CHAPTER 1
INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Group rings and group representation theory are two closely related topics with the latter providing much of the motivation for the work done in the former. However, the historical order of development was rather the reverse: some of the earlier theorems in group representations were discovered via interest in the structure of group rings (see [Ha] and [Mi]). In an account on the early history of group rings, Milies expounds that the roots of the notion of group ring should be sought in the theory of hypercomplex systems, a subject developed from the concepts of quaternions by Sir William Rowan Hamilton (see [Mi]). It was at that early stage when most of the basic concepts were not yet formulated that group rings first appeared, albeit implicitly, in a paper by Arthur Cayley [Ca]. The reader may refer to [Mi] for a more detailed account of Cayley’s paper.

In more recent times, the inclusion of group ring problems among Kaplansky’s famous lists of twelve problems in ring theory (1957 [Ka1], 1970 [Ka2]), as well as the inclusion of group ring material in the books of Lambek (1966 [Lam]), Ribenboim (1969 [Ri]) and the monograph of Herstein (1968 [He]) provided the main driving forces for the algebraic study of group rings. A number of books devoted totally on group rings have also since been written (see for example Passman (1971 [Pa1], 1977 [Pa2]), Mikhail and Zalesskii (1973 [MiZ]), Bovdi (1974 [Bo]), Sehgal (1978 [Se]) and Karpilovsky (1983 [Kar1], 1987 [Kar2])).
In this thesis all rings are assumed to be associative with identity 1 ≠ 0 unless stated otherwise and all modules are unitary. For any ring \( R \), an \( R \)-module \( M \), sometimes denoted by \( M_R \), means a right \( R \)-module.

Let \( R \) be a ring and let \( G \) be a multiplicative group. The group ring \( RG \) consists of all formal sums of the form \( \sum_{g \in G} r_g \cdot g \) with \( r_g \in R \), where only finitely many \( r_g \neq 0 \).

Letting the elements of \( R \) commute with those of \( G \), the operations of addition, scalar multiplication and multiplication are defined in \( RG \) as follows:

\[
\sum_{g \in G} r_g \cdot g + \sum_{g \in G} s_g \cdot g = \sum_{g \in G} (r_g + s_g) \cdot g ;
\]
\[
\left( \sum_{g \in G} r_g \cdot g \right) a = \sum_{g \in G} (r_g a) \cdot g , \quad a \in R ;
\]
\[
a \left( \sum_{g \in G} r_g \cdot g \right) = \sum_{g \in G} (ar_g) \cdot g , \quad a \in R ;
\]
\[
\left( \sum_{h \in G} r_h \cdot h \right) \left( \sum_{k \in G} s_k \cdot k \right) = \sum_{g \in G} \left( \sum_{hk = g} r_h s_k \right) \cdot g .
\]

If \( 1_R \) and \( 1_G \) are the identity elements of \( R \) and \( G \) respectively, then \( RG \) has identity element \( 1 = 1_R \cdot 1_G \). The identifications of \( g \in G \) with \( 1_R \cdot g \in RG \) and \( r \in R \) with \( r \cdot 1_G \in RG \) yield a natural embedding of \( G \) and \( R \) into \( RG \). With these identifications, \( r_g \cdot g \) can be written as \( r_g g \).

By routine verification, we know that \( RG \) is a ring. In fact, \( RG \) is a free \( R \)-module with elements of \( G \) as a basis.

In the remaining sections of this chapter, we shall review some basic results on groups and rings, most of which will be needed in succeeding chapters. We also fix notations and terminologies for later use.
1.2 Some basic results on group rings

Let $R$ be a ring and let $G$ be a group. Let $RG$ denote the group ring of $G$ over $R$. For any element $r = \sum_{g \in G} r_g g \in RG$, the support of $r$ is defined by

$$\text{Supp}(r) = \{ g \in G \mid r_g \neq 0 \}.$$ 

Clearly, $\text{Supp}(r)$ is a finite subset of $G$. Note that $r = 0$ if and only if $\text{Supp}(r) = \emptyset$. The support group of $r$, $S.G.(r)$, is the subgroup generated by $\text{Supp}(r)$. The norm of $r$, $\delta(r)$, is defined to be $\delta(r) = \sum_{g \in G} r_g$.

A mapping $\omega$ from the lattice of subgroups of $G$ to the lattice of right ideals of the group ring $RG$ is defined as follows: If $H$ is a subgroup of $G$, then $\omega H$ is the right ideal of $RG$ generated by $\{ 1 - h \mid h \in H \}$. In particular, if $H = G$, then $\omega G = \Delta$ is called the augmentation ideal of $RG$.

Proposition 1.2.1 (Connell, [Co])

Let $R$ be a ring, $G$ a group and $H$ a subgroup of $G$.

(i) If $H$ is generated by $\{ h_i \}$, then $\omega H$ is generated by $\{ 1 - h_i \}$.

(ii) $\omega H$ is a two-sided ideal of $RG$ if and only if $H$ is a normal subgroup of $G$.

