INVENTORY MODELS FOR DETERIORATING ITEMS WITH POWER PATTERN DEMAND RATE

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INVENTORY MODELS FOR DETERIORATING ITEMS WITH POWER PATTERN DEMAND RATE

ABSTRACT

Inventory management has become a prevalent topic in the field of operational research over some decades now. It cuts across many areas like management sciences, statistics and engineering. However, research focusing on inventory model with power demand pattern is quite limited. Demand patterns are defined as different ways by which products are removed out of inventory to supply customers demand during the schedule period. Power demand pattern permits suiting the demand to a more practical situation. In this research, four deterministic inventory model for deteriorating items with power demand pattern has been developed. We considered in the first models an inventory model for deteriorating item with power demand pattern and time-dependent holding cost without shortages, in the second model, we considered an inventory model for linear time-dependent deteriorating rate and time-varying demand with shortages completely backlogged, while in the third model, an inventory model for delayed deteriorating items with power demand considering shortages and lost sales. Linear deteriorating inventory policy for products with power demand pattern and variable holding cost considering shortages is developed in the final model with some additional features. In all the models, the objective is to determine the optimal replenishment strategy for the proposed inventory model to minimise the total inventory cost per unit time. Mathematical formulation for and analysis of the inventory problems were developed within the framework of the model assumptions. A system of differential equations incorporating initial and boundary conditions are given for the proposed inventory policy, and the problem solved using Microsoft@Excel@ Solver and maple software 2018 to obtain the optimal solutions for all the models. Numerical examples are given at the end of each developed inventory model to establish the robustness and effectiveness of the models. Moreover, the sensitivity analysis of each model was carried out to see the effects of various changes in some possible parameters combination of the inventory policy.

Keywords: Inventory control, Power demand pattern, Deterioration, Shortages, Linearly, Non-instantaneous.

MODEL INVENTORI UNTUK ITEMS YANG MEROSOT DENGAN KADAR PERMINTAAN CORAK KUASA

ABSTRAK

Pengurusan inventori telah menjadi satu topik yang sangat popular dalam bidang penyelidikan operasi sejak beberapa dekad yang lalu. Ia meliputi beberapa bidang seperti sains pengurusan, stastistik dan kejuruteraan. Walau bagaimanapun, penyelidikan yang memberi tumpuan kepada model inventori dengan permintaan corak kuasa agak terhad. Corak permintaan didefinasikan sebagai cara yang berbeza di mana produk dikeluarkan daripada inventori untuk dibekalkan kepada pelanggan dalam tempoh yang dijadualkan. Permintaan corak kuasa boleh menyesuaikan permintaan kepada situasi yang lebih praktikal. Dalam penyelidikan ini, empat model inventori deterministik untuk item yang semakin merosot dengan permintaan corak kuasa telah dibentuk. Dalam model pertama, kami mempertimbangkan satu model inventori untuk item yang semakin merosot dengan permintaan corak kuasa dan kos pemegangan bergantung kepada masa tanpa kekurangan. Dalam model kedua, kami mempertimbangkan satu model inventori di mana kadar merosot bergantung kepada masa secara linear, permintaan corak kuasa dan kekurangan dipenuhi sepenuhmya. Untuk model ketiga, kami membentuk satu model inventori dengan kemerosotan tertangguh, permintaan corak kuasa dengan kekurangan dan hilang jualan. Model terakhir kami menggabungkan kesemua ciri tiga model di atas dimana corak merosot adalah linear, permintaan corak kuasa, kos pemegangan berubah dan kekurangan dibenarkan. Dalam semua model, objektifnya adalah untuk menentukan strategi pengisian semula dalam model inventori yang optimum supaya meminimumkan jumlah kos inventori seunit masa. Perumusan matematik dan analisis masalah inventori telah dirumuskan berdasarkan andaian model. Satu sistem persamaan pembeza dengan syarat awal dan sempadan telah diberikan untuk menjelaskan inventori yang dicadangkan dan masalah telah diselesaikan dengan perisian Microsoft@Excel@ Solver dan maple 2018 untuk mendapatkan penyelesaian optimum untuk semua model. Contoh berangka telah diberikan di akhir setiap model untuk menunjukkah keberkesanan setiap model itu. Selain itu, analisis kepekaan telah dijalankan untuk melihat kesan beberapa perubahan yang mungkin bagi setiap model inventori yang telah dibentuk

Kata kunci: Kawalan inventori, Permintaan corak kuasa, Kemerosotan, KekuranganLinear, Tidak serta merta.

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LIST OF ABBREVIATIONS

EOQ	:	Economic order quantity
EPQ	:	Economic production quantity
HC	:	Holding cost
OC	:	Ordering cost
OC	:	Opportunity cost
ODE	:	Ordinary differential equation
PC	:	Purchasing cost
S C	:	Shortage cost
TC	:	Total cost
W.R.T	:	With respect to

LIST OF APPENDICES

CHAPTER: INTRODUCTION

1.1 Introduction

This introductory chapter aims to outlines the background of the research, set the statement of the problem, state the research purpose and relevance of the study, discuss the research outputs and outcome, indicate the methodology for the research employed in this research work and introduce the organisation follows in the thesis.

1.2 Background of the Study

The target of inventory management is most appropriately described as when the inventory should be replenished and how much should be added to the inventory. This situation has led to inventory problems consisting of making optimal decisions to either minimise or maximise the total cost of the inventory system.

Inventory in general terms can be defined as physical stocks of good kept in store to meet future demand. It also means a usable, but idle resources that have economic value in material management. It is essential to keep physical stocks in the system to meet anticipated demand, failure to do can lead to non-availability of materials/goods when in need which will lead to disruption in production or services to be delivered. According to Muckstadt et al. (2010), we daily experience inventory in a different form at a very specific point in time. For a household, we stock our kitchen with different types of foods items not necessarily needed at that specific point in time, but essential to be used in future time. Manufacturers stock their facilities with inventories of finished goods, raw materials, and work in progress. Wholesalers and retailers stores are filled with materials in excess to prevent stock-out when the products and the materials are needed. Hospital and health centres stocked drugs and other related materials to be used in the nearest future. Security personnel stocked their armouries with different equipments and materials to maintain laws and order in the society when the need arises. These and many other more examples tell us that inventory exists as a result of policy and technology.

One of the most vital component of an inventory model is demand. We keep inventories in order to meet the demand, fulfilled order, and satisfied requirement. Inventory problems subsist as a result of demand; if not, inventory problem will not take place. Over decades, inventory problems have resolved round the case of demand been monotonic or a fixed function. Demand is said to be constant at the fully developed stage of the product's life cycle while it is monotonic at the onset or last stage of the life cycle of the items. Majority of the available inventory models deals with these kind of demand functions. In reality, this is almost impossible. It is better to assume that, the demand of an items is dependent on time/time-varying.

Deterioration is a common phenomenon in inventory model. Some existing models deals with non-deterioration items while others deal with instantaneous deterioration, and still others considered delayed deteriorating items. In this technological age, some items will maintained their freshness over a period of time before they goes bad. It will be appropriate to consider this type of delayed deterioration which is a very important in inventory model.

In these recent days, varying holding cost is engaging the attention of many researchers as a result of its important in maintaining inventory. It is usually assumed that holding/carrying cost is known and always considered as constant. On the contrary, when inventory is stored for later usage, then it is rather important to preserve the physical states of the inventory at hand. To be current and up-to date with the present market reality, time-dependent holding/carrying cost is very essential to consider in inventory model problems.

In inventory model, when the shortages happen, it is presumed that the whole demand is either totally lost or totally backlogged. Realistically, this situation is not true. During the stock-out time, certain consumers are ready to hold on for the succeeding refill and accept their orders at the termination of shortage time. Other will prefer alternative way of purchasing from other available source. Partial backlogging or lost sales best describes this type of situation in the modelling of inventory. In many cases, consumers who once suffered from stock-out may not buy the items again from the individual suppliers, and they usually turn to another stores to buy the items. As a result of this, a bulky percentage of the transactions are lost, resulting in a little profit. This reason make partial backlogging an important factor to be considered when setting up mathematical inventory model.

Organization or business enterprise can profit immensely from making use of mathematical models for inventory. Mathematical models are use to maximize profits and/or to minimize cost and to forestall having a dormant commodity. The models help the business to discover the optimal inventory period of times and to also recommend the total amount of the product that must be ordered or produced to reduce the cost. For any business enterprise or organization to benefit maximally from mathematical models, firstly, mathematical model that will be formulated must put into consideration several key factors. Secondly, using the model they evaluate the order quantity and optimal cycle times. Lastly, the business enterprise or organization must make use of computer often to keep the record of the inventory positions, costs and other factors so that they can modify the models, if they so desire.

The target of this thesis is to develop various mathematical inventory models with power demand patterns and time-dependent deterioration rate and time-varying holding cost which are some of the gaps discovered in the available academic literature.

1.3 Statement of the Problem

The purpose of this research work/thesis is to improve on the present state of knowledge in the subject area of mathematical modelling of inventory control and management through developing theoretically sound and empirically feasible generalised inventory model framework to help the managers of inventory to determine the optimal order and production size that minimise the total inventory costs. The purpose is demonstrated in four salient objectives:

1. To propose an inventory model for linearly time-dependent deteriorating items with power pattern, shortages and time-varying demand.
2. To propose an inventory model for delayed deteriorating items with power demand considering shortages and lost sales.
3. To propose an inventory model for deteriorating items with time-varying demand rate and time- dependent holding cost without considering shortages
4. To propose an inventory model for a linear inventory policy for items with power demand pattern and variable holding cost considering shortages.

1.4 Aims and Objectives of the Research

The impetus of this research is to develop/advance new models and to extend the previous research in the economic order quantity model by adding dependent linearly deteriorating items that have power pattern to determine the optimal minimum cost. It is of interest to consider a model with time-dependent holding cost as against the previous models, which assumed that holding cost is constant. Also, this research will examine delayed deteriorating items with power demand considering shortages and lost sales. Moreover, a linear deteriorating inventory policy for items with power demand pattern and time-dependent holding cost with shortages will be addressed. Besides, the outcome of the models will lead to recommendations on how to exploit these models in practical and theoretical ways.

1.5 Relevance of this study

The research is relevant to both academic and industry in the sense that, in educational, it will improve the present state of knowledge and understanding on the field of inventory management and also provide a guide for subsequent research work, and in the industrial area, it will help to minimise/maximise the costs and the profits which is the sole aims of every industry.

1.6 Research outputs and outcomes

The outcomes of this research will be enumerated below:

1. To develop an inventory model for deteriorating item with power demand pattern and time-varying holding cost as against the usual constant holding cost.
2. To develop the inventory model with linearly dependent deteriorating items as against the usual assumption that the deterioration is constant.
3. To develop a new model for linear deteriorating inventory policy for items with power demand pattern and variable holding cost considering shortages from all the three models mentioned above.
4. To develop an economic order quantity model for delayed deteriorating items with power demand pattern considering shortages and lost sales.

1.7 Research Methodology

This research work is a modelling research work that has to be carried out mainly by the following steps:

1. Literature Review

Comprehensive literature review were carried out under the following subtopic:

- Classical EOQ (economic order quantity models)
- Economic production quantity model
- Inventory models with constant deteriorating rate
- Economic order quantity (EOQ) model with power demand pattern
- Economic production quantity (EPQ) model with power demand pattern
- Deteriorating inventory model with shortages partially backlogged
- Inventory model for deteriorating items with power demand and shortages completely backlogged

2. Decision Variables

Analysis of the inventory and formulation of the problem are developed within the framework of assumptions made for the model to determine the inventory policies of the system. The decision variables were identified and solutions provided.

3. Model Building

The formulated inventory problems are represented by the system of ordinary differential equations (ODE) with both initial and boundary conditions.

4. Developing Algorithms and Solution Approach

Mathematical models are developed for each of the different inventory problems. The aims are to establish the inventory policy that minimises/maximises the total cost/profits per unit time of the formulated models. Solvers like Maple, Excel will be used to solve the non-linear equations concerning decision variables. Moreover, some numerical examples will be presented to illustrate the applications of the developed models.

5. Performance Measure

Sensitivity analysis on the decision variables with regards to changes in the parameter of the model will be carried out. Specific conclusions will be made at the end of each develop models.

1.8 Organisation of the Thesis

The contents of the thesis will be divided into six chapters.

Chapter One: Presented introduction of this work. This chapter begins with the introduction of the research, background of the study, statement of the problem, aims and objectives of the research, relevance of this study, research outputs and outcomes, research methodology and organisation of the thesis.

Chapter Two: Discusses literature review about the work. This chapter reviews the published works of literature related to mathematical modelling on inventory models with time-varying demand, inventory models with deteriorating items, inventory models with power demand pattern, inventory models for non-instantaneous (delayed) deteriorating items, inventory models with shortages and production inventory models for deteriorating items and a time-dependent deteriorating inventory model for items with power demand and variable holding cost considering shortages.

Chapter Three: Discusses the development of the inventory model for deteriorating items with power demand pattern and time-dependent holding cost without shortages. This chapter deals with the inventory model for deteriorating items with time-varying holding cost. Shortages are not considered in the policy. The chapter started with an introduction and followed by assumptions and notation employed in building the model. Mathematical formulation and solution to the formulated problem were given. Moreover, numerical examples and sensitivity analysis to show the effects of various changes in some possible parameters was performed. The chapter ends with concluding observations.

Chapter Four: Proposes an inventory model for linearly time-dependent deteriorating items with power pattern, shortages and time-varying demand. This chapter is concerned with the development of an EOQ model for linearly time-dependent deteriorating items. The chapter begins with an introduction follows by modelling assumption, mathematical model formulation was developed with the solution provided. Furthermore, numerical illustration was given to determine the accuracy of the model. Also, a sensitivity analysis was carried out along with concluding remarks at the end of the chapter.

Chapter Five: Proposes a linear deteriorating inventory model for items with power demand pattern and variable holding cost considering shortages. This chapter presents an inventory model for items that possess a deteriorating linear rate with holding cost, which is also time-dependent. The demand is assumed to follow a power demand pattern. The chapter begins with the introduction, notation and assumptions for the model is also provided. The mathematical formulation of the problem follow this, and the solution method is given. A numerical example illustrates the model and graphical representation of sensitivity analysis to show the influence of various changes in some possible parameters is provided based on the numerical example. Concluding observations is given at the termination of the chapter.

Chapter Seven: Highlights the overall summary, conclusion and next step of research. This section deals with the overview the entire work, the outcome with research contributions and further research direction for future action.

References, List of Publications and Appendix The thesis ends with references, list of publications and appendix arrange in that order.

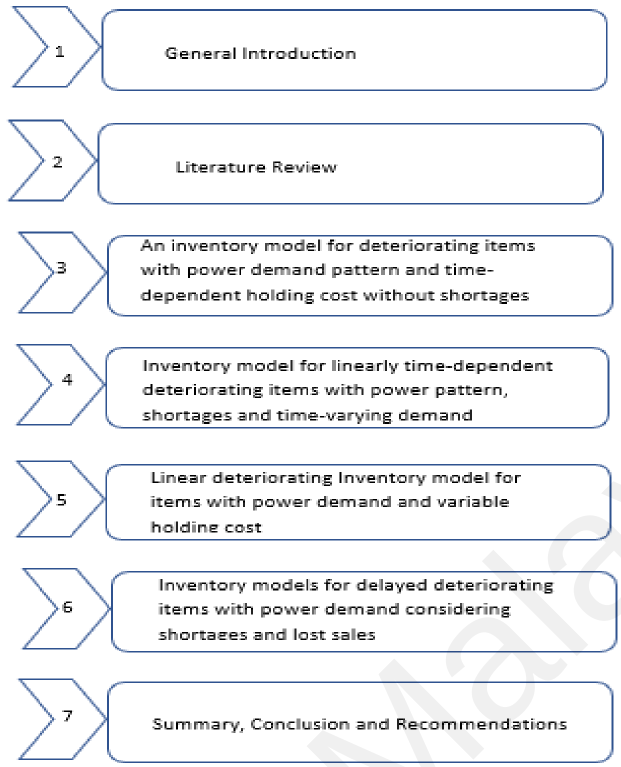


Figure 1.1: An illustrative outline of the organisation of the thesis.

CHAPTER 2: REVIEW OF LITERATURE

2.1 Introduction

The investigation of academic publications related to inventory models has engaged the attention of many researchers and has been discussed extensively over the last many decades. Numerous research works have been produced where various inventory models were introduced and developed. Most of these works centred significantly on the development of solutions of economic order quantity models. Different aspects of EOQ models such as deterioration, shortages, instantaneous or non-instantaneous replenishment, different type of demand has been discussed and analysed.

In this section, a review of the literature of both the EOQ and EPQ model related to this thesis will be discussed here.

2.1.1 Inventory Model with Time-Varying Demand

The first (EOQ) economic order quantity formula was propounded by Harris (1913) in 1913, and since then, many researchers have analysed different kind of inventory system, modifying some of the assumptions in the first model. It is assumed in an economic order quantity model that the demand for goods is unchanging. Nevertheless, this presumption is not truly popular or realistic in practices, and it will be preferable to take into consideration that the demand varies with time. The review of inventory models with time-varying demand is critical since it permits to appropriately modelling the behaviour and evolution of the inventory according to Sicilia et al. (2014).

Several researchers have worked on inventory models in which the demand varies with time. Donalson (1977) analysed an inventory replenishment models with traditional no-shortage strategy for a linear drift in demand using the method of calculus to obtain optimal time of replenishment. Silver (1979) studied an inventory replenishment selection rule for a linear trend in demand. Ritchie (1984) analysed a simple solution method for an inventory model with demand linearly growing. He extended the time planning horizon of Donaldson analytical solution.

Mitra et al. (1984) designed a technique for determining an economic order quantity model with decreasing and increasing linear trend in demand. Goswami & Chaudari (1991) considered a model for an economic order quantity with a linear direction in demand considering shortages. Goswami & Chaudari (1992) proposed an order-level inventory model for degenerating items without shortages. The deterioration rate is time-varying, and the demand rate is dependent upon time. Hariga (1993) discussed on solution procedure for solving an inventory replenishment policy with a linear trend in demand for both increasing and decreasing markets. Teng (1996) developed a model for a deterministic inventory replenishment system with a linear trend in demand. Zhao (2001) proposed a heuristics to solve a replenishment system when the shift in demand is linearly diminishing. Lo et al. (2002) developed a standard no-shortages inventory replenishment policy for linearly increasing and falling trend in demand, and he provided a particular solution method to derive an optimum solution for the system. Goyal & Giri (2003) developed a heuristics procedure for evaluating replenishment space of time for an inventory with a linear diminishing demand rate over a definable horizon.

Yang et al. (2004) proposed a solution procedure to solve a non-linear replenishment policy with decreasing demand. Zhou (2003) analysed a deterministic replenishment policy for a multiples warehouses with demand varying with time at a decreasing rate, shortages were allowed in the policy. Zhou et al. (2004) analysed a deterministic time-varying demand lot-sizing model with waiting-time-dependent partial backlogging. Teng et al. (2005) considered a deterministic economic production quantity model with demand and cost varying with time. Jiafu et al. (2008) examined a combined heuristics for determining order quantity under demand that changes with time.

Omar & Yeo (2009) developed an inventory model that fulfilled a continuous time-varying demand. Zhou et al. (2016) presented an optimum production inventory model for an integrated multiple-stage supply chain with time-varying demand. Escuin et al. (2017) proposed an inventory replenishment system under stochastic time-varying demand. Benkherouf et al. (2017) investigated a limited horizon inventory control problems for two complementary products. The demand is assumed to be time-dependent. Other notable and recent publication in this area are: Kumar (2019).

2.1.2 Inventory Models with Deteriorating Items

Deterioration is one of the terms that cannot be overlooked in inventory management. It is a known fact that almost all items deteriorate over a given period. In most items, the rate at which the items deteriorate is so insignificant that there is little need to consider its values to determine the economic lot sizes. On the other hand, there are area such as the production of chemical like turpentine, alcohol etc., electronics components such as computer parts, chips, resistors, capacitors, touch screen monitors etc., perishable foods such as bread, milk, meats, vegetables etc., that the deterioration which may occur during reasonable storage period is sufficiently considerable that such loss cannot be ignored.

Deterioration can, therefore, be described as decay, spoilage or damage that inhibits the commodities from being utilised for its intended objective. Deterioration can be categorised into two types which are: (i) Process deterioration which affects the inventory system, by either raising the operating costs or raises the probability of failure and (ii) Product deterioration which reduces the on-hand inventory level or reduces the customers demand.

Deteriorating items are the commodity that has a fixed and short period of the lifetime. Examples include, among others: yoghurt, meats, fish, fruits, vegetables, drugs, and so forth.

The study of deteriorating inventory models problem for items started with Whitin (1957) who proposed a fashion item that deteriorates at the close of the depository period. Ghare (1963) was the first to model a negative exponential decaying inventory. They observed that decay follow a pattern in which at any given period, it is proportional to the magnitude of the stock at the beginning of the period. Covert & Philip (1973) expanded the last model by taking into consideration deterioration as a Weibull distribution .

Mishra (1975) presented a production lot-size inventory model for degenerating items with a fixed and variable deterioration rate. Dave (1979) studied an inventory model for a discontinuous-in- time order level for degenerating items with instantaneous replenishment.

Also, Dave & Patel (1981) considered an inventory model for commodities that have a constant deterioration and the demand is time proportional with immediate replacement. Mak (1982) proposed a production lot-size inventory model for an exponentially decaying items considering shortages. He obtained an approximate solution for the optimal production lot-size, the production period, total inventory cycle time and the average total cost. Nahmias (1982) provided the first detailed reviews on the problem of finding a satisfactory ordering policies for fixed lifetime decomposable inventory and continuously exponential decay inventory.

Chowdhury & Chaudhuri (1983) carried out an order-level procedure for deteriorating products considering rates of replenishment as finite and allowed shortage. Elsayed & Teresi (1983) proposed an analysis for degenerating inventory model for commodities with shortages allowed. Two types of models were examined. Model 1 has a deterministic demand with finite rate of production while model 2 has a random probabilistic demand and deterioration rate is two-parameter Weibull distribution. Hollier & Mak (1983) proposed an inventory replenishment approach for decaying products. Constant deterioration rate was assumed, and the demand rate is decreasing and exponentially negative. Raafat et al. (1991)

analysed an inventory model for deteriorating products with constant decline. He obtained an alternative method for Mak1982 production method where he got a correct average total cost expression for the production lot-size system.

An inventory model for constant deteriorating items with inventory-level-dependent demand rate and constant demand was investigated by Giri et al. (1996). Bhunia & Maiti (1998) considered an inventory model with a finite rate of replenishment and dependent on the instantaneous inventory level for deteriorating items. Two models were expanded, one with shortages and other without shortages, and the price of deterioration and demand are found to be a direct increasing function of time. Deng et al. (2007) analysed and examined some earlier work on inventory models for degenerating items with ramp types demand rate. He corrected the errors in the previous calculations and proposed an efficient and thorough method of obtaining an optimal solution. Some excellent survey on recent trend in modelling continuously deteriorating items were carried out by (Raafat 1991; Goyal & Giri (2001); Li et al. 2010; Bakker et al. 2012; Janssen et al. 2016). Banerjee & Agrawal (2017) developed an inventory model for degenerating items in which the selling price of the product depend directly on the freshness of the product. Shortage was considered as a lost sale.

It has been discovered empirically that bankruptcy and life expectancy of many products can best be demonstrated in term of Weibull distribution. This empirical discovery has motivated many researchers to represent the time to deteriorate of an item by a Weibull distribution. An EOQ model in which the rate of deterioration follows a two-parameter Weibull distribution (WB) without shortages was formulated by Covert & Philip (1973). This model was further generalised by Philip (1974) considering three-parameter Weibull distribution. Sarkar et al. (2013) proposed a deteriorating inventory model for commodities with a finite rate of production and time-varying demand over a limited planning horizon. (Sanni & Chukwu 2013; Jain 2016) presented an inventory model with gradient-type demand

and three-parameter Weibull distribution deterioration considering shortages. Chakraborty et al. (2018) proposed two warehouse inventory model with gradient type demand rate, considering deterioration rate as three-parameter Weibull distribution with acceptable delay in payment accommodating shortages that are backlogged partially. Also, (Jalan 1996; Wu 2001; Rajeswari & Vanjikkodi 2012; Mukhopadhyay et al. 2005) derived an economic order quantity (eoq) model for commodities with two-parameters Weibull distribution deterioration and demand is assumed to be linearly increasing with shortages.

Other interesting works in this direction can be found in (Kaliraman 2019; Mishra 2016a; Ritha & Vinoline 2018; Santhi & Karthikeyan 2017; Shaikh et al. 2019; Tuan et al. 2017; Singh et al. 2018; Pramanik & Maiti 2019).

2.1.3 Inventory Models with Power Demand Pattern

Demand pattern is defined as distinct methods by which items are removed out of inventory during the scheduling session to satisfy the demand of the customers. The demand pattern is said to be uniform if the rate of demand is unchanged during all the inventory cycles. One of the advantages of demand pattern is that it enables suiting the demand for more practical situations. Thus, the pattern permits representing the behaviour of demand when it is uniformly distributed throughout the period, and also to reflect sales in different phases of the product life cycle in the market. For example, the demand for inventory increases overtime during the growing period and a decrease in the decline phase.

Many researchers have developed an inventory model that the demand follows a power pattern. Datta & Pal (1988) seem to be the first to examine an order level inventory model for items with power demand pattern and the deterioration rate is variable. The demand is considered as both probabilistic and deterministic. Gupta & Jauhari (1995) established an EOQ model for deteriorating items with power demand pattern considering the permissible delay in payment. Abdul-Jalbar et al. (2009) presented a two-level echelon inventory model for items with power demand and shortages. They assumed that the manner at which items are taken out from the inventory to the retailers follows a power pattern.

Also, Singh et al. (2009) proposed an inventory model for deteriorating items with power demand pattern considering holding cost as an incremental function of time under-inflation and allowed shortages. Sarbjit & Shivraj (2011) in their paper derived an inventory model for deteriorating products with both probabilistic and deterministic power demand pattern. The rate of deterioration is considered as a variable, incorporating inflation and permissible delay in payment. (Krishnaraj & Ramasamy (2012); Mishra et al. (2012)) proposed an inventory model for perishable items with power demand pattern under two-parameter Weibull distribution deterioration with or without shortages.

An inventory model for non-instantaneous Weibull distribution deterioration items under power demand pattern was investigated by Palanivel & Uthayakumar (2014). They considered shortages in their model and assumed holding cost is a linearly increasing function of time during storage. Rajeswari et al. (2015) examined a fuzzy inventory system for items that deteriorate with power demand pattern and shortages are partially backlogged. Pradhan et al. (2016) proposed an inventory model for deteriorating items under two-parameter Weibull distribution and power demand pattern. San-Jose et al. (2017b) examined an optimum inventory system for items that deteriorate with power demand pattern. They proposed a three-parameter Weibull distribution deterioration rate with shortages partially backlogged.

Also, San-Jose et al. (2017a) propounded an optimal inventory policy for degenerating items with power demand pattern with shortages partially backlogged. Sharmila & Uthayakumar (2018) investigated a two-warehouse inventory model for decaying items with power demand pattern. The holding cost is time-varying and trade credit is offered.

2.1.4 Inventory Models for Non- Instantaneous(delayed) Deteriorating Items.

Many researchers believe that the deterioration begins in an instant as the retailer receives the commodity. However, this is not always true since some products will have a period to retain the freshness of their original quality. During this period, there is no deterioration taken place. Wu et al. (2006) described this situation as "non-instantaneous deterioration", and he was the earliest to introduced and analysed an optimum replenishment model for non-instantaneous degenerating products under dependent stock demand with partial backlogging. In reality, this type of situation does occur in some commodities such as fruits, first-hand vegetable and so forth which possess a short life-span to maintain a fresh quality, during this interval, there is almost no spoilage. Shortly, after this interval, some of the products begin to degenerate.

The impression that the deterioration commences from the instant of arrival in stock, for such type of items can mislead retailers to make inaccurate replenishment policy as a result of overestimate of the total relevant cost. It is, therefore, significant to examine the inventory problem for the non-instantaneous degenerating item in the field of inventory management.

Ouyang et al. (2006) proposed an appropriate inventory model for non-instantaneous deteriorating items considering permissible delay in payment. The work generalised many previous models that are related to their work. Chung (2009) provided a full proof on the solution technique for non-instantaneous deteriorating products with permissible delay in payment. He corrected some errors in the previous works. Geetha & Uthayakumar (2010) propounded an inventory model for a non-instantaneous degenerating products considering delay in payment with shortages partially backlogged.

