CHAPTER 4
ON RIGHT NONSINGULAR GROUP RINGS
AND SOME RELATED TOPICS

4.1 Introduction

This final chapter has two major goals - firstly, to study right nonsingular group rings and secondly, to study relations between the prime and Jacobson radicals and right singular ideals of various group rings.

It is not difficult to see from the definition of right nonsingular rings that integral domains are right nonsingular. We shall see in section two that regular rings and simple commutative rings are also right nonsingular (see Propositions 4.2.2 and 4.2.3). In section two we show that for a commutative ring $R$, the prime radical must be contained in its singular ideal with both of them coinciding if $R$ has the additional property of satisfying the a.c.c. on principal annihilator ideals.

Next in section three we shall study some conditions under which a group ring is nonsingular. Among other things, we show that for an ordered group $G$, the group ring $RG$ is nonsingular if and only if $R$ is nonsingular. In particular, if $R$ is a commutative ring, we shall see that $RG$ is semiprimitive if and only if $R$ is nonsingular.

In the final section we shall determine relations between the singular ideal and the prime and Jacobson radicals of a group ring. We show that for $R$ a commutative ring which satisfies the a.c.c. on principal annihilator ideals and $G$ an ordered group, $J(RG) = P(RG) = P(R)G = Z(R)G$. We then deduce that if $R$ is a commutative ring
satisfying the a.c.c. on principal annihilator ideals and $G$ is a free abelian group, then


4.2 Some results on nonsingular rings

First, we consider an example which shows that the right singular ideal of a ring need not coincide with its left singular ideal.

Example 4.2.1

Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix} = \left\{ \begin{pmatrix} a & b + 2\mathbb{Z} \\ 0 & c + 2\mathbb{Z} \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$.

For $\alpha = \begin{pmatrix} a & b + 2\mathbb{Z} \\ 0 & c + 2\mathbb{Z} \end{pmatrix}$, $\beta = \begin{pmatrix} u & v + 2\mathbb{Z} \\ 0 & w + 2\mathbb{Z} \end{pmatrix} \in R$, we define

$$\alpha + \beta = \begin{pmatrix} a + u & (b + v) + 2\mathbb{Z} \\ 0 & (c + w) + 2\mathbb{Z} \end{pmatrix}$$

and

$$\alpha \beta = \begin{pmatrix} au & (av + bw) + 2\mathbb{Z} \\ 0 & cw + 2\mathbb{Z} \end{pmatrix}.$$

Then by routine verification, we have that $R$ is a ring with zero element

$$0_R = \begin{pmatrix} 0 & 0 + 2\mathbb{Z} \\ 0 & 0 + 2\mathbb{Z} \end{pmatrix}.$$

Consider the element $\delta = \begin{pmatrix} 2 & 0 + 2\mathbb{Z} \\ 0 & 0 + 2\mathbb{Z} \end{pmatrix} \in R$. It is not difficult to check that

$$\delta' = \delta' = \begin{pmatrix} 0 & a + 2\mathbb{Z} \\ 0 & b + 2\mathbb{Z} \end{pmatrix} \mid a, b \in \mathbb{Z}.\]
Now note that every nonzero right ideal of $R$ takes one of the following forms:

(i) \[
\begin{pmatrix}
J & \mathbb{Z}/2\mathbb{Z} \\
0 & \mathbb{Z}/2\mathbb{Z}
\end{pmatrix} = \left\{ \begin{pmatrix} j & a + 2\mathbb{Z} \\
0 & b + 2\mathbb{Z} \end{pmatrix} \middle| j \in J; \ a, b \in \mathbb{Z} \right\};
\]

(ii) \[
\begin{pmatrix}
J & \mathbb{Z}/2\mathbb{Z} \\
0 & \mathbb{Z}/2\mathbb{Z}
\end{pmatrix} = \left\{ \begin{pmatrix} j & a + 2\mathbb{Z} \\
0 & 0 + 2\mathbb{Z} \end{pmatrix} \middle| j \in J; \ a \in \mathbb{Z} \right\};
\]

(iii) \[
\begin{pmatrix}
0 & \mathbb{Z}/2\mathbb{Z} \\
0 & 0
\end{pmatrix} = \left\{ \begin{pmatrix} 0 & a + 2\mathbb{Z} \\
0 & 0 \end{pmatrix} \middle| a \in \mathbb{Z} \right\};
\]

(iv) \[
\begin{pmatrix}
0 & 0 \\
0 & \mathbb{Z}/2\mathbb{Z}
\end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\
0 & b + 2\mathbb{Z} \end{pmatrix} \middle| b \in \mathbb{Z} \right\},
\]

where $J$ is a right ideal of $\mathbb{Z}$. It is clear that

\[\delta' \cap \begin{pmatrix}
J & \mathbb{Z}/2\mathbb{Z} \\
0 & \mathbb{Z}/2\mathbb{Z}
\end{pmatrix} \neq \{0_R\}\]

and

\[\delta' \cap \begin{pmatrix}
J & \mathbb{Z}/2\mathbb{Z} \\
0 & \mathbb{Z}/2\mathbb{Z}
\end{pmatrix} \neq \{0_R\}.\]

for any right ideal $J$ of $\mathbb{Z}$. Furthermore, if $I$ is a right ideal of $R$ of the form (iii) or (iv), it is clear that

\[\delta' \cap I \neq \{0_R\}.\]

