CHAPTER 3

ON STRONGLY π-REGULAR RINGS

3.1 Introduction.

The main purpose of this chapter is to study strongly π-regular rings; in particular, conditions which are necessary and sufficient for a ring to be strongly π-regular. We have seen in Theorem 2.2.5 that if x is a strongly π-regular element in a ring R, then there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. Shirley [Sh] has shown that elements in a strongly π-regular ring can be factored into the product of two commuting elements, one of which is invertible and the other a near idempotent. By using this result, we shall obtain several properties of strongly π-regular elements and rings.

We begin in Section 2 of this chapter by studying some properties of strongly π-regular rings. In Section 3, we are concerned with equivalent conditions for strongly π-regular rings. In particular, a proof of the result of Shirley [Sh] mentioned in the preceding paragraph shall be given. We also offer another characterization of strongly π-regular rings, that is, we prove that every element in a strongly π-regular ring can be written as a sum of two commuting elements, one of which is strongly regular and the other nilpotent (Theorem 3.3.5). This characterization, to the best of our knowledge, is not available in the literature. In Section 4, we determine necessary and sufficient conditions for a ring with central idempotents to be strongly π-regular. Some of the results in this section overlap with those of Badawi in [B2]. The proofs given here are however different from those in [B2].

Next in Section 5, we obtain necessary and sufficient conditions for a group ring to be strongly π-regular. In particular, we show that for a ring R and a group G, the group ring RG is strongly π-regular if and only if $(R/P)G$ is strongly π-regular for every prime ideal P of R.

Finally, in Section 6 we study Euler and exact-Euler rings. Badawi [B3] has shown that a ring R is Euler if and only if R is strongly π-regular and $U(R)$ is a torsion group. He also showed that R is exact-Euler if and only if R is strongly π-regular and $\text{Nil}(R)$, $U(R)$ are of bounded index. We shall provide different proofs of these results in Section 6.

3.2 Some properties of strongly π-regular rings.

Let R be a ring. We say that an ideal I of R is *strongly π-regular* provided that for each $x \in I$, there exist a positive integer n and an element $y \in I$ such that $x^n = x^{n+1}y$ and $xy = yx$.

In the first proposition of this section, we show that strongly π-regular is a property inherited by ideals.

Proposition 3.2.1. *Let R be a strongly π-regular ring. If I is an ideal of R, then I is strongly π-regular.*

Proof. Let $x \in I$. Since R is strongly π-regular, then by Theorem 2.2.5 there exist $y \in R$ and a positive integer n such that $x^n = x^{n+1}y$ and $xy = yx$. If $n = 1$, let $z = xy^2$. Then $z \in I$, $xz = zx$ and $x = x^2z$. If $n \geq 2$, let $z = x^{n-1}y^n$. Then $z \in I$, $xz = zx$ and

$$x^n = x^{n+1}y = \cdots = x^{n+n}y^n = x^{n+1}(x^{n-1}y^n) = x^{n+1}z.$$

Thus I is strongly π-regular. $\square$

A subring of a strongly π-regular ring is however not necessarily strongly π-regular. For example, $\mathbb{Z} \leq \mathbb{Q}$ and $\mathbb{Q}$ is strongly π-regular but $\mathbb{Z}$ is not strongly π-regular.

Proposition 3.2.2. *Let $J \leq K$ be ideals in a commutative ring R. Then K is strongly π-regular if and only if J and K/J are both strongly π-regular.*

Proof. If K is strongly π-regular, then obviously K/J is strongly π-regular. Since J is an ideal of R (hence of K) and K is strongly π-regular, so is J (by Proposition 3.2.1). Conversely, assume that J and K/J are both strongly π-regular. Given any $x \in K$, we have $x^n + J = x^{n+1}y + J$ for some positive integer n and some $y \in K$. Thus $x^n - x^{n+1}y \in J$ and consequently,

$$(x^n - x^{n+1}y)^m = (x^n - x^{n+1}y)^{m+1}z$$

for some positive integer m and some $z \in J \leq K$. Then

$$x^{nm}(1 - xy)^m = x^{n(m+1)}(1 - xy)^{m+1}z$$

and it follows that

$$\begin{aligned} & x^{nm}[1 - m(xy) + \cdots + (-1)^m(xy)^m] \\ = \; & x^{n(m+1)}[1 - (m+1)(xy) + \cdots + (-1)^{m+1}(xy)^{m+1}]z. \end{aligned}$$

Thus

$$\begin{aligned} x^{nm} = \; & x^{nm+1}[my - \cdots - (-1)^m y(xy)^{m-1}] \\ & + x^{nm+1}[x^{n-1}(1 - (m+1)(xy) + \cdots + (-1)^{m+1}(xy)^{m+1})]z \end{aligned}$$

and hence,

$$x^{nm} = x^{nm+1}[my - \cdots - (-1)^m y(xy)^{m-1} + x^{n-1}(1 - (m+1)(xy) + \cdots + (-1)^{m+1}(xy)^{m+1})z].$$

Therefore K is strongly π-regular. $\square$

Corollary 3.2.3. *Let R be a commutative ring and J an ideal in R. If J and R/J are both strongly π-regular, then R is strongly π-regular.*

Proposition 3.2.4. *Any finite subdirect product of a commutative strongly π-regular ring is strongly π-regular.*

Proof. It suffices to consider the case of a ring R which is a subdirect product of two strongly π-regular rings. In this case, R has ideals J and K such that $J \cap K = \{0\}$ and R/J and R/K are both strongly π-regular. Observe that

$$J \cong J/(J \cap K) \cong (J + K)/K \leq R/K.$$

Since R/K is strongly π-regular, it follows from Proposition 3.2.1 that $(J+K)/K$ is strongly π-regular and hence, so is J. Then since R/J is strongly π-regular, so is R according to Corollary 3.2.3. □

By Theorem 2.3.6, we have that if R is a commutative reduced ring, then every unit regular element of R is strongly π-regular. In the next proposition we show that this result is still true even if R is not reduced.

