Chapter 1

Introduction

As a branch of discrete mathematics, graph theory has now become a very popular topic in Mathematics. Graphs serve as mathematical models to analize successfully many concrete real-world problems. Certain problems in physics, chemistry, engineering, communications and computer technology can be formulated as problems in graph theory. Also, many branches of mathematics, such as matrix theory, group theory, topology, and probability, have interactions with graph theory.

The objectives of this thesis are to

(i) investigate those smallest regular graphs with given girths and having small crossing numbers;

(ii) classify 5-regular graphs according to their crossing numbers and with given number of vertices.

This thesis is divided into 4 chapters. Chapter 1 contains the basic definitions concerning graphs and some theorems which will be used in the subsequent chapters of the thesis.

In Chapter 2, we give a survey on the problem of finding smallest regular graphs with given girths, in which all known small cages are given. Different variations of the cage problem are also discussed and some related questions are posed.

A variation of the cage problem is discussed in Chapter 3, where we determine those smallest regular graphs with given girths and crossing numbers. In particular, we investigate those smallest regular graphs with given girths and crossing number c, for $c \in \{0, 1, 2\}$.

Finally, a classification of 5-regular graphs according to their crossing numbers and with given number of vertices is discussed in Chapter 4. In particular, it is shown that there exist no 5-regular graphs on 12 vertices with crossing number one. This, together with a result in Chapter 3, implies that the minimum number of vertices in a 5-regular graph with girth three and crossing number one is 14.

1.1 Preliminaries

In this chapter, we present some definitions and theorems which will be frequently referred to throughout this thesis. The basic notations, terminology used are commonly found in most graph theory text books. For those terms and definitions not included here, the reader may refer to [27], [51], [15] and [56].

1.2 Graphs

A **simple graph** G consists of a finite non-empty set $V(G)$ of elements called **vertices** and a finite (possibly empty) set $E(G)$ of distinct unordered pairs of distinct vertices of $V(G)$ called **edges**. We call $V(G)$ the **vertex set** and $E(G)$ the **edge set** of G. The number of vertices of G is the **order** of G, and the number of edges of G is the **size** of G. (These are sometimes written as $|V(G)|$ and $|E(G)|$ respectively.)

The edge $e = (u, v)$ is said to join the vertices u and v. If $e = (u, v)$ is an edge of a graph G, then u and v are **adjacent vertices**, while u and e are **incident**, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are **adjacent edges**. It is convenient to henceforth denote an edge by uv or vu rather than by (u, v).

If one allows more than one edge (but yet a finite number) between the same pair of vertices in a graph, the resulting structure is a **multigraph**. Such edges are called **multiple edges**. A **loop** is an edge that joins a vertex to itself. Unless otherwise mentioned, the graphs we consider are simple graphs with no loops and multiple edges.

The **degree of a vertex** v in a graph G is the number of edges of G incident with v, which is denoted by $deg_G(v)$ or simply $deg(v)$ if G is clear from the context. A vertex is called **even** or **odd** according to whether its degree is even or odd. A vertex of degree 0 in G is called an **isolated vertex** and a vertex of degree 1 is an **end vertex** of G. If u is adjacent to v, then u is a **neighbor** of v. The **neighborhood** of v, denoted by $N_G(v)$ or simply $N(v)$, is the set of all vertices of G adjacent to v.

An **independent set of vertices** in G is a set of vertices of G, no two of which are adjacent.

A sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers where $d_1 \leq d_2 \leq \cdots \leq d_n$ is called a **degree sequence** of a graph G if the vertices of G can be labelled as $v_1, v_2, \ldots, v_n$ so that $deg(v_i) = d_i$ for all i.

If all of the vertices of G have the same degree r, then G is **regular of**

degree r or r**-regular**. A 3-regular graph is a **cubic graph**.

The following result, due to Leonhard Euler in 1736, is called the **handshaking lemma**.

Theorem 1.1 (Handshaking Lemma) *Let G be a graph of order n and size m where $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $\sum_{i=1}^{n} deg(v_i) = 2m$.* □

Two graphs G_1 and G_2 are **isomorphic** (written $G_1 \cong G_2$) if there exists a one-to-one mapping ϕ, called an **isomorphism** from $V(G_1)$ onto $V(G_2)$, such that ϕ preserves adjacency of vertices; that is $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$.