Therefore, $\delta'$ is essential in $R$ and hence, $\delta \in Z_i(R)$. To show that $\delta \in Z_i(R)$, we consider

\[
\begin{pmatrix}
\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\
0 & \mathbb{Z}/2\mathbb{Z}
\end{pmatrix} = \left\{ \begin{pmatrix} z & 0 + 2\mathbb{Z} \\
0 & 0 + 2\mathbb{Z} \end{pmatrix} \middle| z \in \mathbb{Z} \right\}.
\]
By routine verification, \( \left( \begin{array}{c} \mathbb{Z} \\ 0 \end{array} \begin{array}{c} \{0 + 2\mathbb{Z}\} \\ \{0 + 2\mathbb{Z}\} \end{array} \right) \) is a left ideal of \( R \) and it is clear that

\[ \delta' \cap \left( \begin{array}{c} \mathbb{Z} \\ 0 \end{array} \begin{array}{c} \{0 + 2\mathbb{Z}\} \\ \{0 + 2\mathbb{Z}\} \end{array} \right) = \{0_R\}. \]

Hence, \( \delta' \) is not an essential left ideal of \( R \) and it follows that \( \delta \not\in Z_i(R) \). ■

Remark: The ring \( R \) described in Example 4.2.1 can in fact be found in [Pa3, p.260].

From now onwards, all nonsingular rings considered here are right nonsingular.

Lemma 4.2.1

Let \( R \) be a ring. Then \( Z(R) \) contains no nonzero idempotent.

Proof: Let \( e \) be an idempotent in \( Z(R) \). Since \( e = e^2 \), we have \( e' = (e^\lor)^\lor \) and by Lemma 2.5.8, \( e' \cap eR = \{0\} \). Then since \( e' \) ess \( R \), it follows that \( eR = \{0\} \). In particular, \( e = 0 \).

Hence, \( Z(R) \) contains no nonzero idempotent. ■

Proposition 4.2.2

If \( R \) is a regular ring, then \( R \) is nonsingular.

Proof: Let \( a \in Z(R) \). Since \( R \) is regular, there exists an element \( x \in R \) such that \( axa = a \). We thus have \( (xa)^2 = xa \). Moreover, \( a^\lor \subseteq (xa)^\lor \) implies that \( (xa)^\lor \) ess \( R \); that is, \( xa \in Z(R) \). Since \( xa \) is an idempotent, it follows from Lemma 4.2.1 that \( xa = 0 \). Thus \( a = axa = 0 \) and \( Z(R) = \{0\} \) as required. ■

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Proposition 4.2.3

If $R$ is a field, then $R$ is nonsingular.

**Proof:** Since $R$ is simple and commutative, the only essential ideal in $R$ is $R$ itself. Then since $R$ is dense, it follows from Proposition 2.5.7 that $R$ is nonsingular.

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Proposition 4.2.4

Let $R$ be a commutative ring. If $N$ is a nil ideal of $R$, then $a^* \text{ess } R$ for all $a \in N$.

**Proof:** If $N = \{0\}$, then $N^* = R$ and the proof is complete. Suppose that $N \neq \{0\}$. Let $a$ be a nonzero element of $N$ and let $I$ be a nonzero ideal of $R$. If $aI = \{0\}$, then $I \subseteq a^*$ and hence, $a^* \cap I = I \neq \{0\}$. On the other hand, if $aI \neq \{0\}$, then since $a$ is nilpotent, there is a smallest positive integer $n$ such that $a^n I \neq \{0\}$ and $a^{n+1} I = \{0\}$. Since $a(a^n I) = \{0\}$, so $\{0\} = a^n I \subseteq a^* \cap I$. It follows that $a^* \text{ess } R$.

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Corollary 4.2.5

Let $R$ be a commutative ring. Then $Z(R)$ contains all nil ideals of $R$. In particular, $P(R) \subseteq Z(R)$.

**Proof:** Let $N$ be a nil ideal of $R$. By Proposition 4.2.4, $a^* \text{ess } R$ for all $a \in N$. Hence, $a \in Z(R)$ for all $a \in N$ and it follows that $N \subseteq Z(R)$. Since $P(R)$ is a nil ideal of $R$, the final assertion holds.