Proposition 3.2.5. *Let R be a commutative ring. If R is unit regular then R is strongly π-regular.*

Proof. Let $x \in R$. Since R is unit regular, there exists an invertible element $u \in R$ such that $xux = x$. Note that

$$\begin{aligned} x^n &= (xux)^n \\ &= x^n u^n x^n \\ &= x^{2n} u^n \\ &= x^{n+1}(x^{n-1}u^n) \\ &= x^{n+1} z \end{aligned}$$

where $z = x^{n-1}u^n \in R$. Therefore x is strongly π-regular and it follows that R is strongly π-regular. □

In [FS1, Theorem 2.1], Fisher and Snider showed that a ring R is strongly π-regular if each prime factor of R is strongly π-regular. The converse of this is true from the following proposition.

Proposition 3.2.6. *Every homomorphic image of a strongly π-regular ring is strongly π-regular.*

Proof. Let R, S be rings and $f : R \to S$ a (ring) epimorphism. Suppose that R is strongly π-regular and let $v \in S$ be arbitrary. Since f is epi, there exists an element $x \in R$ such that $f(x) = v$. Then since R is strongly π-regular, there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. It follows that

$$v^n = f(x^n) = f(x^{n+1}y) = f(x)^{n+1}f(y) = v^{n+1}f(y)$$

and hence, S is also strongly π-regular. □

We thus have

Theorem 3.2.7. *Let R be a ring. Then R is strongly π-regular if and only if R/P is strongly π-regular for each prime ideal P of R.*

It has been shown in [Go, Theorem 1.14] that the center of a regular ring is regular. We now show that the same can be said about centers of strongly π-regular rings.

Theorem 3.2.8. *The center of a strongly π-regular ring is strongly π-regular.*

Proof. Let R be a strongly π-regular ring with center $Z(R)$, and let $x \in Z(R)$. Then there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$

and $xy = yx$. Let $z = x^n y^{n+1}$. Then

$$\begin{aligned} x^n &= x^{n+1}y = x^{n+2}y^2 = \cdots = x^{2n}y^n = x^{2n+1}y^{n+1} = x^{n+1}(x^n y^{n+1}) \\ &= x^{n+1}z \end{aligned}$$

and $xz = zx$. Now given any $r \in R$, we have

$$\begin{aligned} zr &= (x^n y^{n+1})r \\ &= y^{n+1} r x^n \\ &= y^{n+1} r (x^{2n+1} y^{n+1}) \\ &= (x^{2n+1} y^{n+1}) r y^{n+1} \\ &= x^n r y^{n+1} \\ &= r x^n y^{n+1} \\ &= rz. \end{aligned}$$

Thus $z \in Z(R)$ and hence, $Z(R)$ is strongly π-regular. □

Let R be a ring and let $r \in R$. An element $s \in R$ is said to be a *semi-inverse* of r if $r = r^2 s$, $s = s^2 r$ and $rs = sr$. It may be shown that if $r \in R$ has a semi-inverse then its semi-inverse is unique.

Proposition 3.2.9. *Let R be a strongly π-regular ring and $x \in R$. Then*

(i) *there exist a positive integer n and an element $z \in R$ such that $x^n z^n$ is an idempotent;*

(ii) *there exist a positive integer n and an element $z \in R$ such that $x(1-x^n z^n)$ is nilpotent;*

(iii) *x^n has a semi-inverse for some positive integer n.*

Proof. Since R is strongly π-regular, it follows from Theorem 2.2.5 that there exist an element $z \in R$ and a positive integer n such that $x^n = x^{n+1}z$ and

$xz = zx$. Then

$$x^n = x^{n+1}z = x^{n+2}z^2 = \cdots = x^{2n}z^n$$

and hence,

$$(x^n z^n)^2 = (x^{2n}z^n)z^n = x^n z^n.$$

Thus $x^n z^n$ is an idempotent which proves (i). Since $x^n z^n$ is an idempotent, so is $1 - x^n z^n$. Therefore

$$\begin{aligned}[x(1 - x^n z^n)]^n &= x^n(1 - x^n z^n)^n \\ &= x^n(1 - x^n z^n) \\ &= x^n - x^{2n}z^n = 0,\end{aligned}$$

which proves (ii). For the proof of (iii), we note that

$$\begin{aligned}x^{2n}(z^{2n}x^n) &= (x^{2n}z^n)(z^n x^n) \\ &= x^n z^n x^n \\ &= x^{2n}z^n \\ &= x^n\end{aligned}$$

and

$$\begin{aligned}(z^{2n}x^n)^2 x^n &= z^{2n}x^n(z^{2n}x^{2n}) \\ &= z^{2n}x^n(x^{2n}z^n)z^n \\ &= z^{2n}x^n(x^n z^n) \\ &= z^{2n}(x^{2n}z^n) \\ &= z^{2n}x^n.\end{aligned}$$

