CHAPTER 1

Introduction and Definitions

1.1 Preliminaries and Definitions

A *graph* G is an ordered pair (V, E) where $V = V(G)$ is a non-empty set of vertices and $E = E(G)$ is a set of edges. Two vertices are *adjacent* if they are joined by an edge. An edge joining a vertex to itself is called a *loop*. A *simple* graph is a graph without loops or multiple edges. Unless otherwise stated, all graphs are simple. The edge joining the two adjacent vertices a and b in a graph G is denoted by (a, b). A graph with just one vertex and no edges is a *trivial* graph and all other graphs are *non-trivial*. A *null graph* is a graph with no edges. The *degree* of a vertex v in a graph G, denoted by $\deg(v)$, is the number of edges incident to v.

The vertex v is an *isolated vertex* if $\deg(v) = 0$ and is an *end-vertex* if $\deg(v) = 1$. A graph G is *k-regular* if $\deg(v) = k$ for all $v \in V(G)$. A *regular* graph is one that is k-regular for some k.

Let $G = (V, E)$ be a graph. A graph $H = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. H is a *proper subgraph* of G if $V' \subset V$ or $E' \subset E$. A *spanning subgraph* of G is a subgraph H of G such that $V' = V$. A subgraph H of G is an *induced subgraph* of G if for two vertices $a, b \in V'$, $(a, b) \in E(H)$ if and only if $(a, b) \in E(G)$. In Figure 1.1, G_1 is a spanning subgraph of G; G_2 is an induced subgraph of G.

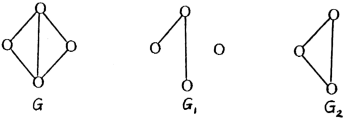


Figure 1.1

Two graphs G and H are *isomorphic*, written $G \cong H$, if there exists a one to one correspondence between their vertex sets which preserves adjacency. Figure 1.2 shows two isomorphic graphs.

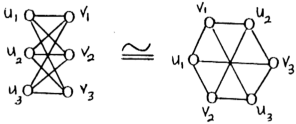


Figure 1.2

A *path of a graph* is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ such that all the v_i's, $0 \le i \le n$ and e_j's, $1 \le j \le n$ are distinct. The *length* of a path is the number of edges that lies along that path. The *distance* $d(u, v)$ between two vertices u and v is the length of the shortest path joining them if any.

A graph is *connected* if every two vertices are joined by a path, otherwise it is *disconnected.* The *complement* of a graph G, denoted by $\bar{G}$, has $V(G)$ as its vertex set, and two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

A *cycle* with n vertices, denoted by C_n, is a connected graph where each of its vertices is of degree 2. Note that C_n is 2-regular. A *complete graph* with n vertices, denoted by K_n, has every pair of its vertices adjacent. Note that K_n is $(n-1)$-regular and that a C_3 is also a K_3. A *bipartite graph* G is a graph whose vertex set can be partitioned into two subsets X and Y such that vertices in X are adjacent only to vertices in Y and vice versa. If every vertex of X is adjacent to every vertex of Y, then G is a *complete bipartite graph.* If G is a complete bipartite graph with $|X| = m$ and $|Y| = n$, we write $G = K_{m,n}$. A *star* is a complete bipartite graph $K_{1,n}$.

A maximal connected subgraph of G is a *connected component* or *component* of G. A *cut-vertex* of a graph is a vertex whose removal increases the number of components in G. An *acyclic* graph contains no cycles. A *tree* with n vertices, denoted by T_n, is a connected acyclic graph. A *path* with n vertices, denoted by P_n, is a tree with precisely two end-vertices. Any graph without cycles is a *forest*, so the components of a forest are trees. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. If $\kappa(G) \geq n$, then G is *n-connected*. In Figure 1.3, the graph G_3 is 1-connected and u is a cut-vertex; the graph G_4 is 2-connected and removing the vertices v and w will result in a disconnected graph with two components. Note that $\kappa(T_n) = 1$, $\kappa(C_n) = 2$, $\kappa(K_n) = n - 1$ and $\kappa(K_{m,n}) = \min\{m, n\}$.