Moreover, $R\left(\frac{G}{H}\right) \cong \frac{RG}{\omega H}$ if $H$ is a normal subgroup of $G$. In particular,

$$R \cong \frac{RG}{\Delta}.$$
(iii) If \( J \) is a right ideal of \( R \), then \( JG \) is a right ideal of \( RG \). Conversely, if \( J' \) is a right ideal of \( RG \), then \( J' \cap R \) is a right ideal of \( R \).

(iv) If \( J \) is a two-sided ideal of \( R \), then \( JG \) is a two-sided ideal of \( RG \) and \[ \frac{R}{J}G \cong \frac{RG}{JG} \]

(v) For any element \( r \in RG, \ r \in \Delta \) if and only if \( \delta(r) = 0 \).

1.3 Some results on group theory

Let \( G \) be a group. The order of \( G \) is denoted by \( |G| \). We say that \( G \) is a **torsion** group if every element of \( G \) is of finite order. On the other hand, \( G \) is a **torsion-free** group if no element of \( G \) except the identity is of finite order. A group \( G \) is said to be a **locally finite** group if every finitely generated subgroup of \( G \) is finite. Clearly, a locally finite group is a torsion group while the converse is not always true as has been shown by Golod [Gol]. However, an abelian group is torsion if and only if it is locally finite. A group \( G \) with \( |G| > 1 \) is a **prime** group if it has no finite normal subgroup except the trivial subgroup \( \{1\} \). Thus by definition a prime group must be infinite. Clearly, an abelian group \( G \) is prime if and only if \( G \) is torsion-free.

Let \( \{G_i\}_{i=1}^m \) be a finite collection of groups. By the **external direct product** \( \prod_{i=1}^m G_i \) is meant the group consisting of all \( m \)-tuples \( (g_1, ..., g_m) \), where \( g_i \in G_i, i = 1, ..., m \), with multiplication defined by

\[
(g_1, ..., g_m) \cdot (g'_1, ..., g'_m) = (g_1 g'_1, ..., g_m g'_m).
\]
We say that $G$ is an *infinite cyclic* group if there is an element $g \in G$ of infinite order such that $G$ is generated by $g$. A group $G$ is *free abelian* if $G$ is a direct sum of infinite cyclic groups. Thus a free abelian group is torsion-free. It is well-known that a finite abelian group $G$ is the direct product of a finite number of cyclic groups of prime power order. If $G$ is a finitely generated abelian group of infinite order, then $G = F \times H$, where $F$ is a nontrivial free abelian group and $H$ is the direct product of a finite number of cyclic groups of prime power order.

A group $G$ is said to be an *ordered* group if the elements of $G$ are linearly ordered with respect to a transitive binary relation $\leq$ and for any elements $h_1, h_2, g \in G$, $h_1 \leq h_2$ implies that $h_1 g \leq h_2 g$ and $gh_1 \leq gh_2$. This means that for any elements $h_1, h_2, g_1, g_2 \in G$, if $h_1 \leq h_2$ and $g_1 \leq g_2$, then $h_1 g_1 \leq h_2 g_2$. It is known (see [Karl, p.40]) that every torsion-free abelian group (hence, every prime abelian group) is an ordered group. It follows that every free abelian group is an ordered group.

A group $G$ is said to satisfy the *ascending chain condition*, or a.c.c. for subgroups if every ascending chain of distinct subgroups of $G$, say $H_1 \subsetneq H_2 \subsetneq \ldots \subsetneq H_n \subsetneq \ldots$ is necessarily finite. If every nonempty collection of subgroups of a group $G$ has a maximal member, then $G$ is said to have the *maximum condition* for subgroups. It is well-known that a group $G$ has the a.c.c. for subgroups if and only if $G$ has the maximum condition for subgroups if and only if the group $G$ itself and all its subgroups are finitely generated.
1.4 Direct sums and products

Throughout this section, let $I$ be an index set. Let $\{R_i\}_{i \in I}$ be a family of rings and consider the set $R = \{(r_i)_{i \in I} \mid r_i \in R_i \text{ for all } i \in I\}$. Then $R$ is a ring by the following definitions of addition and multiplication:

\[
(x_i)_{i \in I} + (y_i)_{i \in I} = (x_i + y_i)_{i \in I}, \\
(x_i)_{i \in I} \cdot (y_i)_{i \in I} = (x_i y_i)_{i \in I},
\]

where $x_i, y_i \in R_i$ for all $i \in I$. We shall refer to $R$ as the direct product of the family $\{R_i\}_{i \in I}$ and write $R = \prod_{i \in I} R_i$. The zero element of $R$ is $(0_i)_{i \in I}$, where $0_i$ is the zero element of $R_i$ and the identity element of $R$ is $(1_i)_{i \in I}$, where $1_i$ is the identity element of $R_i$.