Thus x^n has a semi-inverse. □

Proposition 3.2.10. *Let R be a ring with no zero divisors. If R is strongly π-regular and x is a nonzero element in R, then*

(i) $xy = 0$ *if and only if* $yx = 0$ *for any* $y \in R$;

(ii) x *is invertible.*

Proof. (i) Suppose that $xy = 0$. Then

$$(yx)^2 = y(xy)x = y0x = 0.$$

Since R has no zero divisors, this implies that $yx = 0$. The same argument works conversely.

(ii) By Theorem 2.2.5, there exist $y \in R$ and a positive integer n such that $x^n = x^{n+1}y$ and $xy = yx$. Note that

$$x^n(1 - xy) = x^n - x^{n+1}y = 0.$$

Since R has no zero divisors and $x \neq 0$, it follows that $1 - xy = 0$. Thus $xy = 1$ and since $yx = xy$, we also have $yx = 1$. Therefore x is invertible, as asserted. □

By Proposition 3.2.10 we readily have

Corollary 3.2.11. *A strongly π-regular ring with no zero divisors is a division ring.*

Proposition 3.2.12. *Let R be a ring and $x \in R$. If x is strongly π-regular, then there exists an idempotent $e \in R$ such that $x^n = ex^n$ is strongly regular for some positive integer n. In particular, there exist an idempotent $e \in R$ and a positive integer n such that $x^n = ex^n$ is strongly π-regular.*

Proof. Since x is strongly π-regular, it follows by Theorem 2.2.5 that there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$.

Let $e = x^n y^n$. Then

$$e^2 = (x^n y^n x^n) y^n = x^n y^n = e \in Id(R)$$

and

$$e x^n = x^n y^n x^n = x^{2n} y^n = x^n.$$

Note that

$$(e x^n)^2 y^n = e^2 x^{2n} y^n = e x^n.$$

Therefore, $x^n = e x^n$ is strongly regular. Since strongly regular elements are strongly π-regular, the last assertion follows easily. □

In the next proposition, we will show that the converse of Proposition 3.2.12 is true.

Proposition 3.2.13. *Let R be a ring and $x \in R$. If there exists a positive integer n such that x^n is strongly regular, then x is strongly π-regular.*

Proof. Since x^n is strongly regular, it follows from Theorem 2.2.2 that there exists $y \in R$ such that $x^n = x^{2n} y$ and $x^n y = y x^n$. Then

$$x^n = x^{n+1}(x^{n-1} y) = (y x^{n-1}) x^{n+1};$$

hence x is strongly π-regular. □

As a consequence of Propositions 3.2.12 and 3.2.13 we have the following corollary.

Corollary 3.2.14. *Let R be a ring and $x \in R$. An element x is strongly π-regular if and only if x^n is strongly regular for some positive integer n.*

3.3 Some equivalent conditions for strongly π-regular rings.

We begin this section with some necessary and sufficient conditions for a ring R to be strongly π-regular as obtained by Shirley [Sh].

Theorem 3.3.1. *Let R be a ring. Then R is strongly π-regular if and only if for every element $x \in R$, there exist elements u, $g \in R$ such that*

(i) *u is invertible;*

(ii) *g is a near idempotent;*

(iii) *$ug = gu = x$.*

Proof. Suppose that R is a strongly π-regular ring and let $x \in R$. By Theorem 2.2.5, there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. Note that

$$x^n = x^{n+1}y = x^{n+2}y^2 = \cdots = x^{2n}y^n = x^n y^n x^n.$$

Let $e = x^n y^n$. It is clear that $e \in Id(R)$ and the elements x, y and e commute with each other. Let $u = xe + (1-e)$ and $g = e + x(1-e)$. Then we have

$$\begin{aligned} ug = gu &= [e + x(1-e)][xe + (1-e)] \\ &= exe + x(1-e)^2 \\ &= xe + x - xe = x. \end{aligned}$$

Note that

$$\begin{aligned} g^n &= [e + x(1-e)]^n \\ &= e^n + x^n(1-e)^n \\ &= e + x^n(1-e) \\ &= e + x^n - x^n e = e \in \mathrm{Id}(R). \end{aligned}$$

Thus, g is a near idempotent. To show that u is invertible, let $c = ye + (1 - e)$. Since

$$\begin{aligned} uc = cu &= [ye + (1-e)][xe + (1-e)] \\ &= yexe + (1-e)^2 \\ &= yxe + 1 - e \\ &= yx(x^n y^n) + 1 - e \\ &= (x^{n+1}y)y^n + 1 - e \\ &= x^n y^n + 1 - e \\ &= e + 1 - e = 1, \end{aligned}$$

so u is invertible.