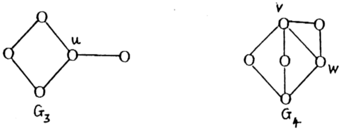


Figure 1.3

1.2 Colouring of Graphs and Chromatic Number

A *colouring* of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colour. The set of all vertices with a same colour forms a *colour class*. An *n-colouring* of a graph G partitions the vertex set $V(G)$ into n colour classes.

A graph is called *n-colourable* if it has an n-colouring. The *chromatic number* $\chi(G)$ of a graph G is the minimum number k such that G is k-colourable, i.e $\chi(G) = k$ if G is k-colourable but not $(k-1)$-colourable. The chromatic numbers of the three graphs shown in Figure 1.4 are 4, 4 and 3 in that order.

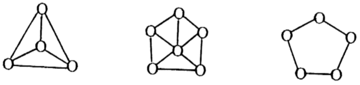


Figure 1.4

A graph G is *critical* if $\chi(G - v) < \chi(G)$ for every vertex v in G. The three graphs shown in Figure 1.4 are all critical.

Theorem 1.2.1. (i) $\chi(K_p) = p$

(ii) $\chi(K_{m,n}) = 2$

(iii) $\chi(C_{2n}) = 2;\ \ \chi(C_{2n+1}) = 3,\ n \geq 1.$

Theorem 1.2.2. *A graph is 2-colourable (bipartite) if and only if it contains no cycle of odd length.*

Since all trees are acyclic, we have the corollary.

Corollary 1.2.3. *Every tree is 2-colourable.*

If $\chi(G) = n$ and every n-colouring of G induces the same partition of $V(G)$ into n colour classes, then G is said to be *uniquely n-colourable.*

Theorem 1.2.4 (Theorem 12.17 of [35], pg. 139). *Every uniquely n-colourable graph is $(n-1)$-connected.*

Corollary 1.2.5. *In the n-colouring of a uniquely n-colourable graph, the subgraph induced by any k colour classes $2 \leq k \leq n$, is $(k-1)$-connected.*

Consider the graph in Figure 1.5. This graph is uniquely 3-colourable because every 3-colouring has the same partition $\{v_0\}$, $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$ of $V(G)$.

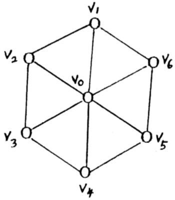


Figure 1.5

1.3 Chromatic Polynomials

Given λ colours, the number of λ-colourings of G is called the *chromatic polynomial* of G and is denoted as $P(G;\lambda)$. This concept was first introduced by Birkhoff [3] in 1912 in order to tackle the four colour problem (now the four colour theorem) and was further developed by Read and Tutte.

The chromatic polynomial is a polynomial of degree n with integer coefficients, where n is the number of vertices in G. For example, consider the complete graph K_n. Choose a vertex in K_n. There are λ ways of colouring this vertex. Picking another vertex we have $\lambda - 1$ colours with which it can be coloured. Pick another vertex; it is adjacent to both vertices already coloured, and can therefore be coloured in $\lambda - 2$ ways. We continue in this way; the last vertex can be given any of the remaining $\lambda - (n-1)$ colours. Hence

$$P(K_n;\lambda) = \lambda(\lambda-1)\dots(\lambda-n+1) = (\lambda)_n.$$

Also, if G is a null graph on n vertices, denoted by N_n, any vertex can be given any of the λ colours and therefore $P(N_n;\lambda) = \lambda^n$.

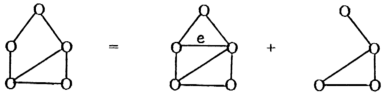
The following results are useful in finding the chromatic polynomial of a graph. For a given graph G, let $G+e$ be the graph obtained from G by adding an edge e to two non-adjacent vertices in G and $G \circ e$ be the graph obtained from G by contracting the edge e and removing all but one multiple edges in the resulting graph.

