Hence, $M_2(\mathbb{F}_2)$ is not strongly regular.

It has been shown by Tominaga [T1] that if R is π-regular and $\mathrm{Nil}(R)$ is of bounded index, then $M_n(R)$ is a strongly π-regular ring for each $n \geq 1$. Since strongly π-regular rings are π-regular, we easily have the following:

Proposition 2.4.1. *If R is a strongly π-regular ring and Nil(R) is of bounded index, then $M_n(R)$ is a strongly π-regular ring for each $n \geq 1$.*

It has been conjectured that the ring of $n \times n$ matrices over a strongly π-regular ring is strongly π-regular (see [Sh, p.9]). To date, not much progress has been made towards proving this conjecture. It is however not difficult to show explicitly that the ring of 2×2 matrices over a field is strongly π-regular as in the following:

Let $\mathbb{F}$ be a field. Clearly, if $A \in M_2(\mathbb{F})$ is nonsingular, then A is strongly π-regular. We are thus left with the task of showing that singular matrices are strongly π-regular. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{F})$ with $det(A) = ad - bc = 0$.

Case I: $a \neq -d$. Since $a + d \in \mathbb{F} \setminus \{0\}$, so $(a+d)^{-1}$ exists. Let $X = \begin{bmatrix} (a+d)^{-1} & 0 \\ 0 & (a+d)^{-1} \end{bmatrix} \in M_2(\mathbb{F})$. Note that

$$\begin{aligned} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{bmatrix} \begin{bmatrix} (a+d)^{-1} & 0 \\ 0 & (a+d)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & cb+d^2 \end{bmatrix} \begin{bmatrix} (a+d)^{-1} & 0 \\ 0 & (a+d)^{-1} \end{bmatrix} \\ &= A^2 X. \end{aligned}$$

Thus A is strongly regular and hence, strongly π-regular.

Case II: $a = -d$. Note that

$$A^2 = \begin{bmatrix} -d & b \\ c & d \end{bmatrix} \begin{bmatrix} -d & b \\ c & d \end{bmatrix} = \begin{bmatrix} d^2+bc & 0 \\ 0 & bc+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since A is a nilpotent element, so A is strongly π-regular.

Since a full matrix ring over a division ring is artinian and artinian rings are strongly π-regular, we in fact have

Proposition 2.4.2. *A full matrix ring over a division ring is strongly π-regular.*

Let S be a commutative ring. A ring R is said to be an *S-algebra* if R is an S-module and

$$(\alpha a)b = \alpha(ab) = a(\alpha b)$$

for all $a, b \in R$, $\alpha \in S$. It is known that if $\mathbb{F}$ is a field, then the matrix ring $M_n(\mathbb{F})$ is an $\mathbb{F}$-algebra.

Lemma 2.4.3. *Let $\mathbb{F}$ be a field and R a finite-dimensional $\mathbb{F}$-algebra. Then R is both right and left artinian (hence, both right and left noetherian).*

Proof. See Lemma 12.17 of [Is]. $\square$

Proposition 2.4.4. *Let $\mathbb{F}$ be a field and let $n > 0$ be an integer. Then $R = \{X \in M_n(\mathbb{F}) \mid X$ is upper triangular$\}$ and $S = \{X \in M_n(\mathbb{F}) \mid X$ is lower triangular$\}$ are strongly π-regular.*

Proof. Note that R and S are both finite-dimensional $\mathbb{F}$-algebras. By Lemma 2.4.3, it follows that R and S are strongly π-regular. $\square$

We note that every strongly π-regular ring is directly finite. Indeed, let R be a strongly π-regular ring and let $x, y \in R$ such that $xy = 1$. Since R is strongly π-regular, then there exist $z \in R$ and a positive integer n such that $y^n = y^{n+1}z$ and $yz = zy$. Then $xy^n = xy^{n+1}z$ and hence, $y^{n-1} = y^n z$. It follows that $xy^{n-1} = xy^n z$ and therefore, $y^{n-2} = y^{n-1}z$. By continuing in the same way, we finally obtain $1 = yz$. Thus,

$$yx = (yx)1 = (yx)(yz) = (yxy)z = yz = 1.$$

Therefore R is directly finite. The following proposition adds weight to the truth of the conjecture that the ring of $n \times n$ matrices over a strongly π-regular ring R is strongly π-regular.

Proposition 2.4.5. *Every full matrix ring over a strongly π-regular regular ring is directly finite.*

Proof. See Corollary 4.4 of [Sh]. $\square$