Conversely, let $x \in R$. Then there exist an invertible element $u \in R$ and a near idempotent $g \in R$ such that $x = ug = gu$. Then $g^{2n} = g^n$ for some positive integer n and therefore,

$$x^n = (gu)^n = g^n u^n = g^{2n}u^{2n}u^{-n} = x^{2n}u^{-n} = x^{n+1}(x^{n-1}u^{-n}).$$

It follows that R is strongly π-regular. □

As an immediate consequence of Theorem 3.3.1 we have

Corollary 3.3.2. *Let R be a strongly π-regular ring. Then every principal right (left) ideal of R is generated by a near idempotent.*

By using Theorem 3.3.1 we also obtain the following two results:

Proposition 3.3.3. *A strongly π-regular ring is unit π-regular.*

Proof. Let R be a strongly π-regular ring and let $x \in R$. By Theorem 3.3.1, there exist a near idempotent $e \in R$ and an invertible element $u \in R$ such that

$x = eu = ue$. Since e is a near idempotent, there exists a positive integer n such that $e^{2n} = e^n$. Note that

$$x^n = (eu)^n = e^n u^n.$$

Then

$$\begin{aligned} x^n u^{-n} x^n &= e^n u^n (u^{-n})(e^n u^n) \\ &= e^{2n} u^n \\ &= e^n u^n \\ &= x^n. \end{aligned}$$

Therefore, x is unit π-regular. □

Corollary 3.3.4. *Let R be a ring. If x is a strongly π-regular element of R, then there exists a positive integer n such that $x^n = eu = ue$ for some $e \in Id(R)$ and $u \in U(R)$.*

Proof. Let $x \in R$ be a strongly π-regular element. By Theorem 3.3.1, there exist a near idempotent $g \in R$ and $v \in U(R)$ such that $x = gv = vg$. Since g is a near idempotent, there exists a positive integer n such that $g^n = g^{2n}$. Let $e = g^n$ and $u = v^n$. Then $e^2 = e \in Id(R)$, $u \in U(R)$ and

$$x^n = g^n v^n = v^n g^n = eu = ue. \qquad \square$$

In the following theorem, we offer another characterization of strongly π-regular rings.

Theorem 3.3.5. *Let R be a ring. Then the following conditions are equivalent:*

(i) *R is strongly π-regular;*

(ii) *Every $x \in R$ can be written in the form $x = a + w$ where a, $w \in R$ such that a is strongly regular, w is nilpotent and $aw = wa$.*

Proof. Suppose first that R is strongly π-regular and let $x \in R$. Then by Theorem 2.2.5, there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. Since

$$x^n = x^{n+1}y = x^{n+2}y^2 = \cdots = x^{2n}y^n,$$

we have that

$$\begin{aligned}(x^{n+1}y^n)^2(x^{n-1}y^n) &= x^{3n+1}y^{3n} \\ &= x^{n+1}(x^{2n}y^n)y^{2n} \\ &= x^{2n+1}y^{2n} \\ &= x(x^{2n}y^n)y^n \\ &= x^{n+1}y^n;\end{aligned}$$

that is, $x^{n+1}y^n$ is strongly regular. Moreover, since x^ny^n is an idempotent, we also have

$$(x - x^{n+1}y^n)^n = x^n(1 - x^ny^n)^n = x^n(1 - x^ny^n) = 0.$$

Thus, $x - x^{n+1}y^n$ is nilpotent. It is clear that $x^{n+1}y^n(x - x^{n+1}y^n) = (x - x^{n+1}y^n)x^{n+1}y^n$. Then since $x = x^{n+1}y^n + (x - x^{n+1}y^n)$, (ii) follows.

Conversely, let $x \in R$ and suppose that $x = a + w$ where a, $w \in R$ such that a is strongly regular, w is nilpotent and $aw = wa$. By Theorem 2.2.2, there exists $b \in R$ such that $a^2b = a$, $ab = ba$ and $bw = wb$. Since

$$x - x^2b = (a + w) - (a + w)^2b = w - w(2a + w)b$$

is nilpotent and $xb = bx$, it follows that $x^n = x^{n+1}z$ for some $z \in R$ with $xz = zx$. Thus x is strongly π-regular. □

Remark 3.3.1. We note that the decomposition in Theorem 3.3.5 is not unique. Indeed, consider the ring $M_2(\mathbb{F}_2)$ of 2×2 matrices over the field $\mathbb{F}_2$. We may write $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in $M_2(\mathbb{F}_2)$ as

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are strongly regular and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are nilpotent.

3.4 Strongly π-regular rings with central idempotents.

In this section, we obtain necessary and sufficient conditions for a ring with central idempotents to be strongly π-regular. First we show the following:

Theorem 3.4.1. *Let R be a strongly π-regular ring with central idempotents. Then Nil(R) is a two-sided ideal of R.*

Proof. Let $z \in \text{Nil}(R)$ and $r \in R$. If $z = 0$, then clearly $rz = zr = 0 \in \text{Nil}(R)$. Assume now that $z \neq 0$. Suppose that $rz \notin \text{Nil}(R)$. Since R is strongly π-regular, we have

$$rz = fu = uf$$

for some near idempotent f and some invertible element u in R. Let n be a positive integer such that f^n is an idempotent and let $e = f^n$. Note that

$$eu^n = f^n u^n = (fu)^n = (rz)^n \neq 0. \tag{3.4.1}$$

Since e is central, ez is a nilpotent element of R. Let m be the smallest positive integer such that $(ez)^m = 0$. Note that $ez \neq 0$. Indeed, if $ez = 0$ we would have

$$0 = (r(ez))^n = e(rz)^n = eu^n$$

which contradicts (3.4.1). Therefore, $m \geq 2$. Now since $n \geq 1$ and $(ez)^m = 0$, so

$$\begin{aligned} 0 &= (r(ez))^n(ez)^{m-1} \\ &= e(rz)^n(ez)^{m-1} \\ &= eu^n(ez)^{m-1} \\ &= u^n(ez)^{m-1}. \end{aligned}$$