Theorem 1.3.1. $P(G;\lambda) = P(G+e;\lambda) + P(G \circ e;\lambda)$.

Proof: Let e be the new edge joining the two non-adjacent vertices a and b in G. The number of ways of colouring G using no more than λ colours where a and b are coloured differently is the number of ways of colouring $G+e$ using no more than λ colours. The number of ways of colouring G using no more than λ colours where a and b are coloured the same is the number of ways of colouring $G \circ e$ using no more than λ colours. Thus $P(G;\lambda) = P(G+e;\lambda) + P(G \circ e;\lambda)$ and the result follows.

We usually let a drawing of a graph to represent its chromatic polynomial. This notational device was introduced by Zykov [78].

Theorem 1.3.1 is illustrated as follows



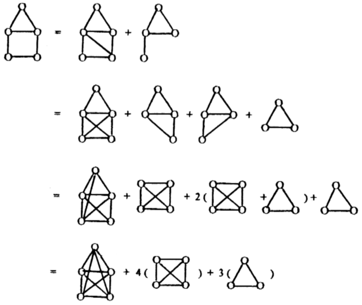
Let $G - e$ be the graph obtained from G by deleting an edge e in G.

Corollary 1.3.2. $P(G;\lambda) = P(G - e;\lambda) - P(G \circ e;\lambda)$.

The recursive formulas in Theorem 1.3.1 and Corollary 1.3.2 allow us to compute the chromatic polynomial of a graph in two ways. We can either use Corollary 1.3.2 to reduce it to null graphs or use Theorem 1.3.1 to reduce it to complete graphs. The former method is more suited to graphs with few edges whereas the latter can be applied more efficiently to graphs with many edges. The process of expressing chromatic polynomial in terms of chromatic polynomials of complete graphs or null graphs is sometimes called "chromatic reduction".

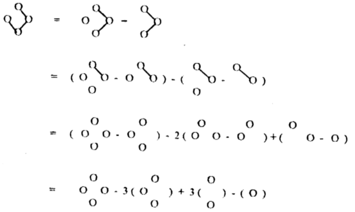
Example 1.1.

(a)



$$= (\lambda)_5 + 4(\lambda)_4 + 3(\lambda)_3$$

(b)



$$= \lambda^4 - 3\lambda^3 + 3\lambda^2 + \lambda$$

Corollary 1.3.3. *Let $P(G;\lambda)$ be the chromatic polynomial of a graph G with $|V(G)| = n$. Then $P(G;\lambda)$ is a polynomial in λ of degree n. Furthermore, it has integer coefficients, with leading term λ^n, constant term zero and the coefficients alternate in sign.*

Proof: By the nature of the chromatic reduction, exactly one graph (either K_n or N_n) will be obtained at the final stage. Hence $P(G;\lambda)$ is a polynomial in λ of degree n with leading term λ^n. Note that if the constant term of $P(G;\lambda)$ is not zero i.e. $P(G;0) \neq 0$, it means that the graph can be coloured with no colour, which is impossible. The fact that the coefficients alternate in sign is proved by induction on the number of vertices and the number of edges. The proof can be found in Read [56].

Corollary 1.3.4. *If G is not N_n, then the sum of the coefficients in $P(G;\lambda)$ is equal to zero.*

Proof: By Corollary 1.3.3, let $P(G;\lambda) = \sum_{i=1}^{n} a_i\lambda^i$. Since G is not N_n, it cannot be coloured with 1 colour, so $P(G;1) = 0$ and the result follows.

Corollary 1.3.5. *Let* $P(G;\lambda) = \sum_{i=1}^{n} a_i\lambda^i$ *be the chromatic polynomial of a graph* G*. Then* $|a_{n-1}| = |E(G)|$.

Proof: In the chromatic reduction, every deletion of an edge contributes 1 to the absolute value of the coefficient of the second term of the chromatic polynomial and the result follows.

