CHAPTER 4

The Chromatic Uniqueness of Edge-Gluing of $K_{2,s}$ and C_m

In this chapter, we prove the chromatic uniqueness of the edge-gluing of $K_{2,s}$ ($s \geq 1$) and C_m ($m \geq 3$), denoted as $K_{2,s} \cup_2 C_m$. This result is also obtained independently by Xu, Liu and Peng [77] using a different method.

Lemma 4.1. *Let $G \sim Y$. Suppose G has at most one triangle. Then $n(C_4^*, G) = n(C_4^*, Y)$ and $-n(C_5^*, G) + n(K_{2,3}, G) = -n(C_5^*, Y) + n(K_{2,3}, Y)$.*

Proof: The proof follows directly from Theorem 2.1.11 and Proposition 3.1.8.

Let G be a connected graph on p vertices and q edges. Then the *cyclomatic number* of G is $q - p + 1$.

Lemma 4.2. *Let G be a connected graph with cyclomatic number c. Then the number of $K_{2,3}$ in G is at most $\binom{c+1}{3}$.*

Proof: By induction on c.

If $c \leq 2$, the result is trivially true. Suppose the result is true for all connected graphs with cyclomatic number c where $c \geq 2$.

Let G be a connected graph with cyclomatic number $c+1$. Then G contains a cycle C. Delete an edge e from C. The resulting graph $G - e$ is connected and has cyclomatic number c. By the induction hypothesis, the number of $K_{2,3}$ in $G - e$ is at most $\binom{c+1}{3}$.

Let $\{K_{2,s_1}, \dots, K_{2,s_t}\}$ denote the set of all subgraphs (which are complete bipartite graphs) in G containing the edge e. Here $s_i \geq 3$ for $i = 1, \dots, t$. Notice that $(s_1 - 1) + \cdots + (s_t - 1) \leq c + 1$. Then the number of $K_{2,3}$ in G

containing the edge e is $\sum_{i=1}^{t}\binom{s_i-1}{2}$. It is a routine exercise to show that this number is no more than $\binom{c+1}{2}$. Consequently, the number of $K_{2,3}$ in G is at most $\binom{c+1}{3}+\binom{c+1}{2}=\binom{c+2}{3}$ and this furnishes the proof.

Lemma 4.3. *Suppose $m_i \geq 3$ is an integer for $i=1,2,\dots,t$. Then*

$$\sum_{i=1}^{t}\binom{m_i}{3} \leq \binom{\left(\sum_{i=1}^{t} m_i\right)-2(t-1)}{3} - 2(t-1).$$

Proof: The lemma is trivially true for $t=1$. It is routine to verify that

$$\binom{m_1}{3}+\binom{m_2}{3} \leq \binom{m_1+m_2-2}{3} - 2.$$

By repeatedly applying the above inequality, the lemma follows.

We can now prove the chromatic uniqueness of $K_{2,s} \cup_2 C_m$. Let H be a graph containing a subgraph of the form $K_{2,l}$ for some $l \geq 2$. Let x be a vertex in $H - K_{2,l}$. Then x is called a *t-vertex* to $K_{2,l}$ if x is adjacent to only two vertices of $K_{2,l}$ so that the resulting subgraph $K_{2,l} \cup \{x\}$ is isomorphic to $K_{2,l+1}$.

Theorem 4.4. *For any $s \geq 1$ and $m \geq 3$, the graph $G = K_{2,s} \cup_2 C_m$ is uniquely determined by its chromatic polynomial.*

Proof: Let Y be a graph such that $Y \sim G$. Then Y is a 2-connected graph on $s+m$ vertices and $2s+m-1$ edges (Proposition 3.1.8). By Theorem 3.1.7, Y contains no K_4-homeomorph as a subgraph because G contains no such subgraph.

When $s=1$, $K_{2,s} \cup_2 C_m$ is the vertex-gluing of K_2 and C_m. It is chromatically unique (see Theorem 3.2.13).

When $s=2$, $K_{2,s} \cup_2 C_m$ is the θ-graph and is chromatically unique (see [10]). For the case $s=3$, the graph $K_{2,3} \cup_2 C_m$ is chromatically unique (see Theorem 3.2.17). So we may assume that $s \geq 4$.

Since G has at most one triangle and $n(K_{2,3}, G) = \binom{s}{3}$, by Lemma 4.1, $n(K_{2,3}, Y) \geq \binom{s}{3}$ if $m \neq 5$ and $n(K_{2,3}, Y) = \binom{s}{3} - 1$ if $m = 5$. In either case, we see that Y contains a subgraph $K_{2,3}$. Let K denote this subgraph.

Let J be the graph $Y - K$ and assume that there are e edges joining K to J. Now note that J has $s + m - 5$ vertices and $2s + m - 7 - e$ edges and so $|E(J)| - |V(J)| = s - e - 2$.

Let $J_1, \ldots, J_k$ be the connected components of J, $k \geq 1$. Suppose there are e_i edges joining K and J_i, $i = 1, \ldots, k$.