We say that $R$ is the external direct sum of $\{R_i\}_{i \in I}$ if

\[
R = \left\{(r_i)_{i \in I} \in \prod_{i \in I} R_i \mid r_i = 0 \text{ for all but finitely many } i \in I \right\}
\]

and in this case we write

\[
R = \bigoplus_{i \in I} R_i. \quad \text{If } I \text{ is a finite set, say } I = \{1, \ldots, n\}, \text{ then } \bigoplus_{i=1}^n R_i = \prod_{i=1}^n R_i.
\]

Let $R$ be a ring and let $\{J_i\}_{i \in I}$ be a collection of right ideals of $R$. We say that $R$ is the internal direct sum of $\{J_i\}_{i \in I}$ and we write $R = \bigoplus_{i \in I} J_i$ if every element $x$ of $R$ can be written uniquely in the form $x = \sum_{i \in I} a_i$, where $a_i \in J_i$ and finitely many $a_i \neq 0$. In other words, $R$ is the internal direct sum of the $J_i$ if and only if $R$ is the sum of the $J_i$ and $x = \sum_{i \in I} a_i = 0$ implies that $a_i = 0$ for all $i \in I$. Equivalently, $R = \bigoplus_{i \in I} J_i$ if and only if
\[ R = \sum_{i \in I} J_i \quad \text{and} \quad J_i \cap \sum_{j \neq i} J_j = \{0\} \quad \text{for all} \quad i \in I. \] In this case, \( J_i \quad (i \in I) \) is said to be a **direct summand** of \( R. \)

Let \( \{M_i\}_{i \in I} \) be a family of \( R \)-modules and consider the set
\[ M = \left\{ (m_i)_{i \in I} \mid m_i \in M_i \text{ for all } i \in I \right\}. \]
Then \( M \) is an \( R \)-module under the following operations:

\[
(x_i)_{i \in I} + (y_i)_{i \in I} = (x_i + y_i)_{i \in I},
\]
\[
(x_i)_{i \in I} \cdot r = (x_i r)_{i \in I},
\]
where \( x_i, y_i \in M_i \) for all \( i \in I \) and \( r \in R \). We say that \( M \) is the **direct product** of the family \( \{M_i\}_{i \in I} \) and write \( M = \prod_{i \in I} M_i \). Now, consider the subset

\[ \oplus_{i \in I} M_i = \left\{ (m_i)_{i \in I} \in \prod_{i \in I} M_i \mid m_i = 0 \text{ for all but finitely many } i \in I \right\}. \]

Then \( \oplus_{i \in I} M_i \) is an \( R \)-submodule of \( \prod_{i \in I} M_i \) and is called the **external direct sum** of \( \{M_i\}_{i \in I} \). Clearly, if \( I \) is a finite set, say \( I = \{1, \ldots, n\} \), then \( \oplus_{i=1}^n M_i = \prod_{i=1}^n M_i \).

**Proposition 1.4.1**

Let \( R_1, \ldots, R_n \) be rings and let \( G \) be a group. Then \( \prod_{i=1}^n R_i G \cong \prod_{i=1}^n R_i G \).

**Proof:** We define \( \phi: \left( \prod_{i=1}^n R_i \right) G \to \prod_{i=1}^n R_i G \) by

\[ \phi\left( \sum_{g \in G} (r_1, \ldots, r_n) g \right) = \left( \sum_{g \in G} r_1 g, \ldots, \sum_{g \in G} r_n g \right), \]
where \( r_i \in R_i, \ i = 1, \ldots , n \). It is straightforward to show that \( \phi \) is a ring isomorphism.

Therefore, \( \left( \prod_{i=1}^{n} R_i \right) G \cong \prod_{i=1}^{n} R_i G \). ■

Proposition 1.4.2

Let \{ \( J_j \) \}_{j \geq 1} \) be a collection of nonzero proper right ideals of a ring \( R \) and let \( G \) be a group. If \( J' = \bigoplus_{j \geq 1} J_j \), then \( J' G = \bigoplus_{j \geq 1} J_j G \).

Proof: Let \( r = \sum_{j \geq 1} J_j G \). Without loss of generality, we may assume that \( r = r_1 + \cdots + r_m \), where \( r_k = \sum_{j=1}^{n_k} a_{k_j} g_j \in J_k G \) with \( g_j \in G \) and \( a_{k_j} \in J_k \) \((1 \leq j \leq n_k, \ 1 \leq k \leq m)\). Let \( n = \max \{ n_1, \ldots , n_m \} \). Let \( b_{k_j} \in J_k \) such that \( b_{k_j} = a_{k_j} \) for \( j = 1, \ldots , n_k \) and \( b_{k_j} = 0 \) for all \( j > n_k, \ k = 1, \ldots , m \). Then \( r = \sum_{j=1}^{n} \left( \sum_{k=1}^{m} b_{k_j} \right) g_j \in J' G \) and so we have \( \sum_{j \geq 1} J_j G \subseteq J' G \).