Suppose G_1 and G_2 are graphs each containing a complete subgraph K_r. The K_r-gluing of G_1 and G_2 is the union of G_1 and G_2 obtained by identifying the two subgraphs K_r. If $r = 1$, the graph thus obtained is a *vertex-gluing* of G_1 and G_2, and is called *edge-gluing* if $r = 2$.

Zykov [78] provides the following formula for finding $P(G;\lambda)$ where G is a K_r-gluing of some graphs G_1 and G_2.

Theorem 1.3.6 [78]. *Let* G *be a* K_r*-gluing of* G_1 *and* G_2*. Then*

$$P(G;\lambda) = \frac{P(G_1;\lambda)P(G_2;\lambda)}{P(K_r;\lambda)}.$$

Theorem 1.3.7. *The chromatic polynomial of a tree with* n *vertices* T_n *is*

$$P(T_n;\lambda) = \lambda(\lambda-1)^{n-1}.$$

Proof: By induction on n. Assume the result is true for $n = k$. T_{k+1} is a vertex-gluing of T_k and K_2. By Theorem 1.3.6,

$$\begin{aligned} P(T_{k+1};\lambda) &= \frac{P(T_k;\lambda)P(K_2;\lambda)}{P(K_1;\lambda)} \\ &= \frac{\lambda(\lambda-1)^{k-1}\lambda(\lambda-1)}{\lambda} \\ &= \lambda(\lambda-1)^k. \end{aligned}$$

By applying Corollary 1.3.2 and Theorem 1.3.7, it can be shown that

Theorem 1.3.8. $P(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$.

Definition 1.3.9. The *join* of two graphs G_1 and G_2, denoted by G_1+G_2, is the graph obtained from the disjoint union of G_1 and G_2 by joining every vertex of G_1 to every vertex of G_2.

In particular, the *wheel* W_n is the join of K_1 and C_{n-1}. The single vertex K_1 is referred to as the *hub* of the wheel. To compute the chromatic polynomial of W_n, we observe that

Lemma 1.3.10. *Let G be the join of K_1 with a graph H. Then* $P(G;\lambda) = \lambda P(H;\lambda-1)$.

Proof: The vertex K_1 can be coloured in λ ways. The number of ways to colour the graph H using no more than $(\lambda-1)$ colours is $P(H;\lambda-1)$. Hence $P(G;\lambda) = \lambda P(H;\lambda-1)$.

Using Lemma 1.3.10 and Theorem 1.3.8, we have

Theorem 1.3.11. $P(W_n;\lambda) = \lambda\big((\lambda-2)^{n-1} + (-1)^{n-1}(\lambda-2)\big)$.

Let $f(\lambda) = \sum\limits_{i\geq 0} a_i(\lambda)_{m-i}$ and $g(\lambda) = \sum\limits_{i\geq 0} b_i(\lambda)_{n-i}$. Then define $f(\lambda)\circ g(\lambda)$ as $f(\lambda)\circ g(\lambda) = \sum\limits_{r=i+j} a_ib_j(\lambda)_{m+n-r}$. Here, the operation $\circ$ is referred to as *umbral multiplication.*

Let $P(G_1;\lambda)$ and $P(G_2;\lambda)$ be the chromatic polynomials expressed as sums of falling factorials, that is

$$P(G_1;\lambda) = a_0(\lambda)_m + a_1(\lambda)_{m-1} + a_2(\lambda)_{m-2} + \cdots + a_m(\lambda)_0$$

$$P(G_2;\lambda) = b_0(\lambda)_n + b_1(\lambda)_{n-1} + b_2(\lambda)_{n-2} + \cdots + b_n(\lambda)_0.$$

Then

$$P(G_1;\lambda)\circ P(G_2;\lambda) = a_0b_0(\lambda)_{m+n} + (a_0b_1 + a_1b_0)(\lambda)_{m+n-1}$$
$$+ (a_0b_2 + a_1b_1 + a_2b_0)(\lambda)_{m+n-2} + \cdots + a_mb_n(\lambda)_0.$$

The following theorem is due to Zykov [78].

