SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS USING ITERATIVE HOMOTOPY ANALYSIS METHOD

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FACULTY OF SCIENCE
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KUALA LUMPUR

2021

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DISSERTATION SUBMITTED IN FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

INSTITUTE OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
UNIVERSITI MALAYA
KUALA LUMPUR

2021

UNIVERSITI MALAYA

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Name of Degree: **MASTER OF SCIENCE**

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SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS USING ITERATIVE HOMOTOPY ANALYSIS METHOD

ABSTRACT

Solving fractional differential equations (FDEs) using homotopy analysis method has been a challenging issue. It is mainly related to the difficulty of evaluation and low convergence rate at necessarily high order approximations. This research proposes iterative homotopy analysis method (IHAM) to fill the gap in the existing approach. At each iteration, an optimal convergence control parameter is computed that corresponds to a global minimum error of solution. When a fractional derivative of a function is much harder to evaluate, especially those involving both left-handed and right-handed fractional derivatives, it is possible to approximate the function through Taylor expansion. Numerical comparisons were conducted through various FDEs selected from the literature. The presented results conclude that IHAM delivers faster convergence and better accuracy. We hope that this study will initiate further research to identify other properties that would lead to a stable analytical method for FDEs.

Keywords: iterative homotopy analysis method, fractional differential equations, left-handed and right-handed fractional derivatives

PENYELESAIAN PERSAMAAN PEMBEZAAN PECAHAN MENGGUNAKAN KAEDAH ANALISIS HOMOTOPI BERULANG

ABSTRAK

Penyelesaian persamaan pembezaan pecahan (FDE) menggunakan kaedah analisis homotopy telah menjadi isu yang mencabar. Ini terutamanya berkaitan dengan kesukaran penilaian dan kadar penumpuan yang rendah pada penghampiran yang semestinya tinggi. Kajian ini mencadangkan kaedah analisis homotopi berulang (IHAM) untuk mengisi jurang dalam kaedah itu yang sedia ada. Pada setiap lelaran, nilai optimum bagi parameter kawalan penumpuan dikira yang bersamaan dengan ralat minimum penyelesaian global. Apabila pembeza pecahan untuk fungsi lebih sukar untuk dinilai, terutamanya yang melibatkan kedua-dua pembeza pecahan tangan kiri dan tangan kanan, adalah mungkin untuk menghampiri fungsi itu melalui pengembangan Taylor. Perbandingan berangka dilakukan melalui pelbagai FDEs yang dipilih dari kesusasteraan. Hasil yang dibentangkan menyimpulkan bahawa IHAM memberikan penumpuan lebih cepat dan ketepatan yang lebih baik. Kami berharap kajian ini akan memulakan penyelidikan lebih lanjut untuk mengenal pasti sifat-sifat lain yang akan membawa kepada kaedah analisis stabil untuk FDEs.

Kata kunci: kaedah analisis homotopi berulang, persamaan pembezaan pecahan, pembeza pecahan tangan kiri and tangan kanan

ACKNOWLEDGEMENTS

I heartily thank my academic supervisors, Professor Dr. Kurunathan A/L Ratnavelu and Dr. Kumaresan Nallasamy, for their continuous encouragement, guidance, and valuable suggestions. They steered me through obstacles in my research, besides writing this dissertation.

I also gratefully acknowledge the financial support of Kementerian Pendidikan Tinggi Malaysia. This accomplishment wouldn't be possible without them.

Last but not least, my very profound gratitude goes to my family, who have never lost faith in me. They offer me unfailing supports throughout my years of postgraduate study. Their contributions to this work, though invisible, are invaluable to me.

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LIST OF SYMBOLS AND ABBREVIATIONS

${}_{a}^{C}D_{t}^{\alpha}$	:	α^{th} Caputo fractional derivative
${}_{a}^{J}D_{t}^{\alpha}$	:	α^{th} Jumarie fractional derivative
${}_{a}^{RL}I_{t}^{\alpha}$	:	α^{th} Riemann-Liouville fractional integral
${}_{a}^{RL}D_{t}^{\alpha}$	:	α^{th} Riemann-Liouville fractional derivative
h	:	convergence control parameter
$\Gamma(z)$	:	Euler gamma function
$E_{\alpha,\beta}(z)$	:	generalized Mittag-Leffler function
ϵ	:	mean squared residual error
BTE	:	Bagley-Torvik equation
Eq.	:	equation
FBVP	:	fractional boundary value problem
FC	:	fractional calculus
FDE	:	fractional differential equation
FIVP	:	fractional initial value problem
FOCP	:	fractional optimal control problem
FRDE	:	fractional Riccati differential equation
IHAM	:	iterative homotopy analysis method
OHAM	:	optimal homotopy analysis method

LIST OF APPENDICES

CHAPTER 1: INTRODUCTION

We are familiar with the concept of integral and derivative in elementary calculus. Fractional calculus (FC) is the calculus that allows integral and derivative of arbitrary order. A fractional differential equation (FDE) is a generalized differential equation involving the fractional derivative. In the past few decades, considerable interests in FDEs stimulate the emergence of new applications in science. Although they are the generalization of ordinary differential equations, however, until recently, an effective general method for solving such equations was not readily accessible. This dissertation is devoted to a method for solving various FDEs.

In the past decades, many scientists have developed various analytical and numerical methods for solving FDEs but need particular attention. Among these methods, optimal homotopy analysis method (OHAM) is probably the efficient procedure. In particular, it provides us freedom on a choice of the auxiliary linear operator and a way to control convergence. We also specify few improvements to eliminate a limitation that we encounter when solving FDEs with OHAM.

1.1 Problem statements

Indeed, finding an analytical solution of FDE is even harder than solving a standard ordinary differential equation. As seen in many treatises of FC, both left-handed and right-handed fractional derivatives are non-local convolution integrals, depending on past values and future values of a function, respectively. Unfortunately, FDEs involve both operators simultaneously introduce an undesirable problem, as illustrated in fractional variational calculus and fractional optimal control problems (FOCPs). In particular, the integrability issue causes difficulty with the analytical evaluation of function value in finding the solution. Tackling the problem will have a significant practical benefit for the development of FC.

Undoubtedly, a key attribute to the success of OHAM is a convergence control parameter. However, due to a limited number of the parameter, we are still far away from reaching satisfactory accuracy to a solution of FDE that requires higher order approximations. Even if the number increase, it negatively impacts CPU performance as well. As we've already experienced, the same parameter gets repeatedly used during iteration, then calculating its value comes back from the starting point in series. The series may suffer from vanishing terms at higher order approximations, consequently, would lead to a lack of convergence region. Investigating the issue can potentially develop more robust OHAM in practice to solve FDE efficiently. For this purpose, we propose a novel approach called iterative homotopy analysis method (IHAM).

1.2 Objectives

Motivated by the above problem statements, a primary goal is to demonstrate the validity of IHAM for solving FDE and exploring the scope for further improvement. Thus, to achieve the objective, this research is guided by the following question:

What are the possible impacts of solving FDE using IHAM?

Since the aforementioned question may not be practically feasible, we need to narrow it down further to four sub-questions as follows:

i. What issues do existing methods include OHAM, have on analytical solutions for different types of FDE? (CHAPTER 2)

ii. What practical technique can benefit a relevant application to solve FDE better? (CHAPTER 2)

iii. How can a convergence control parameter in IHAM be derived at each iteration to ensure its convergence? (CHAPTER 3)

iv. How effective are IHAM compared to OHAM at increasing convergence criteria of FDE solution? (CHAPTER 4)

In summary, this research targets these currently unanswered questions and contributes knowledge that future research can build on.

1.3 Dissertation organization

We hope that this dissertation will be of interest to potential readers of this topic. For this reason, after this CHAPTER 1: INTRODUCTION, the rest is organized following a conventional structure.

CHAPTER 2: LITERATURE REVIEW recalls an essential knowledge of FC that will be used in this dissertation, inclusive of the interplay between left-handed and right-handed fractional operators. In this framework of FC, we consider fractional initial value problems (FIVPs), fractional boundary value problems (FBVPs), and fractional optimal control problems (FOCPs) for practically relevant tasks, as well as critical reviews of existing studies on these common FDEs. On the other hand, we describe the idea of OHAM and its shortcomings one would encounter.

CHAPTER 3: METHODOLOGY details the implementation of IHAM. Unlike OHAM, we treat the convergence control parameter differently into its solution.

CHAPTER 4: RESULTS AND DISCUSSION is devoted to a comparison between OHAM and our proposed IHAM attempted to solve six widely used FDE examples selected from the literature studied in CHAPTER 2. For each problem, we also discuss in detail outcomes and issues concerning obtained results.

CHAPTER 5: CONCLUSION eventually summarizes the significance of our research findings and suggests future work for research.

REFERENCES list particular papers and books that we believe are the most relevant cited in this dissertation.

LIST OF PUBLICATIONS AND PAPERS PRESENTED provides a collection of articles that reflect our original works.

APPENDIX A at the end of this dissertation consists of MATHEMATICA codes for reproducing the results obtained in CHAPTER 4.

CHAPTER 2: LITERATURE REVIEW

Fractional calculus (FC) is a generalization of classical calculus that allows derivative and integral to arbitrary order, see (Miller & Ross, 1993; Oldham & Spanier, 1974) for a detailed study. It was initiated by a discussion between L'Hospital and Leibniz in 1695. An interested reader may consult (Oldham & Spanier, 1974) as well as (Miller & Ross, 1993) for a historical survey of FC. For three centuries, FC developed mainly as a pure mathematical realm. However, in recent decades, its applications in numerous diverse fields have gained wide attention and importance. Examples include viscoelasticity, anomalous diffusion, heat conduction, signal processing, control theory, dynamic systems, chaos and fractals, and so forth (Sun et al., 2018). One reason behind the recent interest in FC is a nonlocal property of the fractional differentiation process. The advantage of fractional derivative becomes apparent in modeling a hereditary property of viscoelastic material, as well as a history dependence behavior of anomalous diffusion (Podlubny, 1998).

2.1 Laplace transform

As one would expect, since a fractional operator is defined commonly by convolution operation, it is hard to evaluate the convolution integral, for instance, singularity behavior at the terminal of integration (Oldham & Spanier, 1974). The use of Laplace integral transform that we shall exploit to convert the fractional operator as convolution integral into a multiplication of algebra (see e.g. Theorem 2.5, Theorem 2.7, and Theorem 2.10). It facilitates relevant application easily to solve fractional differential equation (FDE) involving the operator, which can then be inverse transform through a table of Laplace-transform pairs. Therefore, we recall the following definition for Laplace transform.

Definition 2.1 Let $f(t)$ be an integrable function on $\mathbb{R}_{\geq 0}$. Then, integral $\mathcal{L}\{f(t), t; s\}$ of $s \in \mathbb{C}$ defined by

$$\mathcal{L}\{f(t), t; s\} = \int_0^\infty e^{-st} f(t)\, dt$$

is called Laplace transform of $f(t)$, converges absolutely in the complex half-plane $\Re(s) > 0$ (Podlubny, 1998).

2.2 Special functions

Besides elementary functions, it is noteworthy that certain higher transcendental functions play a vital role in FC. For convenience, we present here the definitions with several essential properties that we need in the sequel.

2.2.1 Gamma function

Undoubtedly, the gamma function removes a discrete nature of the factorial function for fractional integral. Therefore, we start by recalling the following definition.

Definition 2.2 If $z \in \mathbb{C}$ with $\Re(z) > 0$, then

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx, \tag{2.1}$$

known as Euler's gamma function, converges absolutely. Its useful property is a recurrence relationship in the following theorem.

Theorem 2.1 *For $z > 0$, the following equation holds*

$$\Gamma(z+1) = z\,\Gamma(z).$$

Proof. Integration by parts applies to Eq. (2.1) concludes the proof (Oldham & Spanier, 1974). ■

2.2.2 Mittag-Leffler function

While the gamma function is a generalized factorial function, the Mittag-Leffler function is a generalized exponential function. During the last decade, the Mittag-Leffler function comes to prominence due to its role played in FC. For its detailed account of properties and applications, the reader may refer to the paper (Mainardi & Gorenflo, 2000). Let us recall the following definition for the Mittag-Leffler function.

Definition 2.3 If $z \in \mathbb{C}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\beta \in \mathbb{C}$, then the generalized Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

A useful fact of $E_{\alpha,\beta}(z)$ is associated with its appearance in deriving analytical solutions of FDE via the Laplace transform method. Hence the following theorem is of importance.

Theorem 2.2 *Under the assumption that the k^{th} derivative of $E_{\alpha,\beta}(z)$ exists, the following Laplace transform pair with $s \in \mathbb{C}$ holds*

$$\mathcal{L}\left\{t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm c\, t^{\alpha}), t; s\right\} = \frac{k!\, s^{\alpha-\beta}}{(s^{\alpha} \mp c)^{k+1}}, \tag{2.2}$$

where its k^{th} derivative $E_{\alpha,\beta}^{(k)}(z)$, $k = 0, 1, 2, \ldots$ is given by

$$E_{\alpha,\beta}^{(k)}(z) = \frac{d^k}{dz^k} E_{\alpha,\beta}(z).$$

Proof. For details of the proof, see (Podlubny, 1998). ■

2.3 Fractional operators

There are several definitions related to fractional derivative and fractional integral (de Oliveira & Machado, 2014). In this section, we will focus on Riemann-Liouville, Caputo, and Jumarie definitions. These are the most frequently used for applications. For the sake of brevity, our discussion is restricted to a real variable and function.

2.3.1 Riemann-Liouville fractional integral and derivative

Starting with generalized Cauchy's repeated integration formula, we introduce the following definition of α^{th} Riemann-Liouville fractional integrals.

Definition 2.4 Suppose that $f: [a, b] \to \mathbb{R}$ is a function, $\alpha \in \mathbb{R}_{>0}$, and $\Gamma(\alpha)$ is Euler gamma function. Then, left-handed α^{th} Riemann-Liouville fractional integral of the function $f(t)$ with origin a is defined by

$$ {}_{a}^{RL}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{2.3}$$

Similarly, right-handed α^{th} Riemann-Liouville fractional integral of the function $f(t)$ with origin b is defined by

$$ {}_{t}^{RL}I_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau.$$

In light of such fractional integrals, one has the following α^{th} fractional derivatives.