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Proposition 4.2.6

Let $R$ be a commutative ring. Then $R$ is semiprime if and only if $R$ is nonsingular.
Proof: Assume that \( R \) is semiprime. We first note that for any ideals \( I, J \) of \( R \), 
\[(I \cap J) \subseteq IJ \subseteq I \cap J.\]
We thus have \( I \cap J = \{0\} \) if and only if \( IJ = \{0\} \). Now suppose that \( Z(R) \neq \{0\} \). Then there exists a nonzero element \( a \in R \) such that \( a' \) ess \( R \). Since \( R \) is commutative, we have \( a' = a' = a^* \). It is clear that \( a^*(aR) = \{0\} \). Thus \( a^* \cap aR = \{0\} \) which implies that \( aR = \{0\} \); a contradiction. Therefore, \( R \) must be nonsingular. The converse is clear by Corollary 4.2.5. ■

The following proposition shows that the reverse of the inclusion in Corollary 4.2.5 holds if \( R \) satisfies the a.c.c. on right annihilator ideals.

Proposition 4.2.7 (Mewborn and Winton, [MeW])

If \( R \) is a ring satisfying the a.c.c. on right annihilator ideals, then \( Z(R) \) is nilpotent.

In particular, \( Z(R) \subseteq P(R) \).

Proof: Clearly, if \( Z(R) = \{0\} \), then \( Z(R) \subseteq P(R) \). Now suppose that \( Z(R)^{k+1} \neq \{0\} \) for some positive integer \( k \). Consider the set

\[S = \left\{ x' \mid x \in Z(R), Z(R)^k x \neq \{0\} \right\}.\]

Clearly, \( S \neq \emptyset \). Since \( R \) has the maximum condition on right annihilator ideals, we may choose an element \( a \in Z(R) \) so that \( a' \) is maximal in \( S \). Now let \( b \in Z(R) \). By the definition of \( Z(R) \), \( b' \cap aR \neq \{0\} \). This implies that there exists an element \( x \in R \) such that \( ax \neq 0 \) but \( bax = 0 \). Therefore \( a' \subseteq (ba)' \). By the maximality of \( a' \) in \( S \), we must have \( Z(R)^k ba = \{0\} \). Since \( b \) is an arbitrary element of \( Z(R) \), so \( Z(R)^{k+1} a = \{0\} \). Then
since $Z(R)^k a \neq \{0\}$, it follows that $(Z(R)^k)^{\prime} \subseteq (Z(R)^{k+1})^{\prime}$. Now if $Z(R)$ is not nilpotent (hence, $Z(R)^n \neq \{0\}$ for any positive integer $n$), then we would have an infinite ascending chain

$$Z(R)^{\prime} \subseteq (Z(R)^2)^{\prime} \subseteq \cdots \subseteq (Z(R)^k)^{\prime} \subseteq (Z(R)^{k+1})^{\prime} \subseteq \cdots$$

of right annihilator ideals in $R$; a contradiction. Therefore, $Z(R)$ must be nilpotent. \[\Box\]

Lemma 4.2.8

Let $R$ be a ring. If $I$ is a right $T$-nilpotent ideal of $R$, then $I \subseteq P(R)$.

**Proof:** Suppose that $I \not\subseteq P(R)$. Then there exists an element $x_1 \in I \setminus P(R)$. If $lx_1 \subseteq P(R)$, then $x_1, Rx_1 \subseteq lx_1 \subseteq P(R)$. That is, $x_1, Rx_1$ is contained in every prime ideal of $R$. It follows that $x_1$ is contained in every prime ideal of $R$ and hence, $x_1 \in P(R)$; a contradiction. Thus $lx_1 \not\subseteq P(R)$ and so there exists an element $x_2 \in I$ such that $x_2x_1 \in I \setminus P(R)$. By the same argument as above, we can show that $lx_2x_1 \not\subseteq P(R)$. Hence there exists an element $x_3 \in I$ such that $x_3x_2x_1 \in I \setminus P(R)$. By continuing in this way, we obtain a sequence of elements $\{x_i\}_{i \geq 1}$ in $I$ such that $x_n \cdots x_1 \neq 0$ for all $n \geq 1$. This contradicts the right $T$-nilpotence of $I$. Therefore $I \subseteq P(R)$. \[\Box\]

A right ideal $I$ of a ring $R$ is said to be a principal right annihilator ideal if $I = a^\prime$ for some $a \in R$. 

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Proposition 4.2.9 (Ferrero and Torner, [FeT])

If $R$ is a ring satisfying the a.c.c. on principal right annihilator ideals, then $Z(R)$ is right $T$-nilpotent. In particular, $Z(R) \subseteq P(R)$.