Moreover, if \( r = \sum_{j=1}^{n} \left( \sum_{k=1}^{m} b_{k_j} \right) g_j = 0 \), then \( \sum_{k=1}^{m} b_{k_j} = 0 \) for all \( j \). By the directness of \( \sum_{j \geq 1} J_j \), \( b_{k_j} = a_{k_j} = 0 \) for all \( k_j \). Thus \( r = \sum_{j=1}^{n} a_{k_j} g_j = 0 \) for all \( k \). Hence, \( \sum_{j \geq 1} J_j G \) is direct.

For the reverse inclusion, let \( r' \in J' G \). Then \( r' = r_1 g_1 + \cdots + r_m g_m \) for some \( r_k = \sum_{j=1}^{n_k} a_{k_j} \in J' \) and \( g_k \in G \), where \( a_{k_j} \in J_j \) \((1 \leq j \leq n_k, \ 1 \leq k \leq m)\) and \( a_{k_j} \notin J_j \) for \( i \neq j \). Let \( n = \max \{ n_1, \ldots , n_m \} \). Write \( r_k = \sum_{j=1}^{n_k} a_{k_j} \) with \( a_{k_j} = 0 \) for all \( j > n_k \),
$k = 1, \ldots, m$. We then have \( r' = \sum_{k=1}^{m} a_{k_1}g_k + \cdots + \sum_{k=1}^{m} a_{k_n}g_k \) with \( \sum_{k=1}^{m} a_{k_j}g_k \in J_jG, \)
\[ j = 1, \ldots, n. \] Thus \( r' \in \bigoplus_{i \in I} J_iG. \) The equality \( J'G = \bigoplus_{i \in I} J_iG \) then follows.

By using arguments similar to those in the proof of Proposition 1.4.2, the following result can be proven:

**Proposition 1.4.3**

Let \( \{ R_i \}_{i \in I} \) be a collection of rings and let \( G \) be a group. Let \( J_i \) be a nontrivial right ideal of \( R_i \), \( i \geq 1 \). Then \( \left( \bigoplus_{i \in I} J_i \right) G \cong \bigoplus_{i \in I} J_iG. \)

### 1.5 Idempotents and annihilators

Let \( R \) be a ring. An element \( e \in R \) is said to be an idempotent if \( e^2 = e \). Let \( a \in R \). The right annihilator of \( a \), written \( a' \), is defined as \( \{ x \in R \mid ax = 0 \} \) while the left annihilator of \( a \), written \( a' \), is defined as \( \{ x \in R \mid xa = 0 \} \). For any nonempty subset \( B \subseteq R \), we define the right annihilator of \( B \) to be \( B' = \{ x \in R \mid Bx = \{0\} \} \) and the left annihilator of \( B \) to be \( B' = \{ x \in R \mid xB = \{0\} \} \). With these definitions, \( a' \) and \( B' \) are right ideals of \( R \) while \( a' \) and \( B' \) are left ideals of \( R \). If \( R \) is commutative, we denote the annihilator of \( a \) as \( a^* \) and the annihilator of \( B \) as \( B^* \). A right ideal \( A \) of \( R \) is said to be a right annihilator ideal if \( A = B' \) for some \( B \subseteq R \). In a similar manner, a left ideal \( A \) of \( R \)
is a left annihilator ideal if $A = B'$ for some $B \subseteq R$. We abbreviate $(B')'$ as $B''$ and similarly $(B')^\prime$ as $B''$. 

For any right ideals $I_1$ and $I_2$ of $R$, one easily establishes that $(I_1 + I_2)' = I_1' \cap I_2'$. For if $x \in I_1' \cap I_2'$, then $a_i x = a_2 x = 0$ for all $a_i \in I_i$, $i = 1, 2$. So we have $(a_1 + a_2)x = a_1 x + a_2 x = 0$, which implies that $x \in (I_1 + I_2)'$. Thus $I_1' \cap I_2' \subseteq (I_1 + I_2)'$.

For the reverse inclusion, note that since $I_1 \subseteq (I_1 + I_2)$ and $I_2 \subseteq (I_1 + I_2)$, so we have $I_1' \supseteq (I_1 + I_2)'$ and $I_2' \supseteq (I_1 + I_2)'$, respectively. Hence, $(I_1 + I_2)' \subseteq I_1' \cap I_2'$ and the equality $(I_1 + I_2)' = I_1' \cap I_2'$ follows. We note however that for any elements $a_1, a_2 \in R$, the containment $a_1' \cap a_2' \subseteq (a_1 + a_2)'$ always holds but it is not true in general that $a_1' \cap a_2' = (a_1 + a_2)'$. Analogous results hold for left annihilators.

Proposition 1.5.1 (Translation Lemma)

Let $R$ be a ring. If $I$, $J$ are right ideals of $R$ and $a \in R$ such that $a' \cap (I + J) = \{0\}$, then $a(I \cap J) = aI \cap aJ$.