An inventory model for non-instantaneous decaying items weighing the effect of preservation technology investment on inventory parameters was presented by Dye (2013). Shah et al. (2013) studied an optimising and marketing inventory model for non-instantaneous degenerating items in which the advertisement and selling price depends on the demand rate. Maihami & Abadi (2012) presented a joint control and pricing inventory model for non-instantaneous deteriorating items considering the permissible delay in payment and shortages are partially backlogged.

Also, Maihami & Kamaladi (2012) examined an inventory control model and joint pricing system for a non-instantaneous degenerating product with price and time-dependent demand. They incorporated shortages in the model that is partially backlogged. Soni (2013) developed an optimum replenishment model for non-instantaneous deteriorating items with permissible delay in payment for items that have the price and stock dependent demand. Wu et al. (2014) commented on the work of Soni. They modified and corrected the deficiency found in their work. Mahmoudinezhad et al. (2014) proposed an inventory model for non-instantaneous degenerating products with permissible delay in payment under imperfect quality and inflation.

An optimising and replenishment system for non-instantaneous deteriorating items under stochastic demand and promotional efforts was developed by Maihami & Karimi (2014). Chang et al. (2015) investigated an inventory model for optimum pricing and ordering policies for non-instantaneous deteriorating items with order size-dependent and permissible delay in payment. Zhang et al. (2015) formulated an optimal dynamic pricing and replenishment cycle model for non-instantaneous deteriorating items under inventory level and price-dependent demand. Anchal et al. (2016) discussed on inventory model for non-instantaneous deteriorating items considering trade credit financing facility with shortages partially backlogged. They thought the deterioration rate as the variable with linear demand.

Analysis of sensitivity of an inventory model for a non-instantaneous and time-varying deterioration without shortages was carried out by Malik et al. (2016). Other recent works in this direction were carried out by following authors: (Sharma et al. 2016; Tawari et al. 2016;

Udayakumar & Geetha 2018; Rangarajan & Karthikeyan 2017; Li et al. 2019; Jaggi et al. 2018).

2.1.5 Inventory Models With Shortages

In inventory management, shortages do occur as a result of the low level of stock and the demand cannot be fully satisfied. Shortages can also occur as a result of the stock not been persistently documented (periodic inspection), or the stock is ordered behind schedule or when the requested quantity arrived late.

Backlogging occurs as a result of shortages. Researchers sometimes do assume partial backlogging while others considered full backlogging. Completely backlogged happens when the customers are ready to wait until the arrival of the future order; otherwise, the customers leave the system. Partial backlogging occurs when in a specific situation in the time of stock-out period, the customers are not willing or cannot stay; hence, their demand is satisfied from other sources.

The duration of discontinuing time for replenishment is the primary factor in calculating backlogging, and it is discovered that the lingering the waiting time, the lower the backlogging rate. Customers who experience shortages may not repurchase the goods from the respective suppliers and turn to another source to buy their products. As a consequence of this, a colossal proportion of sale is lost, leading to a dwindling profit. Partial backlogging is, therefore, a necessary factor to consider in inventory management.

Numerous researchers have occupied their concentration on inventory models with shortages under partial backlogging. Datta & Pal (1991) formulated an inventory model under-inflation and time-value of money considering the demand rate as linear time-dependent with shortages allowed. Abad (1996) presented a generalised model of dynamic pricing and lot-sizing model for perishable products with shortages, and shortages are backlogged partially. Chakrabati & Chaudhuri (1997) presented a deteriorating inventory replenishment model for products with a linear slope in demand under a finite time horizon with shortages allowed in every cycle. Chung & Tsai (1997) developed an algorithm to solve an EOQ model for degenerating items with the demand that has a linear trend and allowable shortages. They corrected the shortcoming in the previous model.

An integrated inventory model for perishable items considering the effect of pricing, advertisement and backorder on the profit of the system was carried out by Luo (1998). San-Jose et al. (2009) proposed a generalised EOQ inventory model with shortage allowed and partially backlogged. An optimal pricing and lot-sizing model for perishable items under limited production incorporating partial back-ordering and lost sale was formulated by Abad (2003). Yang (2004) derived an inventory model for degenerating items for two-warehouse under-inflation and allowed shortages. They obtained an optimal solution that is unique and less expensive to operate. Chen et al. (2007) studied an optimum replenishment model for demand that is time-varying, continuous and deterministic with shortages in product life cycles. Taleizadeh et al. (2013a) addressed a deterministic inventory control model for perishable product with special sale and shortages.

Also, Taleizadeh et al. (2013b) considered an EOQ model with partial delayed payment and shortages that are partially backlogged. Jain (2016) proposed an EOQ inventory model for products with demand that is a ramp type, considering degeneration as three-parameter Weibull distribution with shortages. Later, Rasel (2017) developed a deterministic inventory model with two-parameter Welbull distribution deterioration, and the demand rate is a gradient type function of time. Shortage is permitted and wholly backlogged. Recently, Pervin et al. (2018) suggested a deterministic inventory control model for deteriorating products with time-dependent demand and time-varying holding cost, and the deterioration rate is Weibull distribution. Shortages were allowed in the system.

2.1.6 Inventory Model with Time-Dependent Holding Cost

Holding cost has been considered in the economic ordered quantity model as a constant function of time. In reality, holding cost can also vary with time since it is usually connected with the storage of an item until usage or store inventory.

In this direction, Naddor (1966) seems to be the first to establish a derivation of total inventory cost for a demand rate that is constant for a lot size system by considering holding cost in the form $q^m t^n$ where q is the amount of stock held for a time t and m, n are non-negative integers. Muhlemann & Valtis-Spanopoulos (1980) modified the traditional EOQ model formula by considering the rate of demand as a constant and the holding cost as a variable function and expressed them as a percentage of the average value of the capital invested in the stock. Later, Weiss (1982) presented a generalised economic order quantity model by taking the unit cost, selling price, demand rate and set up cost as a constant parameter and holding cost is considered as a non-linear function of the duration of time the item remained in stock.

Also, Baker & Urban (1988) propounded an inventory model for an item with the demand rate considered as inventory level dependent. The model maximises the average profit per unit time by taking the optimum order level and order point as the decision variables. Goh (1994) proposed a deterministic, continuous inventory model for a distinct item. The rate of demand is inventory level-dependent, and the holding cost varies with time. Two possible cases were examined. The first for a non-linear function of the duration of time the item is held in inventory and second for the non-linear function of the amount of the on-hand inventory.

Furthermore, Giri & Chaudari (1998) came up with a deterministic perishable inventory model with demand rate depending on stock, and holding costs are treated in two way viz: a non-linear function of the time distance for which the item held in stock and functional form of the amount of the on-hand inventory. Shao et al. (2000) established an inventory model for the optimum target for a process under multiple markets and holding cost is considered as a variable function of time.

Other authors that considered holding cost as variable are (Alfares, 2007; Pando et al. (2012, 2013); Sazvar et al., 2013; Ferguson et al., 2007; Tripathi, 2013; Mishra et al. 2013; Amutha & Chandrasekaran, 2013; Alfares & Ghaithan, 2016; Shukla et al. 2017; Uthayakumar & Karuppasamy, 2017; Ghasemi & Afshar, 2013).

Besides, Alfares & Ghaithan (2019) carried out a thorough research paper review that classified the formulation of EPQ and EOQ models under the presumption of variable holding cost. Different types of holding costs are considered, which includes time-dependent variable holding cost, stock dependent variable holding cost and multiple dependent variables holding cost. Moreover, San-Jose et al. (2019) propounded an inventory model for an item in which the demand rate is dependent on selling price and time power function. Holding cost is presumed to be the power function of time. The model maximises the total inventory profit per unit time by considering the inventory cycle and selling price as the decision variables.

2.1.7 Production Inventory Models for Deteriorating Items

Economic production order quantity (EPQ) models are applicable for the instances where the company acquires its stock over some period or where the products are manufactured internally rather than obtained. Several researchers have deliberated on deteriorating production inventory models for items.

Mishra (1975) seems to be the first researcher who presented a production lot-size inventory model for degenerating products in which the deteriorating rate is both time-varying and constant. Shortages are not allowed in the model. Other notable works in this direction are (Duan et al, 2018; Goyal & Gunasekaran, 1995; Widyadana & Wee, 2011; Taleizadh et al. 2015; Tai 2013; Bukhari & El-Gohary 2012).

In the above models, shortages are not put into consideration, Balkhi & Benkherouf (1996) established a production lot-size inventory models for degenerating products under random time-varying demand and production rate. Shortages are allowed. Samanta (2016) formulated an economic production quantity inventory model for degenerating items with shortage tolerated. A deteriorating production inventory model for product with finite designing horizon and linear time-varying demand were presented by (Khanra (2016); Sana et al. (2004)). Shortage is allowed and wholly backlogged. Mishra (2012) proposed a production inventory model for degenerating products in which demand, production, deterioration and other logistic depend on time with shortages allowed, and wholly backlogged.

Krishnamoorhi & Sivashankari (2016) considered a three-level production inventory model for deteriorating products with the rate of production shown as a variation. Shortage is permitted. A sustainable economic production quantity inventory model that incorporate various form of shortages was studied by Taleizadeh et al. (2018).

Bard & Moore (1990) studied a production planning inventory model considering demand as variable. A production inventory planning model for accepting an order when the demand is not satisfied was proposed by Aouam et al. (2018). Chen et al. (2014) put together the deteriorating economic production quantity inventory model for commodities with up-stream complete trade credit and permissible delay in payment.

A production inventory model with price and quality decisions in production was presented by Jalali et al. (2019). Mokhtari et al. (2017) derived a computational approach to the economic production quantity inventory model for the perishable product in which the demand rate is probabilistic and stock-dependent. Shortages are accepted and wholly backlogged. Pal et al. (2015) proposed a deteriorating economic production quantity inventory model for product with gradient type demand and putting over-valuation into consideration under fuzziness. The rate of deterioration is two-parameter Weibull distribution, and time horizon is finite, and shortages are allowed.

Viji & Karthikeyan (2018) proposed an economic production quantity inventory model for three-stage of production with the deterioration rate following a two-parameter Weibull distribution. Shortages are accepted. Zhao et al. (2016) presented an optimal production quantity inventory model for a multi-stage supply chain. The demand rate is the time variable, and the model was considered over a limited planning horizon. Shah &Vaghela (2018) reviewed economic order quantity (EPQ) deteriorating inventory model for commodities in which demand depend on price, subject to two-level marketing credit financing.

2.1.8 Literature Gap

To the best of our understanding, there are no/few mathematical inventory models developed under the following headings which this thesis intended to address.

- Deteriorating inventory model for items with time-varying demand rate and time-dependent holding cost with no shortages.
- Inventory model for items with linearly time-dependent deteriorating rate and time-dependent holding cost with power pattern demand considering shortages.
- Delayed deteriorating inventory model with power pattern demand rate and lost sale with shortages partially backlogged.
- Inventory policy for items with linear time-dependent deteriorating rate, variable holding cost and power demand form rate with partially backlogged shortages.

CHAPTER 3: AN INVENTORY MODEL FOR DETERIORATINGITEMS WITH TIME-VARYING DEMAND RATE AND TIME-DEPENDENT HOLDING COST WITHOUT SHORTAGES.

3.1 Introduction

In the management of inventory, many mathematical inventory models have been developed over many decades to contain different scenarios. Some models considered where the holding or carrying cost, set-up cost, demand rate as a fixed function concerning time. All this has led to a wide variation of model in the economic ordering quantity and economic production quantity when compared with the real situation in the world under inventory problems.

It is, therefore, necessary that the model and assumptions imposed on the models should be carefully considered in a way to get closer to the real situation as much as possible.

In this chapter, we developed a deteriorating inventory model for items with time-dependent holding cost as against the fixed holding cost which is at a time very far from present reality. Other works that had been done in this area include that of Singh (2017); Mishra et al. (2013). However, this model considered time-varying demand which is of power pattern which makes it more suitable for products which are just gaining recognition in the market.

3.2 Assumptions and notation

Assumptions

The development of the model is based on the following assumptions:

1. The demand rate of the items is represented by power pattern and is a continuous function of time

2. Lead time is negligible/ trivial
3. The rate of deterioration of items is constant
4. Shortages are not permitted
5. The rate of replenishment is infinite
6. Planning horizon is finite

Notation

The following notation is essentials in this chapter.

- T is the length of the inventory cycle (time)
- Q is the order quantity/units
- $I(t)$ is the inventory level at time t
- r is the average demand per inventory cycle.
- n is the demand pattern index (n is non-negative)
- K is the ordering cost / $ /order
- h is the holding cost per unit $ / time
- C is the cost per deteriorated unit/ $ /order
- SC is the set-up cost/cycle
- DC is the deteriorating cost/time units/cycle
- HC is the holding cost is a linear function of time (t) = a + b t,

  (a and b are non-negative)
- TC is the total cost of the inventory model/time units

3.3 Mathematical model formulation

Employing the assumptions and notation above, Figure.3.1 gives the graphical representation of the inventory model. Q is the ordering quantity at the beginning of the planning horizon. The inventory model is derived mathematically as follows:

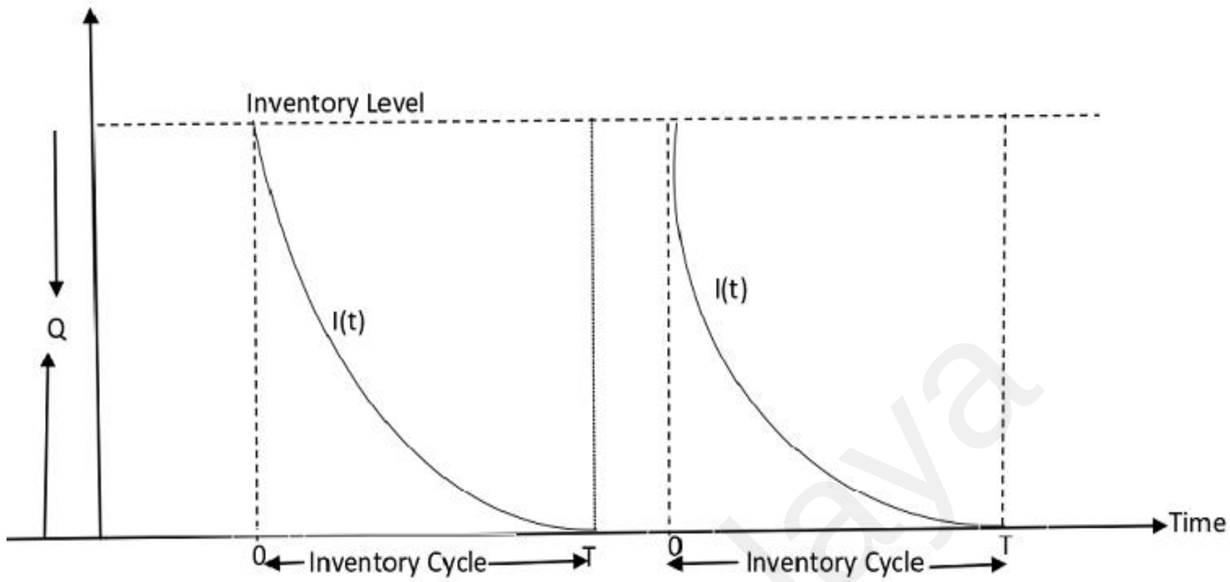


Figure 3.1: Graphical illustration of Inventory Model.

The instantaneous inventory level $I(t)$ at any time t during the cycle time T is given by:

$$\frac{dI(t)}{dt} + \alpha I(t) = -D \ ; \qquad 0 \le t \le T \tag{3.1}$$

$$where \quad D = \frac{rt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}}$$

Using integrating factor = $e^{\alpha t}dt$

$$I'(t)e^{\alpha t} = -\frac{rt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}}e^{\alpha t}$$

$$I'(t)e^{\alpha t} = -\frac{rt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}}\left[\int e^{\alpha t}t^{\frac{1}{n}-1}dt\right]$$

Using power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdot$$

$$e^{\alpha t} = 1 + \alpha t + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!}$$

Since α is small, we can truncate the series at the third terms

$$I(t)e^{\alpha t} = -\frac{r}{nT^{\frac{1}{n}-1}}\left[\int t^{\frac{1}{n}-1}dt + \alpha\int t^{\frac{1}{n}}dt + \frac{\alpha^2}{2}\int t^{\frac{1}{n}+1}dt\right]$$

$$I(t)e^{\alpha t} = -\frac{r}{nT^{\frac{1}{n}-1}}\left[nt^{\frac{1}{n}} + \frac{\alpha n t^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 n t^{\frac{1}{n}+2}}{2(2n+1)}\right]$$

$$I(t)e^{\alpha t} = -\frac{r}{nT^{\frac{1}{n}-1}}\left[nt^{\frac{1}{n}} + \frac{\alpha n t^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 n t^{\frac{1}{n}+2}}{4n+2}\right] + C$$

Applying the boundary condition: $I(T) = 0$, we have:

$$0 = -\frac{r}{nT^{\frac{1}{n}-1}}\left[nt^{\frac{1}{n}} + \frac{\alpha n t^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 n t^{\frac{1}{n}+2}}{4n+2}\right] + C$$

$$C = -\frac{r}{nT^{\frac{1}{n}-1}}\left[nt^{\frac{1}{n}} + \frac{\alpha n t^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 n t^{\frac{1}{n}+2}}{4n+2}\right]$$

From this, we have:

$$I(t)e^{\alpha t} = \frac{r}{T^{\frac{1}{n}-1}}\left[(T^{\frac{1}{n}} - t^{\frac{1}{n}}) + \frac{\alpha}{n+1}(T^{\frac{1}{n}+1} - t^{\frac{1}{n}+1}) + \frac{\alpha^2}{4n+2}(T^{\frac{1}{n}+2} - t^{\frac{1}{n}+2})\right]$$

$$I(t) = \frac{re^{-\alpha t}}{T^{\frac{1}{n}-1}}\left[\left(T^{\frac{1}{n}} - t^{\frac{1}{n}}\right) + \frac{\alpha}{n+1}\left(T^{\frac{1}{n}+1} - t^{\frac{1}{n}+1}\right) + \frac{\alpha^2}{4n+2}\left(T^{\frac{1}{n}+2} - t^{\frac{1}{n}+2}\right)\right]$$

$$0 \le t \le T \tag{3.2}$$

The optimum order quantity level is given by $I(0) = Q$

$$Q = \frac{r}{T^{\frac{1}{n}-1}}\left[T^{\frac{1}{n}} + \frac{\alpha}{n+1}T^{\frac{1}{n}+1} + \frac{\alpha^2}{4n+2}T^{\frac{1}{2}+2}\right] \tag{3.3}$$

The total cost (TC) per unit time is made up of the following cost components:

1. The holding cost HC per cycle $[0, T]$ is given by:

$$HC = \frac{1}{T}\int_0^T (a + bt)I(t)dt$$

$$HC = \frac{1}{T}\int_0^T (a + bt)\left[\frac{re^{-\alpha t}}{T^{\frac{1}{n}-1}}\left[(T^{\frac{1}{n}} - t^{\frac{1}{n}}) + \frac{\alpha}{n+1}(T^{\frac{1}{n}+1} - t^{\frac{1}{n}+1})\right.\right.$$

$$\left.\left. + \frac{\alpha^2}{4n+2}(T^{\frac{1}{n}+2} - t^{\frac{1}{n}+2})\right]\right]dt$$

$$HC = \frac{1}{T}\int_0^T (a+bt)\left[\frac{r(1-\alpha t)}{T^{\frac{1}{n}-1}}\left[(T^{\frac{1}{n}} - t^{\frac{1}{n}}) + \frac{\alpha}{n+1}(T^{\frac{1}{n}+1} - t^{\frac{1}{n}+1})\right.\right.$$

$$\left.\left.+ \frac{\alpha^2}{4n+2}(T^{\frac{1}{n}+2} - t^{\frac{1}{n}+2})\right]\right] dt$$

$$HC = \frac{r}{T^{\frac{1}{n}}}\int_0^T (a+bt)\left[(1-\alpha t)(T^{\frac{1}{n}} - t^{\frac{1}{n}})\right.$$

$$\left.+ \frac{\alpha(1-\alpha t)}{n+1}(T^{\frac{1}{n}+1} - t^{\frac{1}{n}+1}) + \frac{\alpha^2(1-\alpha t)}{4n+2}(T^{\frac{1}{n}+2} - t^{\frac{1}{n}+2})\right]$$

$$HC = \frac{r}{T^{\frac{1}{n}}}\int_0^T (a+bt)\left[T^{\frac{1}{n}} - t^{\frac{1}{n}} - \alpha t T^{\frac{1}{n}} + \alpha t^{\frac{1}{n}+1} + \frac{\alpha T^{\frac{1}{n}+1}}{n+1} - \frac{\alpha t^{\frac{1}{n}+1}}{n+1}\right.$$

$$\left.- \frac{\alpha^2 t T^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 t^{\frac{1}{n}+2}}{n+1} + \frac{\alpha^2 T^{\frac{1}{n}+2}}{4n+2} - \frac{\alpha^2 t^{\frac{1}{n}+2}}{4n+2} - \frac{\alpha^3 t T^{\frac{1}{n}+1}}{4n+2} + \frac{\alpha^3 t^{\frac{1}{n}+3}}{4n+2}\right]$$

$$HC = \frac{r}{T^{\frac{1}{n}}}\left[\int_0^T aT^{\frac{1}{n}} dt - \int_0^T a t^{\frac{1}{n}} dt - \int_0^T \alpha a t T^{\frac{1}{n}} dt + \int_0^T \frac{\alpha a t^{\frac{1}{n}+1}}{n+1} dt\right.$$

$$+ \int_0^T \frac{\alpha a T^{\frac{1}{n}+1}}{n+1}(1-\alpha t) dt + \int_0^T \frac{\alpha^2 a t^{\frac{1}{n}+2}}{n+1} dt + \int_0^T \frac{\alpha^2 a T^{\frac{1}{n}+2}}{4n+2} dt$$

$$- \int_0^T \frac{\alpha^2 a t^{\frac{1}{n}+2}}{4n+2} dt - \int_0^T \frac{\alpha^3 a T^{\frac{1}{n}+1}}{4n+2} dt + \int_0^T \frac{\alpha^3 a t^{\frac{1}{n}+3}}{4n+2} dt + \int_0^T bT^{\frac{1}{n}} dt$$

$$- \int_0^T b t^{\frac{1}{n}+1} dt - \int_0^T b\alpha T^{\frac{1}{n}} t^2 dt + \int_0^T \frac{\alpha b t^{\frac{1}{n}+2}}{n+1} dt + \int_0^T \frac{\alpha^2 b t^{\frac{1}{n}+3}}{n+1} dt$$

$$+ \int_0^T \frac{\alpha b t T^{\frac{1}{n}+1}}{n+1}(1-\alpha t) dt + \int_0^T \frac{\alpha^2 b t T^{\frac{1}{n}+2}}{4n+2} dt - \int_0^T \frac{\alpha^2 b t^{\frac{1}{n}+3}}{4n+2} dt$$

$$\left.- \int_0^T \frac{\alpha^3 b t^2 T^{\frac{1}{n}+1}}{4n+2} dt + \int_0^T \frac{\alpha^3 b t^{\frac{1}{n}+4}}{4n+2} dt\right]$$

Upon integration and simplification, we arrive at the following equation:

$$HC = \frac{rh}{T^{\frac{1}{n}}}\left[aT^{\frac{1}{n}+1} - \frac{anT^{\frac{1}{n}+1}}{n+1} - \frac{\alpha a n T^{\frac{1}{n}+2}}{n+1} + \frac{\alpha a n T^{\frac{1}{n}+2}}{(2n+1)(n+1)} + \frac{\alpha a T^{\frac{1}{n}+2}}{n+1}\right.$$

$$- \frac{\alpha^2 a n T^{\frac{1}{n}+3}}{(2(n+1)} + \frac{\alpha^2 a n T^{\frac{1}{n}+3}}{(n+1)(3n+1)} + \frac{\alpha^2 a T^{\frac{1}{n}+3}}{4n+2} - \frac{\alpha^2 a n T^{\frac{1}{n}+3}}{(3n+1)(4n+3)}$$

$$- \frac{\alpha^3 a T^{\frac{1}{n}+3}}{2(4n+2)} + \frac{\alpha^3 a n T^{\frac{1}{n}+4}}{(4n+1)(4n+2)} + \frac{bT^{\frac{1}{n}+2}}{2} - \frac{bnT^{\frac{1}{n}+2}}{2n+1} - \frac{\alpha b T^{\frac{1}{n}+3}}{3}$$

$$+\frac{\alpha bnT^{\frac{1}{n}+3}}{(3n+1)(n+1)}+\frac{\alpha bT^{\frac{1}{n}+3}}{2(n+1)}-\frac{\alpha^2 bT^{\frac{1}{n}+4}}{3(n+1)}+\frac{\alpha^2 bnT^{\frac{1}{n}+4}}{(4n+1)(n+1)}$$

$$+\frac{\alpha^2 bT^{\frac{1}{n}+4}}{2(4n+2)}-\frac{\alpha^2 bnT^{\frac{1}{n}+4}}{(4n+2)(4n+1)}-\frac{\alpha^3 bT^{\frac{1}{n}+4}}{3(4n+2)}+\frac{\alpha^3 bnT^{\frac{1}{n}+5}}{(4n+2)(5n+1)}\Bigg] \tag{3.4}$$

The number of deteriorated unit during the cycle period $(0, T)$ is given by:

$$Ndu = Q - \int_0^T D(t)dt, where \qquad D(t) = \frac{rt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}}$$

$$Ndu = Q - rT$$

$$= \frac{r}{T^{\frac{1}{n}-1}}\left(T^{\frac{1}{n}} + \frac{\alpha}{n+1}T^{\frac{1}{n}+1} + \frac{\alpha^2}{4n+2}T^{\frac{1}{n}+2}\right) - rT$$

$$Ndu = \frac{r\alpha T^2}{n+1} + \frac{\alpha^2 rT^3}{4n+2}$$

2. Therefore, the deterioration cost is given by:

$$DC = \frac{C_1}{T}\left(\frac{\alpha rT^2}{n+1} + \frac{\alpha^2 rT^3}{4n+2}\right)$$

$$DC = \frac{\alpha C_1 rT}{n+1} + \frac{\alpha^2 rC_1 T^2}{4n+2} \tag{3.5}$$

3. Ordering cost/set-up cost (SC) per cycle $[0, T]$ is given by:

$$SC = \frac{K}{T} \tag{3.6}$$

The total cost per unit time is given by:

$$TC = HC + DC + SC$$

$$TC = \Bigg[arhT - \frac{arhnT}{n+1} - \frac{\alpha rahnT^2}{n+1} + \frac{\alpha rahnT^2}{(2n+1)(n+1)} + \frac{\alpha rhaT^2}{n+1} - \frac{\alpha^2 arhnT^3}{2(n+1)}$$

$$+\frac{\alpha^2 arhnT^3}{(3n+1)(n+1)} + \frac{\alpha^2 rahT^3}{4n+2} - \frac{\alpha^2 rahnT^3}{(4n+3)(3n+1)} - \frac{\alpha^3 rhaT^3}{2(4n+2} + \frac{\alpha^3 rahnT^4}{(4n+1)(4n+2)}$$

$$+\frac{brhT^2}{2} - \frac{rbhnT^2}{2n+1} - \frac{\alpha rbhT^3}{3} + \frac{\alpha rhbnT^3}{(3n+1)(n+1)} + \frac{\alpha rhbT^3}{2(n+1)} - \frac{\alpha^2 rbhT^4}{3(n+1)}$$

$$+\frac{\alpha^2 rhbT^4}{(4n+1)(n+1)}+\frac{\alpha^2 rbhT^4}{2(4n+2}-\frac{\alpha^2 rbhnT^4}{(4n+1)(4n+2}-\frac{\alpha^3 rbhT^4}{3(4n+2}+\frac{\alpha^3 rbnhT^5}{(4n+2)(5n+1)}$$

$$+\frac{\alpha C_1 rT}{n+1}+\frac{\alpha^2 rC_1T^2}{4n+2}+\frac{K}{T}\Big] \tag{3.7}$$

The necessary and sufficient conditions for the total cost TC to be minimised is that:

$$\frac{\partial(TC)}{\partial T}=0 \quad and \quad \frac{\partial^2(TC)}{\partial T^2}\geq 0 \quad for\ all \quad T>0 \tag{3.8}$$

3.4 Method of Solution

We present in this section, a way to calculate the inventory system that minimises the total cost with unit time as given in Equation (3.7). Here, we find the first partial derivative of TC concerning the decision variable T:

$$\frac{\partial(TC)}{\partial T}=arh-\frac{arhn}{n+1}-\frac{2hran\alpha}{n+1}+\frac{2hran\alpha T}{(2n+1)(n+1)}+\frac{2hra\alpha T}{n+1}$$

$$-\frac{3hra\alpha^2T^2}{2(n+1)}+\frac{3ranh\alpha^2T^2}{(n+1)(3n+1)}+\frac{3rah\alpha^2T^2}{(4n+2)}-\frac{3ranh\alpha^2T^2}{(4n+3)(3n+1)}$$

$$-\frac{3rah\alpha^3T^2}{2(4n+2)}+\frac{4hnra\alpha^3T^3}{(4n+2)(4n+1)}+bhrT-\frac{2hnrb\alpha^2T}{(2n+1)}-rhb\alpha T^2+\frac{3rbnh\alpha T^2}{(n+1)(3n+1)}$$

$$+\frac{3rbh\alpha T^2}{2(n+1)}-\frac{4rbh\alpha^2T^3}{3(n+1)}+\frac{4bhr\alpha^2T^3}{(4n+1)(n+1)}+\frac{2rb\alpha^2T^3}{(4n+2)}-\frac{4bhnr\alpha^2T^3}{(4n+1)(4n+2)}$$

$$-\frac{4bhr\alpha^3T^3}{3(4n+2)}+\frac{5bhnr\alpha^3T^4}{(4n+2)(5n+1)}+\frac{C_1\alpha r}{n+1}+\frac{2C_1rT\alpha^2}{4n+2}-\frac{K}{T^2}=0 \tag{3.9}$$

3.5 Numerical illustration

The proposed model is more evident by considering some numerical example in this section. In this example, we evaluated the solution of the inventory problem, calculate the schedule period and order quantity level.