**Proof:** Suppose on the contrary that $Z(R)$ is not right $T$-nilpotent. Then there is a sequence $\{x_i\}_{i \geq 1}$ of elements in $Z(R)$ such that $x_n \cdots x_1 \neq 0$ for all $n \geq 1$. Since $x'_1 \subseteq (x_2 x_1)' \subseteq \cdots$ is an ascending chain of principal right annihilator ideals, it follows from the hypothesis that there exists $m \in \mathbb{N}$ with $(x_m x_{m-1} \cdots x_1)' = (x_{m-1} \cdots x_1)'$. Now since $x_m \in Z(R)$ and $x_{m-1} \cdots x_1 \neq 0$, so we have $x'_m \cap (x_{m-1} \cdots x_1)R \neq \{0\}$. This means that there exists an element $y \in R$ with $x_{m-1} \cdots x_1 y \neq 0$ but $x_m x_{m-1} \cdots x_1 y = 0$; a contradiction. Therefore, $Z(R)$ must be right $T$-nilpotent. The final assertion follows from the preceding lemma. ■

Since nilpotent and right $T$-nilpotent ideals are nil, the following result can be deduced easily from Proposition 4.2.7 or Proposition 4.2.9.

**Corollary 4.2.10**

If $R$ is a ring satisfying the a.c.c. on right annihilator ideals, then $Z(R)$ is nil. Furthermore, if $R$ is semiprime, then $R$ is nonsingular.

**Proposition 4.2.11**

If $R$ is a commutative ring satisfying the a.c.c. on principal annihilator ideals, then $Z(R) = P(R)$.
Proof: The result follows easily from Proposition 4.2.9 and Corollary 4.2.5. ■

Let $R$ be a ring. We say that $R$ is a right Goldie ring if $R$ satisfies the a.c.c. on right annihilator ideals and $R$ contains no infinite direct sum of nonzero right ideals. In other words, a right Goldie ring is a right finite dimensional ring with a.c.c. on right annihilator ideals. The left Goldie ring is defined similarly by replacing the word "right" with "left".

Proposition 4.2.12

If $R$ is a right Noetherian ring, then $R$ is a right Goldie ring.

Proof: Clearly $R$ satisfies the a.c.c. on right annihilator ideals. Also by Proposition 2.5.10, we know that $R$ is right finite dimensional. Hence, $R$ is a right Goldie ring. ■

As a consequence of Propositions 4.2.11 and 4.2.12 we have the following corollaries.

Corollary 4.2.13

If $R$ is a commutative Goldie ring, then $Z(R) = P(R)$.

Corollary 4.2.14

If $R$ is a commutative Noetherian ring, then $Z(R) = P(R)$. 

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4.3 On necessary and sufficient conditions for a group ring to be nonsingular

In [Sn], Snider showed that if $\mathbb{F}$ is a field of characteristic 0 and $G$ is a group, then the singular ideal $Z(\mathbb{F}G)$ is $\{0\}$. If $\mathbb{F}$ has positive characteristic and $G$ is solvable, Brown showed that $Z(\mathbb{F}G) \neq \{0\}$ if and only if $G$ has a finite two-step subnormal subgroup of order divisible by $p$ (see [Br1]). Brown also showed in the same paper that if $\mathbb{F}$ has positive characteristic and $G$ is an FC-hypercentral group, then $Z(\mathbb{F}G) \neq \{0\}$ if and only if $G$ has a finite normal subgroup of order divisible by $p$. In a later paper, Brown showed that for any group $G$ and any field $K \supseteq \mathbb{F}$, $Z(KG) = \{0\}$ if and only if $Z(KG) = \{0\}$ (see [Br2]). In this section we shall look at other circumstances under which a group ring is nonsingular. We first make the following simple observation:

Proposition 4.3.1

Let $R$ be a ring, $G$ a group and $H$ a normal subgroup of $G$. If $Z(RG) = \{0\}$, then $Z(RH) = \{0\}$.

Proof: Let $a \in Z(RH)$. Then $L' \text{ ess } RH$, where $L' = a_{RH}$. By Proposition 2.3.8, $L'RG \text{ ess } RG$. Note that $aL' = \{0\}$ implies that $aL'RG = \{0\}$. Thus $L'RG \subseteq a'$, which implies that $a' \text{ ess } RG$; that is, $a \in Z(RG) = \{0\}$. We thus have $Z(RH) = \{0\}$. ■

By taking $H = \{1\}$ in Proposition 4.3.1, we have the following result:
Corollary 4.3.2

Let $R$ be a ring and let $G$ be a group. If $RG$ is nonsingular, then $R$ is nonsingular.

We note that Corollary 4.3.2 also follows easily from Proposition 2.4.6.

Remark: It is natural to ask if the converse of Proposition 4.3.1 is true. Burgess [Bu] has shown that the converse is indeed true if $G/H$ is an ordered group. We shall see this in Proposition 4.3.7.

Proposition 4.3.3

Let $R$ be a ring and let $G$ be a free abelian group. Then $RG$ is nonsingular if and only if $R$ is nonsingular.

Proof: Note that $Z(RG) = Z(R)G$ by Proposition 2.4.8. Therefore, $Z(R) = \{0\}$ if and only if $Z(RG) = \{0\}$ and the result follows. □

Corollary 4.3.4

Let $R$ be a ring and let $G$ be an infinite cyclic group. Then $RG$ is nonsingular if and only if $R$ is nonsingular.

