CHAPTER 5

(s,2)-RINGS AND RINGS WITH STABLE RANGE ONE

5.1 Introduction.

In this final chapter we shall study (s,2)-rings and rings with stable range one. In particular, we are interested in how these rings are related to various 'regular' rings. For convenience in this chapter, we shall refer to all invertible elements as just units.

In Section 2 we shall study conditions under which strongly regular and strongly π-regular rings are (s,2)-rings. We shall also study some related rings such as semicommutative rings, (s,2)-π-rings and exact-(s,2)-π-rings.

In Section 3 we study some important features of rings with stable range one. We shall see that every strongly π-regular ring has stable range one (Theorem 5.3.5). We also give a proof different from the ones available in the literature that a regular ring has stable range one if and only if it is unit regular.

5.2 On (s,2)-rings.

A ring R is said to be an *(s,2)-ring* (see [FS2] or [He]) if every element in R is a sum of two units of R. It has been shown by Ehrlich [Eh] that if R is a unit regular ring and 2 is a unit in R, then R is an (s,2)-ring. As a consequence of this result, we make the following observations:

Proposition 5.2.1. *Let R be a strongly regular ring. If 2 is a unit in R, then R is an (s,2)-ring.*

Proof. This is obvious since every strongly regular ring is unit regular. □

Proposition 5.2.2. *Let R be a strongly π-regular regular ring. If 2 is a unit in R, then R is an (s,2)-ring.*

Proof. This is also obvious since every strongly π-regular regular ring is unit regular (see [GM, Theorem 5.8]). □

A natural question to ask is whether a ring R is an (s,2)-ring if R is only strongly π-regular. Fisher and Snider [FS2] showed the following:

Theorem 5.2.3. *Let R be a strongly π-regular ring. If 2 is a unit in R, then R is an (s,2)-ring.*

Proof. See Theorem 3 of [FS2]. □

As an immediate consequence, we have

Corollary 5.2.4. *Let R be a ring with prime factors artinian. If 2 is a unit in R, then R is an (s,2)-ring.*

Proof. This follows easily from Theorem 5.2.3 and the fact that rings with prime factors artinian are strongly π-regular (see [FS1, Theorem 2.1]). □

Note that since every abelian π-regular ring is strongly π-regular, we also have the following result:

Corollary 5.2.5. *Suppose that R is an abelian π-regular ring and 2 is a unit of R. Then R is an (s,2)-ring.*

In the case where R is abelian π-regular and 2 is a nonnilpotent element of R, we have the following result of A. Badawi [B2].

Theorem 5.2.6. *Suppose that R is abelian π-regular and 2 is a nonnilpotent element of R. Then there exists $e \in Id(R)$ such that $e \neq 0$ and every element in eR is a sum of two units of R.*

Proof. See Theorem 7 of [B2]. □

Before continuing, we pause to consider the following two propositions.

Proposition 5.2.7. *Let R be a ring. If R has a strongly π-regular element which is a sum of two units, then there exists a near idempotent $e \in R$ which is also a sum of two units.*

Proof. Let x be a strongly π-regular element of R which can be written as a sum of two units. By Theorem 3.3.1, there exist a near idempotent $e \in R$ and an invertible element $u \in R$ such that $x = eu$. By the hypothesis, $x = v + w$ for some v, $w \in U(R)$. It follows that

$$e = (v + w)u^{-1} = vu^{-1} + wu^{-1}$$

is a sum of two units. □

Proposition 5.2.8. *Let R be a ring. If every near idempotent of R is a sum of two units, then so is every strongly π-regular element of R.*

Proof. Let x be a strongly π-regular element of R. Then by Theorem 3.3.1, $x = eu$ for some near idempotent $e \in R$ and some $u \in U(R)$. Since $e = v + w$ for some v, $w \in U(R)$, it follows that

$$x = eu = (v + w)u = vu + wu$$

is a sum of two units in R. □

A natural question to ask is whether subrings of (s,2)-rings are (s,2)-rings. The following result tells us when this is true.

Proposition 5.2.9. *Let R be an (s,2)-ring and let S be a subring of R. If $U(R) \leq S$, then S is an (s,2)-ring.*

Proof. Let $x \in S$. Then there exist $u, v \in U(R)$ such that $x = u + v$. Since $U(R) \leq S$, it follows that S is an (s,2)-ring. $\square$

Proposition 5.2.10. *Let R be a regular (s,2)-ring and suppose that $U(R)$ is a subring of R. Then R is unit regular.*

Proof. Let $x \in R$. Since R is regular, there exists $y \in R$ such that $x = xyx$. Since R is (s,2), we have $y = u + v$ for some $u, v \in U(R)$. Then since $U(R) \leq R$, it follows that $y = u + v \in U(R)$. Therefore, R is unit regular. $\square$

Since every unit regular ring is regular, the following corollary is obvious.