Example: Considering the subsequent parametric values for the inventory model $K=50$, $a=0.5,\ b=0.01,\ C_1=1.5,\ r=100,\ n=0.5,\ \alpha=0.4,\ h=2.5$ in appropriate units. Taking into consideration these numerical values, from Equation (3.9) we obtain a non-linear equation. Using maple software 2018, to solve the equation, we obtain schedule period T for which the inventory is zero to be 0.562 years; replenishment order quantity Q is 116.240 units, inventory total cost TC is $\$168.19$.

From Equation (7.1), the second partial derivative of the cost function TC with respect to the variable T is positive, which satisfied the sufficient condition i.e.

$$\frac{\partial^2 TC(T)}{\partial T^2} = 2.102696 > 0$$

establishing that it is the minimum point.

Figure 3 shows that the function TC is convex concerning T (Schedule period).

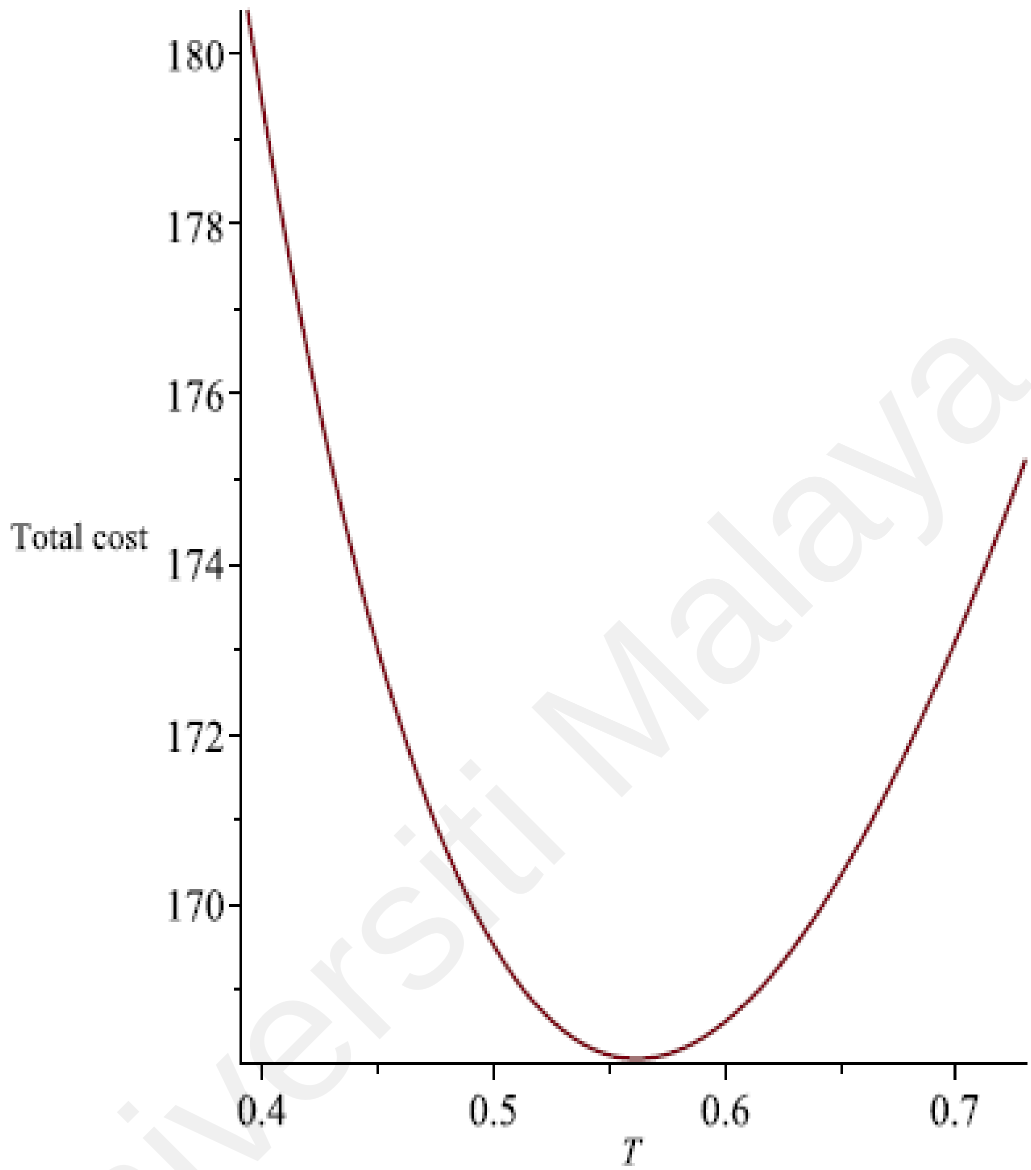


Figure 3.2: Graphical representation of convexity of total cost against *T*.

3.6 Sensitivity analysis

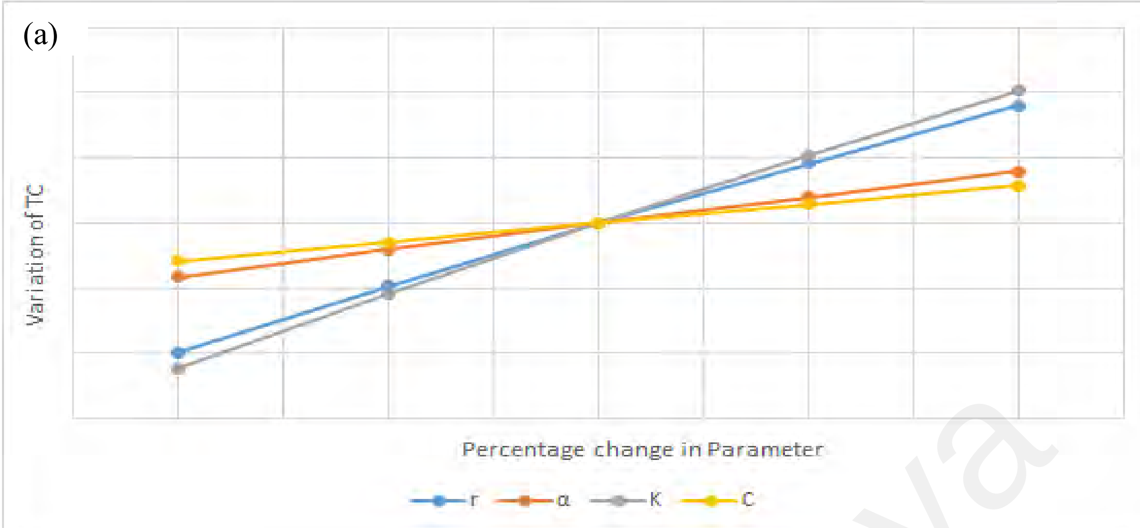
The effect of changes in the values of various parameters r, h, K, C, n, a, b and α is observed in this section on the optimum total cost and the optimum order quantity. The sensitivity analysis is carried out by changing each of the parameters by $+20\%$, $+10\%$, -10%, -20% taking one parameter at a time and keeping other parameters constant. The analysis is built on the example above, and the results are displayed in Table 3.1 and represented graphically by Figures (3.3 -3.5). The following observations are derived from the sensitivity analysis.

1. As the demand rate r is increasing, there is an increase in optimal order quantity Q, and optimal total cost TC, which results in a decrease in the optimal cycle time T. The economic implication or consequence of this is that as demand rate is increasing, there is a need for retailers to order more quantity, to lower the order frequency and inventory cost.
2. As the deteriorating parameter α is increasing, there is an increase in optimal total cost, optimal order quantity and decrease in the optimal cycle time. The implication of this is that an increase in deterioration rate will lead to a rise in the minimum total cost per unit time, and this will decrease the optimal cycle time.
3. As the ordering cost is increasing, there is an increase in the optimal cycle time T, optimal ordering quantity Q and optimal total cost TC. The implication of this is that the retailers should order more quantity when the ordering cost is high to avoid frequent ordering and save cost.
4. As the value of purchasing cost C is increasing, there is a decrease in optimal cycle time T and optimal ordering quantity Q, but there is an increase in the optimal total cost TC. The implication of this is that as the purchasing cost is increasing, the optimal total cost will also be increasing. The retailers can control this situation by shortening the optimal cycle time

and reduce the quantity ordered.

5. An increase in the holding cost h and parameter a and b will result in a decrease in the optimal cycle time and optimal ordering quantity. However, there is an increase in the optimal total cost. Here, retailers should shorten the cycle length of time and amount of order optimal to maintain the inventory cost as low as possible, which can act as an adjustment strategy.

6. An increase in the power index n leads to an increase in the optimal cycle time T and reduces the optimal total cost TC and the optimal ordering quantity Q.



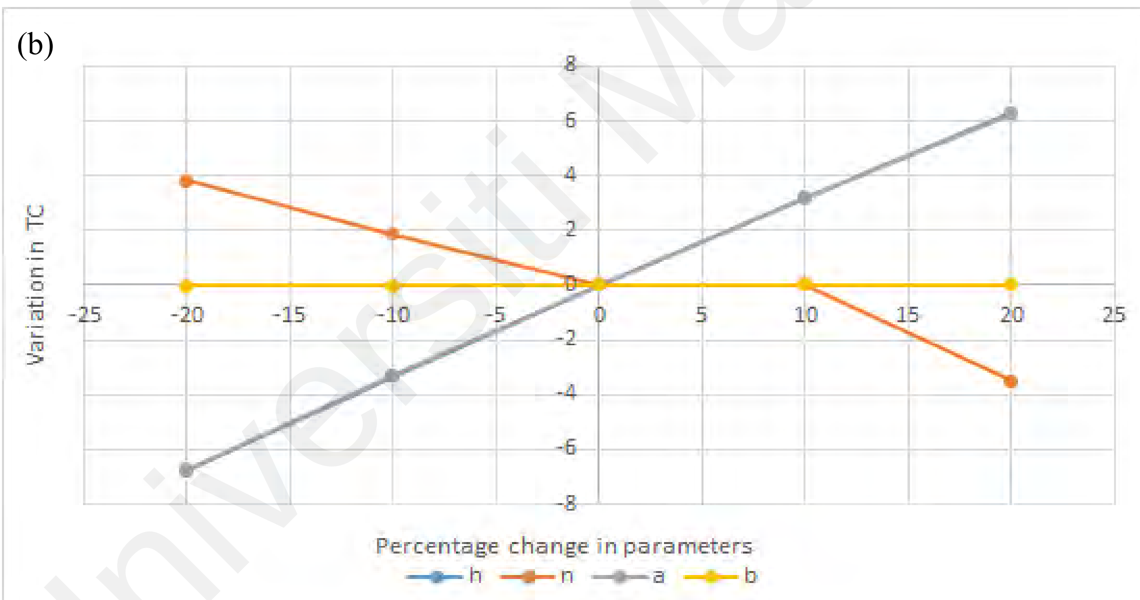
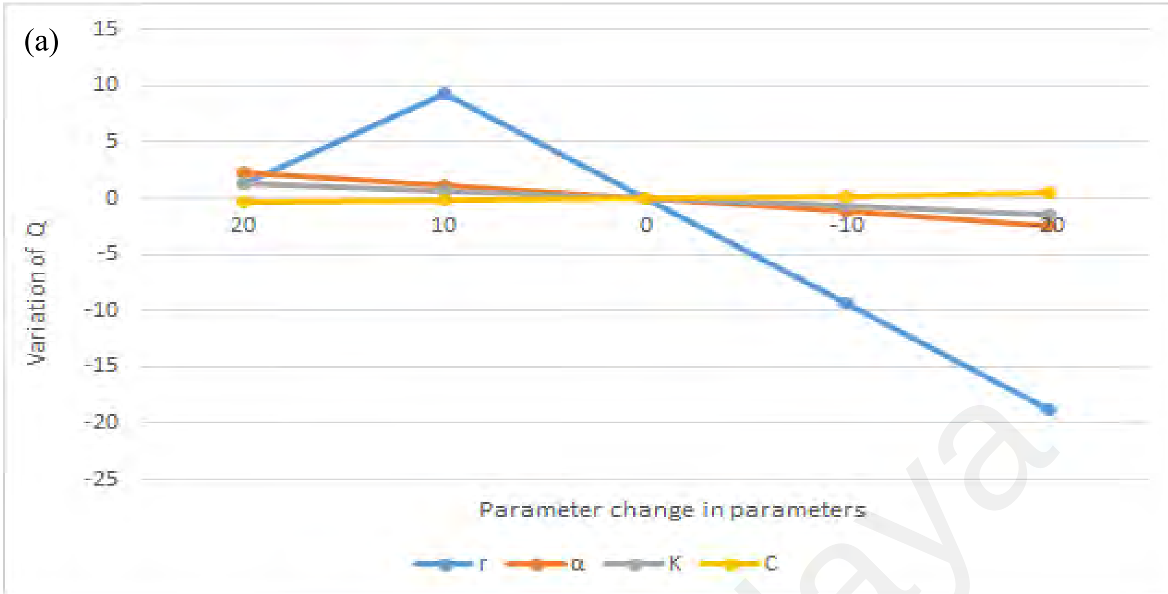


Figure 3.3: Graphical representation of sensitivity analysis of total cost.



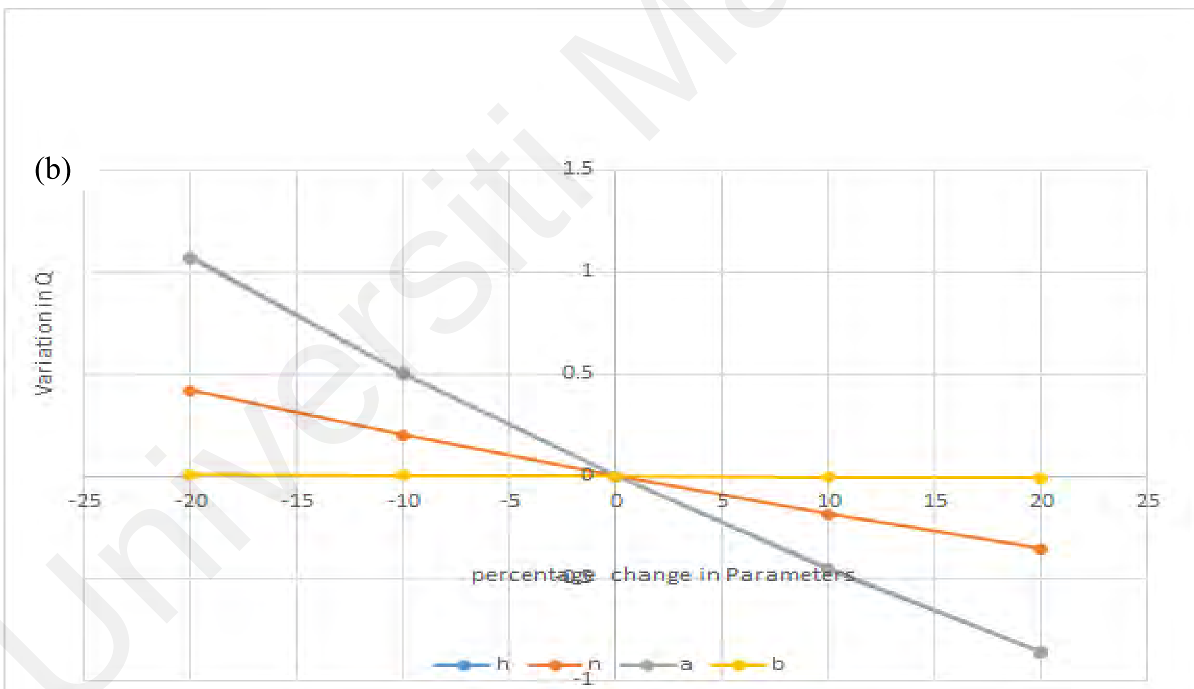
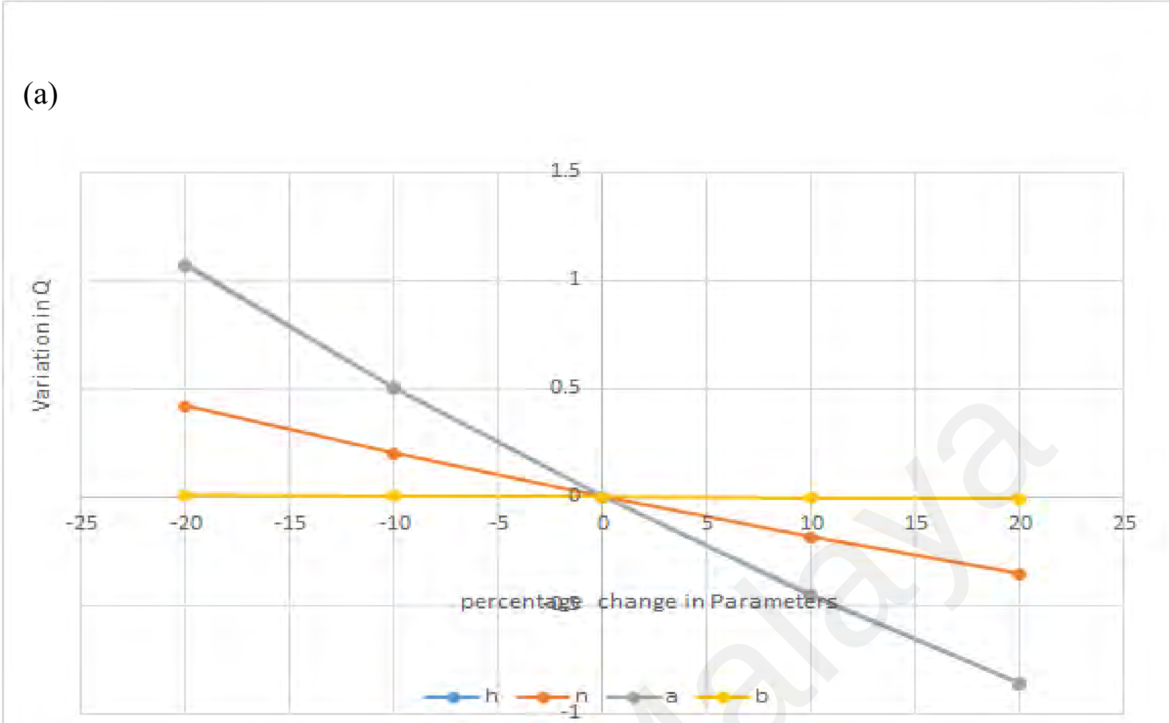


Figure 3.4: Graphical representation of sensitivity analysis of order quantity.



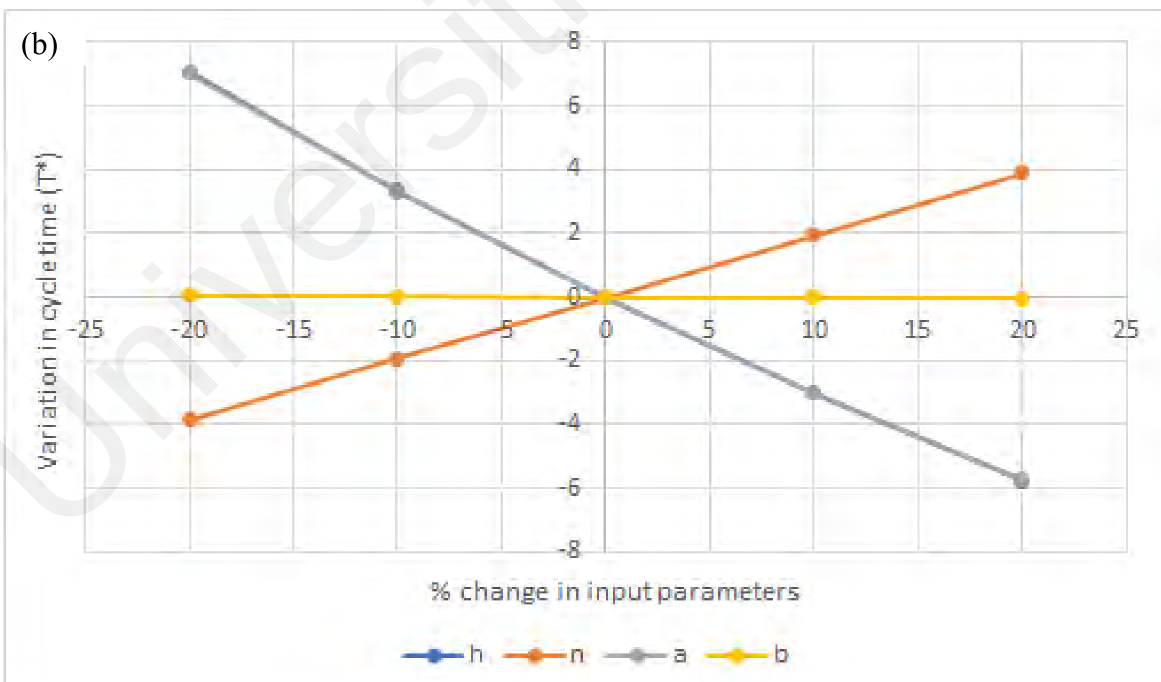


Figure 3.5: Graphical representation of sensitivity analysis of cycle time.

Table 3.1:The consequence of changes in various parameters of the inventory models.

			Change in:					
P*	V*	C*	T	TC	Q	W*	J*	Z*
r	120	20	0.517	183.29	117.838	8.981	1.374	-7.905
	110	10	0.538	175.91	127.057	4.594	9.305	-4.207
	100	0	0.562	168.19	116.240	0	0	0
	90	-10	0.589	160.06	105.383	-4.833	-9.341	4.849
	80	-20	0.621	151.46	94.477	-9.946	-18.722	10.525
α	0.48	20	0.538	174.811	118.878	3.938	2.269	-4.243
	0.44	10	0.549	171.533	117.575	1.989	1.149	-2.194
	0.40	0	0.562	168.187	116.240	0	0	0
	0.36	-10	0.575	164.769	114.869	-2.032	-1.180	2.357
	0.32	-20	0.589	161.274	113.458	-4.110	-2.393	4.898
K	60	20	0.610	185.260	117.742	10.151	1.292	8.526
	55	10	0.586	176.898	117.009	5.179	0.661	4.377
	50	0	0.562	168.187	116.240	0	0	0
	45	-10	0.536	159.074	115.431	-5.418	-0.697	-4.641
	40	-20	0.508	149.491	114.573	-11.116	-1.435	-9.594
C	1.80	20	0.547	172.991	115.794	2.856	-0.384	-2.555
	1.65	10	0.554	170.606	116.012	1.438	0.196	-1.302
	1.50	0	0.562	168.187	116.240	0	0	0
	1.35	-10	0.569	165.733	116.478	-1.459	0.204	1.355
	1.20	-20	0.577	163.243	116.725	-2.939	0.417	2.766
h	3.0	20	0.529	178.777	115.235	6.297	-0.865	-5.770
	2.75	10	0.545	173.572	115.713	3.202	-0.454	-3.022
	2.50	0	0.562	168.187	116.240	0	0	0
	2.25	-10	0.680	162.603	116.827	-3.320	0.505	3.346
	2.0	-20	0.601	156.795	117.486	-6.774	1.072	7.081
n	0.60	20	0.584	162.281	115.827	-3.511	-0.355	3.894
	0.55	10	0.573	168.224	116.025	0.022	-0.185	1.940
	0.50	0	0.562	168.187	116.240	0	0	0
	0.45	-10	0.551	171.316	116.474	1.860	0.201	-1.927
	0.40	-20	0.540	174.576	116.728	3.798	0.419	-3.845
a	0.60	20	0.529	178.738	115.241	6.273	-0.860	-5.732
	0.55	10	0.545	173.551	115.716	3.189	-0.451	-3.001
	0.50	0	0.562	168.187	116.240	0	0	0
	0.45	-10	0.580	162.627	116.823	-3.306	0.501	3.319
	0.40	-20	0.601	156.846	117.475	-6.743	1.062	7.018
b	0.012	20	0.561	168.232	116.232	0.027	-0.007	-0.048
	0.011	10	0.562	168.209	116.236	0.013	-0.004	-0.024
	0.010	0	0.562	168.187	116.240	0	0	0
	0.009	-10	0.562	168.165	116.244	-0.013	0.004	0.024
	0.008	-20	0.562	168.142	116.249	0.027	0.007	0.048

Note**:** P*= Parameters, V*=Values, C*= % Changes, W* = %Change in TC, J*= %Change in Q, Z*= % Change in T.

3.7 Concluding Observations

This model considered a deteriorating inventory model with a constant deterioration rate, and the demand rate follows a power pattern. Shortages are not allowed in the model and holding cost is assumed to be a linear function of time. The power demand pattern rate is chosen because of the new products in the market now in which the demand is dependent upon time. When some products are introduced into the market, the demand may be constant for some time after which the products will gain recognition, and then the need for such products increases. See Figure (3.3a). Examples of such products are android phones, fashions, electronics, computers, etc. Holding cost is presumed to be dependent on time because it has multiple factors that can be allowed to be represented as a linear function of time.

The results from Table 3.1 and Figures (3.3-3.5) show that there is an increase in the total cost as the holding cost is increasing, thereby reducing the schedule period and the ordering quantity. It is also discovered that as the deterioration rate is rising, there is an increase in the total cost, leading to growing in the order quantity and reducing the scheduling period. The model can be useful in the control of inventory of business enterprises that deal with products that have their demand and holding cost dependent on time. The model can be expanded in many directions; for example, shortages can be introduced into the model; another direction is to consider preservative technology and price-dependent demand.

However, in some cases, it is advisable to examine an inventory model with shortages, especially when the holding cost is prohibitive.

CHAPTER 4: AN INVENTORY MODEL FOR LINEARLY TIME-DEPENDENT DETERIORATING ITEMS WITH POWER PATTERN, SHORTAGES, AND TIME-VARYING DEMAND.

4.1 Introduction

In Chapter 3, An inventory model for deteriorating items with time-varying demand rate and time-dependent holding cost without shortages is propounded over a bounded horizon.

Many inventory models are developed under the assumption that the rate of deterioration is constant. It will be more realistic to consider that the price of decay of many commodities will go on increasing with time. This assumption is valid in some way because once any products start to decay, the rate of decaying continues to grow consistently day by day. Although it will not be correct to assume that the deterioration starts immediately, the items are produced. There will be a time when the product will maintain its freshness before the decline set in.

In this chapter, the inventory model for linearly time-dependent deteriorating items with power patterns, shortage, and time-varying demand is considered. Deterioration rate is assumed to be a linear function of time. We developed a new model and extended the work of Rajeswari & Indrani (2015) by examining the inventory cycle time as one of the decision variables. Here the inventory cycle time is not fixed as in the above paper. Inventory cycles depend on two decision variables, that is the time at which the inventory level descends to zero and the length of the scheduling period.

We minimise the total average system cost per unit time. We derived two non-linear equations from the developed model, and we solve the equations using Mathematica and excel solver to obtain the optimal solution. Numerical examples are propounded to establish the application of the model developed, and we use the cases to carry out sensitivity analysis on the impacts of various changes in some possible combinations of model parameters on the decision variables of the mathematical model.