Before continuing, we pause to consider the following useful lemma.
Lemma 4.3.5

Let $R$ be a ring and let $G$ be a group. For any subgroup $H$ of $G$, the mapping

$$\pi_H: RG \rightarrow RH$$

defined by $\pi_H \left( \sum_{g \in G} r_g g \right) = \sum_{g \in H} r_g g$ is an RH-homomorphism.

Proof: Note that the mapping $\pi_H$ can be interpreted as

$$\pi_H \left( \sum_{g \in G} r_g g \right) = \sum_{g \in H} \pi_H(r_g) g,$$

where

$$\pi_H(r_g) = \begin{cases} r_g & \text{if } g \in H, \\ 0 & \text{if } g \not\in H. \end{cases}$$

Let $s = \sum_{g \in G} s_g g$, $s' = \sum_{g \in G} s'_g g \in RG$. Note that

$$\pi_H(s + s') = \pi_H \left( \sum_{g \in G} (s_g + s'_g) g \right)$$

$$= \sum_{g \in H} (s_g + s'_g) g$$

$$= \sum_{g \in H} s_g g + \sum_{g \in H} s'_g g$$

$$= \pi_H(s) + \pi_H(s').$$

Moreover if $r = \sum_{h \in H} h$, $r' = \sum_{h' \in H} h' \in RH$ and $s = \sum_{k \in G} s_k k \in RG$, then
\[ \pi_H(rs') = \pi_H \left( \sum_{g \in G} \left( \sum_{\stackrel{h \in H}{hkk' = g}} r_h s_k r'_n \right) g \right) \]
\[ = \sum_{g \in H} \pi_H \left( \sum_{\stackrel{h \in H}{hkk' = g}} r_h s_k r'_n \right) g \]
\[ = \sum_{g \in H} \pi_H \left( \sum_{\stackrel{h \in H}{hkk' = g}} r_h s_k r'_n \right) g + \sum_{g \in G \setminus H} \pi_H \left( \sum_{\stackrel{h \in H}{hkk' = g}} r_h s_k r'_n \right) g \]
\[ = \sum_{g \in H} \pi_H \left( \sum_{\stackrel{h \in H}{hkk' = g}} r_h s_k r'_n \right) g \]
\[ = \sum_{g \in H} \left( \sum_{\stackrel{g \in G}{hkk' = g}} r_h \pi_H (s_k) r'_n \right) g \quad \text{(see remark below)} \]
\[ = \left( \sum_{\text{all } h} r_h \right) \left( \sum_{g \in G} \pi_H (s_k) k \right) \left( \sum_{h' \in H} r'_n h' \right) \]
\[ = r_\pi_H (s) r'. \]

Therefore, \( \pi_H \) is an RH-homomorphism. \( \blacksquare \)

**Remark:** We know that if \( H \) is a subgroup of a group \( G \), then for any \( h, h' \in H \) and \( k \in G \), \( hkh' \in H \) if and only if \( k \in H \).

Let \( G \) be a group and let \( H \) be a normal subgroup of \( G \) so that \( G/H \) is an ordered group. If \( T = \{ g_i \}_{i \in I} \) is a right transversal for \( H \) in \( G \), then the elements of \( T \) can also be ordered in the following manner: \( g_i < g_j \) if \( H g_i < H g_j \). Recall from Lemma 3.3.5 that if \( T = \{ g_i \}_{i \in I} \) is a right transversal for \( H \) in \( G \), then every element \( a \) of \( RG \) can be uniquely written in the form \( a = \sum_{g \in I} a_i g_i \), where \( a_i \in RH \).
Lemma 4.3.6

Let $R$ be a ring and let $G$ be a group. Let $H$ be a normal subgroup of $G$ so that $G/H$ is an ordered group. Let $T = \{ g_i \}_{i \in I}$ be a right transversal for $H$ in $G$, where the $g_i$'s are in increasing order. For any element $a = \sum_{g_i \in T} a_i g_i \in RG$, where $a_i \in RH$, we define a mapping $\kappa: RG \to RH$ by

$$\kappa(a) = \begin{cases} a_j & \text{if } g_j \text{ is the least } g_i \text{ such that } a_i \neq 0. \\ 0 & \text{if } a = 0. \end{cases}$$

Let $L$ be a nonzero right ideal of $RG$. If $L$ ess $RG$, then $\kappa(L)$ ess $RH$.