Theorem 1.3.12 [78]. Let G_1 and G_2 be any graphs. Then $P(G_1+G_2;\lambda) = P(G_1;\lambda) \circ P(G_2;\lambda)$.

Notice that $C_4 = N_2 + N_2$. Applying Theorem 1.3.12, we see that

$$\begin{aligned} P(C_4;\lambda) &= \{(\lambda)_2 + (\lambda)_1\} \circ \{(\lambda)_2 + (\lambda)_1\} \\ &= (\lambda)_4 + 2(\lambda)_3 + (\lambda)_2 \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda. \end{aligned}$$

In the wheel W_n, the edges that are incident to the hub are called the *spokes* of the wheel, the rest of the edges are the *rims* of the wheel. Note that W_n has $n-1$ spokes and $n-1$ rims. The graph U_n is obtained from the wheel W_n by deleting one of the spokes. This graph is referred to as a wheel with a missing spoke in [17]. Its chromatic polynomial is given by

Theorem 1.3.13. $P(U_n;\lambda) = \lambda\big((\lambda-2)^{n-3}(\lambda^2-3\lambda+3) + (-1)^{n-3}(\lambda-2)\big)$.

1.4 Automorphisms of Graphs

A one-to-one mapping from a finite set onto itself is called a *permutation.* An *automorphism* of a graph G is a permutation α of $V(G)$ which has the property that (u,v) is an edge of G if and only if $(\alpha(u),\alpha(v))$ is an edge of G. In other words, each automorphism α of G is a permutation of the vertices $V(G)$ which preserves adjacency. The set of all automorphisms of G, with the operation of composition, is the *automorphism group* of G, denoted by $\mathrm{Aut}(G)$.

Consider the graph K_4 with the vertices labelled as shown in Figure 1.6(a). Now $\alpha = (12)(34)$ is a permutation on $V(K_4)$ which interchanges 1 and 2 and interchanges 3 and 4. Then the image of K_4 is represented in Figure 1.6(b). Clearly α is an automorphism of K_4.

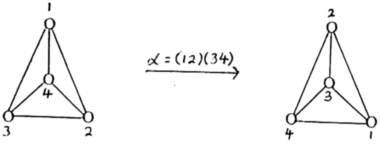


Figure 1.6(a) Figure 1.6(b)

The graph M in Figure 1.7 has only two automorphisms, i.e. $\alpha = 1$ (the identity permutation which fixes every number) and $\beta = (34)$ which interchanges 3 and 4. Here, $\text{Aut}(M) = \{1, (34)\}$.

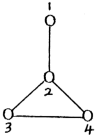


Figure 1.7 The graph M

We say that G is *vertex-transitive* if $\text{Aut}(G)$ acts transitively on $V(G)$, that is given any two vertices u and v, there is an automorphism $\alpha \in \text{Aut}(G)$ such that $\alpha(u) = v$. Note that these two vertices u and v must have the same degree. Hence, if G is not regular, then it is not vertex-transitive. Examples of vertex-transitive graphs are K_m, C_n and the *Petersen graph* shown in Figure 1.8.

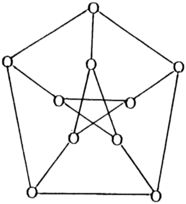


Figure 1.8 Petersen graph

The graph G is *edge-transitive* if given any pair of edges there is an automorphism which sends one into the other. Examples of edge-transitive graphs are K_m, C_n and $K_{m,m}$.

The graph shown in Figure 1.9 is vertex-transitive but not edge-transitive while the graph $K_{m,n}$ with $m \neq n$ is edge-transitive but not vertex-transitive. Examples of regular graphs which are edge-transitive but not vertex-transitive can be found in [6].

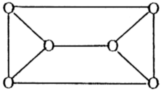


Figure 1.9