4.2 Modelling Assumptions

The inventory model is established based on the following assumptions

1. Deterioration rate is considered as a linear time-dependent
2. Shortages are permitted and are backlogged
3. Demand is considered to be in the form of power demand pattern
4. Time horizon is made to be unbounded/ infinite
5. Lead time is negligible/trivial
6. One item is discussed only in this model
7. The average order is deterministic.

The following notation was used.

- $I(t)$ is the Inventory level of the system
- θ is the Deteriorating rate (θ $lies\ between\ 0\ and\ 1$)
- T is the scheduling period
- t_1 is the time at which the inventory system drops to zero
- M is the initial level of the stock
- m is the re-order points
- A is the ordering cost/unit item /unit time
- h is the carrying cost/unit item/unit time
- b is the backlogging cost /unit item/unit time
- w is the deteriorating cost/unit item/unit time
- p is the purchasing cost/unit item/unit time

- d is the average demand per cycle period
- $D(t)$ is the demand up to the time t. here (t $lies$ $between$ 0 and T)
- The index of demand pattern is $n.$ (n is assumed to be higher than 0)
- Q is the order quantity/unit item/unit time
- $K_1(t_1, T)$ is the average amount carried in the inventory system
- $S_2(t_1, T)$ is the average amount of shortage in the system
- $D_3(t_1, T)$ is the average quantity of deteriorated units
- $P_4(t_1, T)$ is the average quantity of purchased units
- $QC(T)$ is the ordering cost/unit item/unit time
- $HC(t_1, T)$ is the holding cost/unit item /unit time
- $PC(t_1, T)$is the purchasing cost/unit item /unit time
- $DC(t_1, T)$ is the deteriorating cost/ unit item /unit time
- $TC(t_1, T)$ is the total average cost of the inventory system

4.3 Mathematical model formulation

In the development of this model, a situation where inventory is followed by shortages is considered. Here, in this presented model, a cycle can be split up into two periods. $[0,\ T]$ is taken as a single cycle duration. Through $[0,\ t_1]$ the inventory is on the positive side and through $[t_1,\ T]$ the inventory is on the negative side.

Under the given assumptions, the commencing inventory level is M units at time $t = 0$. Throughout the period $t = 0$ to $t = t_1$, the inventory level diminishes, due to demand and deterioration, until it gets to zero level at $t =\ t_1$. During the interval $[t_1,\ T]$ the system experiences shortages which are backlogged to the close of the cycle. At interval $t = T$, the inventory arrives at a maximum shortage level m to clear the backlogged and the inventory level rises again to level M. The system is depicted in Figure 4.1.

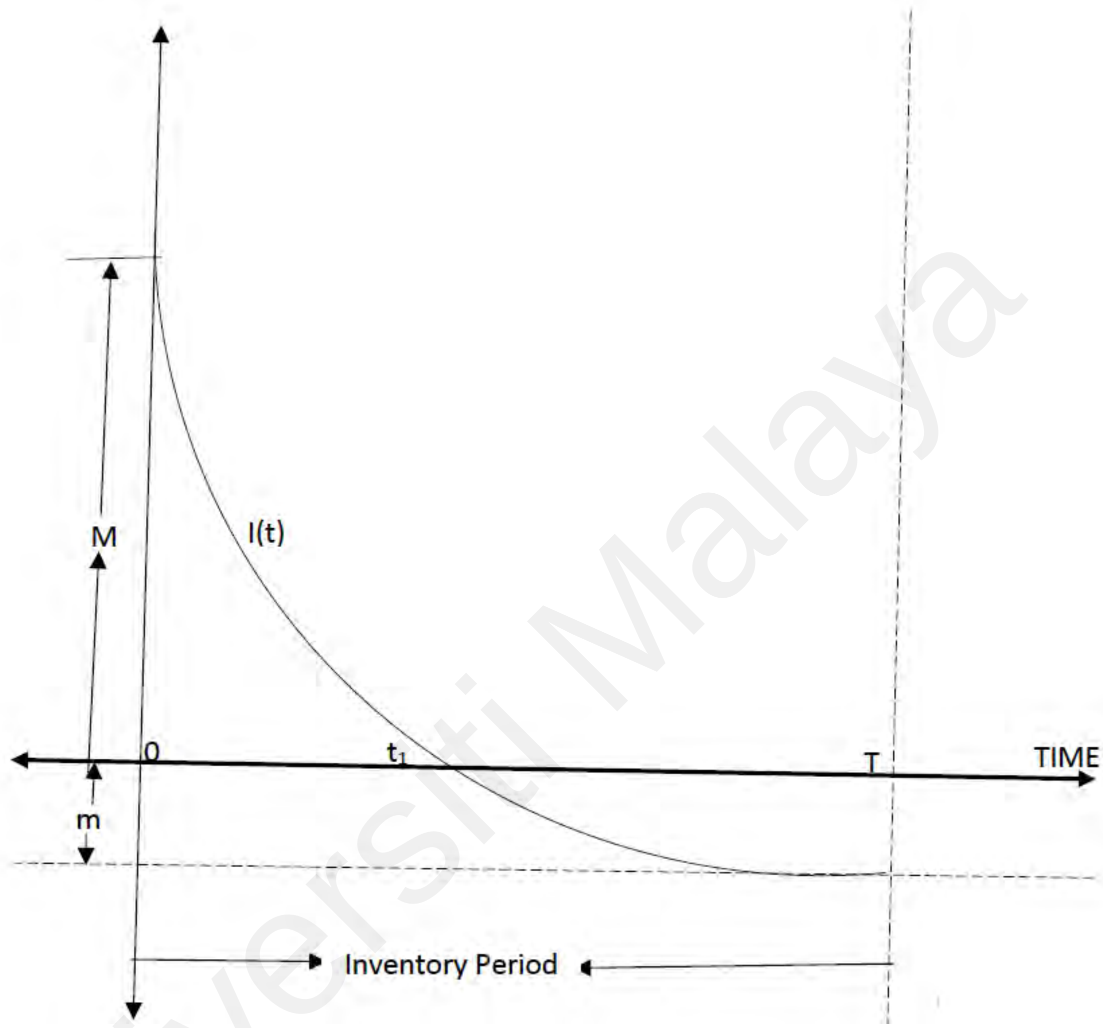


Figure 4.1: Graphical representation of Inventory Model.

The differential equations for the described model are:

$$\frac{dI(t)}{dt} + \theta I(t) = -\frac{dt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}} \quad 0 \leq \ t \leq t_1 \tag{4.1}$$

$$\frac{dI(t)}{dt} = -\frac{dt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}} \quad t_1 \leq t \leq T \tag{4.2}$$

Making use of the boundary conditions $I(0) = M,\ I(t_1) = 0 \ \ and \ \ I(T) = m$

Using the integrating factor $e^{\int \theta t dt} = e^{\frac{\theta t^2}{2}}$

$$I(t)e^{\frac{\theta t^2}{2}} = -\frac{d}{nT^{\frac{1}{n}-1}}\left[\int \ (e^{\frac{\theta t^2}{2}} t^{\frac{1}{n}-1})dt\right]$$

Since θ is small, $0 \leq \theta \leq 1$, we take the first three-term of the power series i.e

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\ldots\ldots..$$

$$e^{\frac{\theta t^2}{2}} = 1 + \frac{\theta t^2}{2} + \frac{(\theta t^2)^2}{8} + \frac{(\theta t^2)^3}{24} + \ldots\ldots\ldots\ldots\ldots..$$

$$I(t)e^{\frac{\theta t^2}{2}} = -\frac{d}{nT^{\frac{1}{n}-1}}\left[\int \ (1 + \frac{\theta t^2}{2} + \frac{\theta^2 t^4}{8})(t^{\frac{1}{n}-1})dt\right]$$

$$I(t)e^{\frac{\theta t^2}{2}} = -\frac{d}{nT^{\frac{1}{n}-1}}\left[nt^{\frac{1}{n}} + \frac{n\theta t^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 n t^{\frac{1}{n}+4}}{8(4n+1)}\right] + C$$

$$using \ \ the \ \ boundary \ \ condition \quad I(0) = M$$

$$M = C, \quad Therefore$$

$$I(t)e^{\frac{\theta t^2}{2}} = M - \frac{d}{nT^{\frac{1}{n}-1}}\left[nt^{\frac{1}{n}} + \frac{n\theta t^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 n t^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

$$I(t) = Me^{-\frac{\theta t^2}{2}} - \frac{de^{-\frac{\theta t^2}{2}}}{T^{\frac{1}{n}-1}}\left[t^{\frac{1}{n}} + \frac{\theta t^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{4.3}$$

From Equation (4.2) , we have:

$$I(t) = -\frac{d}{nT^{\frac{1}{n}-1}}\left[\int \ t^{\frac{1}{n}-1}dt\right]$$

$$I(t) = -\frac{d}{nT^{\frac{1}{n}-1}}\left[nt^{\frac{1}{n}}\right] + C, but \quad I(t_1) = 0$$

$$0 = -\frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}}\right] + C$$

$$C = \frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}}\right]$$

$$I(t) = -\frac{d}{T^{\frac{1}{n}-1}}\left[t^{\frac{1}{n}} - t_1^{\frac{1}{n}}\right] \tag{4.4}$$

$I(t)$ is a persistently decreasing function in the time interval $[0, T]$, therefore the initial net stock level at this interval is procured by substituting the boundary condition $I(t_1)$ into Equation (4.3), we have:

$$0 = Me^{-\frac{\theta t_1^2}{2}} - \frac{de^{-\frac{\theta t_1^2}{2}}}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

$$Me^{-\frac{\theta t_1^2}{2}} = \frac{de^{-\frac{\theta t_1^2}{2}}}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

$$M = \frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{4.5}$$

Replacing Equation (4.5) into Equation (4.3), we have:

$$I(t) = -\frac{de^{-\frac{\theta t^2}{2}}}{T^{\frac{1}{n}-1}}\left[t^{\frac{1}{n}} + \frac{\theta t^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

$$+ \frac{de^{-\frac{\theta t^2}{2}}}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

$$I(t) = -\frac{de^{-\frac{\theta t^2}{2}}}{T^{\frac{1}{n}-1}}\left[\left(t_1^{\frac{1}{n}} - t^{\frac{1}{n}}\right) + \frac{\theta}{2(2n+1)}\left(t_1^{\frac{1}{n}+2} - t^{\frac{1}{n}+2}\right)\right.$$

$$\left. + \frac{\theta^2}{8(4n+1)}\left(t_1^{\frac{1}{n}+4} - t^{\frac{1}{n}+4}\right)\right]$$

$$I(t) = \frac{d(1-\frac{\theta t^2}{2}+\frac{\theta^2 t^4}{8})}{T^{\frac{1}{n}-1}}\left[(t_1^{\frac{1}{n}} - t^{\frac{1}{n}}) + \frac{\theta}{2(2n+1)}(t_1^{\frac{1}{n}+2} - t^{\frac{1}{n}+2})\right.$$

$$\left. + \frac{\theta^2}{8(4n+1)}(t_1^{\frac{1}{n}+4} - t^{\frac{1}{n}+4})\right]$$

$since \quad 0 \leq \theta \leq 1 \quad we \ \ have:$

$$I(t) = \frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} - t^{\frac{1}{n}} - \frac{\theta t^2 t_1^{\frac{1}{n}}}{2} + \frac{\theta^2 t^4 t_1^{\frac{1}{n}}}{8} + \frac{\theta n t^{\frac{1}{n}+2}}{2n+1} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)}\right.$$

$$\left. - \frac{\theta^2 t^2 t_1^{\frac{1}{n}+2}}{4(2n+1)} - \frac{\theta^2 t^{\frac{1}{n}+4}}{8} + \frac{\theta^2 t^{\frac{1}{n}+4}}{4(2n+1)} - \frac{\theta^2 t^{\frac{1}{n}+4}}{8(4n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{4.6}$$

From the boundary condition $I(T) = m$, substitute into Equation (4.4) to get the re-order point:

$$m = -\frac{d}{T^{\frac{1}{n}-1}}\left[T^{\frac{1}{n}} - t_1^{\frac{1}{n}}\right] \tag{4.7}$$

To obtain the lot size Q that replenish the stock, we have:

$Q + m = M \quad \Rightarrow Q = M - m$

$$Q = \frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] + \frac{d}{T^{\frac{1}{n}-1}}\left[T^{\frac{1}{n}} - t_1^{\frac{1}{n}}\right]$$

$$Q = \frac{d}{T^{\frac{1}{n}-1}}\left[T^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{4.8}$$

For this inventory system, the average amount carried depend on the decision variables

t_1 and T

$$K_1(t_1, T) = \frac{1}{T}\int_0^{t_1} I(t)dt$$

$$K_1(t_1, T) = \frac{d}{T^{\frac{1}{n}}}\left[\int_0^{t_1} t_1^{\frac{1}{n}}dt - \int_0^{t_1} t^{\frac{1}{n}}dt - \frac{\theta t_1^{\frac{1}{n}}}{2}\int_0^{t_1} t^2 dt + \frac{\theta^2 t_1^{\frac{1}{n}}}{8}\int_0^{t_1} t^4 dt\right.$$

$$+ \frac{\theta n}{2n+1}\int_0^{t_1} t^{\frac{1}{n}+2}dt + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)}\int_0^{t_1} dt$$

$$-\frac{\theta^2 t_1^{\frac{1}{n}+2}}{4(2n+1)}\int_0^{t_1} t^2 dt - \frac{\theta^2}{8}\int_0^{t_1} t^{\frac{1}{n}+4} dt + \frac{\theta^2}{4(2n+1)}\int_0^{t_1} t^{\frac{1}{n}+4} dt$$

$$\left. -\frac{\theta^2}{8(4n+1)}\int_0^{t_1} t^{\frac{1}{n}+4} dt + \frac{\theta^2}{8(4n+1)}\int_0^{t_1} t_1^{\frac{1}{n}+4} dt\right]$$

$$K_1(t_1,T) = \frac{d}{T^{\frac{1}{n}}}\left[t_1^{\frac{1}{n}+1} - \frac{n t_1^{\frac{1}{n}+1}}{n+1} - \frac{\theta t_1^{\frac{1}{n}+3}}{6} + \frac{\theta^2 t_1^{\frac{1}{n}+5}}{40} + \frac{\theta n^2 t_1^{\frac{1}{n}+3}}{(2n+1)(3n+1)}\right.$$

$$+\frac{\theta t_1^{\frac{1}{n}+3}}{2(2n+1)} - \frac{\theta^2 t_1^{\frac{1}{n}+5}}{12(2n+1)} - \frac{\theta^2 n t_1^{\frac{1}{n}+5}}{8(5n+1)} + \frac{\theta^2 n t_1^{\frac{1}{n}+5}}{4(2n+1)(5n+1)}$$

$$\left. -\frac{\theta^2 n t_1^{\frac{1}{n}+5}}{8(4n+1)(5n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+5}}{8(4n+1)}\right]$$

Upon simplification, we have:

$$K_1(t_1,T) = \frac{d}{T^{\frac{1}{n}}}\left[\frac{t_1^{\frac{1}{n}+1}}{n+1} + \frac{\theta t_1^{\frac{1}{n}+3}}{3(3n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+5}}{15(5n+1)}\right] \tag{4.9}$$

The average shortage across the cycle period is:

$$S_2(t_1,T) = -\frac{1}{T}\int_0^{T} I(t)dt$$

$$S_2(t_1,T) = -\frac{d}{T^{\frac{1}{n}}}\int_0^{T} [t_1^{\frac{1}{n}} - t^{\frac{1}{n}}]dt$$

$$S_2(t_1,T) = \frac{d}{T^{\frac{1}{n}}}\left[\frac{nT^{\frac{1}{n}+1}}{n+1} + \frac{t_1^{\frac{1}{n}+1}}{n+1} - t_1^{\frac{1}{n}}T\right] \tag{4.10}$$

Along the negative inventory cycle, the total quantity of deteriorated units is:

$$D_3(t_1,T) = Q - \int_0^{T} \frac{d t_1^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}} dt$$

$$D_3(t_1,T) = Q - \frac{d}{nT^{\frac{1}{n}-1}}\left[T^{\frac{1}{n}}\right]$$

$$D_3(t_1,T) = \frac{d}{T^{\frac{1}{n}-1}}\left[T^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} - T^{\frac{1}{n}}\right]$$

$$D_3(t_1,T) = \frac{d}{T^{\frac{1}{n}-1}}\left[\frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

To get the average deteriorated unit along the period, we have to divide the total quantity deteriorated along the cycle with T

$$D_3(t_1,T) = \frac{1}{T}.\frac{d}{T^{\frac{1}{n}-1}}\left[\frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

$$D_3(t_1,T) = \frac{d}{T^{\frac{1}{n}}}\left[\frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{4.11}$$

Moreover, the average purchasing units along the inventory is:

$$P_4(t_1,T) = \frac{Q}{T}$$

$$P_4(t_1,T) = \frac{d}{T.T^{\frac{1}{n}-1}}\left[T^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

$$P_4(t_1,T) = \frac{d}{T^{\frac{1}{n}}}\left[T^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{4.12}$$

The average total cost for the inventory system is made up of the following cost components: Ordering cost + holding cost + purchasing cost + deteriorating cost + backlogging cost.

1. Holding cost per unit of time is:

$$HC(t_1,T) = \frac{hd}{T^{\frac{1}{n}}}\left[\frac{t_1^{\frac{1}{n}+1}}{n+1} + \frac{\theta t_1^{\frac{1}{n}+3}}{3(3n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+5}}{15(5n+1)}\right] \tag{4.13}$$

2. Backlogging cost per unit of time is:

$$BC(t_1,T) = \frac{bd}{T^{\frac{1}{n}}}\left[\frac{nT^{\frac{1}{n}+1}}{n+1} + \frac{t_1^{\frac{1}{n}+1}}{n+1} - t_1^{\frac{1}{n}}T\right] \tag{4.14}$$

3. The ordering cost per unit of time is:

$$OC(T) = \frac{A}{T} \tag{4.15}$$

4. The deteriorating cost per unit of time is:

$$DC(t_1, T) = \frac{w(\theta - D(t))}{T}$$

$$DC(t_1, T) = \frac{dw}{T^{\frac{1}{n}}}\left[\frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{4.16}$$

5. The purchasing cost per unit of time is

$$PC(t_1, T) = \frac{PQ}{T}$$

$$PC(t_1, T) = \frac{dp}{T^{\frac{1}{n}}}\left[T^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{4.17}$$

The average total inventory cost per units of time is the sum of the entire cost components given in Equations (4.13 - 4.17)

$$TC(t_1, T) = \frac{A}{T} + \frac{hd}{T^{\frac{1}{n}}}\left[\frac{t_1^{\frac{1}{n}+1}}{n+1} + \frac{\theta t_1^{\frac{1}{n}+3}}{3(3n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+5}}{15(5n+1)}\right] + \frac{bd}{T^{\frac{1}{n}}}\left[\frac{nT^{\frac{1}{n}+1}}{n+1} + \frac{t_1^{\frac{1}{n}+1}}{n+1} - t_1^{\frac{1}{n}}T\right]$$

$$+ \frac{dw}{T^{\frac{1}{n}}}\left[\frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] + \frac{dp}{T^{\frac{1}{n}}}\left[T^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{4.18}$$

We are to find the optimum solution that minimises the total average cost of the function $TC(t_1, T)$ such that 0 lies between t_1 and T and $T > 0$.

4.4 Solution method

To minimise the total relevant cost per unit time, we find the optimal value of the decision variables t_1 and T. We consider the partial derivatives of $TC(t_1, T)$ w.r.t. the decision variables t_1 and T.

$$\frac{\partial TC(t_1, T)}{\partial t_1} = \frac{hd}{T^{\frac{1}{n}}}\left[\frac{t_1^{\frac{1}{n}}}{n} + \frac{\theta t^{\frac{1}{n}+2}}{3n}\right] + \frac{br}{T^{\frac{1}{n}}}\left[\frac{t_1^{\frac{1}{n}}}{n} - \frac{Tt_1^{\frac{1}{n}-1}}{n}\right]$$

$$+\frac{dw}{T^{\frac{1}{n}}}\left[\frac{\theta t_1^{\frac{1}{n}+1}}{2n}\right]+\frac{dp}{T^{\frac{1}{n}}}\left[\frac{\theta t_1^{\frac{1}{n}+1}}{2n}\right]=0 \tag{4.19}$$

and

$$\frac{\partial TC(t_1,T)}{\partial T}=-\frac{A}{T^2}-\frac{hr}{nT^{\frac{1}{n}+1}}\left[\frac{t_1^{\frac{1}{n}+1}}{n+1}+\frac{\theta t_1^{\frac{1}{n}+3}}{3(3n+1)}\right]+\frac{bdn}{n+1}-\frac{bdt_1^{\frac{1}{n}+1}}{n(n+1)T^{\frac{1}{n}+1}}$$

$$+\frac{(1-n)bdt_1^{\frac{1}{n}}}{nT^{\frac{1}{n}}}-\frac{dw}{nT^{\frac{1}{n}+1}}\left[\frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)}\right]-\frac{dp}{nT^{\frac{1}{n}+1}}\left[\frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)}\right]=0 \tag{4.20}$$

Equations (4.19) and (4.20) is a non-linear equation, and we solve for the decision variables t_1 and T when we equate them to zero to get the optimum solution, provided that:

$$\left[\frac{\partial^2\psi(t_1,T)}{\partial t_1^2}\right]\left[\frac{\partial^2\psi(t_1,T)}{\partial T^2}\right]-\left[\frac{\partial^2\psi(t_1,T)}{\partial t\partial T}\right]^2\geq 0. \tag{4.21}$$

4.5 Numerical illustration

In this section, we consider some numerical examples to justify our developed model. In these examples, we find a solution to the inventory problem, evaluate the values of t_1 and T and substitute their values to Equation (4.18) to get the average total cost. We also get the re-order point from Equation (4.7) and lot-size from Equation (4.8).

Example 1:

We examine the following parametric values for the inventory system.

$d=100,\ A=50,\ h=2,\ b=4,\ w=12,\ p=10\ \ n=0.5\ \ \theta=0.1$ with appropriate units.

Here we use maple 2018 to get the optimum solution and check for convexity using Equation (4.12).

$T=0.756$ years, $t_1=0.463$ years, $TC(t_1,T)=1127.52$ units, $Q=75.794$ units, $M=28.561$ units, $m=-47.233$ units.

From Equation (4.21) the Hessian is $H(t_1, T) = 240676.6344$, which is positive; it implies that (t_1, T) is the minimum point.

The convexity of the total cost function and the decision variables are given in Figure (4.2).

Example 2.

We reflect on the same example as in example 1, but we make our $n = 2$ with appropriate units.

$T = 0.903$ years, $t_1 = 0.546$ years, $TC = 1105.44$ units, $Q = 90.488$ units, $M = 70.386$ units, $m = -20.103$ units.

From Equation (4.21), the Hessian is $H(t_1, T) = 143318.229$, which is positive; it implies that (t_1, T) is the minimum point.

Example 3.

We consider the following parametric values for the inventory system as found in Rajeswari & Indrani (2015).

$d = 50,\ A = 250,\ h = 0.5,\ b = 12,\ w = 15,\ p = 8\ n = 2\ \theta = 0.05$ with appropriate units.

Here, we want to make a comparison between our results with the optimal system propounded by the above authors. Solving Equations (4.9 and 4.10) in our propose model and input the same parametric values from the said authors, the following results are obtained:

$T = 2.726432$ years, $t_1 = 2.339401$ years, $TC(t_1, T) = 549.406768$ units, $Q = 139.908330$ units, $M = 122.951587$ units, $m = -10.0459318$ units. From Equation (4.12), the Hessian is: $H(t_1, T) = 5218.42892 > 0$ which is positive, it implies that (t_1, T) is the minimum point.

Bearing in mind that Rajeswari & Indrani (2015) make the scheduling period constant and regarded $T = 1$ year. Their results for the optimum policy are as follows: $t_1 = 0.973038$ years, $Q = 50.215149$ units, and the inventory minimum cost is $\$658.43$. It implies from the results that our optimal policy of $\$549.407$ is better than their own optimal policy of $\$658.43$. It should be noted that the minimum inventory total cost of the said authors is 18% higher than our own model. The reason for the difference is that, they fixed their scheduling period to be $T = 1$ year which did not really give accurate results. More also, other variables from our results are not considered by the Rajeswari & Indrani (2015).

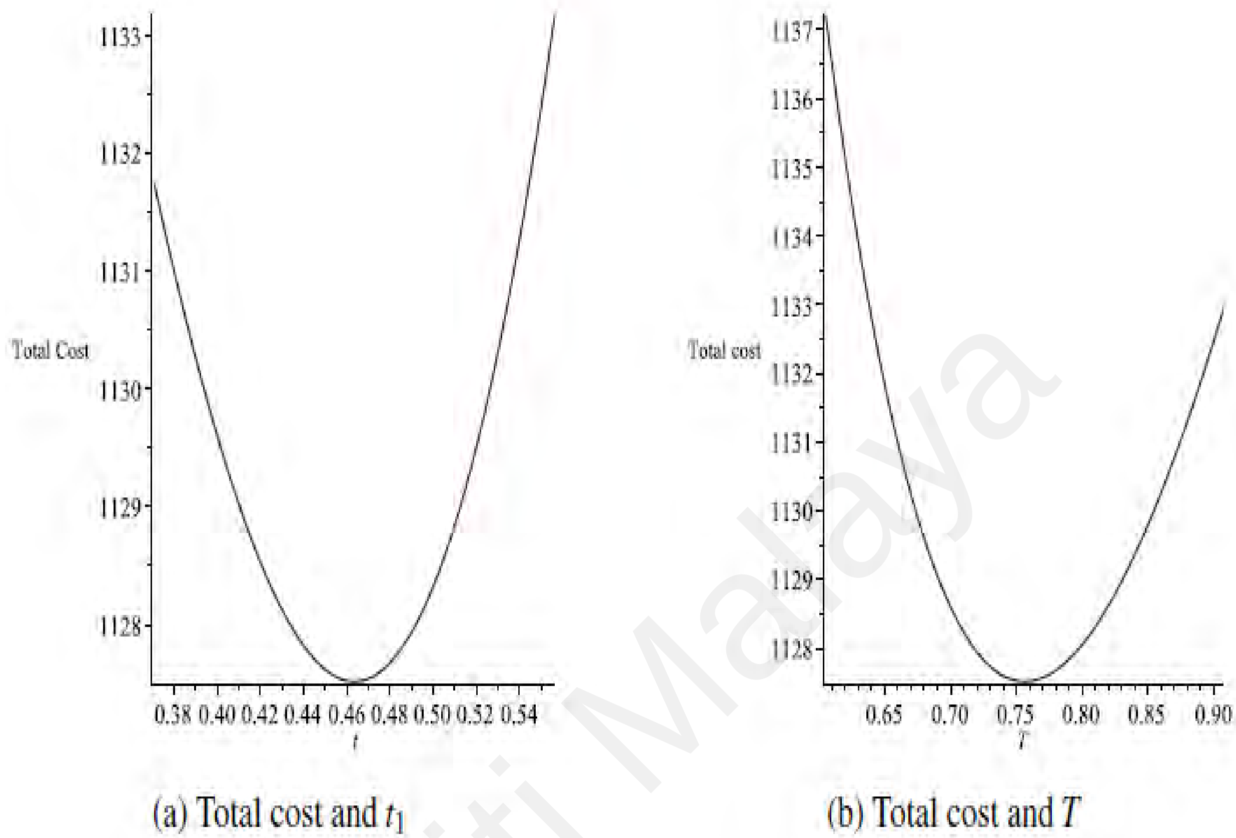


Figure 4.2: Graphical representation of convexity of total cost against cycle and schedule time.