Definition 2.5 Let $f: [a, b] \to \mathbb{R}$ be a function, $n \in \mathbb{Z}_{>0}$, $\alpha \in \mathbb{R}_{>0}$ such that $\alpha \in [n-1, n)$, then left-handed and right-handed α^{th} Riemann-Liouville fractional derivatives of the function $f(t)$ are defined, respectively, by

$$ {}_{a}^{RL}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \tag{2.4} $$

and

$$ {}_{t}^{RL}D_{b}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\, d\tau. $$

Remark 2.1 Unfortunately, α^{th} Riemann-Liouville fractional derivative for constant is non-zero. Furthermore, the operator is only applicable to a problem with boundary conditions equal to zero.

Now, let's consider a property of α^{th} Riemann-Liouville fractional integral, which is necessary for subsequent use. The useful fact that it satisfies the semigroup property in the following theorem.

Theorem 2.3 *For $f\colon [a,b] \to \mathbb{R}$, and $\alpha, \beta \in \mathbb{R}_{>0}$, the following equations hold*

$$ {}_{a}^{RL}I_{t}^{\alpha}\, {}_{a}^{RL}I_{t}^{\beta} f(t) = {}_{a}^{RL}I_{t}^{\alpha+\beta} f(t), \tag{2.5} $$

and

$$ {}_{t}^{RL}I_{b}^{\alpha}\, {}_{t}^{RL}I_{b}^{\beta} f(t) = {}_{t}^{RL}I_{b}^{\alpha+\beta} f(t). $$

Proof. For the proof, see (Hilfer, 2000). ■

A consequence of Theorem 2.3 is the following corollary that the α^{th} Riemann-Liouville fractional derivative provides operation inverse to the same order of α^{th} Riemann-Liouville fractional integral.

Corollary 2.1 *Assume that $f\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{Z}_{>0}$, $\alpha \in \mathbb{R}_{>0}$ such that $\alpha \in [n-1, n)$. Then*

$${}_{a}^{RL}D_{t}^{\alpha}\ {}_{a}^{RL}I_{t}^{\alpha}f(t) = f(t),$$

and

$${}_{t}^{RL}D_{b}^{\alpha}\ {}_{t}^{RL}I_{b}^{\alpha}f(t) = f(t).$$

Proof. By assumption on $\phi(t) = {}_{a}^{RL}I_{t}^{\alpha}f(t)$ and Eq. (2.3), allow us to rewrite Eq. (2.4) as

$${}_{a}^{RL}D_{t}^{\alpha}\phi(t) = \frac{d^n}{dt^n}\,{}_{a}^{RL}I_{t}^{n-\alpha}\phi(t). \tag{2.6}$$

Substituting $\phi(t) = {}_{a}^{RL}I_{t}^{\alpha}f(t)$ into Eq. (2.6), and by semigroup property Eq. (2.5), operator ${}_{a}^{RL}I_{t}^{n-\alpha}$ and ${}_{a}^{RL}I_{t}^{\alpha}$ obviously commute, we obtain

$${}_{a}^{RL}D_{t}^{\alpha}\ {}_{a}^{RL}I_{t}^{\alpha}f(t) = f(t).\ \blacksquare$$

It is well-known that power function is often used in infinite power series of solutions. Therefore, in the following lemma, we provide the α^{th} Riemann-Liouville fractional integral of power function for later applications.

Lemma 2.1 *Let* $\alpha \in \mathbb{R}_{>0}$, $\beta \in \mathbb{R}_{>-1}$ *and* $t \in [a, b]$. *Then,*

$${}_{a}^{RL}I_{t}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \tag{2.7}$$

and

$${}_{t}^{RL}I_{b}^{\alpha}(b-t)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(b-t)^{\beta+\alpha}.$$

Proof. The proof utilizes properties of gamma and beta functions (Oldham & Spanier, 1974). The condition $\beta > 1$ arises as required of positive argument to the beta function.

■

The next corollary asserts that there exists a relation between both ${}_{a}^{RL}I_{t}^{\alpha}$ and ${}_{t}^{RL}I_{b}^{\alpha}$ via fractional integration by parts on $t \in [a, b]$ (Love & Young, 1938). More importantly, it is a fundamental tool to establish optimality conditions for fractional optimal control problems (FOCPs).

Corollary 2.2 *If* $f: [a, b] \rightarrow \mathbb{R}$ *and* $g: [a, b] \rightarrow \mathbb{R}$ *are arbitrary integrable functions for* ${}_{a}^{RL}I_{t}^{\alpha} f(t)$ *and* ${}_{t}^{RL}I_{b}^{\alpha} g(t)$ *of order* $\alpha \in \mathbb{R}_{>0}$*, respectively, then*

$$\int_{a}^{b} f(t)\ {}_{t}^{RL}I_{b}^{\alpha} g(t)\, dt = \int_{a}^{b} g(t)\ {}_{a}^{RL}I_{t}^{\alpha} f(t)\, dt.$$

Proof. For details of the proof, see (Love & Young, 1938). ■

Unfortunately, there is an issue that must be dealt with Definition 2.4. ${}_{a}^{RL}I_{t}^{\alpha}$ performs on past of current value, whereas ${}_{t}^{RL}I_{b}^{\alpha}$ performs on future of current value (Ciesielski & Blaszczyk, 2015, 2017). This problem poses difficulty with a calculation of the current value in finding a solution on finite interval $t \in [a, b]$. Owing to such shortcoming, we consider a function for the α^{th} Riemann-Liouville fractional integral based on its Taylor series as suggested in (Hadamard, 1892; Wei et al., 2017). It provides an elegant way to deal with such a problem, where ${}_{a}^{RL}I_{t}^{\alpha}$ and ${}_{t}^{RL}I_{b}^{\alpha}$ are well defined. The following theorem is an alternate representation for α^{th} Riemann-Liouville fractional integral of an analytic function.

Theorem 2.4 *Assuming that function* $f: [a, b] \rightarrow \mathbb{R}$ *is analytic, and therefore can be represented by its Taylor series. Then, the following interesting facts are readily derived*

$${}_{a}^{RL}I_{t}^{\alpha} f(t) = {}_{a}^{RL}I_{t}^{\alpha} \left(\sum_{k=0}^{\infty} \frac{(t-a)^{k}}{\Gamma(k+1)} f^{(k)}(a) \right),$$

and

$$ {}_{t}^{RL}I_b^{\alpha} f(t) = {}_{t}^{RL}I_b^{\alpha}\left(\sum_{k=0}^{\infty}(-1)^k \frac{(b-t)^k}{\Gamma(k+1)} f^{(k)}(b)\right). $$

Proof. From Eq. (2.3) and repeated integration by parts performed, we deduce

$$ \begin{aligned} {}_{a}^{RL}I_t^{\alpha} f(t) &= \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} f(a) + \frac{1}{\Gamma(\alpha+1)}\int_a^t (t-\tau)^{\alpha} f^{(1)}(\tau)d\tau \\ &= \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} f(a) + \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+2)} f^{(1)}(a) \\ &\quad + \frac{1}{\Gamma(\alpha+2)}\int_a^t (t-\tau)^{\alpha+1} f^{(2)}(\tau)d\tau \\ &\ \vdots \\ &= \sum_{k=0}^{\infty} \frac{(t-a)^{\alpha+k}}{\Gamma(\alpha+k+1)} f^{(k)}(a). \end{aligned} $$

It follows, by exploiting Eq. (2.7), that

$$ {}_{a}^{RL}I_t^{\alpha} f(t) = {}_{a}^{RL}I_t^{\alpha}\left(\sum_{k=0}^{\infty} \frac{(t-a)^k}{\Gamma(k+1)} f^{(k)}(a)\right). \blacksquare $$

Since α^{th} Riemann-Liouville fractional integral is a convolution type pseudo-operator, let's consider its Laplace transform in the following theorem.

Theorem 2.5 *Let* $f: [a,b] \to \mathbb{R}$ *and* $\alpha \in \mathbb{R}_{>0}$. *Then*

$$ \mathcal{L}\{{}_{a}^{RL}I_t^{\alpha} f(t), t-a; s\} = \frac{1}{s^{\alpha}}\mathcal{L}\{f(t+a), t; s\}, \tag{2.8} $$

and

$$ \mathcal{L}\{{}_{t}^{RL}I_b^{\alpha} f(t), b-t; s\} = \frac{1}{s^{\alpha}}\mathcal{L}\{f(b-t), t; s\}. $$

Proof. We start Eq. (2.3) as convolution integral followed by Laplace transform Definition 2.1 with respect to variable $t-a$,

$$\mathcal{L}\{{}^{RL}_{a}I^{\alpha}_{t}f(t), t-a; s\} = \int_{a}^{\infty} e^{-s(t-a)} \int_{a}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\, d\tau\, dt.$$

Next, we apply Fubini's theorem to interchange the integration order, yielding

$$\mathcal{L}\{{}^{RL}_{a}I^{\alpha}_{t}f(t), t-a; s\} = \int_{a}^{\infty} f(\tau) \int_{\tau}^{\infty} e^{-s(t-a)} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\, dt\, d\tau.$$

Finally, performing a change of variable by $u = t - \tau$ followed by $\tau = \nu + a$, we obtain

$$\mathcal{L}\{{}^{RL}_{a}I^{\alpha}_{t}f(t), t-a; s\} = \int_{0}^{\infty} e^{-s\nu} f(\nu + a)\, d\nu \int_{0}^{\infty} e^{-su} \frac{u^{\alpha-1}}{\Gamma(\alpha)}\, du$$

$$= \frac{1}{s^{\alpha}} \mathcal{L}\{f(t+a), t; s\}. \blacksquare$$

Remark 2.2 Even if the left-handed case is considered here, the discussion presented herein can be adapted easily to the right-handed case with only a minor difference (Baleanu et al., 2012).

2.3.2 Caputo fractional derivative

The necessity of including initial and boundary conditions requires a revision of the well-established Riemann-Liouville approach. Therefore, the alternative of fractional derivative introduced by Michele Caputo (Caputo, 1967) as follows:

Definition 2.6 Assume function $f(t)$ has derivative up to integer order n on $[a, b]$. We denote the n^{th} derivative by $f^{(n)}(t)$. Then, for real number $\alpha \in [n-1, n)$, left-handed and right-handed α^{th} Caputo fractional derivatives of $f(t)$ are defined, respectively, by

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \tag{2.9} $$

and

$$ {}_{t}^{C}D_{b}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{t}^{b} (\tau-t)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau. $$

Remark 2.3 The main advantage of Caputo's definition is allowing utilization of physically interpreted initial and boundary conditions as well as its derivative for constant equals to zero.

Having defined α^{th} Riemann-Liouville fractional integral and α^{th} Caputo fractional derivative, let's consider the interaction of both operators. The following theorem shows the fractional integral is inverse to the same order of fractional derivative.

Theorem 2.6 *If $f: [a, b] \to \mathbb{R}$ has $n \in \mathbb{Z}_{>0}$ order continuous derivatives and $\alpha \in \mathbb{R}_{>0}$ such that $\alpha \in [n-1, n)$, then*

$$ {}_{a}^{RL}I_{t}^{\alpha}\, {}_{a}^{C}D_{t}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma(k+1)} f^{(k)}(a), $$

and

$$ {}_{t}^{RL}I_{b}^{\alpha}\, {}_{t}^{C}D_{b}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k (b-t)^k}{\Gamma(k+1)} f^{(k)}(b). $$

Proof. Recall the definition of ${}_{a}^{RL}I_{t}^{\alpha}$ in Eq. (2.3), the right-hand side of Eq. (2.9) has representation takes the following form:

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = {}_{a}^{RL}I_{t}^{n-\alpha} f^{(n)}(t). \tag{2.10} $$

Now, applying operator ${}_{a}^{RL}I_{t}^{\alpha}$ to both sides of Eq. (2.10), then using Theorem 2.3, we deduce that

$$
\begin{aligned}
{}_{a}^{RL}I_{t}^{\alpha}\ {}_{a}^{C}D_{t}^{\alpha}f(t) &= {}_{a}^{RL}I_{t}^{n}f^{(n)}(t) \\
&= f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{k}}{\Gamma(k+1)}f^{(k)}(a)\,. \blacksquare
\end{aligned}
$$

Let us now evaluate the following α^{th} Caputo fractional derivative for power function.

Lemma 2.2 *If* $t \in [a,b]$, $n \in \mathbb{Z}_{>0}$, $\alpha,\beta \in \mathbb{R}_{>0}$ *such that* $\alpha \in [n-1,n)$ *and* $\beta > n-1$, *then*

$$
{}_{a}^{C}D_{t}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha},
$$

and

$$
{}_{t}^{C}D_{b}^{\alpha}(b-t)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-t)^{\beta-\alpha}.
$$

Proof. Analogous to the proof of Lemma 2.1. ■

As we have evaluated earlier Laplace transform of the fractional integral in Theorem 2.5, let us consider Laplace transform for α^{th} Caputo fractional derivative in the following theorem.

Theorem 2.7 *Under the assumption that* n^{th} *differentiable of function* $f\colon [a,b] \to \mathbb{R}$ *and* $\alpha \in [n-1,n)$, *the Laplace transform formulas with* $s \in \mathbb{C}$ *can be developed as follows:*

$$
\mathcal{L}\{{}_{a}^{C}D_{t}^{\alpha}f(t), t-a; s\} = s^{\alpha}\left[\mathcal{L}\{f(t+a), t; s\} - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{s^{k+1}}\right],
$$

and

$$\mathcal{L}\{{}_t^C D_b^\alpha f(t), b-t; s\} = s^\alpha \left[\mathcal{L}\{f(b - t), t; s\} - \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)}{s^{k+1}}\right].$$

Proof. It is a straightforward consequence of Laplace transform on Definition 2.6 proceeds as the proof of Theorem 2.5 in a similar manner, and Laplace transform for $f^{(n)}(t)$. For details of the proof see (Podlubny, 1998). ■

Remark 2.4 For the right-handed case, the discussion presented herein can be similarly shown (Baleanu et al., 2012).

2.3.3 Jumarie fractional derivative

Each fractional derivative has its own drawbacks. Therefore, for instance, the α^{th} Riemann-Liouville fractional derivative for constant is nonzero, and the α^{th} Caputo fractional derivative is limited to a differentiable function (Almeida & Torres, 2011; Atangana & Secer, 2013; de Oliveira & Machado, 2014; Jumarie, 2006, 2007, 2009; Ortigueira & Machado, 2015). To circumvent these drawbacks, Jumarie (2006, 2009, 2013) proposed the following definition that leverages the positive sides of both aforementioned fractional derivatives.