We make the following observations:

(O1) : Each J_i contains at most one t-vertex to K. This is because if there are two t-vertices x_1 and x_2 from J_i to K, then there is a path in J_i connecting x_1 and x_2. This path together with K contains a K_4-homeomorph as a subgraph which is impossible.

(O2) : If $e_i = 2$, then J_i contains a t-vertex only if J_i is an isolated vertex because Y is 2-connected.

Let c_i denote the cyclomatic number of J_i, $i = 1, \ldots, k$. Then $\sum_{i=1}^{k} c_i = s - e - 2 + k$. Consequently, $e \leq s - 2 + k$. Since $e \geq 2k$, it follows that $1 \leq k \leq s - 2$. Let β denote the number of J_i's that are isolated vertices. Then clearly, $\beta \leq k - 1$.

There are two cases that we need to consider.

Case (1): All the J_i's are trees.

Assume that $e_i = 2$ for $i = 1, \ldots, k$. Then $k = s - 2$. From (O2), each isolated vertex of J could be a t-vertex to K and so

$$n(K_{2,3}, Y) \leq \binom{3+\beta}{3} \leq \binom{s}{3}.$$

Clearly, the second inequality holds if $\beta = k - 1$. When $\beta \leq k - 2$, $\binom{3+\beta}{3} < \binom{s}{3} - 1$.

Suppose $\beta = k - 1$. Then one of the J_i, say J_k, is the path on $m - 2$ vertices and $J_1, \ldots, J_{k-1}$ are t-vertices to K. Now, the two edges joining J_k and K are not incident to a common vertex in K or in J_k. Moreover, these two edges must join the two end-vertices of J_k to two adjacent vertices in K. This is because otherwise either Y contains a K_4-homeomorph as a subgraph or $P(Y;\lambda) \neq P(G;\lambda)$. But then $Y \cong G$.

Assume that $e_i \geq 3$ for some i. Then $k \leq s - 3$. Since each isolated vertex in J contributes at most one t-vertex to K, we have

$$n(K_{2,3}, Y) \leq \binom{3+\beta+1}{3} \leq \binom{s}{3}$$

Clearly, the second inequality holds if $\beta = k-1$ and $k = s-3$. When $\beta \leq k-2$, $\binom{4+\beta}{3} < \binom{s}{3} - 1$.

Suppose $\beta = s - 4 = k - 1$. Then one of the J_i, say J_k is a tree on $m - 2$ vertices and $J_1, \ldots, J_{k-1}$ are t-vertices to K. Since one of the end-vertices of J_k is a t-vertex to K, J_k is a path. Now, the other end-vertex of J_k must be adjacent to a vertex in K which is not of degree 2 because otherwise Y contains a K_4-homeomorph as a subgraph. But then $Y \cong G$.

Case (2): Not all the J_i's are trees.

Assume that $J_1, \ldots, J_t$ are not trees and $J_{t+1}, \ldots, J_k$ are trees so that $c_1, \ldots, c_t \geq 1$ and $c_{t+1}, \ldots, c_k = 0$ for some $t \geq 1$.

Consider the subgraph induced by the vertices of $J_i \cup K$. For each i, let H_i denote the graph obtained from $J_i \cup K$ by deleting all the edges in K. Let α_i denote the number of isolated vertices in H_i. Then $\alpha_i \leq 3$.

Let H_i' denote the graph obtained from H_i by deleting all the α_i isolated vertices. Then H_i' is a connected graph with cyclomatic number $c_i+e_i-5+\alpha_i \leq c_i + e_i - 2$. By Lemma 4.2, $n(K_{2,3}, H_i') \leq \binom{c_i+e_i-1}{3}$.

By (O1), since each J_i contributes at most one t-vertex, we have

$$n(K_{2,3}, Y) \leq \binom{k+3}{3} + \sum_{i=1}^{t} \binom{c_i + e_i - 1}{3}$$

$$\leq \binom{k+3}{3} + \binom{\sum_{i=1}^{t}(c_i + e_i - 1) - 2(t-1)}{3} - 2(t-1)$$

by Lemma 4.3.

Now observe that

$$\sum_{i=1}^{t}(c_i + e_i - 1) - 2(t-1) = \sum_{i=1}^{k} c_i + \sum_{i=1}^{t} e_i - 3t + 2$$

$$= (s - e - 2 + k) + e - \sum_{i=t+1}^{k} e_i - 3t + 2$$

$$\leq (s - 2 + k) - 2(k - t) - 3t + 2$$

$$= s - t - k$$

$$\leq s - k - 1.$$

Thus we have

$$n(K_{2,3}, Y) \leq \binom{k+3}{3} + \binom{s-k-1}{3} - 2(t-1)$$

$$\leq \binom{s}{3} - 2t \qquad \text{Lemma 4.3}$$

$$\leq \binom{s}{3} - 2 \qquad \text{because } t \geq 1.$$

This completes the proof of the theorem.