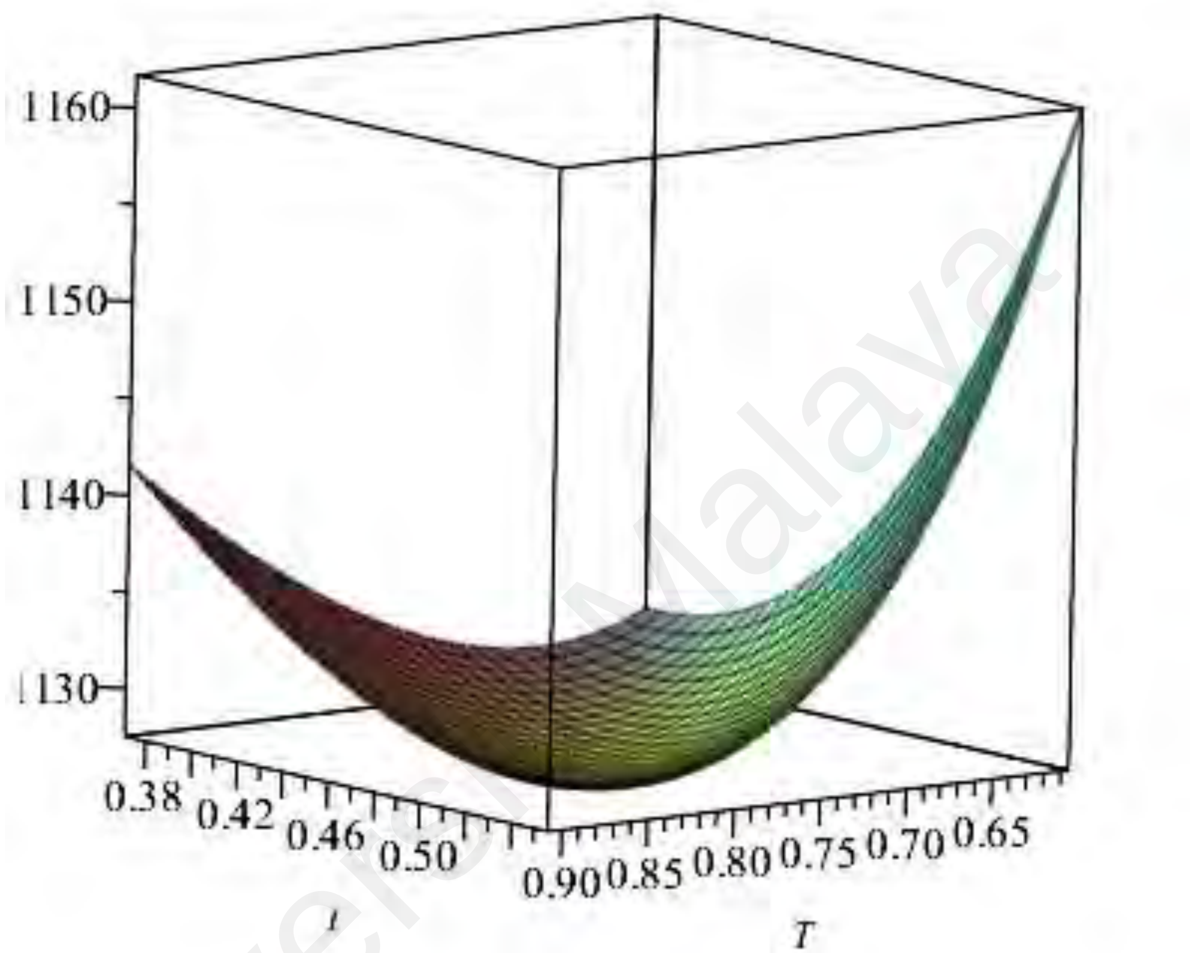


Figure 4.3: Graphical illustration of Convexity of total cost per unit time.

4.6 Sensitivity analysis

A sensitivity analysis of the optimal inventory system is conducted in this section. The effect of changes that occurs in the system policy when the input parameter is varied are carefully considered. Observations are based on the reactions of the t_1, T, Q, M, m, TC against the input parameters d, A, n, h, p, b, w and θ.

By the results from Table 4.1 the following observations are derived:

1. Increase in the value of parameter A leads to increase in the inventory cycle time t_1, scheduling period T, total cost TC, quantity order Q, Initial stock level, M, but decrease in re-order point m, here the optimal decision variables are highly sensitive to change in the parameter A. See Figures (4.4a,4.5a,4.6a,4.7a).

2. Increases in the value of demand rate parameter d lead to decrease in the inventory cycle t_1, scheduling period T and re-order point m, but there is an increase in the inventory total cost TC, quantity order Q, and initial stock level M. Here the decision variables are highly sensitive to change in parameter d. See Figures (4.5, 4.6, 4 .7).

3. An increase in the input parameter h leads to a decrease in the inventory cycle t_1, schedule period T, quantity order Q, and initial stock level M, but there is an increase in the inventory total cost TC. Hence the decision variables are moderately sensitive to change in parameter h. See Figures (4.4, 4.6, 4.7).

4. As the inventory total cost TC, and re-order point m increases, there is a decrease in the inventory cycle time t_1, scheduling period T, quantity order Q and initial stock level M as the value of parameter b increase. Here the optimal decision variables are moderately sensitive to change in b. See Figures (4.4,4.6, 4.8).

5. When the input parameters w and p increase in values, there is a decrease in the inventory cycle time t_1, schedule period T, quantity order Q, initial stock level M, and re-

order level m, also, the inventory total cost increases. The decision variables have a low sensitivity to change in the parameters w and P. See Figures (4.4, 4.7,4.8).

6. In parameter n, as the value increases, there is the corresponding decrease in the inventory cycle t_1, schedule period T, quantity order Q, and re-order level m. However, there is an increase in the inventory total cost TC and initial stock level M. The optimal decision variables are moderately sensitive to change in n. See Figures (4.4,4.5, 4.6,4.7).

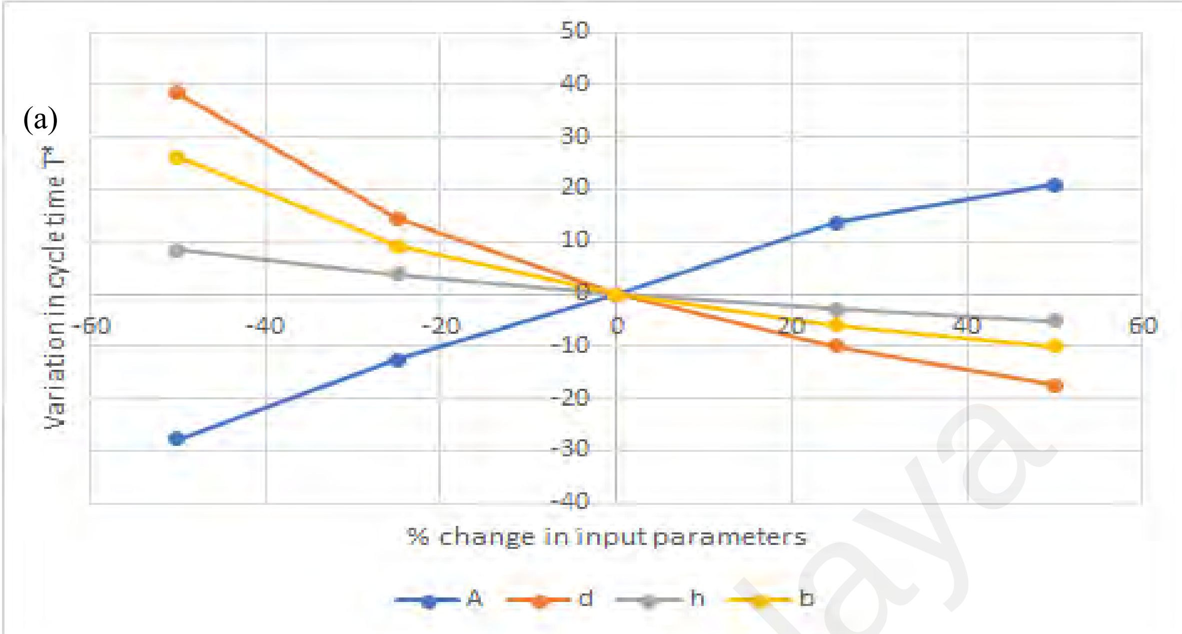
7. An increase in the value of deterioration parameter θ leads to an increase in the inventory total cost TC, but a decrease in all other decision variables. It is observed here that deterioration parameter θ have a very low sensitivity to change. See Figures (4.5,4.7,4.8).

In Table 4.1, below, the following abbreviations are use:

Note: P = parameter, V = values, C =% changes, TC*= %change in TC, Q*= %change in ordering quantity, M*= %change in ordering level, m*= %change in re-order point.

Table 4.1: Outcome of changes in various parameters of the inventory models.

P	V	C	T	t_1	TC*	Q*	M*	m*
			Change in:					
A	75	+50	0.915	0.551	+2.65	+21.060	+17.429	+23.255
	65.5	+25	0.859	0.521	+1.70	+13.584	+11.359	+14.929
	50	0	0.756	0.464	0	0	0	0
	37.5	-25	0.660	0.409	-1.56	-12.728	-11.052	-13.742
	25	-50	0.545	0.341	-1.87	-28.024	-24.892	-29.917
d	150	+50	0.625	0.388	+46.79	+23.794	+27.089	+21.802
	125	+25	0.681	0.421	+23.46	+12.478	+14.172	+11.454
	100	0	0.756	0.464	0	0	0	0
	75	-25	0.866	0.525	-23.63	-14.101	-15.898	-13.014
	50	-50	1.047	0.623	-47.54	-30.713	-34.410	-28.477
h	3.0	+50	0.717	0.386	+0.83	-28.477	-27.137	+5.882
	2.5	+25	0.735	0.421	+0.46	-2.923	-15.147	-4.68
	2.0	0	0.756	0.464	0	0	0	0
	1.5	-25	0.784	0.515	-0.59	+3.736	+19.440	-5.760
	1.0	-50	0.819	0.580	-1.34	+8.565	+4.788	-13.339
b	6.0	+50	0.681	0.478	+1.131	-9.899	+18.254	-26.923
	5.0	+25	0.712	0.472	+0.643	-5.848	+10.294	-15.609
	4.0	0	0.756	0.464	0	0	0	0
	3.0	-25	0.827	0.450	-0.889	+9.227	-13.800	-23.151
	2.0	-50	0.955	0.426	-2.205	+26.078	-33.161	-61.899
w	18	+50	0.746	0.449	+0.103	-1.371	-4.868	+0.744
	15	+25	0.751	0.456	+0.053	-0.714	-2.521	+0.379
	10	0	0.756	0.464	0	0	0	0
	9.0	-25	0.762	0.472	-0.055	+0.779	+2.719	-0.394
	6.0	-50	0.769	0.480	-0.114	+1.634	+5.665	-0.804
p	15	+50	0.748	0.451	+44.431	-1.158	-4.104	+0.624
	12.5	+25	0.752	0.457	+22.217	-0.599	-2.113	+0.317
	10	0	0.756	0.464	0	0	0	0
	7.5	-25	0.761	0.470	-22.219	0.644	+2.251	-0.327
	5.0	-50	0.766	0.477	-44.439	+1.340	+4.655	-0.665
n	0.75	+50	0.762	0.467	-0.130	+0.715	+39.318	-22.628
	0.625	+25	0.755	0.463	-0.013	-0.141	+21394	-13.164
	0.5	0	0.757	0.464	0	0	0	0
	0.375	-25	0.774	0.474	-0.208	+2.300	-26.448	19.684
	0.25	-50	0.835	0.508	-0.933	+10.298	+59.713	+52.633
θ	0.15	+50	0.738	0.438	+0.185	-2.365	-8.496	+1.343
	0.125	+25	0.747	0.450	+0.097	-1.270	+4.517	+0.694
	0.1	0	0.756	0.464	0	0	0	0
	0.075	-25	0.768	0.479	-0.107	1.496	+5.203	-0.745
	0.05	-50	0.782	0.498	-0.226	-3.297	+11.310	-1.549



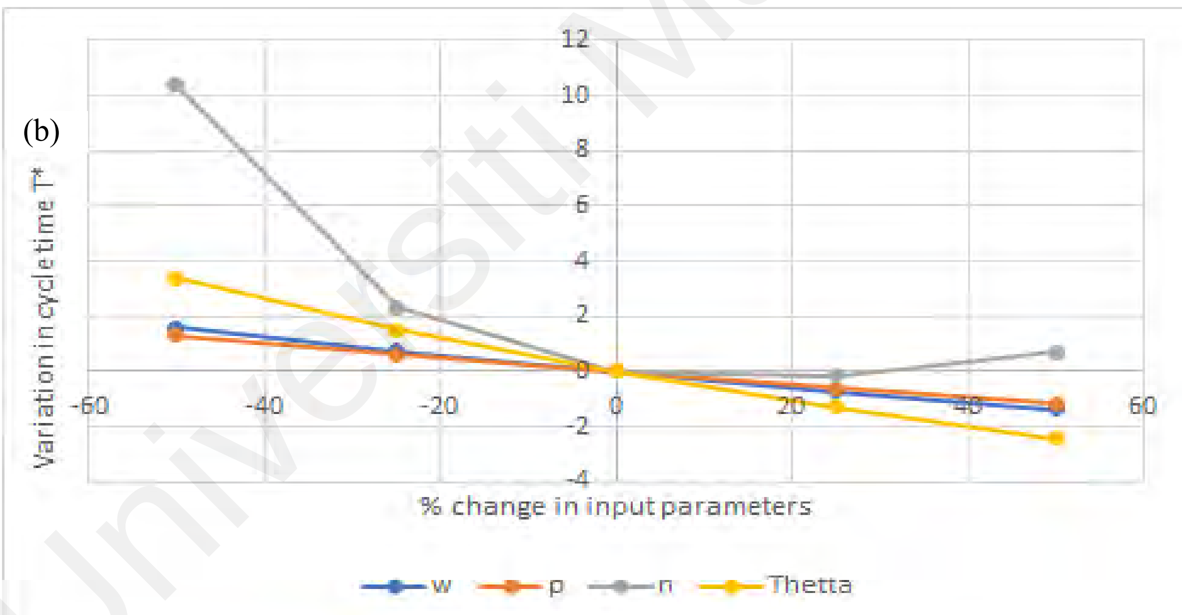
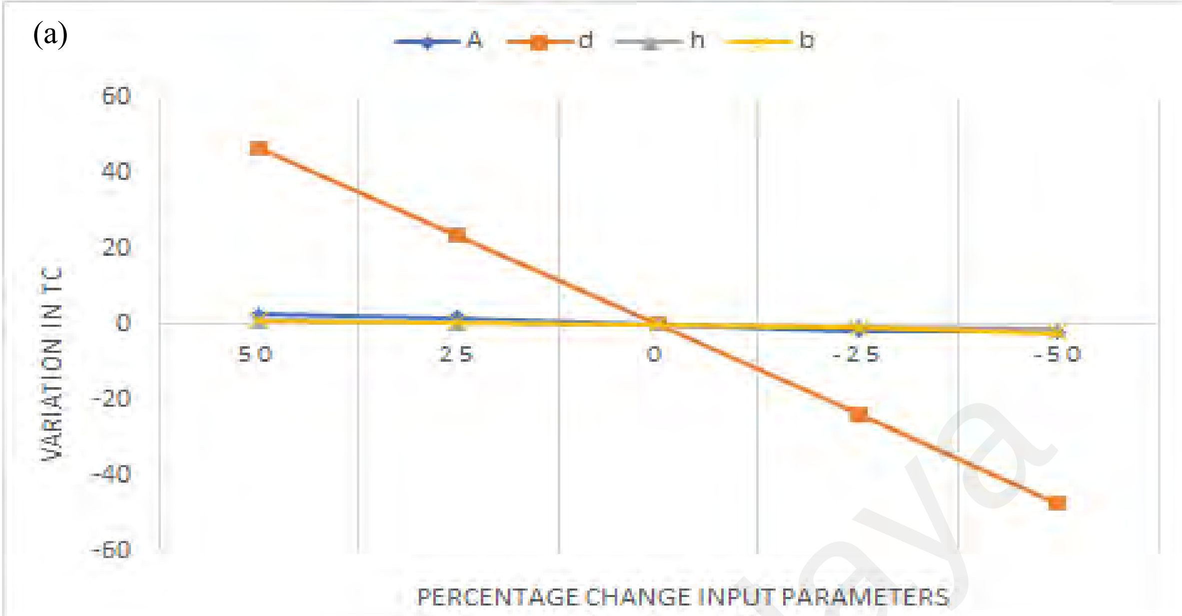


Figure 4.4: Graphical representation of sensitivity analysis of cycle time against percentage change in input parameters.



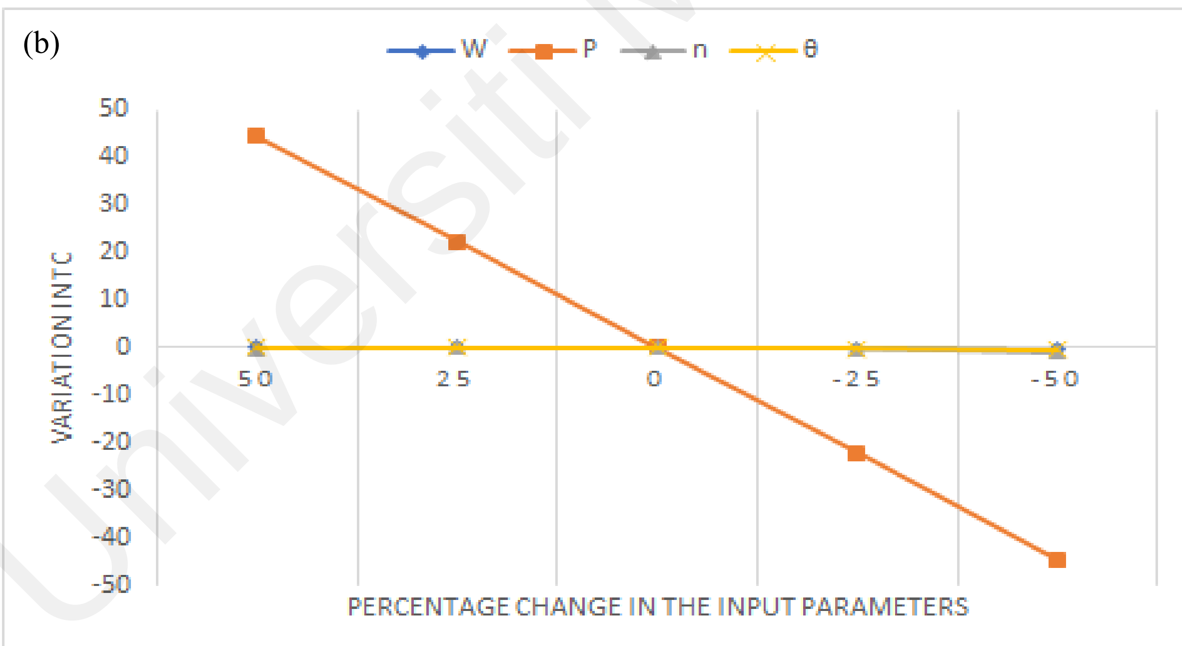
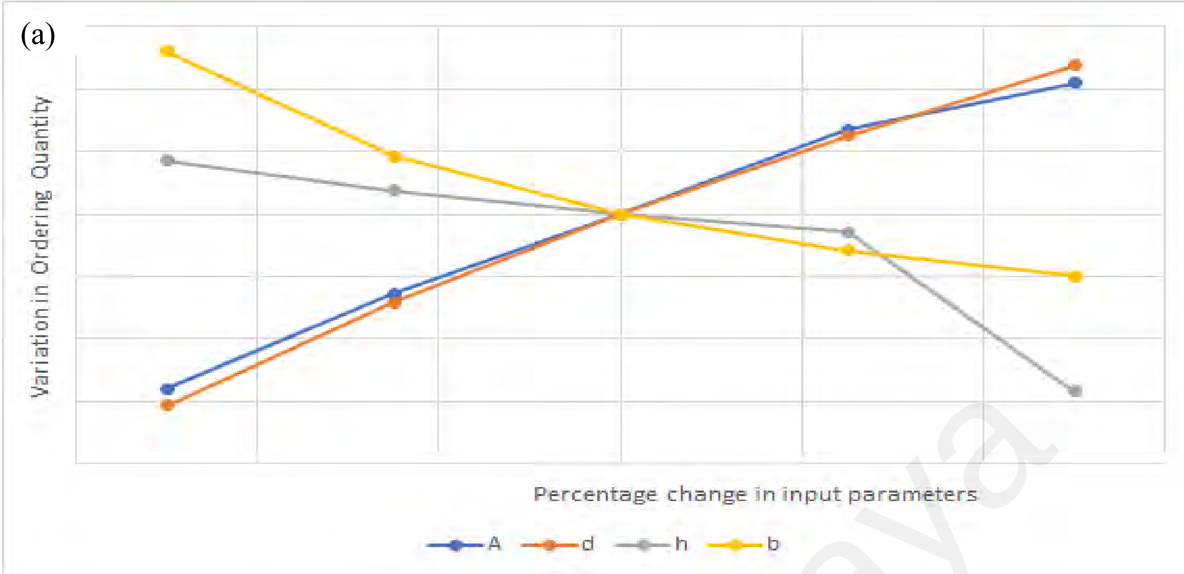


Figure 4.5: Graphical representation of sensitivity analysis of total cost against percentage change in input parameters.



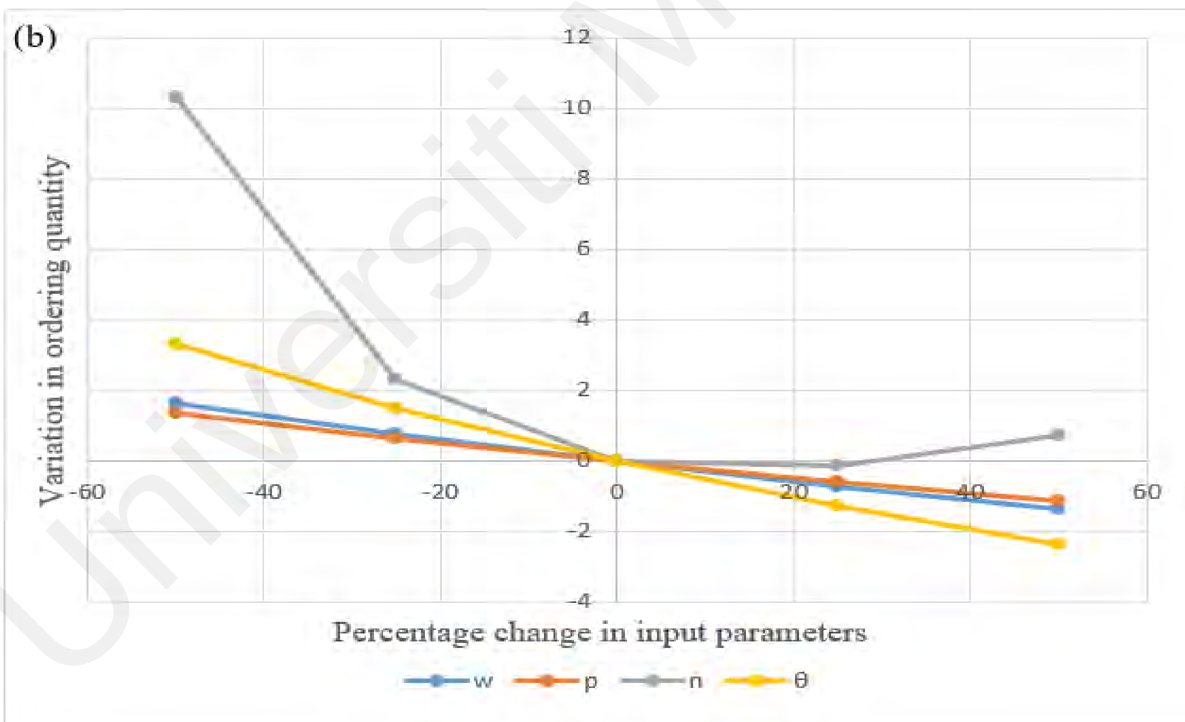


Figure 4.6: Graphical representation of sensitivity analysis of ordering quantity against percentage change in input parameters.

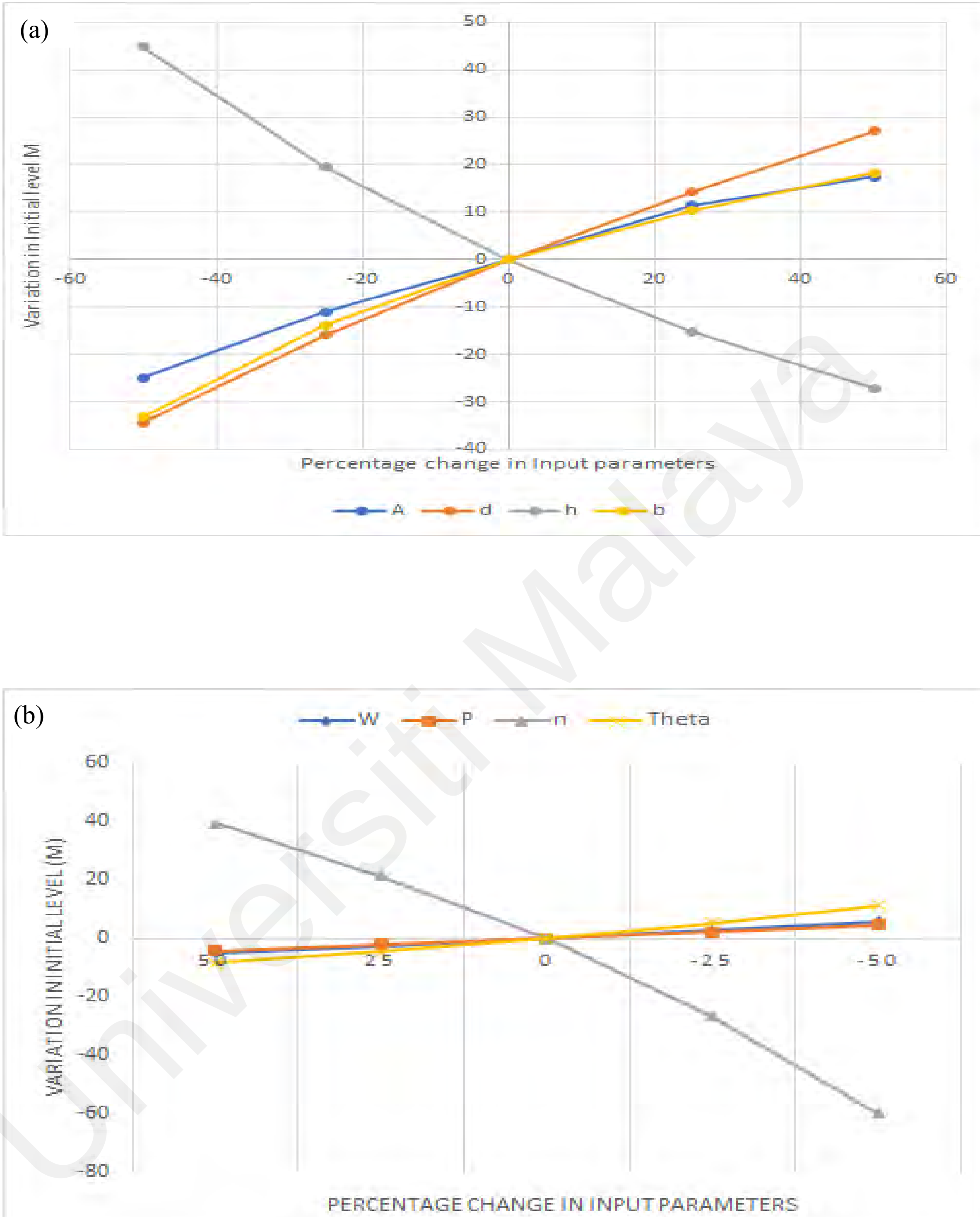


Figure 4.7: Graphical representation of sensitivity analysis of initial-level against percentage change in input parameters.