Proof: Clearly $a(I \cap J) \subseteq aI \cap aJ$. For the other containment, suppose that $aI \cap aJ \not\subseteq a(I \cap J)$. Then there exists an element $x \in aI \cap aJ$ with $x \not\in a(I \cap J)$. That is, $x = ay = az$ for some $y \in I \setminus J$ and $z \in J \setminus I$. Since $a(y - z) = 0$, so $y - z \in a' \cap (I + J) = \{0\}$ and we have $y = z$; a contradiction. Therefore, $a(I \cap J) = aI \cap aJ$. \(\blacksquare\)
Proposition 1.5.2 (Shock, [Sh])

Let $a_1, \ldots, a_n$ be nonzero elements in a ring $R$. Then either $a_k' = a_i'$ for all $i = 1, \ldots, n - 1$ or there exists an element $b \in R$ and an integer $j$, $1 \leq j \leq n$, such that $a_j b \neq 0$ and $(a_j b)' = (a_k b)'$ for any $k \in \{1, \ldots, n\}$ such that $a_k b \neq 0$.

**Proof:** Suppose that $a_k' \neq a_i'$ for some $i \in \{1, \ldots, n - 1\}$. Then the set $A_1 = \{a_i' \mid 1 \leq i \leq n\}$ contains two elements $a_{i_1}'$ and $a_{i_2}'$ such that $a_{i_1}' \neq a_{i_2}'$. Choose $b_1 \in a_{i_1}'$, so that $b_1 \notin a_{i_2}'$. So we have $a_{i_1} b_1 = 0$ but $a_{i_2} b_1 \neq 0$. Next, consider the set $A_2 = \{(a_i b_1)' \mid a_i b_1 \neq 0, 1 \leq i \leq n\}$. Clearly, $a_{i_1} b_1 \in A_2$ but $a_{i_2} b_1 \notin A_2$. Hence $A_2$ has fewer elements than $A_1$. If there exist $(a_{j_1} b_1)'$, $(a_{j_2} b_1)' \in A_2$ such that $(a_{j_1} b_1)' \neq (a_{j_2} b_1)'$ then, without loss of generality, $a_{j_1} b_1 b_2 = 0$ but $a_{j_2} b_1 b_2 \neq 0$ for some $b_2 \in R$. Now consider $A_3 = \{(a_i b_1 b_2)' \mid a_i b_1 b_2 \neq 0, 1 \leq i \leq n\}$. It is not difficult to see that $A_3$ has fewer elements than $A_2$. By repeating the same argument as above, we will finally obtain some $b \in R$ and some integer $j$, $1 \leq j \leq n$, such that $a_j b \neq 0$ and $(a_j b)'' = (a_k b)''$ for any $k \in \{1, \ldots, n\}$ such that $a_k b \neq 0$. \[\blacksquare\]

Proposition 1.5.3 (Connell, [Co])

Let $R$ be a ring and let $G$ be a group.

(i) If $H$ is an infinite subgroup of $G$, then $(\omega H)' = \{0\}$. 

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(ii) If $H$ is a finite subgroup of $G$, then $(oH)' = RG\left(\sum_{h \in H} h\right)$. In particular, if $G$ is finite, then $(oG)' = RG\left(\sum_{g \in G} g\right)$.

1.6 Nil and nilpotent

Let $R$ be a ring. An element $a \in R$ is nilpotent if $a^n = 0$ for some positive integer $n$. A right (left, or two-sided) ideal $I$ of $R$ is nil if every element in $I$ is nilpotent. We say that a right (left, or two-sided) ideal $N$ is nilpotent if $N^n = \{0\}$ for some positive integer $n$. This means that for every choice of $n$ elements $a_1, ..., a_n \in N$, the product $a_1 \cdots a_n$ is zero. Clearly a nilpotent ideal is nil. A two-sided ideal $I$ of $R$ is right $T$-nilpotent if for any infinite sequence $(a_i)_{i \geq 1}$ of elements of $I$, there exists a positive integer $n$ such that $a_n \cdots a_1 = 0$. It is clear that a nilpotent ideal is right $T$-nilpotent and a right $T$-nilpotent ideal is nil.

1.7 Chain conditions

Let $R$ be a ring. $R$ is said to satisfy the ascending chain condition, or a.c.c. on right ideals if for every ascending chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ of right ideals of $R$, there exists a positive integer $n$ (depending on the sequence) such that $I_m = I_n$ for all $m \geq n$.

On the other hand, $R$ is said to satisfy the descending chain condition, or d.c.c. on right ideals if for every descending chain $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of right ideals of $R$, there exists a positive integer $n$ (depending on the sequence) such that $I_m = I_n$ for all $m \leq n$. 

On the other hand, $R$ is said to satisfy the descending chain condition, or d.c.c. on right ideals if for every descending chain $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of right ideals of $R$, there exists a positive integer $n$ (depending on the sequence) such that $I_m = I_n$ for all $m \leq n$.
ideals if for every descending chain $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of right ideals of $R$, there exists a positive integer $n$ (depending on the sequence) such that $I_m = I_n$ for all $m \geq n$.