Definition 2.7 Let $f: [a, b] \to \mathbb{R}$ be a function and $\alpha \in (0, 1)$. Then one has left-handed and right-handed α^{th} Jumarie fractional derivatives of $f(t)$ are defined, respectively, by

$${}_a^J D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{[f(\tau) - f(a)]}{(t-\tau)^\alpha} d\tau, \qquad (2.11)$$

and

$${}_t^J D_b^\alpha f(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{[f(\tau) - f(b)]}{(\tau - t)^\alpha} d\tau.$$

Remark 2.5 As desired, the α^{th} fractional derivative for constant equals zero. Furthermore, $f(a) = f(b) = 0$ is no longer condition for the α^{th} fractional derivative of $f(t)$ continuous on $t \in [a, b]$, and $f(t)$ needs not to be differentiable.

We have proven in Theorem 2.6 that the α^{th} Riemann-Liouville fractional integral is indeed inverse to the α^{th} Caputo fractional derivative. Similarly, the following theorem asserts that the fractional integral is also inverse to the same order of the Jumarie fractional derivative.

Theorem 2.8 *In particular, if* $f: [a, b] \to \mathbb{R}$ *and* $\alpha \in (0, 1)$, *then*

$$ {}_{a}^{RL}I_{t}^{\alpha} \, {}_{a}^{J}D_{t}^{\alpha} f(t) = f(t) - f(a), $$

and

$$ {}_{t}^{RL}I_{b}^{\alpha} \, {}_{t}^{J}D_{b}^{\alpha} f(t) = f(t) - f(b). $$

Proof. Utilizing Eq. (2.3), we can rewrite the right-hand side of Eq. (2.11) to

$$ {}_{a}^{J}D_{t}^{\alpha} f(t) = \frac{d}{dt} \, {}_{a}^{RL}I_{t}^{1-\alpha} [f(t) - f(a)], \tag{2.12} $$

and the fundamental theorem of calculus reads

$$ f(t) - f(a) = {}_{a}^{RL}I_{t}^{1} \frac{d}{dt} f(t). \tag{2.13} $$

We first apply operator ${}_{a}^{RL}I_{t}^{1-\alpha}$ on both sides of Eq. (2.13) and note that operators ${}_{a}^{RL}I_{t}^{1-\alpha}$ and ${}_{a}^{RL}I_{t}^{\alpha}$ commute, we obtain

$$ {}_{a}^{RL}I_{t}^{1-\alpha} [f(t) - f(a)] = {}_{a}^{RL}I_{t}^{1} \, {}_{a}^{RL}I_{t}^{1-\alpha} \frac{d}{dt} f(t). $$

By differentiating on both sides with respect to t implies

$$ {}_{a}^{J}D_{t}^{\alpha} f(t) = {}_{a}^{RL}I_{t}^{1-\alpha} \frac{d}{dt} f(t), $$

with substitution Eq. (2.12). Then, applying operator ${}_{a}^{RL}I_{t}^{\alpha}$ on both sides yields

$$ {}_{a}^{RL}I_{t}^{\alpha} {}_{a}^{J}D_{t}^{\alpha} f(t) = {}_{a}^{RL}I_{t}^{1} \frac{d}{dt} f(t), $$

where we have once again used Theorem 2.3. It follows, by Eq. (2.13), that

$$ {}_{a}^{RL}I_{t}^{\alpha} {}_{a}^{J}D_{t}^{\alpha} f(t) = f(t) - f(a). \blacksquare $$

Analogous to the α^{th} Caputo fractional derivative over power function in Lemma 2.2, the α^{th} Jumarie fractional derivative yields a similar result in the following sense.

Lemma 2.3 *Let $\alpha \in \mathbb{R}_{>0}$ such that $\alpha \in (0,1)$, $\beta \in \mathbb{R}_{>-1}$ and $t \in [a,b]$, we have*

$$ {}_{a}^{J}D_{t}^{\alpha} (t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}, $$

and

$$ {}_{t}^{J}D_{b}^{\alpha} (b-t)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (b-t)^{\beta-\alpha}. $$

Proof. Analogously as in the proof of Lemma 2.1. ■

Now, we deduce the following theorem by integration by parts involving both ${}_{a}^{J}D_{t}^{\alpha}$ and ${}_{t}^{J}D_{b}^{\alpha}$ on $t \in [a,b]$.

Theorem 2.9 *Suppose that $\alpha \in (0,1)$, $f: [a,b] \to \mathbb{R}$ and $g: [a,b] \to \mathbb{R}$ are continuous functions admitting ${}_{a}^{J}D_{t}^{\alpha} f(t)$ and ${}_{t}^{J}D_{b}^{\alpha} g(t)$ respectively, then it holds*

$$\int_a^b [f(t) - f(a)]\, {}_t^J D_b^\alpha g(t)\, dt = \int_a^b [g(t) - g(b)]\, {}_a^J D_t^\alpha f(t)\, dt.$$

Proof. Recalling the rule of fractional integration by parts Corollary 2.2, let us consider $\phi(t) = {}_a^J D_t^\alpha f(t)$ and $\psi(t) = {}_t^J D_b^\alpha g(t)$ admitting ${}_a^{RL} I_t^\alpha \phi(t)$ and ${}_t^{RL} I_b^\alpha \psi(t)$ respectively, one has

$$\int_a^b \phi(t)\, {}_t^{RL} I_b^\alpha \psi(t)\, dt = \int_a^b \psi(t)\, {}_a^{RL} I_t^\alpha \phi(t)\, dt. \tag{2.14}$$

It follows, by Theorem 2.8 that ${}_a^{RL} I_t^\alpha \phi(t) = f(t) - f(a)$ and ${}_t^{RL} I_b^\alpha \psi(t) = g(t) - g(b)$, then performing substitution into Eq. (2.14) concludes the proof. ■

Next, we discuss Laplace transform for the α^{th} Jumarie fractional derivative. For case $\alpha \in (0, 1)$, the following theorem states the result consistent with the Laplace transform of α^{th} Caputo fractional derivative in Theorem 2.7.

Theorem 2.10 *If $f: [a, b] \to \mathbb{R}$ exists for $\alpha \in (0, 1)$, then Laplace transform formulas with $s \in \mathbb{C}$ are given by*

$$\mathcal{L}\left\{{}_a^J D_t^\alpha f(t), t - a; s\right\} = s^\alpha \left[\mathcal{L}\{f(t + a), t; s\} - \frac{f(a)}{s}\right],$$

and

$$\mathcal{L}\left\{{}_t^J D_b^\alpha f(t), b - t; s\right\} = s^\alpha \left[\mathcal{L}\{f(b - t), t; s\} - \frac{f(b)}{s}\right].$$

Proof. We will start with Laplace transform formula for ${}_a^{RL} I_t^{1-\alpha}[f(t) - f(a)]$ in the form:

$$\mathcal{L}\{{}_{a}^{RL}I_{t}^{1-\alpha}[f(t)-f(a)],t-a;\mathrm{s}\}$$

$$=\int_{a}^{\infty}e^{-s(t-a)}\ {}_{a}^{RL}I_{t}^{1-\alpha}[f(t)-f(a)]\,dt.$$

With the aid of integration by parts and Eq. (2.11), we arrive at the following relation

$$\mathcal{L}\{{}_{a}^{J}D_{t}^{\alpha}f(t),t-a;\mathrm{s}\}=s\ \mathcal{L}\{{}_{a}^{RL}I_{t}^{1-\alpha}[f(t)-f(a)],t-a;\mathrm{s}\}.$$

Then, we can derive the result in a similar manner to those in the proof of Theorem 2.5. ■

Remark 2.6 The discussion presented herein is easily adapted to the right-handed case and omitted here for brevity.

As we stated earlier, there are more ways to define fractional derivatives. Thus, we describe fractional differential equations in the next section.

2.4 Fractional differential equations

A fractional differential equation (FDE) is a differential equation through the application of FC. Podlubny (1998) demonstrated that many real-world phenomena could be modeled adequately by FDEs instead of traditional integer order differential equations. Due to the memory effect of FC, FDEs serve as a tool to describe a hereditary property of various processes and materials. They exhibit historical dependence on the process involved. Motivated by their emerging applications in related areas, FDEs are gaining popularity and importance. For more applications, interested readers may refer to the monograph (Sun et al., 2018). We introduce several physical problems here collected from open literature that include nonlinear cases.

Firstly, let's consider famous fractional initial value problems (FIVPs) that have their initial states specified. For instance, Bagley-Torvik equation (BTE) and fractional Riccati differential equation (FRDE). BTE is a linear FDE for modeling viscoelastic materials, whereas FRDE is a class of nonlinear FDE that arises in optimal control problems. On the other hand, we will consider linear fractional boundary value problems (FBVPs) associated with BTE that have values assigned to boundaries of the domain. Furthermore, nonlinear FBVPs form a fascinating class of problem subject to given boundary conditions. Last but not least, we will discuss fractional optimal control problems (FOCPs), which have applications in control systems, e.g., aeronautics, robotics, and economics.

2.4.1 Fractional initial value problems

In the mid-1980s, Bagley and Torvik (1984) formulated FDE for describing immersion of rigid plate in a viscous fluid. For the detailed account of formulation, the reader may refer to the classic paper (Bagley & Torvik, 1984). Indeed, the generalized Bagley-Torvik equation (BTE) is FIVP with $\alpha \in (0, 2]$ involving two derivatives of the following form

$$A(t)\, f^{(2)}(t) + B(t)\, {}_{a}^{C}D_{t}^{\alpha} f(t) + C(t)\, f(t) = g(t), \tag{2.15}$$

coupled with initial conditions

$$f(a) = f_a \qquad \text{and} \qquad f^{(1)}(a) = f_a', \tag{2.16}$$

where $g: [a, b] \to \mathbb{R}$ is given function, $A(t)$, $B(t)$, $C(t)$ are arbitrary coefficient functions, and $f_a,\, f_a^{'} \in \mathbb{R}$ are constants.

Over the last two decades, BTE has been studied extensively in the literature. Initially, Podlubny (1998) used a numerical method to solve the equation in his book (Podlubny, 1998). Since then, many researchers have attempted to approximate the solution, such as homotopy perturbation method (Zolfaghari et al., 2009), Adomian decomposition method (Ray & Bera, 2005b), generalized Taylor collocation method (Çenesiz et al., 2010), optimum homotopy analysis method (Fadravi et al., 2011), Laplace transform (Labecca et al., 2015; Pang et al., 2019), etc. Even though there are several analytical or numerical methods treating this type of equation, in practice it is hardly evaluated, especially for different $g(t)$ and it only can be approximated. Furthermore, there has been no advancement to increase the accuracy further.

We now turn our attention to another example of FIVP, which is fractional Riccati differential equation (FRDE). In fact, FRDE is a generalization of the classical Riccati differential equation (Reid, 1972). In recent years, FRDE has played an important role in applied science, for instance, stochastic processes, optimal control, diffusion problems, etc. For more scientific applications of FRDE, see (Agheli, 2018). Among a variety of fractional derivative definitions, we adopt here Jumarie fractional derivative since it is applicable for non-differentiable function. Jumarie fractional calculus had been applied successfully to fractional Lagrangian mechanics (Jumarie, 2007), fractional variational calculus (Almeida & Torres, 2011), fractional Klein-Gordon (Merdan, 2014), etc. Therefore, we consider the following FRDE with ${}_a^J D_t^\alpha$ for $t \geq a$ (Merdan, 2012)

$$ {}_a^J D_t^\alpha f(t) = A(t) + B(t)\, f(t) + C(t)\, f^2(t), \tag{2.17} $$

along with initial condition

$$ f(a) = f_a, \tag{2.18} $$

where $A(t)$, $B(t)$, $C(t)$ denote given functions, and $f_a \in \mathbb{R}$ is a constant.

Although the exact solution for Eq. (2.17) is not easily accessible, there was always a possible method that approximates the result. Several methods have attracted attention, such as homotopy perturbation method (Odibat & Momani, 2008), homotopy analysis method (Cang et al., 2009), Adomian decomposition method (Momani & Shawagfeh, 2006), variational iteration method (Jafari et al., 2013), differential transform method

(Kumar Bansal & Jain, 2015) among many others. However, their convergence regions are rather small, and their computations are possibly time-consuming at high order approximations.

2.4.2 Fractional boundary value problems

Comparatively speaking, numerous articles are devoted to solving FIVPs, whereas fractional boundary value problems (FBVPs) have received negligible contribution indeed. Recently, FBVPs represent an emerging field that is having a significant impact in various disciplines of science, such as control theory, signal processing, biophysics, and so on. These have created interest concerning a solution for such problems. Zhang (2006) and Abdulla et al. (2016) proved the uniqueness and existence of FBVPs solution. Similarly, Zhao et al. (2011) investigated sufficient conditions for FBVPs having a solution. Therefore, discovering an efficient method for FBVPs solution to is still a significant challenge. Among these methods that have been developed are spline collocation method (Li et al., 2010; Pedas & Tamme, 2014), B-spline method (Azizi et al., 2012), cubic spline method (Zahra & Elkholy, 2013), homotopy analysis method (El-Ajou et al., 2013), homotopy perturbation method (Jafari et al., 2014), hybridizable discontinuous Galerkin method (Karaaslan et al., 2016), quasi-Newton's method (Yun Tao et al., 2016), variation iteration method (Zhao & An, 2017) and others.

Since we are dealing with FBVPs, they can be illustrated elegantly with the help of well-chosen cases from linear and nonlinear problems. Following such ideas, we firstly study linear FBVP associated with BTE (Karaaslan et al., 2016). Our governing equation is analogous to Eq. (2.15), except associated boundary conditions are

$$f(a) = f_a \qquad \text{and} \qquad f(b) = f_b, \tag{2.19}$$

where $f_a, f_b \in \mathbb{R}$ are taken to be constants appearing on the boundary of domain $t \in [a, b]$.

Secondly, we study the following nonlinear FBVP to arbitrary order α on $t \in [a, b]$ in abstract form (Pedas & Tamme, 2014)

$$ {}_a^C D_t^\alpha f(t) = F(t, f(t)), \tag{2.20} $$

with boundary value conditions

$$ f(a) = f_a \qquad \text{and} \qquad f(b) = f_b, \tag{2.21} $$

where $F: [a, b] \times \mathbb{R} \to \mathbb{R}$ is known continuous function, and $f_a, f_b \in \mathbb{R}$ denote arbitrary constants.