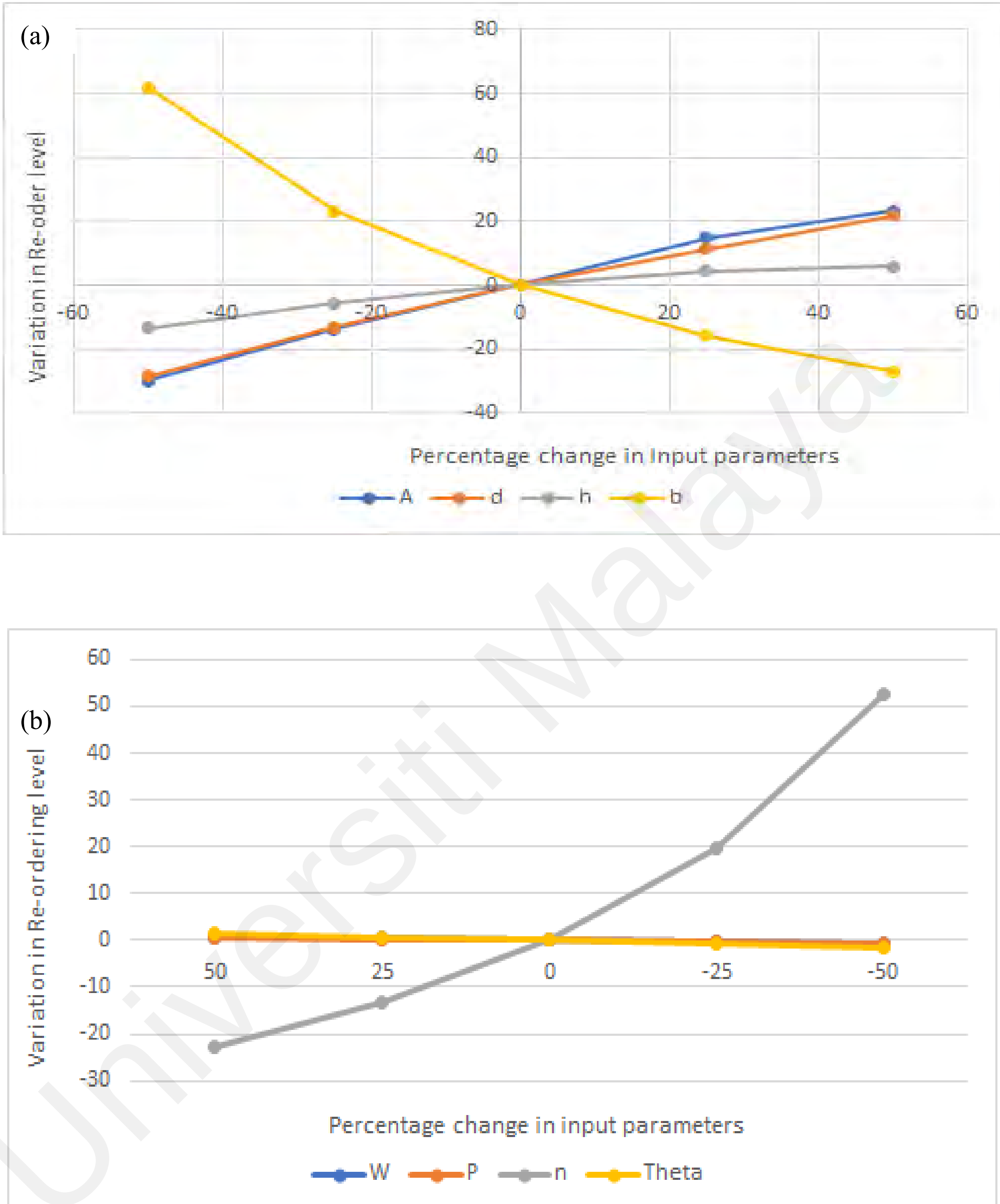


Figure 4.8: Graphical representation of sensitivity analysis of re-ordering cost against percentage change in input parameters.

4.7 Concluding Observations

In this chapter, a deterministic inventory model for deteriorating items with linear deterioration rate has been presented. Demand follows a power pattern and shortages are permitted which are wholly backlogged. The aims are to find an optimum solution that minimises the total average cost. The decision variables here are the time at which the inventory drops to zero and the schedule cycle period. Numerical examples are given, and sensitivity analysis back up with graphical representation is carried out to show how the optimal decision is affected by changes in different parameters in the model. The following are our concluding remarks:

1. When the ordering cost A is increasing, the optimal inventory cost is rising. See Figure 4.6.
2. When the demand rate d is increasing, the optimal cost and the ordering quantity are growing. See Figures (4.5 and 4.6).
3. When the deteriorating rate θ is increasing, the ordering quantity is decreasing, leading to growth in the optimal inventory cost. See Figures (4.6, 4.7).
4. When the power index n is increasing, the inventory ordering quantity is increasing. See Figure (4.6a).

The model is useful to inventory keeping company/organisation that deals with deteriorating item that deteriorates with the passage of time and the demand also varies as the time progresses. Examples of such items include an android mobile phone, computer chips, fashion, electronics etc. The model presented in this chapter provides a basis for various possible extensions. In this direction, future research can enrich the model by adding more realistic assumption like finite replenishment, incorporating, non-instantaneous deterioration, lost sales, product reliability, time value of money, and so on.

When customers required items that are not in stock in the organisation stores, then the customer will go another place (otherwise known as a lost sale) or place an order for the products in another area.

Some organisation are the sole supplier of some product, and they offer a competitive price to their customer, and some also provide a discount for delaying in the delivery of certain items. The acts give these organisation opportunities not to lose the sale when it's inventory eventually drop to zero. Customers, in this case, have to wait for their order to be filled whichever time the new order arrives.

Therefore, shortages are the need/demand that will be fulfilled in some time later than desire. The next model will be formulated base on this fact.

CHAPTER 5: A LINEAR DETERIORATING INVENTORY POLICY FOR ITEMS WITH POWER DEMAND PATTERN AND VARIABLE HOLDING COST CONSIDERING SHORTAGES

5.1 Introduction

In Chapter four, an economic order quantity model for a linearly time-depended deteriorating items with power pattern form, shortages and time-varying demand rate was examined.

In the management of inventory, there are times shortages or stock-out do occur. During this duration, some consumers would be kind enough to, hold back for backlogging, albeit others will not be patient enough to wait, and they will turn elsewhere to meet their demand. As a result of this, opportunity cost arises from the lost sale should be taken into consideration in any inventory model as against a complete backlogging which are very common in inventory modelling.

In this chapter, a linear deteriorating inventory model for products with power demand pattern and variable holding cost will be considered. This model is an extension of our model in chapter three by adding time-depending on the holding cost and time-depending on the deteriorating rate. If $t_d = 0$, we have the same model as in Adaraniwon & Omar (2019) with fixed holding cost. The intention is to minimise the total cost per unit time by optimising the schedule period or cycle time and optimal ordering quantity.

5.2 Notation and Assumptions

The mathematical inventory model for this work is developed established on the following assumptions and notation.

Notation

- T : Length of the inventory cycle.
- t_1: Time at which the inventory deleted to 0.
- $Q_1(t)$: Positive inventory level at time t.
- $Q_2(t)$: Negative inventory level at time t.
- θ: Deteriorating rate $(0 < \theta \leq 1)$
- P: Ordering quantity (units)
- M: Maximum inventory level during the cycle
- N: Maximum inventory level during negative inventory period
- d: Average demand per scheduling period per units per time
- γ: Backlogging rate. $(0 \leq \gamma \leq 1)$
- n: Demand pattern index, (n must be greater than 0)
- A: Ordering cost ($ per order)
- h: Holding cost per unit ($ per /time/unit)
- Z: Purchasing cost per unit ($ per unit).
- K: Cost per shortage unit ($ per unit).
- S: Cost per lost sale unit ($ per unit).
- HC: Holding cost per/time/unit.
- SC: Shortage cost per/time/unit
- LSC: Lost sale cost per/time/unit.
- TC: Total cost of the inventory policy per/time/unit.

Assumptions

1. Demand is a power demand pattern.
2. Shortages are permitted and partially backlogged

3. Deterioration rate of item is a linear function of time
4. Holding cost is time-dependent and taken as $h(t) = h + \beta t$, where $h > 0, \beta > 0$
5. Lead time is negligible.
6. Replenishment is instantaneous.
7. Demand $D(t)$ varies with time and taken as $D(t) = \frac{dt^{\frac{1}{n}}}{nT^{\frac{1}{n}-1}}$, where d is the average demand and the power index is n, $0 < n < \infty$ and $0 \leq t \leq T$. The rate of demand at any given time t is $D'(t)$

5.3 Mathematical Formulation

In the presented model, a cycle can be split up into two periods. In this diagram, $[0, T]$ is considered as a single duration. Through $[0,\ t_1]$ the inventory is on positive side and through $[t_1,\ T]$ the inventory is on the negative side. Let $Q_1(t)$ be the stock level at time t which ranges between $0 \leq t \leq T$. At the onset of the inventory cycle, the maximum inventory level $Q_1(0) = M$ reduces as a result of demand and the process of deterioration also set in for the items. At the interval $t = t_1$, the inventory system gets down to zero level. After that, at the interval $[t_1,\ T]$, shortages occur in the system, and they are backlogged at the end of the cycle. At the interval $t = T$, the system reaches a level N.

The inventory level $Q_1(t)$ and $Q_2(t)$ during the cycle period is described in Figure 5.1.

Based on the above assumptions, the differential equations represent the stock level is given as:

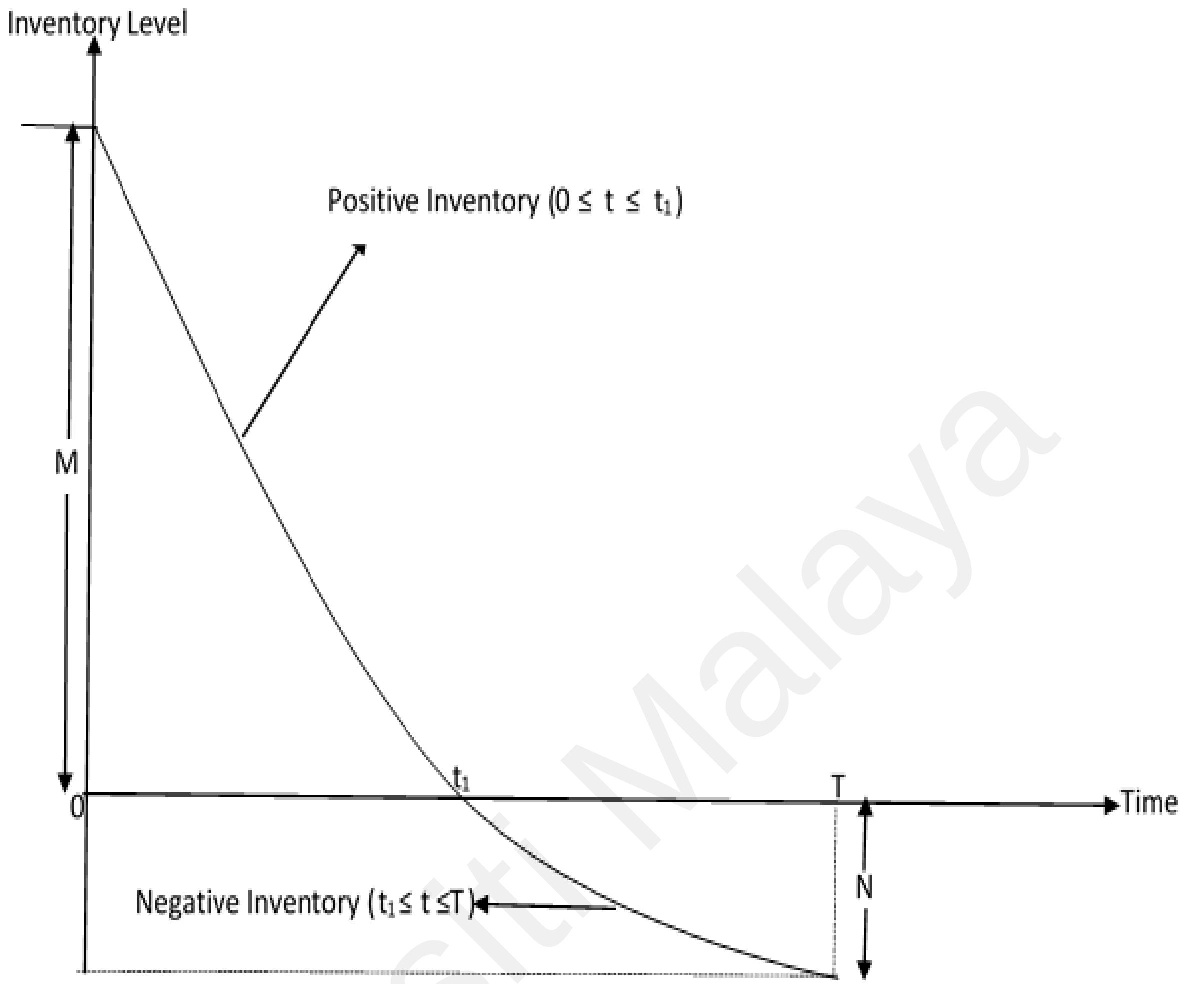


Figure 5.1: Graphical representation of inventory model.

$$\frac{dQ_1(t)}{dt} + \theta t Q(t) = \frac{dt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}}-1} \quad 0 \leq \ t \leq t_1 \qquad (5.1)$$

With the boundary conditions $Q_1(0) = M, Q_1(t_1) = 0 \ \ and \ \ Q_2(t_1) = 0$

Using the integrating factor $e^{\int \theta t dt} = e^{\frac{\theta t^2}{2}}$

$$\frac{Q_1(t)e^{\frac{\theta t^2}{2}}}{dt} + \theta t Q_1(t) e^{\frac{\theta t^2}{2}} = -\frac{d}{nT^{\frac{1}{n}-1}}\left[t^{\frac{1}{n}-1} e^{\frac{\theta t^2}{2}}\right]$$

$$Q_1(t)e^{\frac{\theta t^2}{2}} = -\frac{d}{nT^{\frac{1}{n}-1}}\left[\int t^{\frac{1}{n}-1} e^{\frac{\theta t^2}{2}}\right] dt$$

Since θ is small, $0 \leq \theta \leq 1$, taking the first three expressions of the power series, we have:

$$e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \ldots\ldots\ldots\ldots$$

$$e^{\frac{\theta t^2}{2}} = 1 + \frac{\theta t^2}{2} + \frac{(\theta t^2)^2}{8} + \frac{(\theta t^2)^3}{24} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$Q_1(t)e^{\frac{\theta t^2}{2}} = -\frac{d}{nT^{\frac{1}{n}-1}}\left[\int t^{\frac{1}{n}-1}\left(1 + \frac{\theta t^2}{2} + \frac{\theta^2 t^4}{8}\right)\right] dt$$

$$Q_1(t)e^{\frac{\theta t^2}{2}} = -\frac{d}{nT^{\frac{1}{n}-1}}\left[nt^{\frac{1}{n}} + \frac{n\theta t^{\frac{1}{n}+2}}{2(2n+1)} + \frac{n\theta^2 t^{\frac{1}{n}+4}}{8(4n+1)}\right] + C$$

Making use of the boundary condition $Q_1(0) = M$

$M = C$, therefore:

$$Q_1(t)e^{\frac{\theta t^2}{2}} = M - \frac{d}{nT^{\frac{1}{n}-1}}\left[nt^{\frac{1}{n}} + \frac{n\theta t^{\frac{1}{n}+2}}{2(2n+1)} + \frac{n\theta^2 t^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

$$Q_1(t)e^{\frac{\theta t^2}{2}} = M - \frac{d}{T^{\frac{1}{n}-1}}\left[t^{\frac{1}{n}} + \frac{\theta t^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t^{\frac{1}{n}+4}}{8(4n+1)}\right]$$

$$Q_1(t) = Me^{-\frac{\theta t^2}{2}} - \frac{de^{-\frac{\theta t^2}{2}}}{T^{\frac{1}{n}-1}}\left[t^{\frac{1}{n}} + \frac{\theta t^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t^{\frac{1}{n}+4}}{8(4n+1)}\right] \tag{5.2}$$

During the negative inventory, the policy is described by the equation:

$$\frac{Q_2(t)}{dt} = -\gamma D(t) \quad t_1 \leq \ t \leq T \tag{5.3}$$

$$Q_2(t) = -\frac{\gamma d}{nT^{\frac{1}{n}-1}}\int t^{\frac{1}{n}-1} dt$$

$$Q_2(t) = -\frac{\gamma d}{T^{\frac{1}{n}-1}}\left(t^{\frac{1}{n}}\right) + C$$

Using the boundary condition $Q_2(t_1) = 0$

$$0 = -\frac{\gamma d}{T^{\frac{1}{n}-1}} t_1^{\frac{1}{n}} + C$$

$$C = \frac{\gamma d}{T^{\frac{1}{n}-1}} t_1^{\frac{1}{n}}$$

Therefore:

$$Q_2(t) = -\frac{\gamma d}{T^{\frac{1}{n}-1}} t^{\frac{1}{n}} + \frac{\gamma d}{T^{\frac{1}{n}-1}} t_1^{\frac{1}{n}}$$

$$Q_2(t) = \frac{\gamma d}{T^{\frac{1}{n}-1}} \left[t_1^{\frac{1}{n}} - t^{\frac{1}{n}} \right] \tag{5.4}$$

Q_1 is a steadily decreasing function in the time interval $[0, T]$; therefore the initial net stock level at this interval is obtained by substituting the boundary condition $Q_{t_1} = 0$, into Equation (5.3), we have

$$0 = Me^{-\frac{\theta t_1^2}{2}} - \frac{de^{-\frac{\theta t_1^2}{2}}}{T^{\frac{1}{n}-1}} \left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} \right]$$

$$Me^{-\frac{\theta t_1^2}{2}} = \frac{de^{-\frac{\theta t_1^2}{2}}}{T^{\frac{1}{n}-1}} \left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} \right]$$

$$M = \frac{d}{T^{\frac{1}{n}-1}} \left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} \right] \tag{5.5}$$

The maximum negative inventory per units is given as:

$$N = -Q_2(T)$$

$$N = \frac{\gamma d}{T^{\frac{1}{n}-1}} \left(T^{\frac{1}{n}} - t_1^{\frac{1}{n}} \right) \tag{5.6}$$

The order size during the entire period $[0, T]$ is given as:

$$P = M + N$$

$$P = \frac{\gamma d}{T^{\frac{1}{n}-1}} \left(T^{\frac{1}{n}} - t_1^{\frac{1}{n}} \right) + \frac{d}{T^{\frac{1}{n}-1}} \left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} \right]$$

$$P = \frac{d}{T^{\frac{1}{n}-1}} \left[\gamma T^{\frac{1}{n}} - \gamma t_1^{\frac{1}{n}} + t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} \right] \tag{5.7}$$

The cost of holding inventory occurs at the interval $[0, t_1]$ only; hence, the holding cost during this interval $[0, t_1]$ is obtained as follows:

$$HC = \int_0^{t_1} h(t) Q_1(t) dt$$

$$HC = \int_0^{t_1} (h+\beta t) Q_1 dt$$

$$HC = \int_0^{t_1} (h+\beta t)\left[Me^{-\frac{\theta t^2}{2}} - \frac{de^{-\frac{\theta t^2}{2}}}{nT^{\frac{1}{n}-1}}\left(t^{\frac{1}{n}} + \frac{\theta t^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t^{\frac{1}{n}+4}}{8(4n+1)}\right)\right]dt$$

$$\begin{aligned} HC = \int_0^{t_1} (h+\beta t)\Bigg[& t_1^{\frac{1}{n}} - t^{\frac{1}{n}} - \frac{\theta t^2 t_1^{\frac{1}{n}}}{2} + \frac{\theta^2 t^4 t_1^{\frac{1}{n}}}{8} + \frac{\theta n t^{\frac{1}{n}+2}}{2n+1} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} \\ & - \frac{\theta^2 t^2 t_1^{\frac{1}{n}+2}}{4(2n+1)} - \frac{\theta^2 t^{\frac{1}{n}+4}}{8} + \frac{\theta^2 t^{\frac{1}{n}+4}}{4(2n+1)} - \frac{\theta^2 t^{\frac{1}{n}+4}}{8(4n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}\Bigg] \end{aligned}$$

Upon expansion and some simplifications, we have:

$$\begin{aligned} HC = \frac{d}{T^{\frac{1}{n}-1}}\Bigg[& h t_1^{\frac{1}{n}+1} - \frac{hn t_1^{\frac{1}{n}+1}}{n+1} - \frac{h\theta t_1^{\frac{1}{n}+3}}{6} + \frac{h\theta^2 t_1^{\frac{1}{n}+5}}{40} + \frac{h\theta n^2 t_1^{\frac{1}{n}+3}}{(2n+1)(3n+1)} \\ & + \frac{h\theta t_1^{\frac{1}{n}+3}}{2(2n+1)} - \frac{h\theta^2 t_1^{\frac{1}{n}+5}}{12(2n+1)} - \frac{hn\theta^2 t_1^{\frac{1}{n}+5}}{8(5n+1)} + \frac{hn\theta^2 t_1^{\frac{1}{n}+5}}{4(5n+1)(2n+1)} \\ & - \frac{hn\theta^2 t_1^{\frac{1}{n}+5}}{8(5n+1)(4n+1)} + \frac{h\theta^2 t_1^{\frac{1}{n}+5}}{8(4n+1)} + \frac{\beta t_1^{\frac{1}{n}+2}}{2} - \frac{\beta n t_1^{\frac{1}{n}+2}}{2n+1} - \frac{\beta\theta t_1^{\frac{1}{n}+4}}{8} \\ & + \frac{\beta\theta^2 t_1^{\frac{1}{n}+6}}{48} + \frac{\beta\theta^2 n^2 t_1^{\frac{1}{n}+4}}{(2n+1)(4n+1)} + \frac{\beta\theta t_1^{\frac{1}{n}+4}}{4(2n+1)} - \frac{\beta\theta^2 t_1^{\frac{1}{n}+7}}{16(2n+1)} - \frac{\beta\theta^2 n t_1^{\frac{1}{n}+6}}{8(6n+1)} \\ & + \frac{\beta\theta^2 n t_1^{\frac{1}{n}+6}}{8(6n+1)} + \frac{\beta\theta^2 n t_1^{\frac{1}{n}+6}}{4(2n+1)(6n+1)} - \frac{\beta\theta^2 n t_1^{\frac{1}{n}+6}}{8(4n+1)(6n+1)} + \frac{\beta\theta^2 t_1^{\frac{1}{n}+6}}{16(4n+1)}\Bigg] \end{aligned} \tag{5.8}$$

Purchase cost is obtained thus:

$$PC = Z\left(M + \int_{t_1}^{T} \gamma D(t)\right)dt$$

$$PC = \frac{Zd}{T^{\frac{1}{n}-1}}\left(t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} + \frac{\gamma}{nT^{\frac{1}{n}-1}}\left(\int_{t_1}^{T} t^{\frac{1}{n}-1}dt\right)\right)$$

$$PC = \frac{Zd}{T^{\frac{1}{n}-1}}\left(t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} + \frac{\gamma}{T^{\frac{1}{n}-1}}\left(T^{\frac{1}{n}} - t_1^{\frac{1}{n}}\right)\right)$$

$$PC = \frac{Zd}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} + \gamma\left(T^{\frac{1}{n}} - t_1^{\frac{1}{n}}\right)\right] \tag{5.9}$$

Shortages as a result of stock out is stockpile in the policy during the interval $[t_1,\ T]$. The policy attains the optimum level of shortage at $(t = T)$; hence, the overall shortage cost at this period obtained thus:

$$SC = K \int_{t_1}^{T} -Q_2(t)dt$$

$$SC = K \int_{t_1}^{T} -\frac{\gamma d}{T^{\frac{1}{n}-1}}(t_1^{\frac{1}{n}} - t^{\frac{1}{n}})dt$$

$$SC = -\frac{K\gamma d}{T^{\frac{1}{n}-1}} \int_{t_1}^{T} (t_1^{\frac{1}{n}} - t^{\frac{1}{n}})dt$$

$$SC = \frac{Kd\gamma}{T^{\frac{1}{n}-1}} \left[T t_1^{\frac{1}{n}} - \frac{nT^{\frac{1}{n}+1}}{n+1} - \frac{t_1^{\frac{1}{n}+1}}{n+1} \right] \tag{5.10}$$

As a result of stock out during (t_1, T), Shortage is stockpile, but not all consumers are willing to stand by for the next lot size to emerge. Hence this culminates in some loss of sale, which accounts for the loss in profits.

Lost sale cost is calculated as follows:

$$LSC = S \int_{t1}^{T} (1-\gamma)D(t)dt$$

$$LSC = S \int_{t1}^{T} (1-\gamma)\frac{dt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}}dt$$

$$LSC = \frac{Sd(1-\gamma)}{nT^{\frac{1}{n}-1}} \left[\int_{t_1}^{T} t^{\frac{1}{n}-1} dt \right]$$

$$LSC = \frac{Sd(1-\gamma)}{nT^{\frac{1}{n}-1}} \left[nT^{\frac{1}{n}} - nt^{\frac{1}{n}} \right]$$

$$LSC = \frac{Sd(1-\gamma)}{T^{\frac{1}{n}-1}} \left[T^{\frac{1}{n}} - t^{\frac{1}{n}} \right] \tag{5.11}$$

The total cost for the inventory system is made up of the following cost components

TC = Ordering cost + Holding cost + Purchase cost + Shortage cost + Lost sale cost /T

$$TC = \frac{d}{T^{\frac{1}{n}-1}} \left[ht_1^{\frac{1}{n}+1} - \frac{hnt_1^{\frac{1}{n}+1}}{n+1} - \frac{h\theta t_1^{\frac{1}{n}+3}}{6} + \frac{h\theta^2 t_1^{\frac{1}{n}+5}}{40} + \frac{h\theta n^2 t_1^{\frac{1}{n}+3}}{(2n+1)(3n+1)} \right.$$

$$+\frac{h\theta t_1^{\frac{1}{n}+3}}{2(2n+1)}-\frac{h\theta^2 t_1^{\frac{1}{n}+5}}{12(2n+1)}-\frac{hn\theta^2 t_1^{\frac{1}{n}+5}}{8(5n+1)}+\frac{hn\theta^2 t_1^{\frac{1}{n}+5}}{4(5n+1)(2n+1)}$$

$$-\frac{hn\theta^2 t_1^{\frac{1}{n}+5}}{8(5n+1)(4n+1)}+\frac{h\theta^2 t_1^{\frac{1}{n}+5}}{8(4n+1)}+\frac{\beta t_1^{\frac{1}{n}+2}}{2}-\frac{\beta n t_1^{\frac{1}{n}+2}}{2n+1}-\frac{\beta\theta t_1^{\frac{1}{n}+4}}{8}$$

$$+\frac{\beta\theta^2 t_1^{\frac{1}{n}+6}}{48}+\frac{\beta\theta^2 n^2 t_1^{\frac{1}{n}+4}}{(2n+1)(4n+1)}+\frac{\beta\theta t_1^{\frac{1}{n}+4}}{4(2n+1)}-\frac{\beta\theta^2 t_1^{\frac{1}{n}+7}}{16(2n+1)}-\frac{\beta\theta^2 n t_1^{\frac{1}{n}+6}}{8(6n+1)}$$

$$+\frac{\beta\theta^2 n t_1^{\frac{1}{n}+6}}{8(6n+1)}+\frac{\beta\theta^2 n t_1^{\frac{1}{n}+6}}{4(2n+1)(6n+1)}-\frac{\beta\theta^2 n t_1^{\frac{1}{n}+6}}{8(4n+1)(6n+1)}+\frac{\beta\theta^2 t_1^{\frac{1}{n}+6}}{16(4n+1)}\Bigg]$$

$$+\frac{Zd}{T^{\frac{1}{n}}}\left[t_1^{\frac{1}{n}}+\frac{\theta t_1^{\frac{1}{n}+2}}{2(2n+1)}+\frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)}+\gamma\left(T^{\frac{1}{n}}-t_1^{\frac{1}{n}}\right)\right]$$

$$+\frac{Kd\gamma}{T^{\frac{1}{n}}}\left[Tt_1^{\frac{1}{n}}-\frac{nT^{\frac{1}{n}+1}}{n+1}-\frac{t_1^{\frac{1}{n}+1}}{n+1}\right]$$

$$+\frac{Sd(1-\gamma)}{T^{\frac{1}{n}}}\left[T^{\frac{1}{n}}-t^{\frac{1}{n}}\right]+\frac{A}{T} \tag{5.12}$$

5.4 Solution Method

We propound an approach to evaluate the inventory policy that minimises the total inventory cost per unit time in this section. From Equation (5.12), we find the first partial derivative of $TC(T, t_1)$ concerning the decision variables T and t_1:

We obtain:

$$\frac{\partial TC(t_1,T)}{\partial t_1} \quad and \quad \frac{\partial TC(t_1,T)}{\partial T}$$

To minimise the total cost $TC(t_1, T)$ per unit time, the optimum value of T and t_1 can be procured by solving the equations

$$\frac{\partial TC(t_1,T)}{\partial t_1}=0 \quad and \quad \frac{\partial TC(t_1,T)}{\partial T}=0 \tag{5.13}$$