**Proof:** First, we show that $\kappa(L) = \{ \kappa(a) \mid a \in L \}$ is a right ideal of $RH$. Let $a = a_k g_k + \sum_{g_i \in T} a_i g_i, b = b_j g_j - \sum_{g_i \in T} b_j g_j \in L$, where $g_k$ is the least $g_i$ such that $a_i \neq 0$ and $g_j$ is the least $g_j$ such that $b_j \neq 0$. By definition, $\kappa(a) = a_k$ and $\kappa(b) = b_l$. Since the elements of $T$ are ordered, $g_k < g_j$ implies that $g_k g_k^{-1} g_j < g_j g_k^{-1} g_k$. Also, $L$ is a right ideal of $RG$ implies that $a g_k^{-1} g_i \in L$. We thus have that $a g_k^{-1} g_i + b \in L$ and therefore, $a_k + b_l = \kappa ag_k^{-1} g_i + b \in \kappa(L)$. It is clear that $\kappa(a + b) = \kappa(a) + \kappa(b)$. Using the same argument as in the proof of Lemma 4.3.5, we have that if $r \in RH$, then $\kappa(a)r = \kappa(ar) \in \kappa(L)$. Hence, $\kappa(L)$ is a right ideal of $RH$.

Now suppose that $\kappa(L) \cap J = \{ 0 \}$ for some right ideal $J$ of $RH$. If $J \neq \{ 0 \}$, then $JRG \neq \{ 0 \}$. Note that $JRG = \left\{ \sum_{g_i \in T} a_i g_i \mid a_i \in J \right\}$. Since $L$ ess $RG$, so $L \cap JRG \neq \{ 0 \}$.

Thus there exists a nonzero element $b \in L \cap JRG$. But $\kappa(b) \in \kappa(L) \cap J = \{ 0 \}$ implies that $b = 0$; a contradiction. Therefore, $J = \{ 0 \}$ and we have $\kappa(L)$ ess $RH$ as desired.
We are now ready to prove

Proposition 4.3.7 (Burgess, [Bu])

Let $R$ be a ring and let $G$ be a group. Let $H$ be a normal subgroup of $G$ so that $G/H$ is an ordered group. If $Z(RH) = \{0\}$, then $Z(RG) = \{0\}$.

Proof: Let $\kappa$ be defined as in the previous lemma. Suppose that $Z(RG) \neq \{0\}$. Then there exists a nonzero element $a \in Z(RG)$ such that $a' \in RG$. Let $L = a'$. Then we have $ax = 0$ for all $x \in L$. Let $a = \sum_{i \in I} a_i g_i, x = \sum_{j \in J} x_j g_j$ with $\kappa(a) = a_p$ and $\kappa(x) = x_q$.

By the ordering of transversal, $a_p g_p x_q g_q = 0$ and hence, $a_p g_p \kappa(x) = 0$. It follows that $a_p g_p \kappa(L) = \{0\}$. Since $H$ is normal, there exists $a'_p \in RH$ such that $a_p g_p = g_p a'_p$; hence, $a'_p \kappa(L) = \{0\}$. By the previous lemma we know that $\kappa(L) \text{ ess } RH$. Then since $\kappa(L) \subseteq (a'_p)'$, it follows that $a'_p \text{ ess } RH$; hence, $a'_p \in Z(RH)$. Note that $a'_p \neq 0$ since $a_p \neq 0$. We thus have $Z(RH) \neq \{0\}$; a contradiction. Therefore, $Z(RG)$ must be equal to zero.

The following corollaries are easy consequences of the preceding result and Corollary 4.3.2.
Corollary 4.3.8

Let $R$ be a ring and let $G$ be an ordered group. Then $RG$ is nonsingular if and only if $R$ is nonsingular.

Corollary 4.3.9

Let $R$ be a ring and let $G$ be a torsion-free abelian group. Then $RG$ is a nonsingular ring if and only if $R$ is a nonsingular ring.

We close this section with a number of related results. According to Proposition 4.2.6, a commutative ring is nonsingular if and only if it is semiprime. Consequently, we have the following results:

Proposition 4.3.10

Let $R$ be a commutative ring and let $G$ be a group. Then $RG$ is semiprime if and only if $R$ is nonsingular and the order of no finite normal subgroup of $G$ is a zero divisor in $R$.

Proof: The result follows easily from Theorem 1.8.2 and Proposition 4.2.6. $lacksquare$

Corollary 4.3.11

Let $R$ be a commutative ring and let $G$ be an abelian group. Then $RG$ is nonsingular if and only if $R$ is nonsingular and the order of no finite subgroup of $G$ is a zero divisor in $R$. 
Proof: Clearly if $R$ is a commutative ring and $G$ is an abelian group, then $RG$ is commutative and the result follows from the previous proposition.

Proposition 4.3.12

Let $R$ be a commutative ring and let $G$ be a prime abelian group. Then $RG$ is nonsingular if and only if $R$ is nonsingular.

Proof: The result follows from Theorem 1.8.3.

Proposition 4.3.13

Let $R$ be a commutative ring and let $G$ be an ordered group. Then $RG$ is semiprimitive if and only if $R$ is nonsingular.

Proof: The result follows from Theorem 1.8.4.

Corollary 4.3.14

Let $R$ be a commutative ring and let $G$ be a torsion-free abelian group. Then $RG$ is semiprimitive if and only if $R$ is nonsingular.