Corollary 5.2.11. *Let R be an (s,2)-ring and suppose that $U(R)$ is a subring of R. Then the following conditions are equivalent:*

(i) *R is regular;*

(ii) *R is unit regular.*

As mentioned earlier in this section, Ehrlich [Eh] has shown that a ring R is an (s,2)-ring if R is unit regular and 2 is a unit in R. In the following proposition we show that the converse of this is true if $U(R) \leq R$.

Proposition 5.2.12. *Let R be a ring and suppose that $U(R)$ is a subring of R. Then R is an (s,2)-ring if and only if R is unit regular and 2 is a unit of R.*

Proof. It suffices to show the necessity part. Suppose that R is an (s,2)-ring. Given any $x \in R$, there exist $u, v \in U(R)$ such that $x = u + v$. Since $U(R) \leq R$, so $x \in U(R)$ and hence, x^{-1} exists. Clearly, $xx^{-1}x = x$ and therefore R is unit regular. Moreover, $2 = 1 + 1 \in U(R)$. $\square$

Proposition 5.2.13. *Let R be a ring. If $U(R)$ is a subring of R, then the following conditions are equivalent:*

(i) *R is strongly regular;*

(ii) *R is strongly π-regular;*

(iii) *R is an (s,2)-ring.*

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): By Theorem 3 of [FS2].

(iii) $\Rightarrow$ (i): Let $x \in R$. Since R is an (s,2)-ring, there exist $u, v \in U(R)$ such that $x = u + v$. Then since $U(R) \leq R$, it follows that $x \in U(R)$ and hence, x^{-1} exists. Clearly, $x = x^2x^{-1} = x^{-1}x^2$. Thus, x is strongly regular. □

Let R be a ring and $a \in R$. If for every $b \in R$, there exist $r, s \in R$ such that $ab = ra$ and $ba = as$, then a is called *semicommutative.* If every element of R is semicommutative, then R is called a semicommutative ring. In the case of semicommutative rings, the following results are known:

Theorem 5.2.14. *Let R be a semicommutative ring. A π-regular ring R is an (s,2)-ring if and only if every element in $Id(R)$ is a sum of two units of R.*

Proof. See Theorem 4 of [B1]. □

Corollary 5.2.15. *Let R be a semicommutative π-regular ring. If $2 \in U(R)$, then R is an (s,2)-ring.*

Proof. See Corollary 2 of [B1]. □

We say that a ring R is an *(s,2)-π-ring* if for each element x in R, there is a positive integer $n \geq 1$ such that x^n is a sum of two units of R. If there is a fixed positive integer $n \geq 1$ such that x^n is a sum of two units of R for each x in R, then R is called an *exact-(s,2)-π-ring.*

Theorem 5.2.16 (Badawi, [B3]).

(i) *Let R be a strongly π-regular ring. Then R is an (s,2)-π-ring if and only if every element in $Id(R)$ is a sum of two units of R. In particular, if $2 \in U(R)$, then R is an (s,2)-π-ring.*

(ii) *Let R be an exact-Euler ring. Then R is an exact-(s,2)-π-ring if and only if each element in $Id(R)$ is a sum of two units of R. In particular, if $2 \in U(R)$, then R is an exact-(s,2)-π-ring.*

Proof. (i) Let $x \in R$. By Theorem 3.3.1, there exist a unit $u \in R$ and a near idempotent $g \in R$ such that $x = gu = ug$. Let n be a positive integer such that $g^n = g^{2n}$ and let $e = g^n$. Then $e \in Id(R)$ and $x^n = eu^n$. Therefore, R is an (s,2)-π-ring if each $e \in Id(R)$ is a sum of two units of R. It is clear by definition that if R is an (s,2)-π-ring, then each $e \in Id(R)$ is a sum of two units of R. Next, suppose that $2 \in U(R)$. By Theorem 5.2.3, R is an (s,2)-ring; hence, an (s,2)-π-ring.

(ii) Let R be an exact-Euler ring. Then there exists a positive integer n such that $x^n = e \in Id(R)$ for each $x \in R$. Thus, R is an exact-(s,2)-π-ring if each element in $Id(R)$ is a sum of two units of R. The converse of this is clear by definition of an exact-(s,2)-π-ring. Next, suppose that $2 \in U(R)$. Since R is strongly π-regular (by Theorem 3.6.3), it follows from Theorem 5.2.3 that R is an (s,2)-ring. In particular, every element in $Id(R)$ is a sum of two units of R; hence, R is an exact-(s,2)-π-ring. □

5.3 Rings with stable range one.

A ring R is said to have *stable range one* if for any a, $b \in R$ satisfying $aR+bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. By Vaserstein [V1, Theorem 1], this definition is left-right symmetric.