Provided that Equation (5.12) satisfies the following conditions:

$$\left(\frac{\partial^2 TC(t_1,T)}{\partial t_1^2}\right) \quad > 0 \quad and \quad \left(\frac{\partial^2 TC(t_1,T)}{\partial T^2}\right) \quad > 0 \tag{5.14}$$

$$\left(\frac{\partial^2 TC(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC(t_1,T)}{\partial t_1 \partial T}\right)^2 > 0 \tag{5.15}$$

$$\begin{aligned}
\frac{\partial TC(t_1,T)}{\partial t_1} = \frac{d}{T^{\frac{1}{n}}}\Bigg[& \frac{h(n+1)t^{\frac{1}{n}}}{n} - ht_1^{\frac{1}{n}} - \frac{h\theta(3n+1)t_1^{\frac{1}{n}+2}}{6n} + \frac{h\theta^2(5n+1)t_1^{\frac{1}{n}+4}}{40n} \\
& + \frac{h\theta n t_1^{\frac{1}{n}+2}}{2n+1} + \frac{h\theta(3n+1)t_1^{\frac{1}{n}+2}}{2n(2n+1)} - \frac{h\theta^2(5n+1)t_1^{\frac{1}{n}+4}}{2n(2n+1)} - \frac{h\theta^2 t_1^{\frac{1}{n}+4}}{8} \\
& + \frac{h\theta^2 t_1^{\frac{1}{n}+4}}{4(2n+1)} - \frac{h\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} + \frac{h\theta^2(5n+1)t_1^{\frac{1}{n}+4}}{8n(4n+1)} + \frac{\beta(2n+1)t_1^{\frac{1}{n}+1}}{2n} - \beta t_1^{\frac{1}{n}+1} \\
& - \frac{\beta\theta(4n+1)t_1^{\frac{1}{n}+3}}{8n} + \frac{\beta\theta^2(6n+1)t_1^{\frac{1}{n}+5}}{48n} + \frac{\beta\theta^2 n t_1^{\frac{1}{n}+3}}{(2n+1)} + \frac{\beta\theta(4n+1)t_1^{\frac{1}{n}+3}}{4n(2n+1)} \\
& - \frac{\beta\theta^2(7n+1)t_1^{\frac{1}{n}+6}}{16n(2n+1)} - \frac{\beta\theta^2 t_1^{\frac{1}{n}+5}}{8} + \frac{\beta\theta^2 t_1^{\frac{1}{n}+5}}{4(2n+1)} - \frac{\beta\theta^2 t_1^{\frac{1}{n}+5}}{8(4n+1)} + \frac{\beta\theta^2(6n+1)t_1^{\frac{1}{n}+5}}{16n(4n+1)}\Bigg] \\
& \frac{Zd}{T^{\frac{1}{n}}}\left[\frac{t_1^{\frac{1}{n}-1}}{n} + \frac{\theta t_1^{\frac{1}{n}+1}}{2n} + \frac{\theta^2 t_1^{\frac{1}{n}+3}}{8n} - \frac{\gamma t_1^{\frac{1}{n}-1}}{n}\right] - \frac{K\gamma d}{T^{\frac{1}{n}}}\left[\frac{T t_1^{\frac{1}{n}-1}}{n} - \frac{t_1^{\frac{1}{n}}}{n}\right] \\
& - \frac{Sd(1-\gamma)}{T^{\frac{1}{n}}}\left(\frac{t_1^{\frac{1}{n}-1}}{n}\right) + \frac{A}{T} = 0
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
\frac{\partial TC(t_1,T)}{\partial T} = \frac{-A}{T^2} - \frac{d}{nT^{\frac{1}{n}+1}}\Bigg[& ht^{\frac{1}{n}+1} - \frac{hnt_1^{\frac{1}{n}+1}}{n+1} + \frac{h\theta t_1^{\frac{1}{n}+3}}{6} + \frac{h\theta^2 t_1^{\frac{1}{n}+5}}{40} + \frac{hn^2\theta t_1^{\frac{1}{n}+3}}{(2n+1)(3n+1)} \\
& + \frac{h\theta t_1^{\frac{1}{n}+3}}{4n+2} - \frac{h\theta^2 t_1^{\frac{1}{n}+5}}{4n+2} - \frac{hn\theta^2 t_1^{\frac{1}{n}+5}}{4n+8} + \frac{hn\theta^2 t_1^{\frac{1}{n}+5}}{4(2n+1)(5n+1)} - \frac{hn\theta^2 t_1^{\frac{1}{n}+5}}{8(4n+1)(5n+1)} \\
& + \frac{h\theta^2 t_1^{\frac{1}{n}+5}}{8(4n+1)} + \frac{\beta t_1^{\frac{1}{n}+2}}{2} - \frac{\beta n t_1^{\frac{1}{n}+2}}{2n+1} - \frac{\beta\theta t_1^{\frac{1}{n}+4}}{8} + \frac{\beta\theta^2 t_1^{\frac{1}{n}+6}}{48} + \frac{\beta\theta^2 n^2 t_1^{\frac{1}{n}+4}}{(2n+1)(4n+1)} \\
& + \frac{\beta\theta t_1^{\frac{1}{n}+4}}{8n+4} - \frac{\beta\theta^2 t_1^{\frac{1}{n}+7}}{32n+16} - \frac{\beta\theta^2 n t_1^{\frac{1}{n}+6}}{48n+8} + \frac{\beta n\theta^2 t_1^{\frac{1}{n}+2}}{4(2n+1)(6n+1)} + \frac{\beta\theta^2 t_1^{\frac{1}{n}+6}}{64n+16} \\
& - \frac{\beta\theta^2 n t_1^{\frac{1}{n}+2}}{8(4n+1)(6n+1)}\Bigg] - \frac{ZD}{nT^{\frac{1}{n}+1}}\left[t_1^{\frac{1}{n}} + \frac{\theta t_1^{\frac{1}{n}+2}}{4n+2} + \frac{\theta^2 t_1^{\frac{1}{n}+4}}{8(4n+1)} + \gamma T^{\frac{1}{n}} - \gamma t^{\frac{1}{n}}\right]
\end{aligned}$$

$$+\frac{Zd\gamma}{nT}+\frac{K\gamma d}{nT^{\frac{1}{n}+1}}\left[Tt_1^{\frac{1}{n}}-\frac{nT^{\frac{1}{n}+1}}{n+1}-\frac{t_1^{\frac{1}{n}+1}}{n+1}\right]-\frac{Kd\gamma(t_1^{\frac{1}{n}}-T^{\frac{1}{n}})}{T^{\frac{1}{n}}}$$

$$+\frac{Sd(1-\gamma)}{nT}-\frac{Sd(1-\gamma)(T^{\frac{1}{n}}-t_1^{\frac{1}{n}})}{nT^{\frac{1}{n}+1}} \tag{5.17}$$

Equations (5.16) and (5.17) are highly non-linear; the values of t_1 and T are solved for the optimal values to obtain minimum total inventory cost per unit time. Maple software 2018 and Excel was utilised to get the values of the decision variables.

5.5 Numerical Examples

Here, we give an example to demonstrate the results derived from the linear deteriorating inventory policy for products with power demand pattern and variable holding coat considering shortages

Example 5.51

The subsequent parametric values are considered for the inventory policy in their respective units

$A = 500$, $d = 100$, $h = 0.4$ units, $\beta = 15$, $K = \$10$ per units, $S = \$8$ per units, $Z = \$12$ per units, $\theta = 0.8$, $n = 0.5$, $\gamma = 0.6$

Solving Equation (5.16) and Equation (5.17), The optimum value of $T = 1.671$ and $t_1 = 0.593$

Make use of these values of t_1 and T, the second derivatives can be found. Hence $\frac{\partial^2 TC(t_1,T)}{\partial t_1^2} = 977.479 > 0$ and $\frac{\partial^2 TC(t_1,T)}{\partial T^2} = 294.161 > 0$,

$\frac{\partial^2 TC(t_1,T)}{\partial T \partial t_1} = -254.961$. Therefore from Equation (5.15), we have:

$\frac{\partial^2 TC(t_1,T)}{\partial t_1^2} * \frac{\partial^2 TC(t_1,T)}{\partial T^2} - \left(\frac{\partial^2 TC(t_1,T)}{\partial T \partial t_1}\right)^2 = 222531.2104$. T and t_1 minimises the total inventory average cost since they both satisfies the necessary and sufficient condition

Equation (5.14) and Equation (5.15).

When the values of T and t_1 are substituted into Equation (5.12), the total cost $TC(t_1, T) = 1627.69$. and $M = 22.6001,\ P = 110.210\ N = 87.610$

To further establish that the solution is correct, the total cost function is plotted against some values of t_1 and $T,$ which give us a strictly convex graph as shown in Figures (5.2, 5.3).

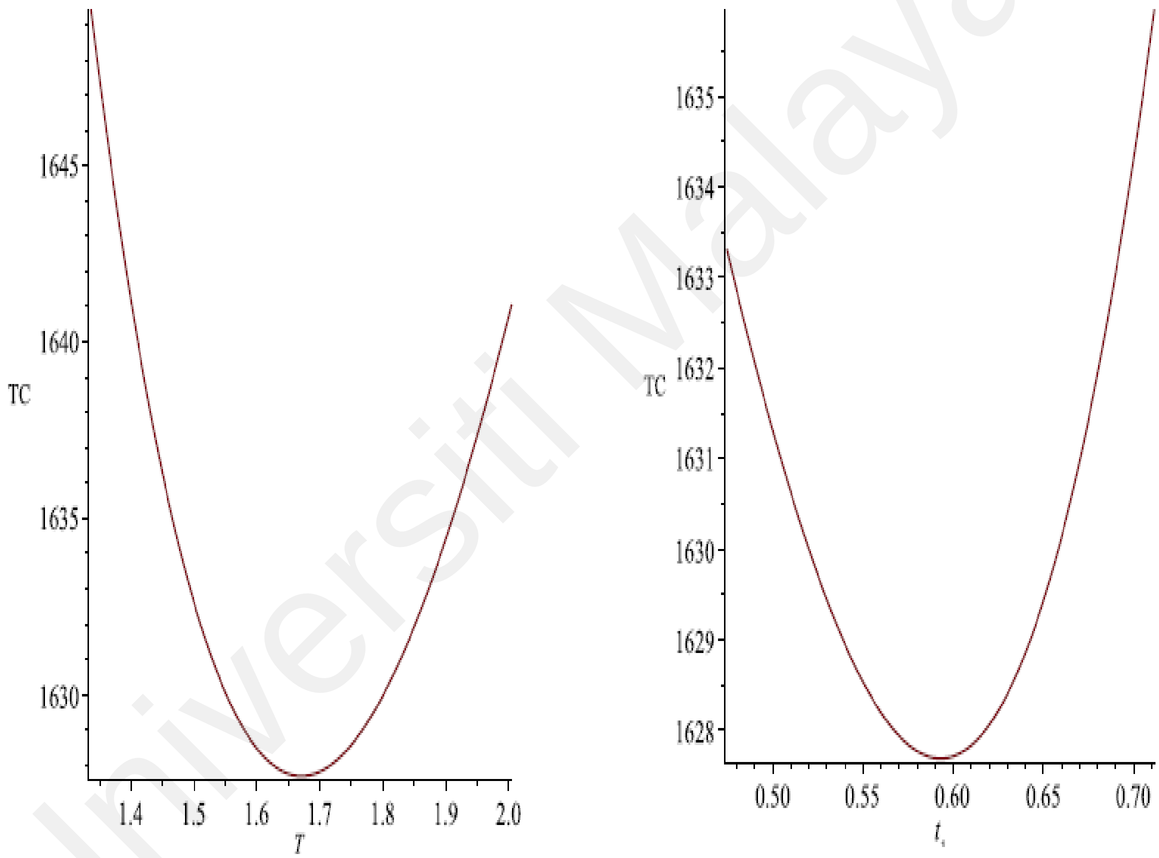


Figure 5.2: Graphical representation of convexity of total cost against schedule and cycle time (a) Total cost and t_1, (b) Total cost and T.

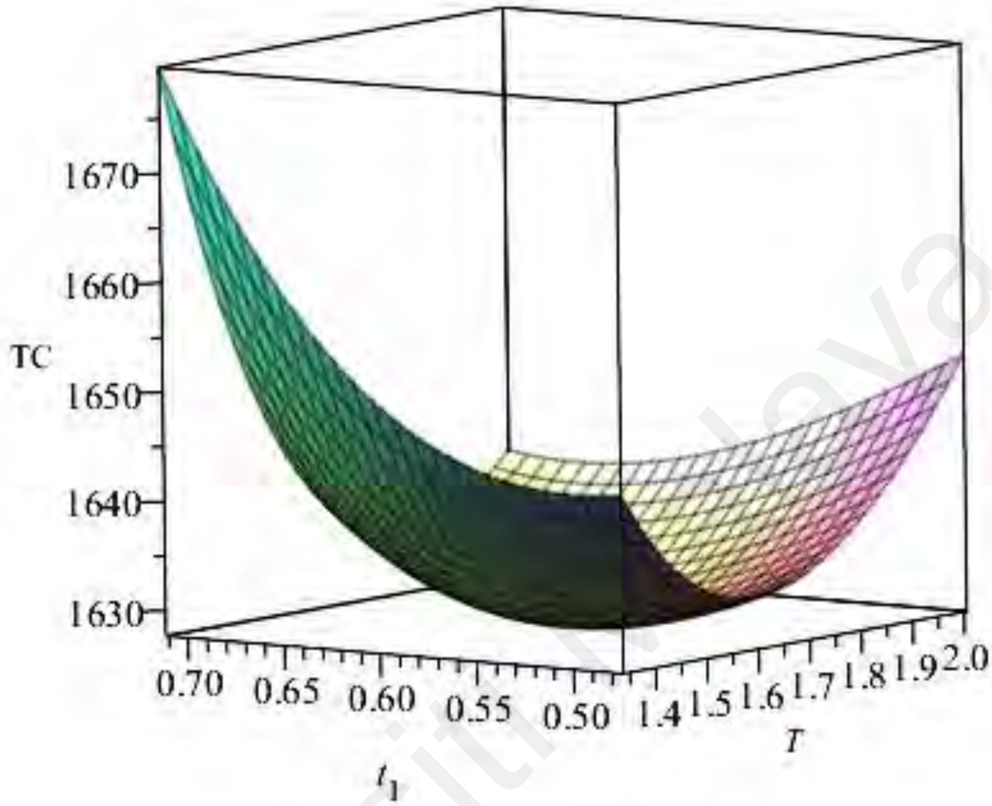


Figure 5.3: Graphical representation of convexity of total cost per unit time (Total cost and $(\boldsymbol{T}, \boldsymbol{t_1})$).

It is evident from Figures (5.2 and 5.3), that the total cost function is strictly convex, showing us that the optimal value of t_1 and T can be derived with the aid of the total cost function of the policy as long as the total inventory cost per unit time is the minimum.

Table 5.1: Sensitivity Analysis of the Parameters in the Inventory model.

			Change in:						
P*	V*	C*	T	T*	t_1	$TC(t_{1,T})$	U*	P	G*
	120	20	1.535	-8.107	0.557	1890.85	16.168	121.791	10.509
	110	10	1.598	-4.328	0.574	1759.87	8.121	116.116	5.360
d	100	0	1.671	0	0.593	1627.69	0	110.209	0
	90	-10	1.755	5.028	0.615	1494.11	-8.206	104.038	-5.599
	80	-20	1.854	10.967	0.639	1358.89	-16.514	97.559	-11.478
	600	20	1.817	8.781	0.630	1685.00	3.521	119.613	8.532
	550	10	1.746	4.537	0.613	1656.95	1.798	115.072	4.412
	500	0	1.671	0	0.593	1627.69	0	110.21	0
A	450	-10	1.591	-4.773	0.572	1597.03	-1.884	105.082	-4.653
	400	-20	1.507	9.823	0.549	1564.75	-3.867	99.64	-9.591
	12	20	1.545	-7.532	0.609	1674.79	2.894	104.155	-5.486
	11	10	1.603	-4.042	0.601	1650.51	1.402	106.927	-2.971
	10	0	1.671	0	0.593	1627.69	0	110.201	0
K	9	-10	1.750	4.745	0.583	1602.87	-1.525	114.152	3.585
	8	-20	1.845	10.408	0.572	1575.71	-3.193	118.967	7.95
	9.6	20	1.658	-0.774	0.617	1683.23	3.412	190.489	0.253
	8.8	10	1.665	-0.359	0.605	1655.56	1.712	110.373	0.148
S	8.0	0	1.671	0	0.593	1627.69	0	110.21	0
	7.2	-10	1.676	0.305	0.580	1599.62	-1.724	109.998	-0.192
	6.4	-20	1.680	0.557	0.567	1571.37	-3.460	109.74	-0.426
	0.48	20	1.670	-0.014	0.591	1628.10	0.025	110.112	-0.089
	0.44	10	1.671	-0.007	0.592	1627.90	0.013	110.16	-0.045
h	0.40	0	1.671	0	0.593	1627.69	0	110.21	0
	0.36	-10	1.671	0.007	0.594	1627.48	-0.013	110.259	0.044
	0.32	-20	1.671	0.014	0.595	1611.04	-1.023	110.309	0.001
	18	20	1.662	-0.490	0.568	1630.88	0.196	108.821	-1.260
	16.5	10	1.666	-0.257	0.580	1629.35	0.102	109.481	-0.661
β	15	0	1.671	0	0.593	1627.69	0	110.210	0
	13.5	-10	1.675	0.285	0.607	1625.88	-0.111	111.021	0.736
	12	-20	1.681	0.604	0.622	1623.89	-0.233	111.929	1.560
	14.4	20	1.678	0.429	0.538	1784.49	9.633	108.613	-1.449
	13.2	10	1.675	0.261	0.566	1706.46	4.839	109.416	-0.720
Z	12	0	1.671	0	0.593	1627.69	0	110.210	0
	10.8	-10	1.664	-0.392	0.620	1548.11	-4.889	110.965	0.685
	9.6	-20	1.655	-0.926	0.647	1467.62	-9.834	111.683	1.337
	0.96	20	1.663	-0.459	0.572	1630.16	0.152	108.284	-0.840
	0.88	10	1.667	-0.238	0.582	1628.96	0.078	108.734	-0.432
θ	0.8	0	1.671	0	0.593	1627.69	0	110.21	0
	0.72	-10	1.675	0.256	0.604	1626.34	-0.083	110.712	0.455
	0.64	-20	1.680	0.533	0.616	1624.91	-0.171	111.243	0.937

Table 5.1, continued.

			Change in:						
P*	V*	C*	T	T*	t_1	$TV(t_1, T)$	U*	P	G*
γ	0.72	20	1.535	-8.10	0.625	1711.88	5.172	119.77	8.674
	0.66	10	1.599	-4.310	00.610	1671.07	2.665	115.253	4.576
	0.60	0	1.671	0	0.593	1627.69	0	110.210	0
	0.54	-10	1.754	4.972	0.573	1581.48	-2.839	104.603	-5.086
	0.48	-20	1.851	10.812	0.550	1532.13	-5.871	98.389	-10.726
n	0.60	20	0.987	-40.948	0.384	1728.46	6.181	67.926	-38.367
	0.55	10	1.647	-1.430	0.587	1636.77	0.558	110.634	0.385
	0.50	0	1.671	0	0.593	1627.69	0	110.210	0
	0.45	-10	1.703	1.955	0.601	1616.03	-0.716	110.288	0.071
	0.40	-20	1.751	4.812	0.613	1601.09	-1.635	111.262	0.883

Note: P = Parameter, V=Values, C = %Change, U*= %change in TC, G* = %change in P.

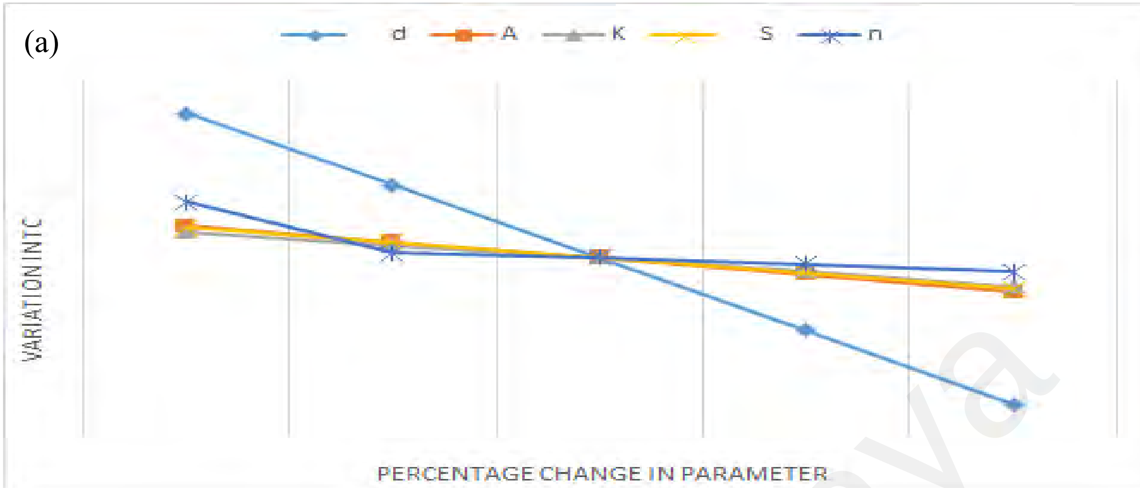
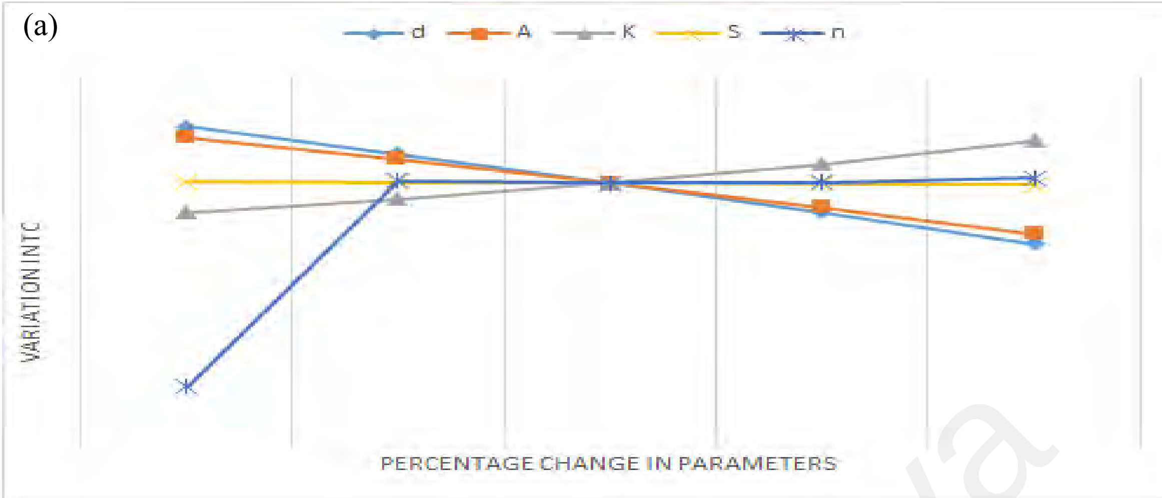


Figure 5.4: Graphical representation of sensitivity analysis of total cost against percentage change in input parameters.



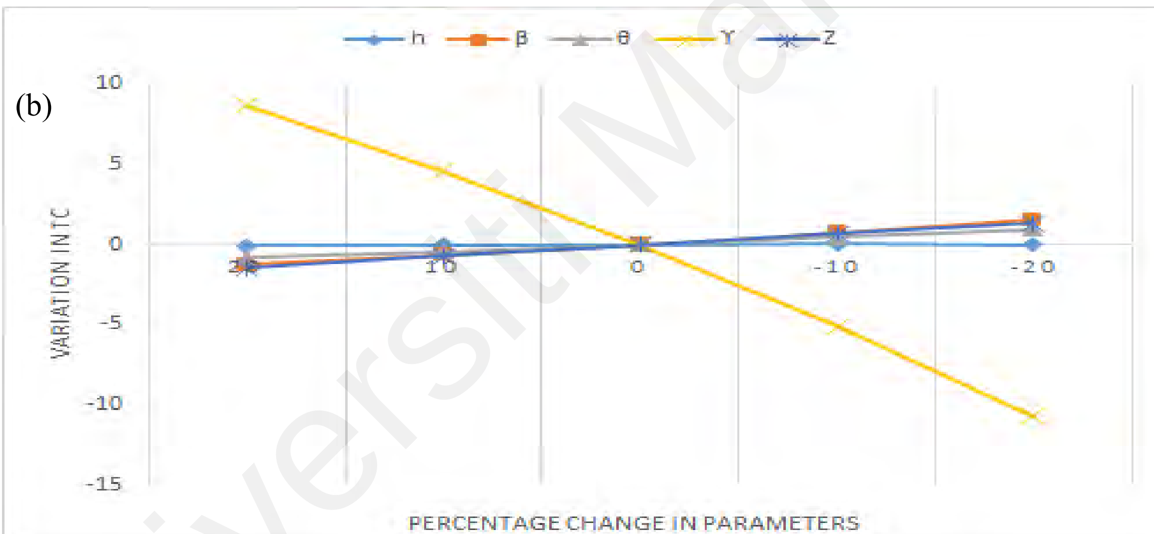


Figure 5.5: Graphical representation of sensitivity analysis of ordering quantity against percentage change in input parameters.

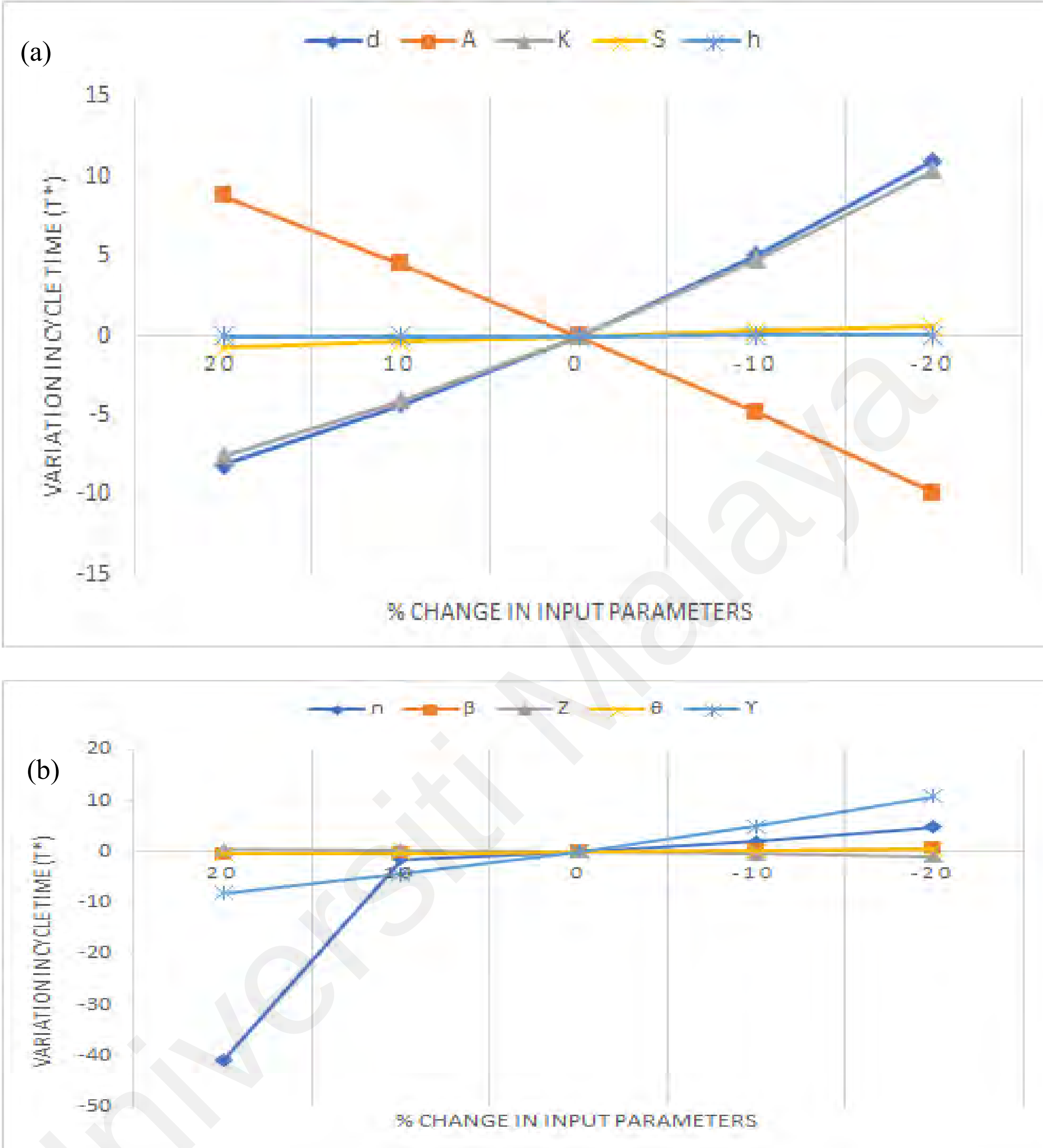


Figure 5.6: Graphical representation of sensitivity analysis of cycle time against percentage change in input parameters.