Proof: Since a torsion-free abelian group is ordered, the result follows easily from the previous proposition.

Proposition 4.3.15

Let $R$ be a commutative ring and let $G$ be an abelian group such that $G$ is not torsion. Then $RG$ is semiprimitive if and only if $R$ is nonsingular and the order of no finite subgroup of $G$ is a zero divisor in $R$.

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4.4 Some relations between $Z(RG)$, $P(RG)$ and $J(RG)$

Our goal here is to determine relations between $Z(RG)$, $P(RG)$ and $J(RG)$. We show that under certain conditions $Z(RG)$ and the two radicals coincide. It is worth noting that the singular ideal $Z(R)$ is not a radical in the sense that $R/\sqrt[\down]{Z(R)}$ need not be nonsingular (see Passman [Pa3], p.261, Problem 7 for an example). Another example of this (which is a modification of an example by Fisher [Fi1]) is as follows:

Example 4.4.1

Let $R$ be the ring generated over the integers by the noncommuting elements $x_1, x_2, \ldots$ subject to the conditions that $x_i x_j = 0$ for $j \geq i$. Then $x_i' = R$ and for any $i \geq 2$, $x_i'$ is the right ideal of $R$ generated by $x_i, x_{i+1}, \ldots$. Since $\{0, x_i\}$ is a right ideal of $R$ and $x_i' \cap \langle 0, x_1 \rangle = \{0\}$ for any $i \geq 2$, so $x_i \not\in Z(R)$ for $i \geq 2$. It is not difficult to see that any monomial of the form

$$x_{j_1} x_{j_2} \cdots x_{j_n},$$

where $j_1 > j_2 > \ldots > j_n \geq 2$ is not contained in $Z(R)$. Since $(Rx_1)' = R$ is essential in $R$, so $Rx_1 \subseteq Z(R)$ and we have that $Z(R)$ is the ideal of $R$ generated by $x_1$.

Now since $x_2 \not\in Z(R)$, so $x_2 + Z(R)$ is a nonzero element of $R/\sqrt[\down]{Z(R)}$. Since $x_2 x_1 \in Z(R)$, so $x_1 + Z(R) \in (x_2 + Z(R))'$. Then since $x_2 x_j = 0$ for all $j \geq 2$, we have
that \((x_2 + Z(R))' = R/Z(R)\). Therefore \((x_2 + Z(R))'\) is essential in \(R/Z(R)\) and hence,
\[x_2 + Z(R) \in Z(R/Z(R))'.\] Since \(x_2 + Z(R)\) is nonzero, so \(Z(R/Z(R))' \neq \{0 + Z(R)\}\). ■

Lemma 4.4.1

Let \(R\) be a ring. If \(K\) is a two-sided ideal of \(R\) such that \(R/K\) is semiprime, then \(K \supseteq P(R)\).

Proof: First, we observe that if \(\psi\) is a ring homomorphism from \(R\) to \(R'\), then \(\psi(P(R)) \subseteq P(R')\). Indeed, let \(P'\) be a prime ideal of \(R'\). Then the mapping \(\psi': R \to R'/P',\) defined by \(\psi'(a) = \psi(a) + P', \ a \in R,\) is a ring homomorphism. Note that \(\ker \psi'\) is a prime ideal of \(R\). So we have \(P(R) \subseteq \ker \psi'\) and therefore, \(\psi(P(R)) \subseteq P'\).

Since \(P(R')\) is the intersection of all the prime ideals of \(R'\), it follows that \(\psi(P(R)) \subseteq P(R')\). Now substitute \(R'\) with \(R/K\). We then have \(\psi(P(R)) \subseteq P(R/K) = \{0 + K\}\). That is, if \(a \in P(R)\), then \(\psi(a) \in P(R/K) = \{0 + K\}\).

Thus \(a \in K\) and consequently \(P(R) \subseteq K\). ■

Using arguments similar to those in the proof of the preceding lemma, we have the following result:
Lemma 4.4.2

Let $R$ be a ring. If $K$ is a two-sided ideal of $R$ such that $\frac{R}{K}$ is semiprimitive, then $K \supseteq J(R)$.

The following results up to Corollary 4.4.6 have appeared in Groenewald’s Ph.D. thesis [Gr].

Proposition 4.4.3

Let $R$ be a ring and let $G$ be a group. Then $P(R)G = P(RG)$ if and only if the order of no finite normal subgroup of $G$ is a zero divisor in $\frac{R}{P(R)}$.

Proof: Suppose that the order of no finite normal subgroup of $G$ is a zero divisor in $\frac{R}{P(R)}$. From Proposition 1.8.1(ii), we have $P(R)G \subseteq P(RG)$. For the reverse inclusion, since $\frac{RG}{P(R)G} \equiv \left(\frac{R}{P(R)}\right)G$ and $\frac{R}{P(R)}G$ is semiprime by Theorem 1.8.2, so $\frac{RG}{P(R)G}$ is semiprime. It then follows from Lemma 4.4.1 that $P(RG) \subseteq P(R)G$.