But since u^n is invertible in R, it follows that $(ez)^{m-1} = 0$; contradicting the fact that m is the smallest positive integer such that $(ez)^m = 0$. Therefore we must have $rz \in \mathrm{Nil}(R)$. Similarly, it can be shown that $zr \in \mathrm{Nil}(R)$.

Next let $x, y \in \mathrm{Nil}(R)$. Suppose that $x + y \notin \mathrm{Nil}(R)$. Since R is strongly π-regular, there is a positive integer n such that

$$(x+y)^n = gv = vg \tag{3.4.2}$$

for some idempotent g and some invertible element v in R. Since $x+y \notin \mathrm{Nil}(R)$, so g cannot be nilpotent. By (3.4.2) we have that

$$g(x+y)^{n-1}x = gv(1 - v^{-1}(x+y)^{n-1}y). \tag{3.4.3}$$

From the preceding paragraph we know that $v^{-1}(x+y)^{n-1}y \in \mathrm{Nil}(R)$. We thus have that $1 - v^{-1}(x+y)^{n-1}y$ is invertible. Then since g is central and not nilpotent, the element $gv(1 - v^{-1}(x+y)^{n-1}y)$ cannot be nilpotent. But this contradicts (3.4.3) since $g(x+y)^{n-1}x \in \mathrm{Nil}(R)$. Hence we must have $x + y \in \mathrm{Nil}(R)$. This completes the proof that $\mathrm{Nil}(R)$ is a two-sided ideal of R. □

As a consequence of Theorem 3.4.1, we have

Corollary 3.4.2. *Let R be a strongly π-regular ring with central idempotents. Then $J(R) = \mathrm{Nil}(R)$.*

Proof. Let $x \in J(R)$. Since R is strongly π-regular, there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. Then $x^n(1-xy) = 0$

and since $1 - xy$ has a right inverse in R, it follows that $x^n = 0$. Therefore x is nilpotent and hence, $J(R) \subseteq \text{Nil}(R)$. The reverse inclusion follows from the fact that $\text{Nil}(R)$ is a two-sided ideal of R (by Theorem 3.4.1) and that $J(R)$ contains all nil ideals of R. The equality $J(R) = \text{Nil}(R)$ thus follows. □

We also note the following property:

Proposition 3.4.3. *If R is a strongly π-regular ring with central idempotents, then $R/Nil(R)$ is strongly regular (hence, regular).*

Proof. Let $x+\text{Nil}(R) \in R/\text{Nil}(R)$. Then by Theorem 2.2.5, there exist $y \in R$ and a positive integer n such that $x^{n+1}y = x^n$ and $xy = yx$. Thus $e = x^n y^n \in Id(R)$ and hence, $1 - e \in Id(R)$. Then since

$$x^n = x^{2n}y^n = x^n e = ex^n,$$

it follows that

$$[(1-e)x]^n = (1-e)x^n = 0 \in \text{Nil}(R).$$

Hence $(1-e)x \in \text{Nil}(R)$ and therefore

$$\begin{aligned} x + \text{Nil}(R) &= ex + \text{Nil}(R) = x^{n+1}y^n + \text{Nil}(R) \\ &= [x + \text{Nil}(R)]^2[x^{n-1}y^n + \text{Nil}(R)]. \end{aligned}$$

It follows that $R/\text{Nil}(R)$ is strongly regular (hence, regular). □

To prove the main result in this section we shall need the following lemmas:

Lemma 3.4.4. *Let R be a ring and I a two-sided nil ideal of R. Then idempotents of R/I can be lifted to R.*

Proof. See [La, p.72]. □

Lemma 3.4.5. *A reduced strongly π-regular ring R is strongly regular (hence, regular).*

Proof. Let $x \in R$. By Theorem 3.3.1, there exist a near idempotent $g \in R$ and an invertible element $u \in U(R)$ such that $x = gu = ug$. Since g is a near idempotent, there exists a positive integer n such that $g^n = g^{2n}$. Note that

$$x^n = (gu)^n = g^n u^n = g^{2n} u^n = g^n (g^n u^n) = g^n x^n$$

and $xg = gx$. Thus, $((1 - g^n)x)^n = 0$. Since R is reduced,

$$x = xg^n = x^2 (u^{-1} g^{n-1}).$$

It follows that R is strongly regular (hence, regular). □

The main result in this section is as follows:

Theorem 3.4.6. *Let R be a ring with central idempotents. Then R is strongly π-regular if and only if Nil(R) is a two-sided ideal of R and R/Nil(R) is regular.*

Proof. Suppose that R is strongly π-regular. By Theorem 3.4.1, it follows readily that $\mathrm{Nil}(R)$ is a two-sided ideal of R. Since $R/\mathrm{Nil}(R)$ is reduced strongly π-regular, it follows from Lemma 3.4.5 that $R/\mathrm{Nil}(R)$ is strongly regular and hence, regular.