Theorem 5.3.1 **([V2]).** *If a ring R has stable range one then all one-sided inverses of R are two-sided.*

Proof. See Theorem 2.6 of [V2]. $\square$

Proposition 5.3.2. *A ring R has stable range one if and only if for any a, x, $b \in R$ satisfying $ax + b = 1$, there exists $y \in R$ such that $a + by$ is invertible.*

Proof. Assume that R has stable range one. Let a, x, $b \in R$ satisfy $ax + b = 1$. Then $aR + bR = R$ and by definition, there exists $y \in R$ such that $a + by$ is right invertible. By Theorem 5.3.1, we have that $a + by$ is left invertible. The converse is obvious. $\square$

One of the most important features of the stable range one condition is the cancellation of related modules from direct sums. Evans [Ev, Theorem 2] proved that if the endomorphism ring of a module M has stable range one, then $M \oplus A \cong M \oplus B$ implies that $A \cong B$. The converse of this is however not necessarily true as shown by taking $M = \mathbb{Z}$.

It has been shown in [Fu] and [Go] that a regular ring has stable range one if and only if it is unit regular. By using Proposition 2.2.9 we now provide another proof of this result.

Proposition 5.3.3. *A regular ring R has stable range one if and only if it is unit regular.*

Proof. First, assume that R has stable range one. Given any $a \in R$, there exists $x \in R$ such that $axa = a$. Clearly, $ax + (1 - ax) = 1$. By the assumption on R and by Proposition 5.3.2, there exists $y \in R$ such that $u = a + (1 - ax)y$ is invertible.

Therefore, $axu = ax[a + (1 - ax)y] = axa = a$. It follows that $ax = au^{-1}$ from which we have $au^{-1}a = axa = a$.

Conversely, assume that R is unit regular and suppose that $ax + b = 1$ for some a, x, $b \in R$. By Proposition 2.2.9, we may write $a = eu$, $b = gv$ for some idempotents e, $g \in R$ and some units u, $v \in R$. It follows that

$$e(ux + b) + (1 - e)gv = eux + eb + (1 - e)b = ax + b = 1.$$

Since R is regular, there exists $c \in R$ such that $(1 - e)g = (1 - e)gc(1 - e)g$. Let $f = (1 - e)gc(1 - e)$. We then have

$$\begin{aligned} e(ux + b) + fb &= e(ux + b) + (1 - e)gc(1 - e)gv \\ &= 1 - (1 - e)gv + (1 - e)gv = 1. \end{aligned}$$

Note that $0 = feux = fax = f(1 - b)$, that is, $fb = f$. We also have $e = e1 = e(ax + b) = e(ux + b)$. Thus

$$e + f = e(ux + b) + fb = 1.$$

It is clear that $1 + ebv^{-1}c(1 - e)$ is a unit with inverse $1 - ebv^{-1}c(1 - e)$. Since $e+f = 1$, we have that $e+(1-e)gc(1-e) = 1$, that is, $e+(1-e)gvv^{-1}c(1-e) = 1$. But since $b = gv$, we have $e + (1 - e)bv^{-1}c(1 - e) = 1$ and therefore

$$e + bv^{-1}c(1 - e) = 1 + ebv^{-1}c(1 - e).$$

Since $(1 - e)e = 0$, we can write

$$e + bv^{-1}c(1 - e)[1 + ebv^{-1}c(1 - e)] = 1 + ebv^{-1}c(1 - e).$$

Multiplying on the right by u, we then obtain

$$\begin{aligned} a + bv^{-1}c(1 - e)[1 + ebv^{-1}c(1 - e)]u &= eu + bv^{-1}c(1 - e)[1 + ebv^{-1}c(1 - e)]u \\ &= [1 + ebv^{-1}c(1 - e)]u \\ &\in U(R). \end{aligned}$$

It then follows from Proposition 5.3.2 that R has stable range one. $\square$

A ring R is said to have *unit 1-stable range* if for any x, $y \in R$, there is a unit $u \in R$ such that $x - u$ and $y - u^{-1}$ are both units. Unit 1-stable range is a stronger condition than stable range one as shown in the following:

Proposition 5.3.4. *Let R be a ring. If R satisfies unit 1-stable range, then R has stable range one.*

Proof. Let a, x, $b \in R$ with $ax + b = 1$. By the hypothesis, there exist u, v, $w \in U(R)$ such that $x = u + v$ and $a = w + u^{-1}$. Note that

$$\begin{aligned} a + bv^{-1} &= a + (1 - ax)v^{-1} \\ &= w + u^{-1} + [1 - (w + u^{-1})(u + v)]v^{-1} \\ &= w + u^{-1} + [-wu - wv - u^{-1}v]v^{-1} \\ &= -wuv^{-1} \in U(R). \end{aligned}$$

By Proposition 5.3.2, it follows that R has stable range one. $\square$

The converse of Proposition 5.3.4 is not necessarily true. For example, $\mathbb{Z}/2\mathbb{Z}$ has stable range one but does not have unit 1-stable range. Note that $\mathbb{Z}/3\mathbb{Z}$ has both unit 1-stable range and stable range one.