1.3 Paths and Cycles

A sequence of edges of the form $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ (sometimes abbreviated to $v_0v_1 \cdots v_k$) is a **walk** between v_0 and v_k. The number k (the number of occurrences of edges) is called the **length** of the walk. If these edges are all distinct, then the walk is a **trail**, and if the vertices $v_0, v_1, \ldots, v_k$ are also distinct, then the walk is a **path**. A walk or trail $v_0v_1 \cdots v_k$ is closed if $v_0 = v_k$, and for $k > 0$, a closed walk in which the vertices $v_0, v_1, \ldots, v_{k-1}$ are all distinct is a **cycle**. A cycle is **even** if its length is even; otherwise it is **odd**. A cycle of length n is an n-**cycle**; a 3-cycle is also called a **triangle**. A graph of order n that is a path or a cycle is denoted by P_n or C_n, respectively.

The length of a shortest cycle in a graph G is the **girth** of G and is denoted by $g(G)$ or simply by g if G is clear from the context. If u and v are vertices in G, then the length $d(u, v)$ of any shortest path from u to v is the **distance** between u and v.

1.4 New Graphs from Old

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, then we write $H \subseteq G$. If $V(H) = V(G)$, then H is called a **spanning subgraph** of G. If V_1 is any set of vertices in G, then the **subgraph induced by** V_1 is the subgraph of G obtained by taking the vertices in V_1 and joining those pairs of vertices in V_1 which are joined in G.

If $e \in E(G)$, then $G - e$ is the graph obtained from G by removing the edge e; more generally, $G - \{e_1, \ldots, e_k\}$ is the graph obtained from G by removing the edges $e_1, \ldots, e_k$. Similarly, if $v \in V(G)$, then $G - v$ is the graph obtained from G by removing the vertex v together with all its incident edges.

The deletion of a set of vertices from G is defined analogously.

The **null graph** (or **empty graph**) of order n is the graph with n vertices and no edges.

An **elementary subdivision** of a nonempty graph G is a graph obtained from G by removing some edge $e = uv$ and adding a new vertex w and two edges uw and vw. A **subdivision** of G is a graph obtained from G by a succession of elementary subdivisions (including the possibility of none). Two graphs are **homeomorphic** if they have isomorphic subdivisions.

An **elementary contraction** of a graph G is obtained by identifying two adjacent vertices u and v, that is, by the removal of u and v and the addition of a new vertex w adjacent to those vertices to which u or v was adjacent. A graph G is **contractible** to a graph H if H can be obtained from G by a succession of elementary contractions.

If G_1 and G_2 are disjoint graphs, then their **union** $G = G_1 \cup G_2$ is the graph with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If a graph G consists of $k(\geq 2)$ disjoint copies of a graph H, then we write $G = kH$.

The **cartesian product** $G = G_1 \times G_2$ is the graph with the vertex set $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

The **complement** $\overline{G}$ of a graph G is the graph with the vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ whenever they are not adjacent in G.

1.5 Connectivity

A graph G is **connected** if there is a path joining each pair of vertices of G; a graph that is not connected is called **disconnected**. A maximal connected subgraph of G is called a **component** of G.

If G is a connected graph, and if the graph $G - e$ is disconnected for some edge e, then e is called a **bridge** of G. More generally, a **cutset** in G is a set of edges whose removal disconnects G. If G is connected, its **edge connectivity** $\lambda(G)$ is the size of the smallest cutset in G. A graph G is said to be k-**edge connected** if $\lambda(G) \geq k$.

1.6 Examples of Graphs

A graph is called **complete** if every pair of vertices are adjacent, and the complete graph on n vertices is denoted by K_n.

A **bipartite graph** G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 and a vertex of V_2. A **complete bipartite graph** is a bipartite graph in which each vertex in V_1 is adjacent to every vertex in V_2. If $|V_1| = s$ and $|V_2| = t$, then the complete bipartite graph is denoted by $K_{s,t}$.

The **Platonic graphs** are the graphs corresponding to the vertices and edges of the five regular solids—the tetrahedron, cube, octahedron, dodecahedron and icosahedron (see Figure 1.1).

1.7 Planar Graphs

A graph is **planar** if it can be drawn in the plane such that no edges intersect except at a vertex to which they are both incident. A graph G drawn in the plane in this way is called a **plane graph**. The regions defined by a plane graph of G are called the **faces** of G. The following result, known as **Euler's**

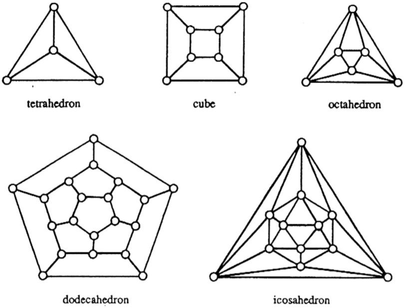

Figure 1.1: Platonic graphs

formula for plane graphs or Euler's polyhedron formula, establishes a connection among the number of vertices, edges, and faces of any plane graph.