To prove the sufficiency part, we let $x \in R$. Since $R/\mathrm{Nil}(R)$ is regular, there exists some $y \in R$ such that $\bar{x}\bar{y}\bar{x} = \bar{x}$ where $\bar{x} = x + \mathrm{Nil}(R)$ and $\bar{y} = y + \mathrm{Nil}(R)$. Clearly, $(\bar{y}\bar{x})^2 = \bar{y}\bar{x}$. By Lemma 3.4.4 there is an idempotent $e \in R$ such that $\bar{e} = \bar{y}\bar{x}$, that is $e - yx \in \mathrm{Nil}(R)$. Thus there exists an integer $m \geq 1$ such that $(e - yx)^m = 0$. Since e is central, $e = zx$ for some $z \in R$. Now $\bar{x} = \bar{x}\bar{y}\bar{x} = \bar{x}\bar{e}$ gives us $x - xe \in \mathrm{Nil}(R)$. Hence there exists some integer $n \geq 1$ with $0 = (x - xe)^n = x^n - x^n e$. Therefore $x^n = x^n e = ex^n = zx^{n+1}$. Thus R is left π-regular and hence, strongly π-regular. □

Remark 3.4.1. It has been shown by Ohori [Oh, Theorem 2(2)] that in an abelian π-regular ring R, $\mathrm{Nil}(R)$ is a two-sided ideal of R and $R/\mathrm{Nil}(R)$ is regular. Since abelian π-regular rings are strongly π-regular with central idempotents and vice versa, Theorem 3.4.6 in this section shows that the converse of Ohori's result is also true. It is also necessary to mention here that Theorems 3.4.1 and 3.4.6 in this section can be found for abelian π-regular rings in [B2, Theorems 2 and 3]. The proofs given in this section are however independent from those in [B2].

We end this section with another characterization of strongly π-regular rings with central idempotents.

Theorem 3.4.7. *Let R be a strongly π-regular ring with central idempotents and let P be a prime ideal of R. Then every element of R/P is nilpotent or invertible.*

Proof. Let $x \in R$, $x \notin P$. Since x is strongly π-regular, we may write

$$x = fu = uf$$

for some near idempotent f and some invertible element u in R. Let n be a positive integer such that $e = f^n$ is an idempotent. Then $x^n = eu^n = u^n e$. Since $(1-e)Re = \{0\} \in P$ and P is a prime ideal, it follows that $e \in P$ or $1 - e \in P$. If $e \in P$, then $x^n = eu^n \in P$; hence, $x + P$ is nilpotent. If $1 - e \in P$, then

$$x^n + P = eu^n + P = (e + P)(u^n + P) = u^n + P$$

is invertible in R/P. It follows that $x + P$ is also invertible in R/P. $\square$

3.5 Strongly π-regular group rings.

In this section we study conditions which are necessary or sufficient for a group ring to be strongly π-regular. Most of the results in this section will appear in [CC]. The main result in this section is as follows:

Theorem 3.5.1. *Let R be a ring and G a group. If $(R/P)G$ is strongly π-regular for every prime ideal P of R, then RG is strongly π-regular.*

Proof. Suppose to the contrary that RG is not strongly π-regular. Then there exists an element $x \in RG$ such that for any positive integer n, $x^n \neq x^{n+1}y$ for any $y \in RG$. Therefore the sequence

$$xRG \supseteq x^2RG \supseteq \cdots \supseteq x^nRG \supseteq x^{n+1}RG \supseteq \ldots$$

of right ideals of RG does not terminate. Let $\mathcal{F}$ be the set of all ideals I of R such that the sequence

$$(x+IG)(RG/IG) \supseteq (x+IG)^2(RG/IG) \supseteq \ldots$$

does not terminate. Note that $\mathcal{F} \neq \emptyset$ since $\{0\} \in \mathcal{F}$. Furthermore, $\mathcal{F}$ is partially ordered by inclusion. Let $(I_\alpha)_{\alpha\in\Omega}$ be a chain of elements of $\mathcal{F}$ and let $J = \cup_{\alpha\in\Omega} I_\alpha$. Clearly, J is an ideal of R and $I_\alpha \subseteq J$ for all $\alpha \in \Omega$. We show that $J \in \mathcal{F}$. Suppose that $J \notin \mathcal{F}$. Then

$$z = x^n - x^{n+1}r \in JG$$

for some $r \in RG$ and some positive integer n. Since Supp (z) is finite, there exists some $\alpha \in \Omega$ such that $z \in I_\alpha G$. It follows that the sequence

$$(x+I_\alpha G)(RG/I_\alpha G) \supseteq (x+I_\alpha G)^2(RG/I_\alpha G) \supseteq \ldots$$

terminates, which is a contradiction. Therefore $J \in \mathcal{F}$ and thus by Zorn's Lemma, $\mathcal{F}$ contains a maximal element M. Since

$$(R/M)G \cong RG/MG$$

is not strongly π-regular, it follows by the hypothesis that M is not a prime ideal. Therefore there exist ideals A, B of R such that