A right (or left) ideal $M$ of $R$ is maximal if $M \neq R$ and $M$ is not strictly contained in any proper right (or left) ideal of $R$. On the other hand, a right (or left) ideal $M$ of $R$ is minimal if $M \neq \{0\}$ and $M$ does not strictly contain any nonzero right (or left) ideal of $R$.

The maximum condition for right ideals is said to hold in a ring $R$ if every nonempty collection of right ideals of $R$ has at least one maximal member while the minimum condition for right ideals is said to hold in $R$ if every nonempty collection of right ideals of $R$ has at least one minimal member. It is well-known that a ring $R$ satisfies the a.c.c. (d.c.c.) on right ideals if and only if the maximum condition (minimum condition) for right ideals holds in $R$.

We say that $R$ is right Noetherian if $R$ satisfies the a.c.c. on right ideals while $R$ is right Artinian if $R$ satisfies the d.c.c. on right ideals. It is well-known that a right Artinian ring is right Noetherian. Left Noetherian and left Artinian rings are similarly defined by replacing the word "right" with "left". A Noetherian ring is a ring which is both right and left Noetherian. Similarly, an Artinian ring is a ring which is both right and left Artinian.

Theorem 1.7.1 (Connell, [Co])

Let $R$ be a ring and let $G$ be a group. Then $RG$ is Artinian if and only if $R$ is Artinian and $G$ is finite.
Theorem 1.7.2 (Connell, [Co])

Let $R$ be a ring and let $G$ be a group.

(i) If $R$ is Noetherian and $G$ is finite, then $RG$ is Noetherian.

(ii) If $RG$ is Noetherian, then $R$ is Noetherian and $G$ has the maximum condition for subgroups.

(iii) If $G$ is an abelian group, then $RG$ is Noetherian if and only if $R$ is Noetherian and $G$ is finitely generated.

Remark: Let $\mathcal{P}$ be a property possessed by a right (left, or two-sided) ideal of $R$ (for example, $\mathcal{P}$ may be nilpotent or nil). In the remainder of this thesis, we shall say that $R$ satisfies the a.c.c. on right (left, or two-sided) $\mathcal{P}$ ideals if for every ascending chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ of right (left, or two-sided) $\mathcal{P}$ ideals of $R$, there exists a positive integer $n$ (depending on the sequence) such that $I_m = I_n$ for all $m \geq n$. We define a ring $R$ as satisfying the d.c.c. on right (left, or two-sided) $\mathcal{P}$ ideals analogously.

1.8 Prime and Jacobson radicals

Let $R$ be a ring. A proper two-sided ideal $P$ of $R$ is said to be prime if for any elements $a, b \in R$, $arb \subseteq P$ implies that $a \in P$ or $b \in P$. The intersection of all prime ideals of $R$, denoted by $P(R)$, is called the prime radical of $R$. It is well-known that $P(R)$ is a nil ideal of $R$ and contains all the nilpotent ideals of $R$. A ring $R$ is prime if $\{0\}$ is a prime ideal of $R$. That is, $R$ is a prime ring if and only if for any nonzero two-sided ideals
$I, J$ of $R$, we have $IJ \neq \{0\}$. We say that $R$ is semiprime if $P(R) = \{0\}$. It is not difficult to show that $R$ is semiprime if and only if $R$ contains no nonzero nilpotent right (left, or two-sided) ideals.

The intersection of all maximal right (or left) ideals of $R$ is called the Jacobson radical of $R$ and is written as $J(R)$. It is well-known that $J(R)$ consists of all $x \in R$ such that for all $r \in R$, $1 - xr$ has a right inverse. Furthermore, $J(R)$ is a two-sided ideal of $R$ which contains all the nil ideals of $R$. In particular, $P(R) \subseteq J(R)$. We say that $R$ is semiprimitive if $J(R) = \{0\}$. A ring $R$ is said to be simple if $\{0\}$ and $R$ are the only two-sided ideals of $R$.

If $R$ is a Noetherian ring, then $P(R)$ is nilpotent. If $R$ is an Artinian ring, then $P(R) = J(R)$ and $J(R)$ is nilpotent.

Proposition 1.8.1 (Connell, [Co])

Let $R$ be a ring and $G$ a group. Let $H$ be a subgroup of $G$. Then

(i) $P(RH) \supseteq RH \cap P(RG)$, with equality if $H$ is contained in the centre of $G$. In particular, if $H = \{1\}$, then $P(R) = R \cap P(RG)$ and therefore

(ii) $P(R)G \subseteq P(RG)$.

(iii) $J(RH) \supseteq RH \cap J(RG)$. In particular, if $H = \{1\}$, then $J(R) \supseteq R \cap J(RG)$. The equality holds if either

(a) $R$ is Artinian, or

(b) $G$ is locally finite.

Thus, if either (a) or (b) is satisfied, we have

(iv) $J(R)G \subseteq J(RG)$.  

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Theorem 1.8.2 (Connell, [Co] & Passman, [Pa2])

Let $R$ be a ring and $G$ a group. Then $RG$ is semiprime if and only if $R$ is semiprime and the order of no finite normal subgroup of $G$ is a zero divisor in $R$.