Remarkably, the FBVP solution may either fails to exist or not unique. Hence, finding the solution in closed form is a nontrivial task. Although we have the previously introduced methods, there is still room for improvement waiting to be carried out.

2.4.3 Fractional optimal control problems

As defined in Agrawal (2004, 2008), fractional optimal control problem (FOCP) generalizes classical optimal control problem, whose dynamic is described by FDE. FOCP requires minimization of functional on both state $x(t)$ and control $u(t)$ variables subject to the constrained dynamic. The following Figure 2.1 shows the components of optimum control configuration.

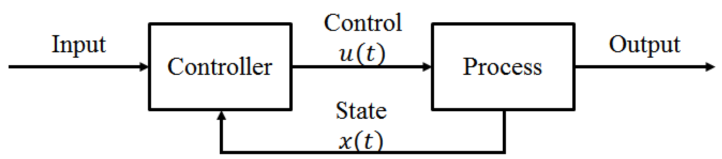

Figure 2.1: Optimum Control Configuration

Over the last two decades, FOCPs have been well investigated by many researchers; see (Agrawal, 2004, 2008; Agrawal & Baleanu, 2007; Alizadeh & Effati, 2018; Baleanu et al., 2009; Ghomanjani, 2016; Heydari et al., 2016; Khader & Hendy, 2012; Lotfi et al., 2011; Sweilam & Al-Ajami, 2015; Tang et al., 2015; Tohidi & Nik, 2015) and the references cited therein. They were developed mainly for Riemann-Liouville and Caputo fractional derivatives.

In the paper (Kamocki, 2014), FOCP using both ${}_{a}^{J}D_{t}^{\alpha}$ and ${}_{t}^{J}D_{b}^{\alpha}$ on $t \in [a, b]$ has proved its uniqueness and existence of a solution. It was shown that the fractional derivatives seem to be more appropriate. Here, we will adopt Jumarie's definition to formulate FOCP, as it has the advantage to deal with boundary conditions.

We begin to demonstrate Hamiltonian formulation for FOCP using Jumarie fractional calculus Definition 2.7. Here, we shall restrict our discussion to the following quadratic performance index, which is integral of state $x(t)$ and control $u(t)$ variables defined in a quadratic form on $t \in [a, b]$. The main problem investigated here is determining $x(t)$ and $u(t)$ that make the following performance index minimum

$$J = \frac{1}{2}\int_{a}^{b} [Q(t)\, x^2(t) + R(t)\, u^2(t)]dt, \tag{2.22}$$

whose initial condition is supplemented by

$$x(a) = x_a,$$

constrained with fractional dynamic

$${}_{a}^{J}D_{t}^{\alpha}x(t) = A(t)\, x(t) + B(t)\, u(t), \tag{2.23}$$

where $Q(t)$, $R(t)$, $A(t)$ and $B(t)$ are arbitrary functions.

Remark 2.7 Note that Eq. (2.22) may also include necessary terminal conditions, which are absent for the sake of simplicity.

To derive necessary optimality conditions, we combine Eqs. (2.22) – (2.23) via Lagrange multiplier technique to introduce augmented performance index as

$$\bar{J} = \frac{1}{2}\int_a^b \left[H(x,u,\lambda,t) - \lambda(t)\, {}_a^J D_t^\alpha x(t)\right] dt, \tag{2.24}$$

where $\lambda(t)$ denotes costate variable, and we define Hamiltonian $H(x,u,\lambda,t)$ as

$$\begin{aligned} H(x,u,\lambda,t) = &\frac{1}{2}[Q(t)\, x^2(t) + R(t)\, u^2(t)] \\ &+ \lambda(t)[A(t)\, x(t) + B(t)\, u(t)]. \end{aligned} \tag{2.25}$$

Then if $x(t)$, $u(t)$ and $\lambda(t)$ are subject to variations, we obtain

$$x'(t) = x(t) + \gamma\, \varphi(t),$$

$$u'(t) = u(t) + \gamma\, \phi(t),$$

and

$$\lambda'(t) = \lambda(t) + \gamma\, \psi(t),$$

where γ is scalar, $\varphi(t)$, $\phi(t)$ and $\psi(t)$ are arbitrary independent variations of $x(t)$, $u(t)$ and $\lambda(t)$, respectively. The condition for Eq. (2.24) takes on an extreme value if

$$\frac{\partial}{\partial \gamma}\int_a^b \left[H(x',u',\lambda',t) - \lambda'(t)\, {}_a^J D_t^\alpha x'(t)\right] dt \bigg|_{\gamma=0} = 0.$$

At $\gamma = 0$,

$$\int_a^b \left[\frac{\partial H}{\partial x}\varphi(t) + \frac{\partial H}{\partial u}\phi(t) + \left(\frac{\partial H}{\partial \lambda} - {}_a^J D_t^\alpha x(t)\right)\psi(t) - \lambda(t)\, {}_a^J D_t^\alpha \varphi(t)\right] dt = 0. \tag{2.26}$$

Using Theorem 2.9, the last integral in Eq. (2.26) can be expressed by

$$\int_a^b \lambda(t)\, {}_a^J D_t^\alpha \varphi(t) dt = \int_a^b \left[\left(\varphi(t) - \varphi(a)\right) {}_t^J D_b^\alpha \lambda(t) + \lambda(b)\, {}_a^J D_t^\alpha \varphi(t)\right] dt,$$

and substituting it into Eq. (2.26) yields

$$\int_a^b \left[\left(\frac{\partial H}{\partial x} - {}_t^J D_b^\alpha \lambda(t)\right)\varphi(t) + \frac{\partial H}{\partial u}\phi(t) + \left(\frac{\partial H}{\partial \lambda} - {}_a^J D_t^\alpha x(t)\right)\psi(t) + \varphi(a)\, {}_t^J D_b^\alpha \lambda(t) - \lambda(b)\, {}_a^J D_t^\alpha \varphi(t)\right] dt = 0.$$

Consequently, the optimality conditions are given by

$$ {}_a^J D_t^\alpha x(t) = \frac{\partial H}{\partial \lambda}, \tag{2.27}$$

$$ {}_t^J D_b^\alpha \lambda(t) = \frac{\partial H}{\partial x}, \tag{2.28}$$

and

$$\frac{\partial H}{\partial u} = 0, \tag{2.29}$$

with boundary conditions

$$x(a) = x_a \qquad \text{and} \qquad \lambda(b) = 0. \tag{2.30}$$

Remark 2.8. It should be mentioned in the present case $\varphi(a) = 0$ since $x(a)$ is specified.

The system of FDEs results in a two-point boundary value problem in which both ${}_a^J D_t^\alpha$ and ${}_t^J D_b^\alpha$ appear. It is similar to the previous work in (Agrawal & Baleanu, 2007), except that fractional derivative definition and appropriate changes must be made to circumvent the problem caused by the boundary conditions. Substituting Eq. (2.25) into Eqs. (2.27) – (2.29) yield

$${}_a^J D_t^\alpha x(t) = A(t)\, x(t) - \frac{B^2(t)}{R(t)}\, \lambda(t), \tag{2.31}$$

$${}_t^J D_b^\alpha \lambda(t) = Q(t)\, x(t) + A(t)\, \lambda(t), \tag{2.32}$$

and

$$u(t) = -\frac{B(t)}{R(t)}\, \lambda(t). \tag{2.33}$$

We must emphasize that Eq. (2.31) consists of ${}_a^J D_t^\alpha$, whereas Eq. (2.32) consists of ${}_t^J D_b^\alpha$, simultaneously. It indicates that the system requires not only past values for the left-handed derivative but also future values for the right-handed derivative (Ciesielski & Blaszczyk, 2015, 2017). This fact implies that finding solutions $x(t)$ and $\lambda(t)$ to the FOCP would be difficult. Using the Taylor series proposed in Theorem 2.4, we are trying to solve Eqs. (2.31) – (2.32) subject to boundary conditions Eq. (2.30). Besides, it follows that $u(t)$ can be obtained through Eq. (2.33) once $\lambda(t)$ is known.

Having presented the selected cases of FDEs for our research interest, we consider FDEs solution in the next section and propose a class of analytical method to enhance its accuracy as well as convergence further.

2.5 Homotopy analysis method

As a consequence of extensive applications of FDEs, much-related research has significantly grown in the last three decades. Unfortunately, most FDEs do not have easily found exact solutions. Therefore, approximate solutions via either analytical or numerical approaches may be the ideal candidate. Well, generally speaking, unforeseen difficulties have arisen in analytical methods, and numeric methods have therefore become a preferred approach (Pooseh et al., 2013). Although solving FDE analytically is not a trivial task, analytical solutions have certain advantages over numerical solutions. For instance, numeric methods use discretization, which has rounding-off errors causing loss of accuracy, whereas analytical solutions are free from the errors (Daftardar-Gejji & Jafari, 2005).

Analytical solutions of many applications where FDEs appear cannot be established. In the last two decades, many researchers have devoted themselves to search for a robust and stable analytical method in solving FDEs. Various methods have been developed, for instance, Adomian decomposition method (Daftardar-Gejji & Jafari, 2005; Ray & Bera, 2005a), variational iteration method (Das, 2009; Odibat & Momani, 2006), homotopy perturbation method (Jafari et al., 2014; Odibat & Momani, 2008; Zolfaghari et al., 2009), etc. The searching for a new approach to writing down analytical solutions of FDEs is an important topic, as it has the potential for much related research. Recently, a semi-analytical technique based on homotopy in topology, called optimal homotopy analysis method (OHAM) has become increasingly popular. This method has been proven sufficient for various research problems, such as Kawahara equation (Wang, 2011), nonlinear partial fractional differential equations (Gepreel & Nofal, 2015), linear optimal control problems (Jia et al., 2017), and so on. For more details on OHAM, one may consult the works presented in (Liao, 2010, 2012). More importantly, OHAM has the following advantages over both perturbation and non-perturbation methods (Liao, 2009):

- Independent of any physical parameter, which is necessary for the perturbation technique.
- Flexibility of auxiliary linear operator.

2.5.1 Optimal homotopy analysis method

The layout of OHAM is now well documented in the literature, and interested readers may refer to the earlier works of (Liao, 2012) for details. First, we consider the subsequent equation in form

$$N[f(t)] = 0, \tag{2.34}$$

where $f(t)$ is a function to be solved for operator N under boundary constraint given in B,

$$B[f(t)] = 0. \tag{2.35}$$

To elucidate OHAM initially on Eq. (2.34), it is intended to construct zeroth order deformation equation

$$(1-q)L[\phi(t;q)] = q\, h\, N[\phi(t;q)], \tag{2.36}$$

where $h \neq 0$ is convergence control parameter, $q \in [0,1]$ is homotopy embedding parameter, L denotes auxiliary linear operator, whose solution $\phi(t;q)$ varies continuously with respect to q. Moreover, the Maclaurin series of $\phi(t;q)$ about q gives the following homotopy series

$$\phi(t;q) = \sum_{k=0}^{\infty} \phi_k(t)\, q^k, \tag{2.37}$$

where k^{th} homotopy derivative $\phi_k(t)$ can be represented by

$$\phi_k(t) = \frac{1}{k!}\frac{\partial^k \phi(t;q)}{\partial q^k}\bigg|_{q=0}.$$

The OHAM provides us freedom on a choice of $L[\phi(t;q)]$, which has a significant impact on the base function of $\phi(t;q)$ in series Eq. (2.37). It is obvious from Eq. (2.36) that $L[\phi_0(t)] = 0$ at $q = 0$ and $\phi(t;1) = f(t)$ at $q = 1$, respectively. We now go about finding expressions for $\phi_k(t)$ one after other in order $k = 1, 2, \ldots$. Upon k^{th} successive differentiation of Eq. (2.36) with respect to q, then setting $q = 0$ after dividing by $k!$ yields so-called k^{th} order deformation equation

$$L[\phi_k(t)] - L[\phi_{k-1}(t)] = \frac{h}{(k-1)!}\frac{\partial^{k-1}}{\partial q^{k-1}} N[\phi(t;q)]\bigg|_{q=0}, \tag{2.38}$$

where the right-hand side of Eq. (2.38) is dependent only upon known results $\phi_0(t)$, $\phi_1(t)$, …, $\phi_{k-1}(t)$. Additionally, it will be helpful to define Eq. (2.38) iteratively in the following way:

$$L[\phi_1(t)] = h\, N[\phi(t;q)]|_{q=0},$$

$$L[\phi_2(t)] - L[\phi_1(t)] = h \frac{\partial}{\partial q} N[\phi(t;q)]\bigg|_{q=0},$$

$$\vdots \tag{2.39}$$

$$L[\phi_{k-1}(t)] - L[\phi_{k-2}(t)] = \frac{h}{(k-2)!} \frac{\partial^{k-2}}{\partial q^{k-2}} N[\phi(t;q)]\bigg|_{q=0},$$

$$L[\phi_k(t)] - L[\phi_{k-1}(t)] = \frac{h}{(k-1)!} \frac{\partial^{k-1}}{\partial q^{k-1}} N[\phi(t;q)]\bigg|_{q=0}.$$

Thus, we can directly derive $\phi_k(t)$ by solving linear terms in Eq. (2.39) subject to the mentioned boundary condition Eq. (2.35). Remarkably, $\phi_k(t)$ mentioned after Eq. (2.39) will depend upon h, which has an impact on the convergence of Eq. (2.37) as proved in the following theorem.

Theorem 2.11 *Suppose that series Eq. (2.37) is convergent at* $q = 1$ *under appropriate convergence control parameter* h *assumption, and thus* $f(t)$ *can be expressed in the form*

$$f(t) = \sum_{k=0}^{\infty} \phi_k(t). \tag{2.40}$$

Proof. According to proof in convergence theorem (Liao, 2003, 2012), if resultant series Eq. (2.40) is absolutely convergent, then it represents the solution of Eq. (2.34).■

As illustrated in (Liao, 2010; Turkyilmazoglu, 2016), we can find optimal h that controls the convergence of series Eq. (2.40). Upon the k^{th} successive iteration, we arrive at the subsequent mean squared residual error

$$\epsilon = \frac{1}{n} \sum_{i=1}^{n} \left(N\left[\sum_{j=0}^{k} \phi_j(t_i) \right] \right)^2, \tag{2.41}$$

for n equally spaced t values over interval $[t_0, t_1]$. Indeed, by requiring ϵ from Eq. (2.41) decays toward zero, we can compute the optimal h to preserve the convergence rate of Eq. (2.40) in a faster manner. Through software *Mathematica 12*, we can use the command **Minimize** to compute the optimal h that achieves a global minimum of ϵ. Nevertheless, it has some drawbacks that arise in the evaluation of ϵ as follows:

- The necessity to solve the accumulation of h from a set of $\phi_k(t),\ k = 1, 2, \ldots$ given in Eq. (2.39) simultaneously leads to an increase in CPU time.
- In particular, ϵ decays toward zero monotonously at higher order approximations since then the convergence rate of solution diminishes substantially in value.