Hence, $P(R)G = P(RG)$.

Conversely, if $P(R)G = P(RG)$, then $\frac{RG}{P(RG)} = \frac{RG}{P(R)G} \equiv \left(\frac{R}{P(R)}\right)G$. It follows that $\left(\frac{R}{P(R)}\right)G$ is semiprime and by Theorem 1.8.2, the order of no finite normal subgroup of $G$ is a zero divisor in $\frac{R}{P(R)}$.

A consequence of this result is as follows:
Corollary 4.4.4

Let $R$ be a ring and let $G$ be a finite group of order $n$. Then $P(R)G = P(RG)$ if and only if $n$ is not a zero divisor in $\frac{R}{P(R)}$.

Proposition 4.4.5

Let $R$ be a ring and let $G$ be a group. Then $P(R)G = P(RG)$ if and only if $\frac{R}{P(R)}G$ is semiprime.

Proof: Assume that $\frac{R}{P(R)}G$ is semiprime. Since $\frac{R}{P(R)G} \cong (\frac{R}{P(R)})G$, so $\frac{RG}{P(R)G}$ is also semiprime and consequently $P(RG) \subseteq P(RG)$ by Lemma 4.4.1. By Proposition 1.8.1(ii), it is always true that $P(R)G \subseteq P(RG)$. Hence, $P(R)G = P(RG)$ as desired. The reverse implication is clear. ■

Corollary 4.4.6

Let $R$ be a ring and let $G$ be a torsion-free group. Then $P(R)G = P(RG)$.

Proof: This is clear from Proposition 4.4.3 since a torsion-free group contains no finite normal subgroup other than the trivial subgroup $\{1\}$. ■

Proposition 4.4.7

Let $R$ be a commutative ring satisfying the a.c.c. on principal annihilator ideals and let $G$ be an ordered group. Then $J(RG) = P(RG) = P(R)G = Z(R)G$. 

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Proof: Since $R$ is a commutative ring satisfying the a.c.c. on principal annihilator ideals, we have $Z(R) = P(R)$ by Proposition 4.2.11 and consequently $Z(R)G = P(R)G$. By Theorem 1.8.4, $\frac{RG}{P(R)G} \cong \frac{R}{P(R)}G$ is semiprimitive. Then by Lemma 4.4.2, $J(RG) \subseteq P(R)G$. But we always have $P(R)G \subseteq P(RG)$ (by Proposition 1.8.1(ii)) and $P(RG) \subseteq J(RG)$. Putting all these information together, we have

$$Z(R)G = P(R)G \subseteq P(RG) \subseteq J(RG) \subseteq P(R)G;$$

hence, the equality $J(RG) = P(RG) = P(R)G = Z(R)G$ follows. ■

Corollary 4.4.8

Let $R$ be a commutative ring satisfying the a.c.c. on principal annihilator ideals and let $G$ be a torsion-free abelian group. Then

$$J(RG) = P(RG) = P(R)G = Z(R)G.$$ 

Proof: The result follows readily from Proposition 4.4.7 since a torsion-free abelian group is ordered. ■

Corollary 4.4.9

Let $R$ be a commutative ring satisfying the a.c.c. on principal annihilator ideals and let $G$ be a free abelian group. Then


Proof: From Proposition 2.4.8, we know that $Z(R)G = Z(RG)$. The rest of the equality follows easily from Corollary 4.4.8 since a free abelian group is torsion-free. ■
Since a Goldie ring has the a.c.c. on right annihilator ideals, the following corollaries are straightforward consequences of the above results.

Corollary 4.4.10

Let $R$ be a commutative Goldie ring and let $G$ be an ordered group. Then

$$J(RG) = P(RG) = P(R)G = Z(R)G.$$  

Corollary 4.4.11

Let $R$ be a commutative Goldie ring and let $G$ be a torsion-free abelian group. Then

$$J(RG) = P(RG) = P(R)G = Z(R)G.$$  

Corollary 4.4.12

Let $R$ be a commutative Goldie ring and let $G$ be a free abelian group. Then


Proposition 4.4.13

Let $R$ be a commutative Noetherian ring and let $G$ be a finite abelian group of order $n$ such that $n$ is not a zero divisor in $R/P(R)$. Then

$$P(RG) = P(R)G = Z(R)G = Z(RG).$$  

Proof: Since $R$ is a commutative Noetherian ring, we have $Z(R) = P(R)$ by Corollary 4.2.14 and consequently $Z(R)G = P(R)G$. From Theorem 1.7.2(i), we know that $RG$ is a commutative Noetherian ring. Then by Corollary 4.2.14 again, $Z(RG) = P(RG)$. By
Corollary 4.4.4 we have that $P(R)G = P(RG)$. Putting all these together gives us the equality $P(RG) = P(R)G = Z(R)G = Z(RG)$. ■