Theorem 1.8.3 (Connell, [Co])

Let $R$ be a ring and let $G$ be a prime group. Then $RG$ is semiprime if and only if $R$ is semiprime.

Theorem 1.8.4 (Connell, [Co])

Let $R$ be a commutative ring and let $G$ be an ordered group. Then $RG$ is semiprimitive if and only if $R$ is semiprime.

Theorem 1.8.5 (Connell, [Co])

Let $R$ be a commutative ring and let $G$ be a finite group of order $n$. Then $RG$ is semiprime (semiprimitive) if and only if $R$ is semiprime (semiprimitive) and $n$ is not a zero divisor in $R$.

Theorem 1.8.6 (Connell, [Co])

Let $R$ be a commutative ring and let $G$ be an abelian group.

(i) If $G$ is torsion, then $RG$ is semiprimitive if and only if $R$ is semiprimitive and the order of every finite subgroup of $G$ is not a zero divisor in $R$. 

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(ii) If $G$ is not torsion, then $RG$ is semiprimitive if and only if $R$ is semiprime and the order of every finite subgroup of $G$ is not a zero divisor in $R$.

1.9 Regularity

Let $R$ be a ring. We say that $R$ is a (von Neumann) regular ring if for every element $a \in R$, there exists $x \in R$ such that $axa = a$.

Theorem 1.9.1 (Auslander, [A] & Connell, [Co])

Let $R$ be a ring and let $G$ be a group. Then $RG$ is regular if and only if

(i) $R$ is regular,

(ii) $G$ is locally finite, and

(iii) the order of every finite subgroup of $G$ is a unit in $R$.

1.10 Polynomial rings

Let $R$ be a ring. A polynomial ring in the indeterminate $x$ with coefficients in $R$ will be denoted by $R[x]$. Let $f(x) = a_nx^n + \cdots + a_0$ be a nonzero polynomial in $R[x]$. If $a_n \neq 0$, then $n$ is called the degree of $f(x)$ and we write $\deg f(x) = n$. The leading term of $f(x)$ is $a_nx^n$ and $a_n$ is the leading coefficient of $f(x)$.

If we adjoin to a ring $R$ the indeterminates $x_1, x_2, \ldots$, then we obtain a polynomial ring $R[x_1, x_2, \ldots]$ consisting of all finite sums $\sum_{i=1}^{n} a_i x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}$, where
\( a_i \in R, \ k_i \in \mathbb{Z}^+, \ i = 1, \ldots, m \). By the \textit{degree} of a term \( ax_1^{k_1}x_2^{k_2}\cdots x_n^{k_n} \), we mean the sum of the exponents \( \sum_{i=1}^{n} k_i \), and we write \( \text{deg} (ax_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}) = \sum_{i=1}^{n} k_i \).

Proposition 1.10.1

Let \( \{J_i\}_{i \in \mathbb{Z}} \) be a collection of nonzero right ideals of a ring \( R \). If \( J' = \bigoplus_{i \in \mathbb{Z}} J_i \), then \( J'[x_1, x_2, \ldots] = \bigoplus_{i \in \mathbb{Z}} J_i[x_1, x_2, \ldots] \).

\textbf{Proof:} Let \( f = f(x_i, x_{i_2}, \ldots, x_{i_n}) \in \bigoplus_{i \in \mathbb{Z}} J_i[x_1, x_2, \ldots] \). We may write \( f = \sum_{i=1}^{m} f_i \), where

\[
 f_i = \sum_{k=1}^{s_i} a_{i_k} x_{i_k} \in J_i[x_1, x_2, \ldots], \quad i, s_i \geq 1, \text{ with } a_{i_k} \in J_i, \ x_{i_k} = x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}, \ b_i \in \mathbb{Z}^+ \text{ for } j = 1, \ldots, n \text{ and } x_{i_r} \neq x_{i_s} \text{ for } r \neq s. \text{ Thus}
\]

\[
 f = \sum_{i=1}^{m} \left( \sum_{k=1}^{s_i} a_{i_k} x_{i_k} \right)
 = \sum_{i=1}^{m} \left( a_{i_1} x_{i_1} + a_{i_2} x_{i_2} + \cdots + a_{i_{s_i}} x_{i_{s_i}} \right)
 + a_{i_1} x_{i_2} + a_{i_2} x_{i_2} + \cdots + a_{i_{s_i}} x_{i_{s_i}}
 + \cdots
 + a_{i_1} x_{i_{s_i}} + a_{i_2} x_{i_{s_i}} + \cdots + a_{i_{s_i}} x_{i_{s_i}}
 = \sum_{i} \bar{a}_i x_i,
\]