$$AB \subseteq M$$

but $A, B \not\subseteq M$. Let $A' = M + A$ and $B' = M + B$. Then M is strictly contained in A' and B', and we also have that

$$A'B' = (M + A)(M + B) \subseteq M.$$

By the maximality of M in $\mathcal{F}$, the sequences

$$(x + A'G)(RG/A'G) \supseteq (x + A'G)^2(RG/A'G) \supseteq \dots$$

and

$$(x + B'G)(RG/B'G) \supseteq (x + B'G)^2(RG/B'G) \supseteq \dots$$

both terminate. Hence there exists a positive integer m such that

$$(x^m + A'G)(RG/A'G) = (x^{2m+1} + A'G)(RG/A'G)$$

and

$$(x^m + B'G)(RG/B'G) = (x^{2m+1} + B'G)(RG/B'G).$$

It follows that

$$x^m - x^{2m+1}s \in A'G$$

and

$$x^m - x^{2m+1}t \in B'G$$

for some s, $t \in RG$. Therefore

$$(x^m - x^{2m+1}s)(x^m - x^{2m+1}t) \in (A'B')G \subseteq MG$$

from which it follows that

$$x^{2m} - x^{2m+1}w \in MG$$

for some $w \in RG$. Hence the sequence

$$(x + MG)(RG/MG) \supseteq (x + MG)^2(RG/MG) \supseteq \dots$$

terminates; contradicting the fact that $M \in \mathcal{F}$. We thus have that RG must be a strongly π-regular ring. □

By Proposition 1.5.2 and Theorem 3.5.1, we readily have

Corollary 3.5.2. *Let R be a ring and G a group. Then RG is strongly π-regular if and only if $(R/P)G$ is strongly π-regular for every prime ideal P of R.*

We now obtain other sufficient conditions for a group ring to be strongly π-regular as follows:

Theorem 3.5.3. *Let R be a ring with artinian prime factors and let G be a locally finite group. Then RG is strongly π-regular.*

Proof. Let P be a prime ideal of R and $x = \sum_{g \in G} r_g g \in (R/P)G$. Let H_x be the subgroup of G generated by the support of x. Since Supp(x) is finite and G is locally finite, it follows that H_x is finite. It is clear that $x \in (R/P)H_x$. We note that $(R/P)H_x$ is strongly π-regular. Indeed, since R/P is artinian and H_x is finite, so $(R/P)H_x$ is artinian (by Theorem 1.5.4); hence, strongly π-regular. Since x is arbitrary in $(R/P)G$, so $(R/P)G$ is strongly π-regular. By Theorem 3.5.1, it follows that RG is strongly π-regular. □

A natural question to ask is whether the converse of Theorem 3.5.3 is true. We show that it is partially true if G is abelian (see Proposition 3.5.5). First we prove the following:

Proposition 3.5.4. *Let R be a ring and G a group. If RG is strongly π-regular, then R is strongly π-regular and G is torsion.*

Proof. From the (ring) isomorphism $R \cong RG/\Delta$, we have that R is strongly π-regular. To show that G is torsion, let $g \in G$, $g \neq 1$. Consider the element $1 - g \in RG$. Note that $1 - g$ does not have a right or left inverse in RG since $1 \notin \Delta$.

Now since RG is strongly π-regular, it follows from Theorem 2.2.5 that there exist a positive integer n and an element $r \in RG$ such that

$$(1 - g)^n = (1 - g)^{n+1} r$$

and

$$(1 - g)r = r(1 - g).$$

If $1 - g$ is nilpotent, then $1 - g$ is a zero divisor and hence by Proposition 6 in [Co], g has finite order. Suppose that $1 - g$ is not nilpotent. Then since

$$(1 - g)^n [1 - (1 - g)r] = 0$$

and $1 - (1 - g)r \neq 0$, so $1 - g$ must also be a zero divisor. Hence, by Proposition 6 in [Co] again we have that g has finite order. □

Since torsion abelian groups are locally finite, we easily have the following from Proposition 3.5.4:

Proposition 3.5.5. *Let R be a ring and G an abelian group. If RG is strongly π-regular, then R is strongly π-regular and G is locally finite.*

We end this section by giving sufficient conditions for a group ring to be unit regular as follows:

Theorem 3.5.6. *Let R be a unit regular ring with all prime factors artinian and let G be a locally finite group such that the order of every finite subgroup of G is a unit in R. Then the group ring RG is unit regular.*

Proof. By Theorem 3.5.3, RG is strongly π-regular. Since unit regular rings are regular, we have from Theorem 1.5.3 that RG is regular. Then since a strongly π-regular regular ring is unit regular, we have the desired result. □

As a partial converse to Theorem 3.5.6 we have the following:

Proposition 3.5.7. *Let R be a ring and G a group. If RG is unit regular, then*

(i) *R is unit regular,*

(ii) *G is locally finite,*

(iii) *the order of every finite subgroup of G is a unit in R.*

Proof. Since a unit regular ring is regular, conditions (ii) and (iii) follow readily from Theorem 1.5.3. Condition (i) follows from the fact that

$$R \cong RG/\Delta$$

and homomorphic images of unit regular rings are unit regular. □

3.6 Euler and exact-Euler rings.

Following Badawi [B3], a ring R is called *Euler* if for every $x \in R$ there exists a positive integer n such that $x^n \in Id(R)$, that is, every element of R is a near idempotent. R is called an *exact-Euler* ring if there exists a positive integer n such that $x^n \in Id(R)$ for every $x \in R$. Clearly, an exact-Euler ring is Euler and as we shall show below, an Euler ring is strongly π-regular. Necessary and sufficient conditions for a ring to be Euler or exact-Euler have been considered by