Owing to such shortcomings, we propose a novel approach in the next chapter.

CHAPTER 3: METHODOLOGY

It is well-known that an auxiliary convergence control parameter in OHAM is an essential attribute to fine-tune convergence. Despite its many documented successes, the approach restricts two convergence control parameters at the most, as suggested in the paper (Liao, 2010). A disadvantage of the method is a failure to enhance the quality of precision in some cases that require high order approximations. Additionally, it is computationally expensive if the number of parameter increase, as shown in (Fan & You, 2013; Niu & Wang, 2010).

Contrary to OHAM, a new idea of constructing homotopy is offered. This method is called iterative homotopy analysis method (IHAM) by introducing a different convergence control parameter at each iteration. In this approach, we can always get accurate enough homotopy series at each iteration. Moreover, our proposed IHAM is computationally more effective than OHAM, as each control convergence parameter at the corresponding iteration can be determined. A background literature search does not indicate that such a method is published so far. As far as we know, this is the only research that employs new IHAM to solve FDEs.

3.1 Iterative homotopy analysis method

To outline our new IHAM, we establish the following zeroth order deformation equation with operator N, solution $\phi(t;q)$, and homotopy embedding parameter $q \in [0,1]$, like those considered in Eq. (2.36)

$$(1-q)L[\psi(t;q)] = q\, N[\phi(t;q)], \tag{3.1}$$

except for auxiliary linear operator L with solution $\psi(t;q)$ subject to mentioned boundary condition Eq. (2.35). Similarly, the Maclaurin series representation of $\psi(t;q)$ becomes

$$\psi(t;q) = \sum_{k=0}^{\infty} \psi_k(t)\, q^k,$$

where k^{th} homotopy derivative $\psi_k(t)$ can be represented by

$$\psi_k(t) = \frac{1}{k!}\frac{\partial^k \psi(t;q)}{\partial q^k}\bigg|_{q=0}.$$

In particular, we have $L[\psi_0(t)] = 0$ by setting Eq. (3.1) at $q = 0$. In addition to this, Eq. (3.1) needs to be differentiated using the same procedure as mentioned in the above OHAM to deduce the k^{th} order deformation equation, namely

$$\mathcal{L}[\psi_k(t)] - L[\psi_{k-1}(t)] = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial q^{k-1}}\mathcal{N}[\phi(t;q)]\bigg|_{q=0}. \qquad (3.2)$$

Interestingly, $\phi_k(t)$ is governed by $\psi_k(t)$ in Eq. (3.2) suggests that there exists a relation between them. It gives rise to the following innovative definition concerning approximation behavior of $\phi_k(t)$ relates to $\psi_k(t)$.

Definition 3.1 Recalling Theorem 2.11, the convergence of solution will be guaranteed fortunately rely on convergence control parameter. Thus both $\phi_k(t)$ and $\psi_k(t)$ are related to each other via

$$\phi_k(t) = h_k\, \psi_k(t),$$

where $h_k \neq 0$ denotes the k^{th} convergence control parameter that corresponds to the k^{th} order deformation equation. Then, in view of Definition 3.1, we can write Eq. (3.2) in an iterative manner as follows:

$$
\begin{aligned}
&L\left[\frac{\phi_1(t)}{h_1}\right] = N[\phi(t;q)]|_{q=0}, \\
&L\left[\frac{\phi_2(t)}{h_2}\right] - L\left[\frac{\phi_1(t)}{h_1}\right] = \frac{\partial}{\partial q} N[\phi(t;q)]\bigg|_{q=0}, \\
&\vdots \\
&L\left[\frac{\phi_{k-1}(t)}{h_{k-1}}\right] - L\left[\frac{\phi_{k-2}(t)}{h_{k-2}}\right] = \frac{1}{(k-2)!}\frac{\partial^{k-2}}{\partial q^{k-2}} N[\phi(t;q)]\bigg|_{q=0}, \\
&L\left[\frac{\phi_k(t)}{h_k}\right] - L\left[\frac{\phi_{k-1}(t)}{h_{k-1}}\right] = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial q^{k-1}} N[\phi(t;q)]\bigg|_{q=0}.
\end{aligned}
\tag{3.3}
$$

We saw in Eq (2.41) that the optimal convergence control parameter is determined by the so-called mean squared residual error ϵ. Surely, we can apply the same concept to h_k, $k = 0, 1, \ldots$ given in Eq (3.3). Contrary to Eq. (2.39), Eq. (3.3) determines each optimal h_k at the corresponding k^{th} iteration independently signifying considerable advantages as follows:

- The present approach avoids heavy accumulation of convergence control parameters during the evaluation of ϵ, resulting in less CPU time.
- As the approximation level gets higher, the growth rate of solution convergence is retained well as a result of rapidly decaying ϵ.

In the next chapter, we will verify the adoption of IHAM for some FDE examples.

CHAPTER 4: RESULTS AND DISCUSSION

In this chapter, we shall demonstrate the practical applicability of the above presented IHAM. We take into consideration widely used examples from the literature described in CHAPTER 2. For each problem, we made a comparison between the previously well-established OHAM and the new IHAM. Thus, we shall concentrate on the evaluation of their convergence criteria within any desired CPU time, which includes computational efficiency as well. These examples are selected because their solution exists in the literature, therefore serve as validation for IHAM. All computations were performed using *Mathematica 12* software on a 64-bit PC with 16 GB RAM and 2.8 GHz CPU.

4.1 Fractional initial value problems

As a first attempt to solve FDEs, let us begin with fractional initial value problems (FIVPs). They are FDEs together with a specified initial condition at starting point.

In the following, we consider Bagley-Torvik equation (BTE) and fractional Riccati differential equation (FRDE). The former being a linear problem, but the latter being a nonlinear problem.

4.1.1 Bagley-Torvik equation

Our first example is similar to that presented in the book (Podlubny, 1998) for which numerical solution is available. Substituting $\alpha = 1.5$, $A(t) = 1$, $B(t) = 0.5$, $C(t) = 0.5$ and $a = 0$ into Eq. (2.15), we have the following BTE

$$f^{(2)}(t) + 0.5\,{}_{0}^{C}D_{t}^{1.5} f(t) + 0.5\,f(t) - g(t) = 0, \tag{4.1}$$

subject to initial conditions Eq. (2.16)

$$f(0) = f^{(1)}(0) = 0, \tag{4.2}$$

with given function

$$g(t) = 8u(t) - 8u(t-1),$$

in terms of Heaviside unit step function $u(t)$. In order to make Eq. (4.1) solvable through both OHAM and IHAM, we choose

$$L[f(t)] = f^{(2)}(t) + 0.5\, f(t),$$

as auxiliary linear operator subject to Eq. (4.2). Thus, we obtain the following first few orders of OHAM and IHAM series:

$$\begin{aligned} f_{\text{OHAM}}(t) = 2.202720 \times 10^{-10}[&t^6 E_{2,-1}(-0.5\, t^2) \\ &- u(t-1)(t-1)^6 E_{2,-1}(-0.5\, (t-1)^2)] \\ &- 2.890560 \times 10^{-8}[t^{5.5} E_{2,-0.5}(-0.5\, t^2) \\ &- u(t-1)(t-1)^{5.5} E_{2,-0.5}(-0.5\, (t-1)^2)] + \cdots \\ f_{\text{IHAM}}(t) = 1.818640 \times 10^{-12}[&t^6 E_{2,-1}(-0.5\, t^2) \\ &- u(t-1)(t-1)^6 E_{2,-1}(-0.5\, (t-1)^2)] \\ &- 8.561400 \times 10^{-10}[t^{5.5} E_{2,-0.5}(-0.5\, t^2) \\ &- u(t-1)(t-1)^{5.5} E_{2,-0.5}(-0.5\, (t-1)^2)] + \cdots \end{aligned}$$

For the sake of comparison, we have plotted Figure 4.1 that displays the mean residual squared error ϵ across CPU time introduced by OHAM and IHAM on interval $[0, 20]$. Perhaps, somewhat surprisingly, the ϵ decay is slow for IHAM, as opposed to OHAM. Like any other method, IHAM has its shortcomings, but it wouldn't fail to converge at higher order approximations, nevertheless.

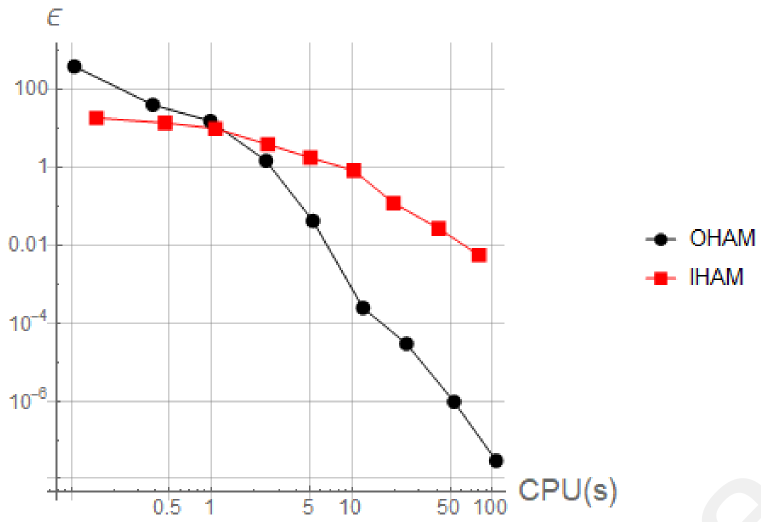

Figure 4.1: Mean squared residual error ϵ for BTE. Circle: OHAM; Square: IHAM.

In addition, with the known solution (Podlubny, 1998), we present absolute error $|\Delta f(t)|$ obtained by both methods in Figure 4.2 together with their mean value in Table 4.1. It turns out that OHAM has better accuracy, which necessitates further improvement of IHAM. One possible reason behind this is due to convergence control parameters of IHAM diminish in value.

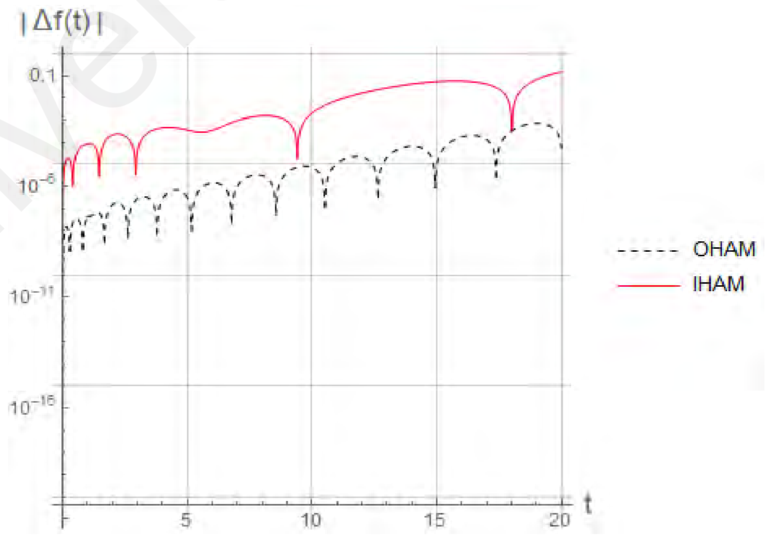

Figure 4.2: Absolute error $|\Delta f(t)|$ for BTE. Dashed line: OHAM; Solid line: IHAM.

Table 4.1: Mean absolute error $\overline{|\Delta f(t)|}$ obtained by OHAM and IHAM for BTE.

Mean absolute error	OHAM	IHAM
$\overline{|\Delta f(t)|}$	7.932006×10^{-5}	2.018109×10^{-2}

4.1.2 Fractional Riccati differential equation

The second example is concerned with the extension of previous work (Cang et al., 2009) on FRDE. Considering case $\alpha \in (0,1)$, $A(t) = 1$, $B(t) = 0$, $C(t) = -1$ and $a = 0$, the corresponding FRDE in Eq. (2.17) becomes

$$ {}_0^J D_t^\alpha f(t) + f^2(t) - 1 = 0, \tag{4.3}$$

along with initial condition Eq. (2.18)

$$f(0) = 0. \tag{4.4}$$

When $\alpha = 1$, Eq. (4.3) is an ordinary Riccati differential equation whose exact solution (Cang et al., 2009) is

$$f(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{4.5}$$

For implementing both OHAM and IHAM to solve Eq. (4.3), we shall pick the following auxiliary linear operator

$$L[f(t)] = {}_0^J D_t^\alpha f(t),$$

with Eq. (4.4) remains satisfied. The first few orders of the OHAM and IHAM series in case $\alpha = 0.5$ are given by

$$f_{\text{OHAM}}(t) = 1.128380\,t^{0.5} - 0.905921\,t^{1.5} + 1.003600\,t^{2.5} - 1.013020\,t^{3.5} + 0.851830\,t^{4.5} - 0.580941\,t^{5.5} + 0.318759\,t^{6.5} + \cdots$$

$$f_{\text{IHAM}}(t) = 1.128330\,t^{0.5} - 0.953956\,t^{1.5} + 1.243200\,t^{2.5} - 1.616110\,t^{3.5} + 1.798410\,t^{4.5} - 1.594700\,t^{5.5} + 1.099010\,t^{6.5} + \cdots$$

From Figure 4.3 (a) and Figure 4.3 (b), we observe an interesting pattern in mean squared residual error ϵ decay behavior of both methods on interval $[0, 5]$ at $\alpha = 0.5$ and $\alpha = 1$, respectively. Although the ϵ decrease for OHAM and IHAM within the desired CPU time, the ϵ of IHAM decays faster to gain higher order approximations. From these figures, it is evident that IHAM consistently improves its convergence.