Theorem 1.2 (Euler's Formula for Plane Graphs) *Let G be a plane graph with $n(\geq 3)$ vertices, m edges and f faces. Then*

$$n - m + f \geq 2$$

where equality holds if and only if G is a connected graph. □

A planar graph G is a **maximal planar graph** or **triangulation** if each face of G is bounded by a triangle.

Theorem 1.3 *If G is a maximal planar graph of order $n \geq 3$ and size m, then*

$$m = 3n - 6.$$

□

Colloraly 1.1 *If G is a planar graph of order $n \geq 3$ and size m, then*

$$m \leq 3n - 6.$$

□

There are two graphs, namely K_5 and $K_{3,3}$, that play an important role in the study of planar graphs.

Theorem 1.4 *The graphs K_5 and $K_{3,3}$ are non-planar.* □

A necessary and sufficient condition for a graph to be planar has been given by Kuratowski [66].

Theorem 1.5 (Kuratowski's Theorem) *A graph G is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.* □

1.8 The Crossing Number of a Graph

The **crossing number** $cr(G)$ of a graph G is the minimum number of pairwise intersections of its edges when G is drawn in the plane. When G is drawn in the plane, we assume that

(a) adjacent edges never cross;

(b) two non-adjacent edges cross at most once;

(c) no edge crosses itself;

(d) no more than two edges cross at a point of the plane.

Finding the crossing number of a graph is one of the many interesting unsolved problems in graph theory. Although the crossing number has been obtained for some graphs in families like K_n and $K_{s,t}$, a general solution for graphs in those families is still unknown. It has been shown by Garey and Johnson [44] that the problem of computing crossing number is NP-complete.

Let G be a graph. The **removal number** of G, denoted $rem(G)$, is defined to be the minimum number of edges in G whose removal results in a planar graph. Obviously $cr(G) \geq rem(G)$. In the event that $rem(G) = 1$, then G contains an edge e such that $G - e$ is planar. Such an edge is called a **p-critical edge** of G.

A few observations will be useful. Clearly a graph G is planar if and only if $cr(G) = 0$. Further, if $H \subseteq G$, then $cr(H) \leq cr(G)$, while if H is a subdivision of G, then $cr(G) = cr(H)$.

We list some results that will be used in the subsequent chapters.

Theorem 1.6 *[47] For $1 \leq n \leq 10$,*

$$cr(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

□

Theorem 1.7 *[62, 113] If s and t are integers ($s \leq t$) and either $s \leq 6$ or $s = 7$ and $t \leq 10$, then*

$$cr(K_{s,t}) = \left\lfloor\frac{s}{2}\right\rfloor\left\lfloor\frac{s-1}{2}\right\rfloor\left\lfloor\frac{t}{2}\right\rfloor\left\lfloor\frac{t-1}{2}\right\rfloor.$$

□

In [52], Harary, Kainen and Schwenk showed that $cr(C_3 \times C_3) = 3$. They also conjectured that $cr(C_n \times C_m) = n(m-2)$ for $3 \leq m \leq n$. The crossing number of $C_n \times C_m$ is known for every $m \geq n$, for each n satisfying $3 \leq n \leq 6$. The cases $n = 3, 4, 5$ and 6 were solved in [86], [7], [63] and [93] respectively.

Theorem 1.8 *[86] For all $t \geq 3$, $cr(C_3 \times C_t) = t$.* □

Theorem 1.9 *[7] For all $t \geq 4$, $cr(C_4 \times C_t) = 2t$.* □

Theorem 1.10 *[63] For all $t \geq 5$, $cr(C_5 \times C_t) = 3t$.* □

Theorem 1.11 *[93] For all $t \geq 6$, $cr(C_6 \times C_t) = 4t$.* □

Finally, we state an upper bound for $cr(\overline{C_n})$ and its exact values for $3 \leq n \leq 10$. (See [48].)

Theorem 1.12 *For the complement $\overline{C_n}$ of an n-cycle,*

$$cr(\overline{C_n}) \leq \begin{cases} \frac{1}{64}(n-3)^2(n-5)^2 & \text{for } n \text{ odd,} \\ \frac{1}{64}n(n-4)(n-6)^2 & \text{for } n \text{ even.} \end{cases}$$

□

Theorem 1.13

$$cr(\overline{C_n}) = \begin{cases} 0 & \text{for } 3 \leq n \leq 6, \\ 1 & \text{for } n = 7, \\ 2 & \text{for } n = 8, \\ 9 & \text{for } n = 9, \\ 15 & \text{for } n = 10. \end{cases}$$

□