In [GM], Goodearl and Menal showed that an algebraic algebra over an infinite field $\mathbb{F}$ has both stable range one and unit 1-stable range. They also conjectured that any algebraic algebra has stable range one. This conjecture has been proven by P. Ara in [Ar]. Ara [Ar] has in fact proven the following stronger and deeper result:

Theorem 5.3.5. *Strongly π-regular rings have stable range one.*

Prior to this effort by Ara [Ar], several mathematicians have made various attempts to link strongly π-regular rings with rings having stable range one. We list here some of the major results which have been obtained.

A *corner* of a ring R is any (non-unital) subring eRe where e is an idempotent in R.

Theorem 5.3.6. *A strongly π-regular ring R has stable range one if and only if every nilpotent regular element of each corner of R is unit regular in that corner.*

Proof. See Theorem 6.1 of [GM]. □

Proposition 5.3.7. *Let R be a strongly π-regular ring. If every element of R is a sum of a unit and a central unit, then R has stable range one.*

Proof. See Corollary 6.2 of [GM]. □

Proposition 5.3.8 ([Yu]). *Let R be a strongly π-regular ring. If*

(i) *all idempotents of R are central, or*

(ii) *2 is invertible in R and every element of R is a sum of an idempotent and a central unit,*

then R has stable range one.

Corollary 5.3.9 ([CY]). *A strongly π-regular ring R has stable range one if and only if every regular element of R is unit regular in R.*

Theorem 5.3.10 ([CY]). *Let R be a strongly π-regular ring. If all powers of every regular element of R are regular, then R has stable range one.*

A major breakthrough in linking strongly π-regular rings with rings having stable range one came via exchange rings. Let R be a ring and $M = M_R$ be a right R-module. Following Crawley and Johnson [CJ], M_R is said to have the *exchange property* if for every right R-module A_R and any two decompositions of A_R

$$A_R = M' \oplus N = \oplus_{i \in I} A_i$$

where $M'_R \cong M_R$, there exist submodules $A'_i \subseteq A_i$ such that

$$A = M' \oplus (\oplus_{i \in I} A'_i).$$

M_R is said to have the *finite exchange property* if the above condition is satisfied whenever the index I is finite.

The study of exchange rings was pioneered by Warfield in [Wa]. Warfield called a ring R an exchange ring if R_R has the exchange property as above and proved that this definition is left-right symmetric. He also proved that for any right R-module M, M has the finite exchange property if and only if $End_R(M)$ is an exchange ring. Nicholson [Ni] showed that a ring R is an exchange ring if and only if all idempotents of R lift modulo left (right) ideals of R. The exchange property of rings is preserved under taking corners and matrix rings. As shown below, π-regular rings (hence, strongly π-regular and regular) rings are exchange rings.

Proposition 5.3.11 ([St]). *A π-regular ring R is an exchange ring.*

Proof. Let $x \in R$. Then there exist $y \in R$ and a positive integer n such that $x^n = x^n y x^n$. Let $e = yx^n$. Then $e = e^2 = (yx^n yx^{n-1})x \in Rx$ and $x^n = x^n yx^n = x^n e \in Re$. Now,

$$1 - x^n = (1 + x + x^2 + \cdots + x^{n-1})(1 - x) \in R(1 - x).$$

Thus, $1 = x^n + (1 - x^n) \in Re + R(1 - x)$. It follows that $R = Re + R(1 - x)$. Then there exist elements t and s in R such that $1 = te + s(1 - x)$. Let $f = e + (1 - e)te$.

Then $f^2 = f$ and $f = kx$ for some $k \in R$. Note that

$$1 - f = (1 - e) - (1 - e)te = (1 - e)(1 - te) = (1 - e)s(1 - x).$$

Thus,

$$f - x = f(1 - x) - (1 - f)x = kx(1 - x) - (1 - e)s(1 - x)x$$

and hence, $f - x \in R(x - x^2)$. Let I be an arbitrary left ideal of R and suppose that $x - x^2 \in I$. Then $f - x \in R(x - x^2) \subseteq I$. Therefore R is an exchange ring. $\square$

Ara [Ar] made use of the fact that a strongly π-regular ring R is an exchange ring to show that any nilpotent regular element of R is unit regular (see [Ar, Theorem 2]). By using Theorem 5.3.6 he then concluded that R has stable range one.