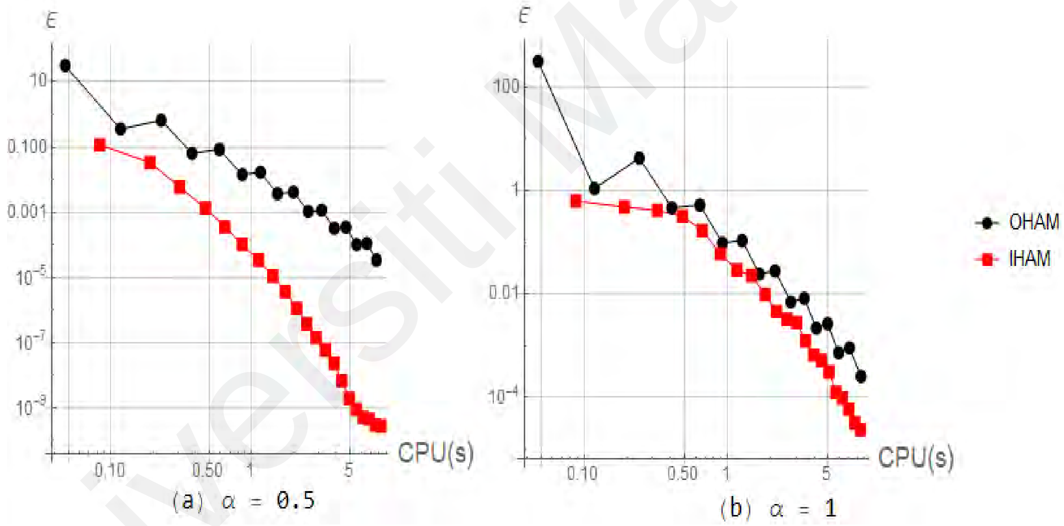

Figure 4.3: Mean squared residual error ϵ for FRDE at (a) $\alpha = 0.5$ and (b) $\alpha = 1$. Circle: OHAM; Square: IHAM.

Additionally, we have plotted absolute error $|\Delta f(t)|$ of both methods using Eq. (4.5) at $\alpha = 1$ in Figure 4.4, and their mean value in Table 4.2. Contrary to OHAM, IHAM yields better accuracy throughout the interval.

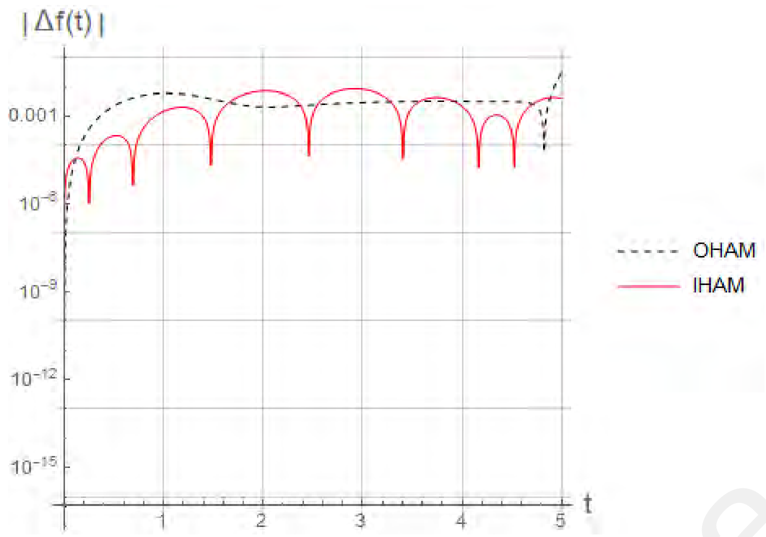

Figure 4.4: Absolute error $|\Delta f(t)|$ for FRDE at $\alpha = 1$. Dashed line: OHAM; Solid line: IHAM.

Table 4.2: Mean absolute error $\overline{|\Delta f(t)|}$ obtained by OHAM and IHAM for FRDE at $\alpha = 1$.

Mean absolute error	OHAM	IHAM
$\overline{\vert\Delta f(t)\vert}$	3.446290×10^{-3}	2.854200×10^{-3}

This example is solved through IHAM for some values of $\alpha \in (0, 1)$ until $\epsilon \leq 10^{-7}$ is satisfied. The obtained results of k^{th} iteration, mean squared residual error ϵ, number of terms M, and CPU time are reported in Table 4.3. Interestingly, the computation of $\alpha < 1$ turns out to be faster, which is proportional to M.

Table 4.3: k^{th} iteration, mean squared residual error ϵ, number of terms M and CPU time for FRDE at various α.

α	k	ϵ	M	CPU (s)
0.25	5	3.348544×10^{-7}	7	3
0.50	10	3.769467×10^{-7}	12	5
0.75	17	4.576198×10^{-7}	19	9
1.00	28	7.077217×10^{-7}	30	15

The $f(t)$ depicted in Figure 4.5 demonstrates that the IHAM solution indeed agrees well with Eq. (4.5) as α closes to 1. Meanwhile, the results with $\alpha < 1$ compare well with those obtained in the literature (Khader, 2013).

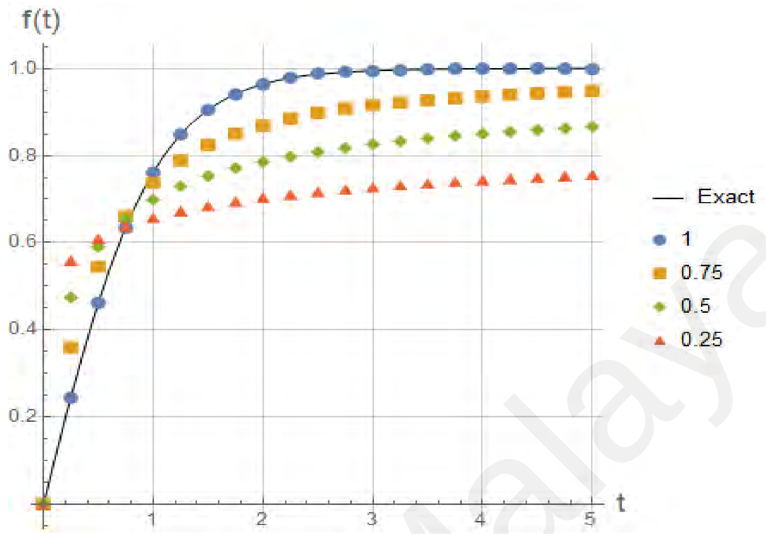

Figure 4.5: Solution for FRDE at various α. Solid line: Exact solution; Circle: $\alpha = 1$; Square: $\alpha = 0.75$; Diamond: $\alpha = 0.5$; Triangle: $\alpha = 0.25$.

4.2 Fractional boundary value problems

In contrast to the FIVPs discussed in Section 4.1, we consider fractional boundary value problems (FBVPs) in this section. They have boundary conditions specified at interval endpoints. Let us illustrate this kind of problem with the following two examples – firstly, linear FBVP is associated with BTE as given by Eq. (2.15), and secondly, nonlinear FBVP has the form of Eq. (2.20).

4.2.1 Linear fractional boundary value problem

We start by recalling the subsequent linear FBVP considered in (Li et al., 2010) on interval $[0, 1]$,

$$f^{(2)}(t) + \sin(t)\, {}_{0}^{C}D_t^{0.5} f(t) + t\, f(t) - g(t) = 0, \tag{4.6}$$

subject to boundary conditions Eq. (2.19)

$$f(0) = f(1) = 0, \tag{4.7}$$

where function $g(t)$ is given by

$$g(t) = t^9 - t^8 + 56t^6 - 42t^5 + \sin(t)\left(\frac{\Gamma(9)}{\Gamma(8.5)}t^{7.5} - \frac{\Gamma(8)}{\Gamma(7.5)}t^{6.5}\right). \tag{4.8}$$

Eqs. (4.6) – (4.8) admit exact solution given by

$$f(t) = t^8 - t^7. \tag{4.9}$$

With a choice of auxiliary linear operator

$$L[f(t)] = f^{(2)}(t),$$

and considering boundary conditions Eq. (4.7) allows us to construct homotopy series via both OHAM and IHAM. Therefore, we have the following first few orders of OHAM and IHAM series:

$$\begin{aligned} f_{\mathrm{OHAM}}(t) = t^8 - t^7 &+ 5.733990 \times 10^{-12}\, t + 1.719750 \times 10^{-11}\, t^{3.5} \\ &+ 1.111320 \times 10^{-11}\, t^4 - 1.013320 \times 10^{-12}\, t^{5.5} \\ &- 1.075210 \times 10^{-10}\, t^6 - 1.085980 \times 10^{-10}\, t^{6.5} + \cdots \\ f_{\mathrm{IHAM}}(t) = t^8 - t^7 &- 1.407520 \times 10^{-12}\, t + 2.543990 \times 10^{-11}\, t^{3.5} \\ &+ 1.643950 \times 10^{-11}\, t^4 - 1.498990 \times 10^{-12}\, t^{5.5} \\ &- 5.203540 \times 10^{-11}\, t^6 - 5.255660 \times 10^{-11}\, t^{6.5} + \cdots \end{aligned}$$

For comparison purposes, let us examine the mean squared residual error ϵ decay described by OHAM and IHAM in Figure 4.6. Notice that both methods generate nearly the same ϵ decrease within the desired CPU time. Hence, both approaches have proven successful for this type of problem.

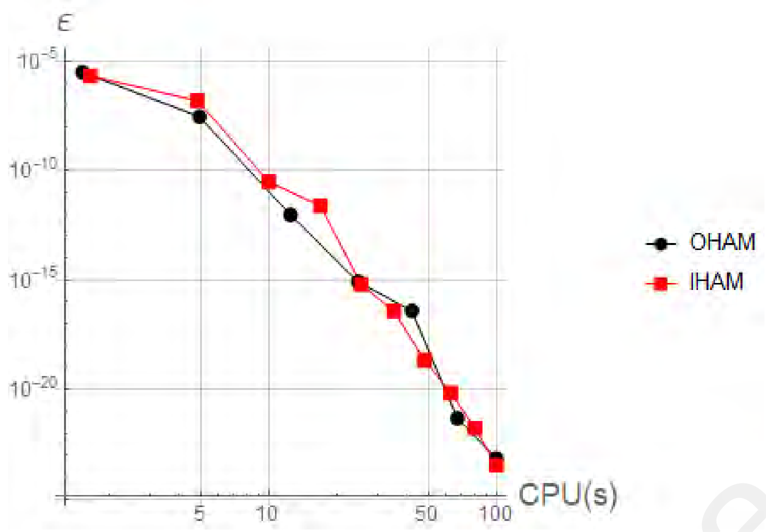

Figure 4.6: Mean squared residual error ϵ for linear FBVP. Circle: OHAM; Square: IHAM.

Since the above comparison does not result in a significant difference, one may try absolute error $|\Delta f(t)|$ of both methods with Eq.(4.9) in Figure 4.7 and their mean value in Table 4.4. Specifically, a closer look at these results reveals that IHAM gives slightly better accuracy compared to OHAM.

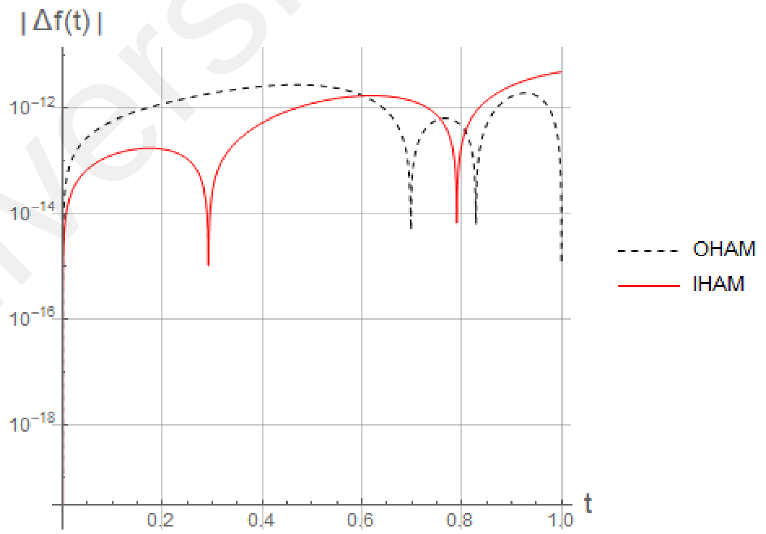

Figure 4.7: Absolute error $|\Delta f(t)|$ for linear FBVP. Dashed line: OHAM; Solid line: IHAM.

Table 4.4: Mean absolute error $\overline{|\Delta f(t)|}$ obtained by OHAM and IHAM for linear FBVP.

Mean absolute error	OHAM	IHAM
$\overline{|\Delta f(t)|}$	1.348667×10^{-12}	1.081129×10^{-12}

4.2.2 Nonlinear fractional boundary value problem

The next example studied here is the following nonlinear FBVP presented in (Pedas & Tamme, 2014; Yun Tao et al., 2016) on interval $[0, 1]$

$$ {}_0^C D_t^{1.5} f(t) - f^3(t) - \frac{\Gamma(2.9)}{\Gamma(1.4)} t^{0.4} + (t^{1.9} - 1)^3 = 0, \tag{4.10}$$

with boundary conditions Eq. (2.21)

$$f(0) = -1 \qquad \text{and} \qquad f(1) = 0. \tag{4.11}$$

The exact solution to this problem is

$$f(t) = t^{1.9} - 1. \tag{4.12}$$

To approximate Eq. (4.12) through both OHAM and IHAM, we can employ auxiliary linear operator

$$L[f(t)] = {}_0^C D_t^{1.5} f(t),$$

subject to boundary conditions Eq. (4.11). Similarly, the first few orders of the OHAM and IHAM series can be produced as follows:

$$f_{\text{OHAM}}(t) = t^{1.9} - 1 - 3.449720 \times 10^{-5}\, t + 1.137230 \times 10^{-4}\, t^{1.5} - 6.742330 \times 10^{-4}\, t^{2.5} + 1.127710 \times 10^{-3}\, t^{3} - 8.176340 \times 10^{-5}\, t^{3.4} + 9.231140 \times 10^{-4}\, t^{3.5} + \cdots$$

$$f_{\text{IHAM}}(t) = t^{1.9} - 1 - 1.576700 \times 10^{-9}\, t + 6.633640 \times 10^{-11}\, t^{1.5} + 1.223510 \times 10^{-8}\, t^{2.5} - 1.389890 \times 10^{-11}\, t^{3} + 6.917660 \times 10^{-12}\, t^{3.4} + 7.869700 \times 10^{-9}\, t^{3.5} + \cdots$$

We focus our attention once again on the mean squared residual error ϵ decay obtained in both methods, as illustrated in Figure 4.8. Remarkably, the ϵ of OHAM decays faster but restricts to low order approximations and hence limited convergence. On the other hand, the ϵ of IHAM steadily decreases toward higher order approximations and results in better convergence.