where \( \bar{a}_i \) is the sum of those \( a_{i_k} \)'s with \( x_{i_k} = x_i \), and \( x_{i_k} \neq x_j \) for \( i \neq j \). Note that for every \( \bar{a}_i \), there is at most one \( a_{i_k} \) appearing in each \( J_i \). Hence, \( \bar{a}_i \in J' \) for all \( t \) and therefore, \( f \in J'[x_1, x_2, \ldots] \). Furthermore, suppose that \( f = \sum_{i=1}^{m} f_i = 0 \). Since \( f \) can be written in the form \( \sum_{i} \bar{a}_i x_i \), we have \( \bar{a}_i = 0 \) for all \( t \). By the directness of \( J' \), we have
\[ a_{i_k} = 0 \text{ for all } i_k. \] Thus \( f_i = \sum_{k=1}^n a_{i_k} x_{i_k} = 0 \text{ for all } i = 1, \ldots, m. \) It follows that

\[ \sum_{i \in I} J_i[x_1, x_2, \ldots] \text{ must be direct.} \]

Conversely, let \( f' \in J'[x_1, x_2, \ldots] \) and let \( a_1, \ldots, a_n \) be the coefficients of \( f'. \)

Say \( a_i = \sum_{k=1}^m a_{i_k} \in J', \) where \( a_{i_k} \in J_k, \) \( a_{j_k} \notin J_j \text{ for } j \neq k, \) \( j, k \in \{1, \ldots, m\}, \)

\( i \in \{1, \ldots, n\} \) and where some of the \( a_{i_k} \)'s may be zero. Let \( m = \max\{m_1, \ldots, m_n\}. \)

Write \( f' = \sum_{i=1}^m f_i, \) where \( f_i \) consists of those terms of \( f' \) with coefficients in \( J_i. \) Then it is clear that \( f_i \in J_i[x_1, x_2, \ldots]. \) Hence, \( f' \in \oplus_{i \in I} J_i[x_1, x_2, \ldots] \) and the result follows. ■

Corollary 1.10.2

Let \( \{J_i\}_{i \geq 1} \) be a collection of nonzero right ideals of a ring \( R. \) If \( J' = \oplus_{i \geq 1} J_i, \) then \( J'[x] = \oplus_{i \geq 1} J_i[x]. \)

Let \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \) be a nonzero element of \( R[x]. \) We say that the right annihilators of the coefficients of \( f(x) \) are equal if \( a_n = a'_i \) for any \( a_i \neq 0, \)

\( i = 0, 1, \ldots, n-1. \)

Proposition 1.10.3 (Shock, [Sh])

For any nonzero polynomial \( p(x) \) in \( R[x], \) there exists an element \( b \in R \) such that \( p(x)b \neq 0 \) and the right annihilators of the coefficients of \( p(x)b \) are equal.
Proof: Let \( p(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x] \). If \( a^*_i = a'_i \) for any \( a_i \neq 0 \), \( i = 0, 1, \ldots, n-1 \), then we choose \( b = 1 \) and the result follows. Suppose that \( a^*_i \neq a'_i \) for some \( a_i \neq 0 \), \( 0 \leq i \leq n-1 \). By Proposition 1.5.2, there exists an element \( b \in R \) and an integer \( j \), \( 0 \leq j \leq n \), such that \( a_j b \neq 0 \) and \( (a_j b)' = (a_k b)' \) for any \( k \in \{0, 1, \ldots, n\} \) such that \( a_k b \neq 0 \). It follows that \( p(x) \) satisfies the required conditions. ■

Proposition 1.10.4 (Shock, [Sh])

Let \( p(x) = a_0 x^k + a_{k+1} x^{k+1} + \cdots + a_{k+n} x^{k+n} \) be a polynomial in \( R[x] \), where \( k \geq 0 \). Assume that \( a_i \neq 0 \) and \( a^*_i \subseteq a'_i \) for \( i = 1, \ldots, n \). Let \( q(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x] \). Then \( p(x)q(x) = 0 \) if and only if \( b_0, b_1, \ldots, b_m \in a'_i \).

Proof: Let \( h(x) = p(x)q(x) \). Then \( h(x) = c_k x^k + c_{k+1} x^{k+1} + \cdots + c_{k+n+m} x^{k+n+m} \), where

\[
\begin{align*}
c_k &= a_k b_0, \
c_{k+1} &= a_k b_1 + a_{k+1} b_0, \ldots, \
c_{k+i} &= \sum_{i=0}^{n} a_{i} b_j, \ldots, 
\end{align*}
\]

\[
c_{k+n+m} = a_{k+n} b_m. 
\]

Note that \( h(x) = 0 \) implies that \( c_{k+i} = 0 \) for all \( i = 0, 1, \ldots, n+m \). Since \( c_k = 0 \), so \( b_0 \in a^*_i \subseteq a'_i \) for \( i = 1, \ldots, n \). Then since \( b_0 \in a^*_{k+1} \), \( c_{k+1} = 0 \) implies that \( b_1 \in a^*_i \subseteq a'_i \) for \( i = 1, \ldots, n \). Accordingly, for \( j = 0, 1, \ldots, m \), we have \( b_j \in a'_i \). The reverse implication is clear. ■