Badawi [B3]. In this section, we provide different proofs for some of the results in [B3] which we state here in Theorems 3.6.1 and 3.6.3. It is clear that if R is a ring, the set of units $U(R)$ forms a multiplicative group. We begin with the following characterization of Euler rings:

Theorem 3.6.1 (Badawi, [B3]). *A ring R is Euler if and only if R is strongly π-regular and $U(R)$ is a torsion group.*

Proof. Suppose that R is Euler. Let $u \in U(R)$. Then there exists a positive integer n such that $u^{2n} = u^n$. Note that

$$u^n = u^{2n-n} = u^{2n}u^{-n} = u^n u^{-n} = 1.$$

Since u is arbitrary in $U(R)$, we obtain that $U(R)$ is torsion. Now let $x \in R$. Then $y = x^n \in Id(R)$ for some positive integer n. Clearly, $x^{2n}y = x^n$ and $xy = yx$. It follows that $x^n = x^{n+1}(x^{n-1}y) = (yx^{n-1})x^{n+1}$ and hence, R is strongly π-regular.

Conversely, suppose that R is strongly π-regular and $U(R)$ is torsion. Let $x \in R$. By Corollary 3.3.4, there exists a positive integer n such that $x^n = ue = eu$ for some $e \in Id(R)$ and $u \in U(R)$. Since $U(R)$ is a torsion group, there exists a positive integer m such that $u^m = 1$. Then

$$x^{nm} = u^m e^m = e^m = e \in Id(R).$$

Thus, R is Euler. □

Proposition 3.6.2. *A subring of an Euler ring is also Euler.*

Proof. Let R be an Euler ring and S a subring of R. For any $x \in S \leq R$, there exists a positive integer n such that $x^n \in Id(R)$. But $x^n \in S$ since S is a subring of R. Hence, $x^n \in Id(S)$ and it follows that S is also Euler. □

We next consider exact-Euler rings.

Theorem 3.6.3 (Badawi, [B3]). *A ring R is exact-Euler if and only if R is strongly π-regular and Nil(R), $U(R)$ are of bounded index.*

Proof. Suppose first that R is exact-Euler. Then R is Euler and it follows readily from Theorem 3.6.1 that R is strongly π-regular. Let $u \in U(R)$ and $x \in \text{Nil}(R)$. Since R is exact-Euler, there is a (fixed) positive integer n such that u^n, $x^n \in Id(R)$. Then

$$u^n = u^{2n}u^{-n} = u^n u^{-n} = 1.$$

Since u is arbitrary in $U(R)$, it follows that $U(R)$ is torsion. Let m be the smallest positive integer such that $x^m = 0$. Clearly, $m \leq n$. Hence, Nil(R) is of bounded index.

Conversely, suppose that R is strongly π-regular and Nil(R), $U(R)$ are of bounded index w, m, respectively. Let $x \in R$. Then there exist a positive integer n and an element $y \in R$ which commutes with x such that $x^n = x^{n+1}y$. Thus, $x^n = x^{2n}y$ and hence,

$$\begin{aligned} x^{n+k} &= x^{2n+k}y = x^{n+k}(x^{n+1}y)y = x^{2n+k+1}y^2 = x^{n+k+1}(x^{n+1}y)y^2 = x^{2n+k+2}y^3 \\ &= \cdots = x^{2(n+k)}y^{k+1} \end{aligned}$$

for any positive integer k. We may thus assume that $x^n = x^{2n}y$ for $n > w$. Since $(x^n y)^2 = x^{2n}y^2 = x^n y$, we have $x^n y \in Id(R)$ and hence, so is $1 - x^n y$. Note that $[x(1 - x^n y)]^n = x^n(1 - x^n y) = 0$. Thus, $[x(1 - x^n y)]^w = 0$ which gives us $x^w(1 - x^n y) = 0$. It follows that $x^w = x^{n+w}y = x^{2w}(x^{n-w}y)$; that is, x^w is strongly regular. As shall be shown in Theorem 4.3.1, $x^w = eu = ue$ for some $e \in Id(R)$ and $u \in U(R)$. Thus, $x^{wm} = e^{wm}u^{wm} = e \in Id(R)$. Since x arbitrary in R, this shows that R is exact-Euler. □

By using arguments similar to those in the proof of Proposition 3.6.2, the following result can be proven.

Proposition 3.6.4. *A subring of an exact-Euler ring is also exact-Euler.*

We have seen in Section 2 that a subring of a strongly π-regular ring R is not necessarily strongly π-regular. However, if in addition $U(R)$ is torsion, we have the following:

Corollary 3.6.5. *Let R be a strongly π-regular ring with $U(R)$ torsion. Then any subring of R is also strongly π-regular.*

Proof. Let S be a subring of R. Since R is Euler (by Theorem 3.6.1), it follows from Proposition 3.6.2 that S is also Euler. Hence, S is strongly π-regular by Theorem 3.6.1. $\square$