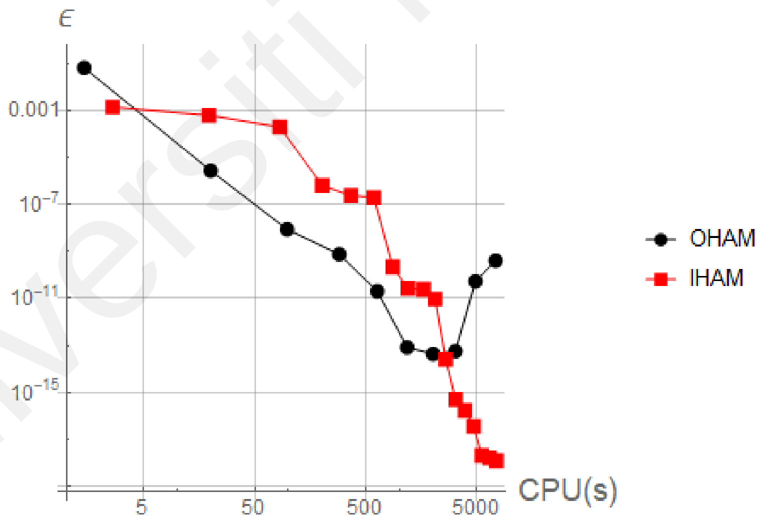

Figure 4.8: Mean squared residual error ϵ for nonlinear FBVP. Circle: OHAM; Square: IHAM.

Furthermore, using Eq. (4.12), it's possible to compare absolute error $|\Delta f(t)|$ of both methods, as well as their mean value. To serve this purpose, we present the results in Figure 4.9 and Table 4.5. One can see that IHAM has the smaller absolute error at higher order approximations, leading to a highly accurate solution.

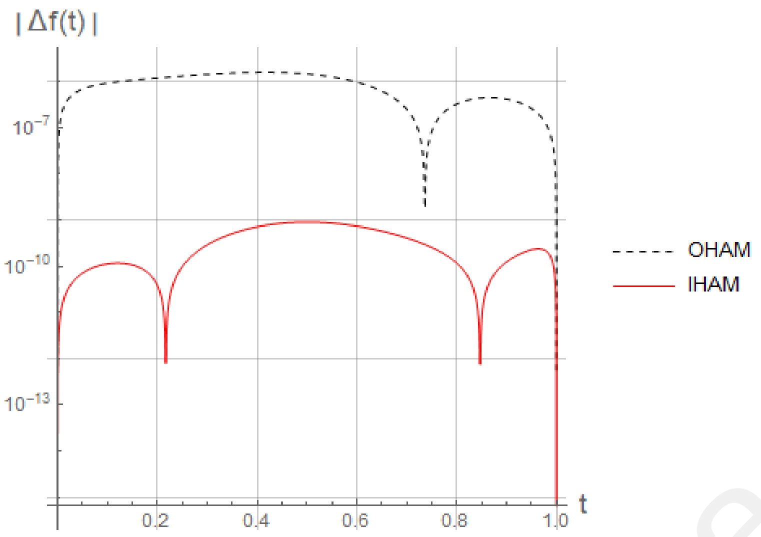

Figure 4.9: Absolute error $|\Delta f(t)|$ for nonlinear FBVP. Dashed line: OHAM; Solid line: IHAM.

Table 4.5: Mean absolute error $\overline{|\Delta f(t)|}$ obtained by OHAM and IHAM for nonlinear FBVP.

Mean absolute error	OHAM	IHAM
$\overline{\lvert\Delta f(t)\rvert}$	8.688650×10^{-7}	3.523985×10^{-10}

4.3 Fractional optimal control problems

Last but not least, we carry out a similar study for fractional optimal control problems (FOCPs), whose dynamic constraint is expressed by FDE. We shall consider the following two famous FOCP examples, time invariant and time variant. The exact solution to these examples for case $\alpha = 1$ can be found in the literature (Agrawal, 1989, 2004; Agrawal & Baleanu, 2007) and references therein.

4.3.1 Time invariant fractional optimal control problem

At first glance, let us consider the subsequent time invariant FOCP. We aim to determine state $x(t)$ and control $u(t)$ that minimize quadratic performance index

$$J = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]dt, \quad (4.13)$$

constrained by dynamic

$$ {}_{0}^{J}D_{t}^{\alpha}x(t) = -x(t) + u(t), $$

with initial condition $x(0) = 1$. Moreover, it follows from Eqs. (2.22) – (2.23) that

$$ Q(t) = R(t) = -A(t) = B(t) = 1. $$

and thus Eqs. (2.31) – (2.33) lead us to

$$ {}_{0}^{J}D_{t}^{\alpha}x(t) + x(t) + \lambda(t) = 0, \tag{4.14} $$

$$ {}_{t}^{J}D_{1}^{\alpha}\lambda(t) + \lambda(t) - \mathrm{x}(t) = 0, \tag{4.15} $$

and

$$ u(t) = -\lambda(t), $$

with boundary conditions Eq. (2.30) as

$$ x(0) = 1 \qquad \text{and} \qquad \lambda(1) = 0. \tag{4.16} $$

We now have to solve Eqs. (4.14) – (4.15) via both OHAM and IHAM involve the subsequent auxiliary linear operators

$$ L_1[x(t)] = {}_{0}^{J}D_{t}^{\alpha}x(t), $$

$$ L_2[\lambda(t)] = {}_{t}^{J}D_{1}^{\alpha}\lambda(t), $$

subject to boundary conditions Eq. (4.16). Consequently, the first few orders of OHAM and IHAM series in case $\alpha = 0.5$ are as follows:

$$x_{\text{OHAM}}(t) = 1 - 1.497660\, t^{0.5} + 1.260480\, t - 0.316724\, t^{1.5} + 0.039199\, t^2 - 0.846082\, t^{2.5} + 0.158191\, t^3 + 2.659930\, t^{3.5} + \cdots$$

$$u_{\text{OHAM}}(t) = -0.353910\,(1-t)^{0.5} + 0.178172\,(1-t) - 0.039766\,(1-t)^{1.5} + 0.017725\,(1-t)^2 - 0.231917\,(1-t)^{2.5} + \cdots$$

$$x_{\text{IHAM}}(t) = 1 - 1.522840\, t^{0.5} + 1.338900\, t - 0.377034\, t^{1.5} + 0.201909\, t^2 - 1.847590\, t^{2.5} + 0.890184\, t^3 + 5.642160\, t^{3.5} + \cdots$$

$$u_{\text{IHAM}}(t) = -0.404320\,(1-t)^{0.5} + 0.355808\,(1-t) - 0.209851\,(1-t)^{1.5} + 0.113846\,(1-t)^2 - 0.550836\,(1-t)^{2.5} + \cdots$$

Figure 4.10 (a - b) show a comparison between mean squared residual error ϵ decay of both methods at $\alpha = 0.5$ and $\alpha = 1$. In general, the ϵ decrease for OHAM and IHAM within the desired CPU time. Nevertheless, IHAM requires less computational effort to achieve higher order approximations, which significantly contributes to improving its convergence.

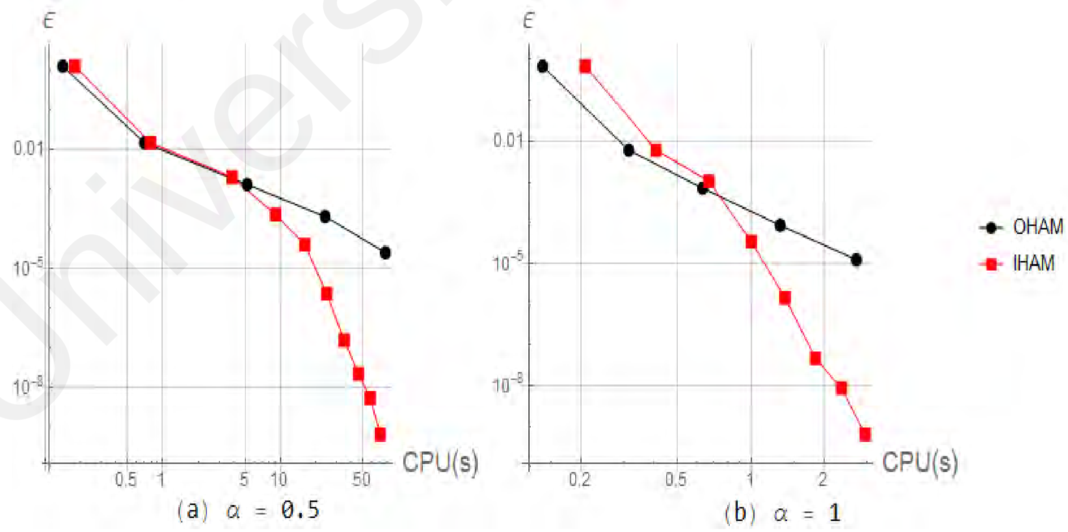

Figure 4.10: Mean squared residual error ϵ for time invariant FOCP at (a) $\alpha = 0.5$ and (b) $\alpha = 1$. Circle: OHAM; Square: IHAM.

Furthermore, Figure 4.11 (a - b) and Table 4.6 signify better accuracy at $\alpha = 1$, where absolute errors ($|\Delta x(t)|$, $|\Delta u(t)|$) of IHAM, including mean values, are smaller than those of OHAM with the exact solution (Agrawal, 2004).

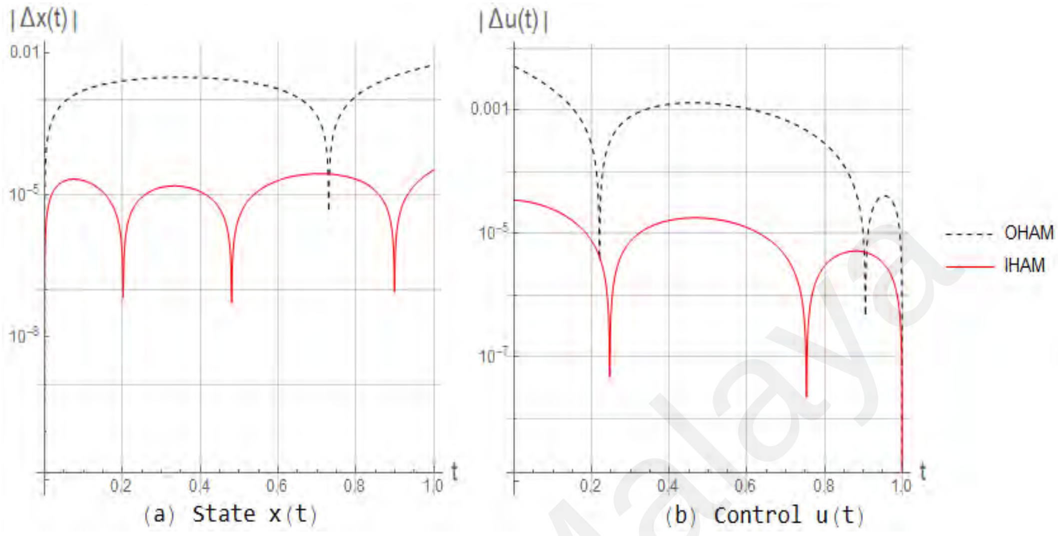

Figure 4.11: Absolute errors $|\Delta x(t)|$ and $|\Delta u(t)|$ for time invariant FOCP at $\alpha = 1$. Dashed line: OHAM; Solid line: IHAM.

Table 4.6: Mean absolute errors $\overline{|\Delta x(t)|}$ and $\overline{|\Delta u(t)|}$ obtained by OHAM and IHAM for time invariant FOCP at $\alpha = 1$.

Mean absolute error	OHAM	IHAM
$\overline{|\Delta x(t)|}$	2.226620×10^{-3}	1.487842×10^{-5}
$\overline{|\Delta u(t)|}$	1.016442×10^{-3}	1.149686×10^{-5}

Carrying out IHAM to solve this example for some values of $\alpha \in (0, 1)$ until $\epsilon \leq 10^{-11}$ is satisfied. We have tabulated Table 4.7 that displays their k^{th} iteration, mean squared residual error ϵ, number of terms M, and CPU time. As expected from Theorem 2.4, $\alpha < 1$ essentially requires more CPU time to generate sufficient M if the desired ϵ is pursued.

Table 4.7: k^{th} iteration, mean squared residual error ϵ, number of terms M and CPU time for time invariant FOCP at various α.

α	k	ϵ	M	CPU (s)
0.25	10	6.485586×10^{-11}	232	190
0.50	10	8.781502×10^{-11}	117	91
0.75	10	8.214418×10^{-11}	207	167
1.00	8	9.650381×10^{-11}	9	3

Figure 4.12 (a) and Figure 4.12 (b) depict $x(t)$ and $u(t)$ of IHAM at various values of α, including the exact solution (Agrawal, 2004) at $\alpha = 1$, respectively. It indicates that the result matches the exact solution when α approaches 1, whereas the solutions with $\alpha < 1$ are similar to those found in the paper (Tang et al., 2015).

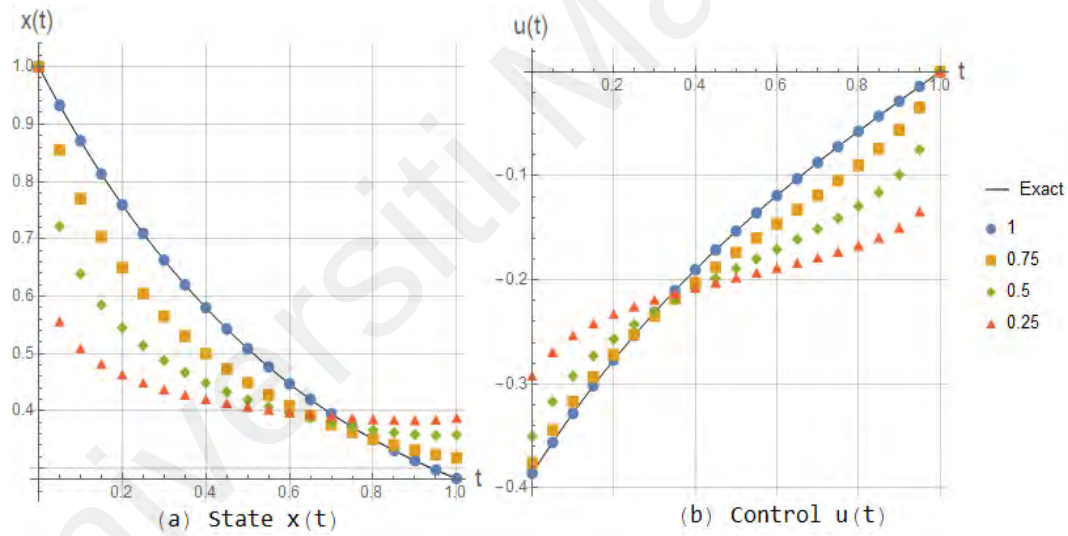

Figure 4.12: Solutions for time invariant FOCP at various α. Solid line: Exact solution; Circle: $\alpha = 1$; Square: $\alpha = 0.75$; Diamond: $\alpha = 0.5$; Triangle: $\alpha = 0.25$.

4.3.2 Time variant fractional optimal control problem

The second example that we shall discuss is the following time variant FOCP with identical performance index Eq. (4.13) and boundary conditions Eq. (4.16), like those considered in the above first example, except that dynamic constraint is given by

$${}_{0}^{J}D_{t}^{\alpha}x(t) = t\,x(t) + u(t).$$

In this case, we have

$$Q(t) = R(t) = B(t) = 1 \qquad \text{and} \qquad A(t) = t,$$

which follows from Eqs. (2.22) – (2.23). Alternatively, we can solve this example in the same manner as the previous one, whereas Eqs. (4.14) – (4.15) in a slightly different form as follows

$${}_{0}^{J}D_{t}^{\alpha}x(t) - t\,x(t) + \lambda(t) = 0, \tag{4.17}$$

$${}_{t}^{J}D_{1}^{\alpha}\lambda(t) - t\,\lambda(t) - x(t) = 0, \tag{4.18}$$

subject to boundary conditions Eq. (4.16). In the same manner, we get the following first few orders of the OHAM and IHAM series in the case of $\alpha = 0.5$:

$$x_{\text{OHAM}}(t) = 1 - 1.102570\,t^{0.5} + 0.768096\,t^{1.5} - 0.271720\,t^{2} - 0.633567\,t^{2.5} + 0.140092\,t^{3} + 2.4932600\,t^{3.5} + 0.064610\,t^{4} + \cdots$$

$$u_{\text{OHAM}}(t) = -0.780401\,(1-t)^{0.5} - 2.991540\,(1-t) + 5.981590\,(1-t)^{1.5} - 1.554350\,(1-t)^{2} - 7.825800\,(1-t)^{2.5} + \cdots$$

$$x_{\text{IHAM}}(t) = 1 - 1.085050\,t^{0.5} + 1.350160\,t^{1.5} - 0.721304\,t^{2} - 1.790550\,t^{2.5} + 0.746757\,t^{3} + 5.946000\,t^{3.5} - 0.858870\,t^{4} + \cdots$$

$$u_{\text{IHAM}}(t) = -1.025280\,(1-t)^{0.5} - 0.911759\,(1-t) + 1.362330\,(1-t)^{1.5} + 1.605520\,(1-t)^{2} - 3.764110\,(1-t)^{2.5} + \cdots$$

Like the previous Example 4.3.1, the same comparison between mean squared residual error ϵ decay of both methods at $\alpha = 0.5$ and $\alpha = 1$ are presented in Figure 4.13 (a - b). Once again, these figures confirmed that the ϵ decay of IHAM is much faster within the desired CPU time. Moreover, IHAM reaches higher order approximations at an accelerated pace and consequently leads to rapid convergence.

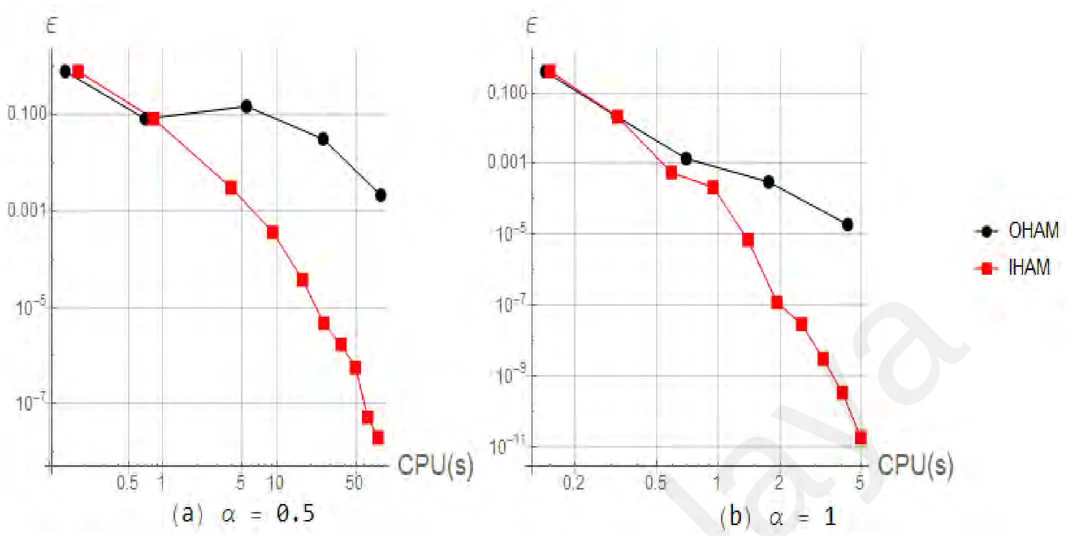

Figure 4.13: Mean squared residual error ϵ for time variant FOCP at (a) $\alpha = 0.5$ and (b) $\alpha = 1$. Circle: OHAM; Square: IHAM.

Apart from the above comparisons, we are also interested in absolute errors ($|\Delta x(t)|$, $|\Delta u(t)|$) of both methods at $\alpha = 1$ and their mean values, as illustrated in Figure 4.14 (a - b) and Table 4.8, respectively. From these figures and table, IHAM has even smaller absolute errors with the exact solution (Agrawal & Baleanu, 2007), signifying rapid convergence of IHAM is further assured.

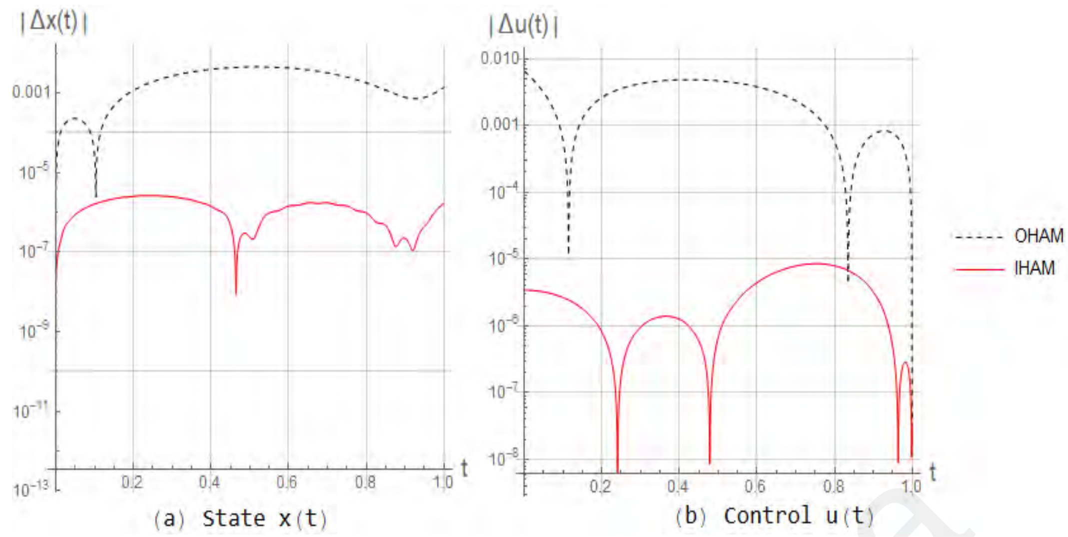

Figure 4.14: Absolute errors $|\Delta x(t)|$ and $|\Delta u(t)|$ for time variant FOCP at $\alpha = 1$. Dashed line: OHAM; Solid line: IHAM.

Table 4.8: Mean absolute errors $\overline{|\Delta x(t)|}$ and $\overline{|\Delta u(t)|}$ obtained by OHAM and IHAM for time variant FOCP at $\alpha = 1$.

Mean absolute error	OHAM	IHAM
$\overline{|\Delta x(t)|}$	2.321960×10^{-3}	1.314811×10^{-6}
$\overline{|\Delta u(t)|}$	2.740796×10^{-3}	3.127456×10^{-6}

As we did in the previous case, this example is solved through IHAM for some values of $\alpha \in (0, 1)$ until $\epsilon \leq 10^{-9}$ is satisfied. The resulting k^{th} iteration, mean squared residual error ϵ, number of terms M and CPU time are tabulated in Table 4.9. In view of Theorem 2.4, extra computation time is required for $\alpha < 1$ generates nearly the equivalent ϵ when larger M is involved.

Table 4.9: k^{th} iteration, mean squared residual error ϵ, number of terms M and CPU time for time variant FOCP at various α.

α	k	ϵ	M	CPU (s)
0.25	20	5.280239×10^{-9}	333	762
0.50	10	6.513002×10^{-9}	123	99
0.75	10	5.491473×10^{-9}	219	174
1.00	7	3.018190×10^{-9}	14	4

Figure 4.15 (a) and Figure 4.15 (b) are clear evidence that the IHAM solution reveals excellent agreement with the exact solution (Agrawal & Baleanu, 2007) as α closes to 1. Additionally, the IHAM solutions at $\alpha < 1$ are comparable to those obtained in the literature (Tang et al., 2017).

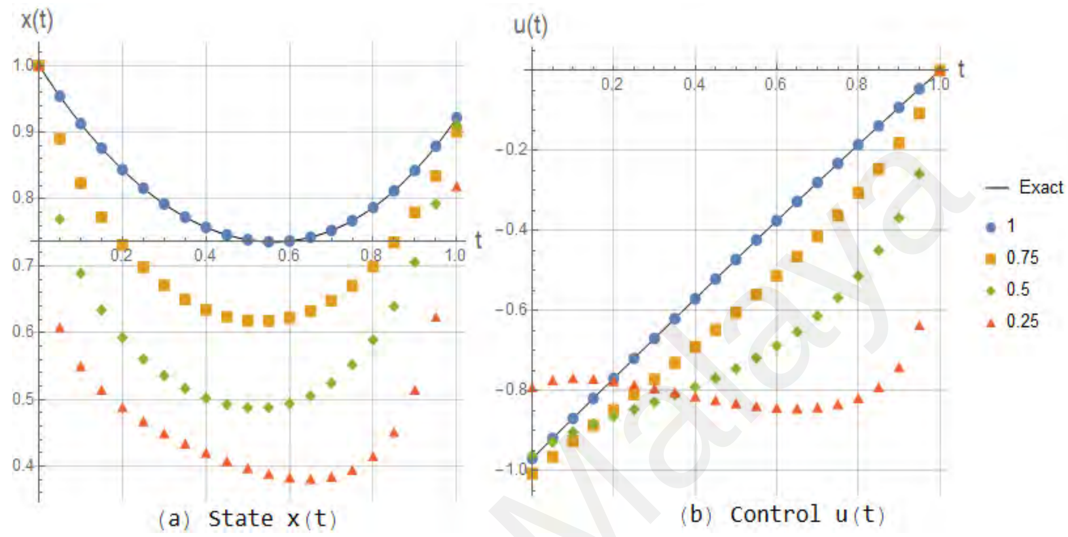

Figure 4.15: Solutions for time variant FOCP at various α. Solid line: Exact solution; Circle: $\alpha = 1$; Square: $\alpha = 0.75$; Diamond: $\alpha = 0.5$; Triangle: $\alpha = 0.25$.

CHAPTER 5: CONCLUSION

This research aimed to identify an effective strategy for solving FDEs analytically. In the beginning, we recalled a fundamental knowledge of FC. Then we looked at some linear and nonlinear problems connected with FIVPs, FBVPs, and FOCPs. Based on our investigation with a survey of existing methods, we can conclude that a low convergence rate at necessarily high order approximations still seems not to have been addressed. This issue inspires us to consider an opportunity for improving the solution.

By means of Laplace transform, a calculation involving fractional operator turns out to be simpler within less computational time. On the other hand, we further developed FOCP by considering α^{th} Jumarie modified Riemann-Liouville fractional derivative. The formulation inherently leads to a system of FDEs consisting of both left-handed and right-handed α^{th} fractional derivatives, and the resulting integration is much harder to evaluate. An alternative way to ease such difficulty is substituting a function being integrated into the integration by its Taylor series. Anyway, the proposed technique greatly benefits a relevant application to solve FDEs better.

Although there exist several approaches to solve FDEs, we consider OHAM discussed in this dissertation could be privileged due to its simplicity. Furthermore, we developed a novel idea called IHAM to address a gap in OHAM by relaxing the limitation of a convergence control parameter. Distinct from OHAM, a different convergence control parameter is attached to a homotopy series at each iteration. This approach offers new insight into the approximation behavior of the solution.

Following such an idea, the presented comparisons with six commonly used FDE examples clearly illustrate a distinction between OHAM and IHAM. The results indicate that IHAM is more robust at further increasing convergence criteria, even though less

CPU time is an advantage. More precisely, this improvement may bring a new way of solving FDEs at least the presented examples point to this fact. Nevertheless, there are some exceptions and room for improvements, such as Example 4.1.1 of BTE.

Based on the implication of results, a future study could extend IHAM to solve other FDEs, for instance, Partial FDEs, Sequential FDEs, etc. This extension is left to potentially perceptive readers who can explore these possibilities. Last but not least, further research is needed to reveal other underlying properties that could strengthen the effectiveness of IHAM. We hope that our proposed approach will be advantageous to the development of FC.

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LIST OF PUBLICATIONS AND PAPERS PRESENTED

List of publications

1. **Lee, M. O.**, Kumaresan, N., & Ratnavelu, K. (2016). Solution of fuzzy fractional differential equations using homotopy analysis method. *MATEMATIKA, 32*(2), 113-119.

2. **Lee, M. O.**, Kumaresan, N., & Ratnavelu, K. (2017). Solution of fuzzy linear fractional order boundary value problems. *Global Journal of Pure and Applied Mathematics, 13*(5), 509-517.