5.6 Results and Discussion

Based on the above computational results, in the example 5.51 and displayed in Table 5.1, the following remarks can be drawn up:

1. A rise in the value of demand rate d produces a decrease in the schedule period T, but there is an increase in the inventory total cost TC and ordering quantity P.
2. It is observed that an increase in ordering cost A leads to a rise in the schedule period, inventory total cost and ordering cost.
3. An increase in the shortage cost K results in a decrease in schedule period and order quantity, but there is an increase in the total inventory cost.
4. Increase in the lost sale S leads to a decrease in the schedule period; the total inventory cost and order quantity increases.
5. Increase in the value of parameter β and holding cost h results to decrease in the schedule period and ordering quantity, but the inventory total cost increases.
6. Increase in the purchasing cost Z leads to increase in the schedule period and total inventory cost; however, there is a decrease in the ordering quantity.
7. Increase in the value of deterioration parameter θ result in a decrease in schedule period and order quantity, but the inventory total cost is increasing.
8. When the backlogging rate γ is growing, there is a rise in the inventory total cost and ordering cost, which results in a decrease in the scheduled period.
9. Finally, the growth of the value of index number n of the power demand pattern results in the rise in the schedule period and inventory total cost; however, there is a decrease in the ordering quantity.

Economic implication of the above results is stated thus:

1. Increase in demand rate d results in an increase in the total cost TC, ordering quantity

P, lower cycle time t_1, and T. Implication of this is that an increase in demand rate will lead to a decrease in the optimal cycle, but results in the higher value of optimal total cost per unit time. This is normal because if the demand rate is taller, the stock will be used up quickly, and the cycle time will decrease.

2. Increase in the value of the deterioration rate θ result in lower value of cycle length T, smaller ordering quantity and increase in the value optimal total cost. The implication of this is that an increase in the deterioration rate will lead to a decrease in the optimal cycle length. The total cost per unit time will increase because when deterioration cost increases, there will be an increase in the total inventory cost per unit time, which will lead to stocks getting finish earlier as a result of lower cycle length.

3. Increase in the values of holding cost h and β lead to an increase in the amount of total cost and a decrease in the value of ordering quantity with lower cycle time and length. This is advantageous to retailers in that when the holding cost is kept at a minimum, the volume of inventory ordering quantity must be reduced and the time for the stock to be used up must also be reduced to minimise the total inventory cost.

5.7 Sensitivity Analysis

Generally, models are formulated to choose some future direction of action. Consequently, the parameters employed would be based on the prediction of future conditions, which unavoidably introduces some element of uncertainty. Based on the above reason, it is always essential to conduct a sensitivity analysis after finding the solution to the model with the assumed values of the chosen parameters. The main reason for this is to identify those parameters that cannot be altered much without changing the optimal solution, we then select a solution which remains a good one over the intervals of possible values of the sensitive parameters. Here, based on the example 5.51, the sensitivity analysis of the decision variables

T^*, the schedule period, G^* ordering quantity, and inventory total cost U^* against changes in the parameters $d, A, K, S, h,\ beta, Z, \theta, \gamma$ and n of the inventory policy is analysed. By varying the values from $+20\%, +10\%, -10\%, -20\%$. One parameter is considered at a time while leaving the other parameters constant.

The results are displayed in Table 5.1 and graphically represented by Figures (5.4-5.6), The main observations of the results and the graphical representations of the sensitivity analysis concerning the parameters are as follows:

1. 20% overestimation in the value of the demand rate d result in increases of U^* and G^* by 16% and 11% respectively, but the decrease in T^* by 8%. 20% underestimation in the demand rate d results in reductions of U^* and G^* by 17% and 11% respectively, but increase in T^* by 11%. Thus U^*, G^* and T^* are moderately sensitive to changes in the values of the parameter d.
2. 20% overestimation in the amount of the ordering cost A leads to increases of U^*, G^* and T^* by 4%, 9% and 9% respectively. On the other hand, 20% underestimation in the demand rate A results in decreases of U^*, G^* and T^* by 4%, 10% and 10% respectively. Thus, G^* and T^* are moderately sensitive, and U^* is lowly sensitive to changes in the values of the parameter A.
3. 20% overestimation in the amount of the shortage cost K leads to increases of U^* by 3% while G^* and T^* decreases by 5% and 8% respectively. However, 20% underestimation in K leads to increases in G^* and T^* by 8% and 10% respectively, while U^* decrease by 3%. Thus, T^* and G^* are moderately sensitive to changes in both overestimation and underestimation, while U^* is less susceptible to changes in the value of parameter K.
4. 20% overestimation in the value of the power pattern index n leads to increases of U^*

by 6% and decreases in G^* and T^* by 41% and 38% respectively. On the other hand, 20% underestimation in n results in increases of G^* and T^* by 1% and 5% respectively while U^* decrease by 2%. Thus, G^* and T^* are highly sensitive to overestimation and U^* is lowly sensitive to changes in overestimation; meanwhile, T^* is less sensitive in underestimation and G^* and T^* are less sensitive in underestimation to changes in the values of the parameter n.

5. 20% overestimation in the amount of the backlogging rate γ leads to increases of U^* and G^* by 5% and 9% respectively, while T^* decrease by 8% . For 20% underestimation in γ results in increases of T^* by 11% and reductions of U^* and G^* by 6% and 11% respectively, Thus T^* is relatively sensitive, G^* is moderately susceptible, and U^* is lowly sensitive to changes in the values of the parameter γ.

6. 20% overestimation in the value of the purchase cost Z leads to an increase in U^* by 10% and decrease in G^* by 1%. Also, 20% underestimation in Z results in decline of U^* by 10% and increase of G^* by 1%. Thus U^* is moderately sensitive, G^* is less sensitive, and T^* is insensitive to changes in the values of the parameter Z.

7. 20% overestimation in the value of the parameter β leads to decrease in G^* by 1%, and 20% underestimation leads to a 2% increase in G^*. Therefore, G^* is less sensitive, and U^* and T^* are insensitive to changes in the value of parameter β.

8. 20% of both overestimation and underestimation in the value of the unit lost sale cost S in U^* are 3% and 3%. Thus U^* is less sensitive, G^* and T^* are insensitive to changes in the value of parameter S.

9. All the decision variables G^*, U^* and T^* are insensitive to both overestimation and underestimation in the parameters h and θ.

5.8 Concluding Observations

In this chapter, a linear deteriorating inventory policy for products with variable holding cost and demand presumed to be in the form of power demand pattern is proposed. The model is an extension of Adaraniwon & Omar (2019) when $t_d = 0$ with the addition of time-dependent deterioration rate and variable holding cost. Shortages are allowed and partially backlogged which captures real-life situation since some retailers will be willing to wait for the arrival of new stock during stock-out patiently, but the longer the waiting time, the possibility of the consumers looking for elsewhere to meet their demand.

The objective of this model is to evaluate the optimal replenishment procedure that minimises the average inventory total cost per unit time. If the deterioration rate were to be constant, the model would be reduced to that of Mishra (2016b) without power demand pattern. Optimum order quantity and optimal replenishment cycle time were derived, and the solution obtained. The outcomes are further established with the aid of numerical example and, sensitivity analysis carried out and depicted graphically of the decision variables with regards to alterations in the input parameters in the model.

The results obtained indicate that the effect of power demand index parameter n on the average minimum cost is quite significant. On thorough examining the influence of the policy input parameters on the decision variables, it was found out that U^* is sensitive to overestimation and underestimation of the parameters d and Z while G^* is susceptible to the overestimation and underestimation of the parameters d, A, K, n, γ. Also, U^* is less sensitive or insensitive to overestimation and underestimation of the parameters A, K, n, β, S, h and θ while G^* is less sensitive or insensitive of the parameters n, Z, β, S, h and θ.

The developed model can be extended further by adding a quantity discount, trade credit, stochastic demand rate, finite replenishment, and so on.

CHAPTER 6: AN INVENTORY MODEL FOR DELAYE DETERIORATING ITEMS WITH POWER DEMAND CONSIDERING SHORTAGES AND LOST SALES

6.1 Introduction

In the previous Chapter 5, an inventory policy for items with linear time-dependent deteriorating rate, variable holding cost and power demand pattern with shortages which are backlogged partially was developed and discussed. In daily business world, shortages do occur, during this period some customers are impatient to wait for backorder and therefore would go to another seller to buy or go for alternative items, while some will be willing to wait for backorder. In the inventory model with dependent stock demand, some researchers assumed shortages should be completely backlogged while some believe that shortages should be partially backlogged.

As a result of extremely competitive market, delivering varieties of items to the consumers owing to globalization, partial backorder is more feasible than complete backorder. Example can be found in high technology items and fashionable commodities with short product life span. The readiness of a consumers to tarry for long for backlogging during the shortage time reduces with the hold-back time. Through the stock-out period, the backorder rate is generally regarded as a non-increasing linear function of backorder replenishment lead time through the amount of shortages. The bigger the expected shortage quantity is, the smaller the backorder rate would be. the left-over fraction of the shortage is lost. This type of backlogging is refers to as time-dependent partial backlogging.

In this chapter, an inventory model for delayed deteriorating items with power demand considering shortages and lost sales is considered. We assumed shortages is partially backorder and remaining is lost. Here, we developed a new model and expanded the work of

Sicilia et al. (2014) by incorporating lost sale.

6.2 Modelling Assumptions and Notation

The following assumptions are used:

1. The inventory system involves a single item.
2. Deterioration takes place after the life span of the items.
3. There is no replenishment or repair taking place for any deteriorating items.
4. The replenishment takes place at a tremendous rate with zero lead time.
5. The demand rate, $D(t)$ at any time t is $D(t) = \frac{dt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}}$ where d means the average demand, n is the pattern index with $0 < n < \infty$.
6. Shortages are accepted with the backlogging rate is depending on the length of the waiting time for the succeeding replenishment. The negative inventory of the backlogging rate is given by $B(t) = \frac{1}{1+\gamma(T-t)}$, where γ is a backlogging parameter $0 \leq t \leq T$ and the waiting time is $(T-t), (t_d \leq t \leq T)$. The unresolved fraction $1 - B(t)$ is considered as lost sales.

The subsequent notations are used in the model

- A is the ordering cost.
- α is the deteriorating rate, $(0 < \alpha < 1)$.
- K_1 is the holding cost.
- K_2 is the deteriorating cost per unit per year.
- K_3 is the shortage cost for backlogged items per unit per year.
- K_4 is the cost of lost sale per unit.
- T is the optimum cycle length.

- t_d is the length of time when the items experience no deterioration.
- t_1 is the length of time when the inventory has no shortage.
- Q is the quantity ordered during a cycle of length T.
- S is the maximum inventory level during $[0, T]$.
- P is the maximum backordered unit during the stock out period.
- $Q_1(t)$ is the level of positive inventory at time t where $0 \leq t \leq t_d$ when there is no deterioration.
- $Q_2(t)$ is the level of positive inventory at time t where $t_d \leq t \leq t_1$ when there is deterioration.
- $Q_3(t)$ is the level of negative inventory at time t where $t_d \leq t \leq T$.
- $\psi(t_1, T)$ is the total cost per unit per time.

6.3 Mathematical Formulation

In this propounded model, a cycle can be separated into three periods. In this diagram, $[0, T]$ is considered as a single cycle duration. During $[0,\ t_d]$ and $[t_d,\ t_1]$ the inventory is on the positive side and $[t_1,\ T]$ is on the negative side of the inventory. The inventory system for the model is given in Figure 6.1. In the beginning, a lot size of Q units enter the system at the beginning of each cycle, where $Q = P + S$. The deterioration will take place after time t_d and reach zero inventory level in time t_1. The shortages occur in the interval $[t_1,\ T]$ and there are partially backlogged and lost sales at the end of cycle time.

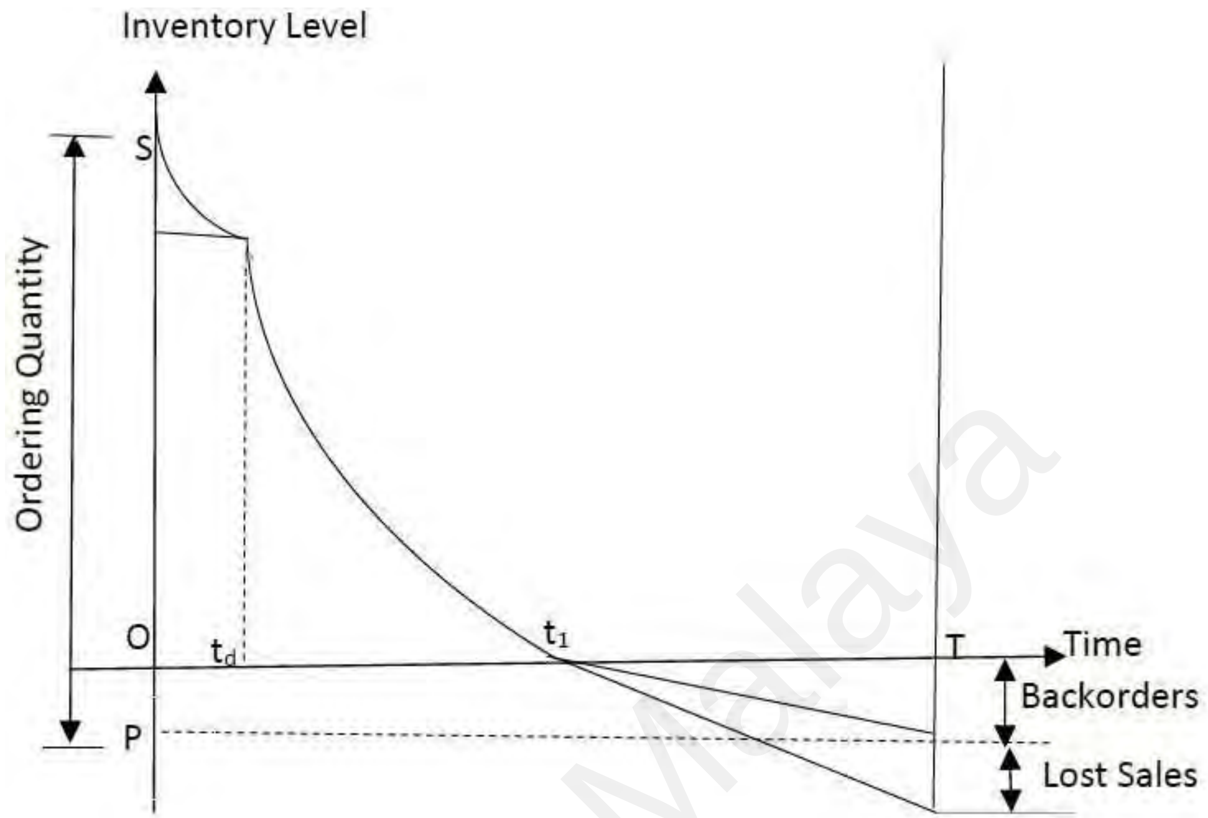


Figure 6.1: Graphical Illustration of Inventory Model.

The following differential equations can describe the inventory system in Figure 6.1,

$$\frac{dI_1(t)}{dt} = -\frac{dt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}}, \quad 0 \le t \le t_d. \tag{6.1}$$

$$\frac{dI_2(t)}{dt} + \alpha I_2(t) = -\frac{dt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}}, \quad t_d \le t \le t_1. \tag{6.2}$$

$$\frac{dI_3(t)}{dt} = \frac{-B(t)}{1+\gamma(T-t)}, \quad t_1 \le t \le T. \tag{6.3}$$

With the boundary conditions $I_1(0) = S, I_2(t_1) = 0, and\ I_3(t_1) = 0$.

Solving the above differential equations, then we have

$$I_1(t) = S - \frac{d}{T^{\frac{1}{n}-1}}[t^{\frac{1}{n}}] \quad 0 \le t \le t_d. \tag{6.4}$$

From Equation (6.2), we have:

$$I_2(t)e^{\alpha t} = -\int \frac{dt^{\frac{1}{n}-1}}{nT^{\frac{1}{n}-1}} e^{\alpha t}dt = -\frac{d}{nT^{\frac{1}{n}-1}}\int e^{\alpha t}t^{\frac{1}{n}-1}dt.$$

We take the first three terms of the exponential function and disregarding the higher terms since it becomes minimal. It follows that:

$$I_2(t)e^{\alpha t} = -\frac{d}{T^{\frac{1}{n}-1}}\left[\int (1+\alpha t+\frac{\alpha^2t^2}{2!})t^{\frac{1}{n}-1}dt\right]$$

$$= -\frac{d}{T^{\frac{1}{n}-1}}\left[\int t^{\frac{1}{n}-1}dt + \alpha\int t^{\frac{1}{n}}dt + \frac{\alpha^2}{2}\int t^{\frac{1}{n}+1}dt\right]$$

$$= -\frac{d}{T^{\frac{1}{n}-1}}\left[t^{\frac{1}{n}} + \frac{\alpha t^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 t^{\frac{1}{n}+2}}{2(2n+1)}\right] + C$$

With the boundary condition $I_2(t_1) = 0$, we have

$$C = \frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} + \frac{\alpha t_1^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)}\right]$$

$$I_2(t) = \frac{de^{-\alpha t}}{T^{\frac{1}{n}-1}}\left[(t_1^{\frac{1}{n}} - t^{\frac{1}{n}}) + \frac{\alpha}{n+1}(t_1^{\frac{1}{n}+1} - t^{\frac{1}{n}+1}) + \frac{\alpha^2}{2(2n+2)}(t_1^{\frac{1}{n}+2} - t^{\frac{1}{n}+2})\right]$$

Expanding further and considering the first three terms, then

$$I_2(t) = \frac{d}{T^{\frac{1}{n}-1}}\Bigg[(1-\alpha t+\frac{\alpha^2t^2}{2})[t_1^{\frac{1}{n}} - t^{\frac{1}{n}}] + \frac{\alpha}{n+1}(1-\alpha t+\frac{\alpha^2t^2}{2})[t_1^{\frac{1}{n}+1} - t^{\frac{1}{n}+1}]$$

$$+\frac{\alpha^2}{2(2n+1)}(1-\alpha t+\frac{\alpha^2t^2}{2})[t_1^{\frac{1}{n}+2} - t^{\frac{1}{n}+2}]\Bigg]$$

$$I_2(t) = \frac{d}{T^{\frac{1}{n}-1}}\Bigg[t_1^{\frac{1}{n}} - t^{\frac{1}{n}} + \frac{n\alpha t^{\frac{1}{n}+1}}{n+1} - \alpha t t_1^{\frac{1}{n}} + \frac{\alpha^2t^2t_1^{\frac{1}{n}}}{2} - \frac{\alpha^2 t^{\frac{1}{n}+2}}{2} + \frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}$$

$$+\frac{\alpha^2 t^{\frac{1}{n}+2}}{n+1} - \frac{\alpha^2 t t_1^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)} - \frac{\alpha^2 t^{\frac{1}{n}+2}}{2(2n+1)}\Bigg] \qquad (6.5)$$

At time t_d, from Figure (6.1), we have $I_1(t_d) = I_2(t_d)$, then

$$S - \frac{d}{T^{\frac{1}{n}}}\left[t_d^{\frac{1}{n}}\right] = \frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} - t_d^{\frac{1}{n}} + \frac{n\alpha t_d^{\frac{1}{n}+1}}{n+1} - \alpha t_d t_1^{\frac{1}{n}} + \frac{\alpha^2 t_d^2 t_1^{\frac{1}{n}}}{2} - \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2} + \frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right.$$

$$\left. + \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{n+1} - \frac{\alpha^2 t_d t_1^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)} - \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2(2n+1)}\right]$$

$$S = \frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} + \frac{n\alpha t_d^{\frac{1}{n}+1}}{n+1} - \alpha t_d t_1^{\frac{1}{n}} + \frac{\alpha^2 t_d^2 t_1^{\frac{1}{n}}}{2} - \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2} + \frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right.$$

$$\left. + \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{n+1} - \frac{\alpha^2 t_d t_1^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)} - \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2(2n+1)}\right] \tag{6.6}$$

Replacing Equation (6.6) into Equation (6.4)

$$I_1(t) = \frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} - t^{\frac{1}{n}} + \frac{n\alpha t_d^{\frac{1}{n}+1}}{n+1} - \alpha t_d t_1^{\frac{1}{n}} + \frac{\alpha^2 t_d^2 t_1^{\frac{1}{n}}}{2} - \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2} + \frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right.$$

$$\left. + \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{n+1} - \frac{\alpha^2 t_d t_1^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)} - \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2(2n+1)}\right] \tag{6.7}$$

In the time of the shortage interval $[t_1,\ T]$, the demand at the time t is partially backlogged, the solution to Equation (6.3) making use of the boundary condition:

$I_3(t_1) = 0$ results in:

$$I_3(t) = \frac{d}{T^{\frac{1}{n}-1}}\left[\left(t_1^{\frac{1}{n}} - t^{\frac{1}{n}}\right)(1-\gamma T) + \frac{\gamma}{n+1}\left(t_1^{\frac{1}{n}+1} - t^{\frac{1}{n}+1}\right)\right] \tag{6.8}$$

The maximum back-ordered inventory P is obtained when $t = T,$ from Equation (6.8)

$$P = -\frac{d}{T^{\frac{1}{n}-1}}\left[\left(t_1^{\frac{1}{n}} - T^{\frac{1}{n}}\right)(1-\gamma T) + \frac{\gamma}{n+1}\left(t_1^{\frac{1}{n}+1} - T^{\frac{1}{n}+1}\right)\right] \tag{6.9}$$

Finally, from Figure 6.1, we have

$$Q = \frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}} - \alpha t_d t^{\frac{1}{n}} + \frac{n\alpha t_d^{\frac{1}{n}+1}}{n+1} + \frac{\alpha t_1^{\frac{1}{n}+1}}{n+1} - \frac{\alpha^2 t_d t_1^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{n+1}\right]$$

$$-\frac{d}{T^{\frac{1}{n}-1}}\left[(t_1^{\frac{1}{n}}-T^{\frac{1}{n}})(1-\gamma T)+\frac{\gamma}{n+1}(t_1^{\frac{1}{n}+1}-T^{\frac{1}{n}+1})\right] \tag{6.10}$$

The total relevant inventory cost per cycle consist of the following cost components:

1. The ordering cost is A.
2. The inventory holding cost is given by:

$$HC=K_1\left[\int_0^{t_d} I_1(t)dt+\int_{t_d}^{t_1} I_2(t)dt\right]$$

$$=\frac{K_1 d}{T^{\frac{1}{n}-1}}\left[\int_0^{t_d}\left(t_1^{\frac{1}{n}}-t^{\frac{1}{n}}+\frac{n\alpha t_d^{\frac{1}{n}+1}}{n+1}-\alpha t_d t_1^{\frac{1}{n}}+\frac{\alpha^2 t_d^2 t_1^{\frac{1}{n}}}{2}-\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2}+\frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right.\right.$$

$$\left.\left.+\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{n+1}-\frac{\alpha^2 t_d t_1^{\frac{1}{n}+1}}{n+1}+\frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)}-\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2(2n+1)}\right)dt\right]$$

$$+\frac{K_1 d}{T^{\frac{1}{n}-1}}\left[\int_{t_d}^{t_1}\left(t_1^{\frac{1}{n}}-t^{\frac{1}{n}}+\frac{n\alpha t^{\frac{1}{n}+1}}{n+1}-\alpha t t_1^{\frac{1}{n}}+\frac{\alpha^2 t^2 t_1^{\frac{1}{n}}}{2}-\frac{\alpha^2 t^{\frac{1}{n}+2}}{2}+\frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right.\right.$$

$$\left.\left.+\frac{\alpha^2 t^{\frac{1}{n}+2}}{n+1}-\frac{\alpha^2 t t_1^{\frac{1}{n}+1}}{n+1}+\frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)}-\frac{\alpha^2 t^{\frac{1}{n}+2}}{2(2n+1)}\right)dt\right]$$

Upon simplification, we have:

$$HC=\frac{K_1 d}{T^{\frac{1}{n}-1}}\left[\frac{\alpha^2 t_d^3 t_1^{\frac{1}{n}}}{3}-\frac{\alpha t_d^2 t_1^{\frac{1}{n}}}{2}+\frac{t_1^{\frac{1}{n}+1}(2-\alpha^2 t_d^2)}{2(n+1)}+\frac{n\alpha t_d^{\frac{1}{n}+2}}{2n+1}+\frac{\alpha t_1^{\frac{1}{n}+2}}{2(2n+1)}\right.$$

$$\left.+\frac{\alpha^2 t_1^{\frac{1}{n}+3}}{6(3n+1)}-\frac{\alpha^2 n^2 t_d^{\frac{1}{n}+3}}{(n+1)(3n+1)}\right] \tag{6.11}$$

3. The deterioration cost is

$$DC=K_2\left[I_2(t_d)-\int_{t_d}^{t_1} B(t)dt\right]$$

$$=\frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}}-t_d^{\frac{1}{n}}+\frac{n\alpha t_d^{\frac{1}{n}+1}}{n+1}-\alpha t_d t_1^{\frac{1}{n}}+\frac{\alpha^2 t_d^2 t_1^{\frac{1}{n}}}{2}-\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2}+\frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right.$$

$$\left.+\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{n+1}-\frac{\alpha^2 t_d t_1^{\frac{1}{n}+1}}{n+1}+\frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)}-\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2(2n+1)}\right]-\frac{d}{T^{\frac{1}{n}-1}}\left[t_1^{\frac{1}{n}}-t_d^{\frac{1}{n}}\right]$$

$$DC = \frac{d}{T^{\frac{1}{n}-1}}\left[\frac{n\alpha t_d^{\frac{1}{n}+1}}{n+1} - \alpha t_d t_1^{\frac{1}{n}} + \frac{\alpha^2 t_d^2 t_1^{\frac{1}{n}}}{2} - \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2} + \frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right.$$

$$\left.+\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{n+1} - \frac{\alpha^2 t_d t_1^{\frac{1}{n}+1}}{n+1} + \frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)} - \frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2(2n+1)}\right] \quad (6.12)$$

4. Shortage cost per cycle as a result of the backlog is given by:

$$SC = K_3\left[\int_{t_1}^{T} - I_3(t)dt\right]$$

$$= \frac{-K_3 d}{T^{\frac{1}{n}-1}}\left[\int_{t_1}^{T}\left((t_1^{\frac{1}{n}} - t^{\frac{1}{n}})(1-\gamma T) + \frac{\gamma}{n+1}(t_1^{\frac{1}{n}+1} - t^{\frac{1}{n}+1})\right)dt\right]$$

$$= \frac{-K_3 d}{T^{\frac{1}{n}-1}}\left[T t_1^{\frac{1}{n}}(1-\gamma T) - \frac{t_1^{\frac{1}{n}+1}}{n+1}(1-2\gamma T) - \frac{\gamma t_1^{\frac{1}{n}+2}}{2n+1}\right.$$

$$\left.+\frac{nT^{\frac{1}{n}+1}}{n+1} + \frac{2n^2\gamma T^{\frac{1}{n}+2}}{(2n+1)(n+1)}\right] \quad (6.13)$$

5. The lost sale cost during the interval $[0.T]$ is given by:

$$LC = K_4\left[\int_{t_1}^{T}\{1 - \frac{B(t)dt}{1+\gamma(T-t)}\}\right]$$

$$LC = \frac{K_4 d}{T^{\frac{1}{n}-1}}\left[\frac{n\gamma T^{\frac{1}{n}+1}}{n+1} + \frac{\gamma t_1^{\frac{1}{n}+1}}{n+1} - \gamma T t_1^{\frac{1}{n}}\right] \quad (6.14)$$

Finally, the total relevant inventory cost per unit time is given by

$\psi(t_1, T) = \frac{1}{T}$ (ordering cost + holding cost + deteriorating cost +

shortage cost and lost sale cost).

$$\psi(t_1,T) = \frac{A}{T} + \frac{K_1 d}{T^{\frac{1}{n}}}\left[\frac{\alpha^2 t_d^3 t_1^{\frac{1}{n}}}{3} - \frac{\alpha t_d^2 t_1^{\frac{1}{n}}}{2} + \frac{t_1^{\frac{1}{n}+1}(2-\alpha^2 t_d^2)}{2(n+1)} + \frac{n\alpha t_d^{\frac{1}{n}+2}}{2n+1} + \frac{\alpha t_1^{\frac{1}{n}+2}}{2(2n+1)}\right.$$

$$\left.+\frac{\alpha^2 t_1^{\frac{1}{n}+3}}{6(3n+1)} - \frac{\alpha^2 n^2 t_d^{\frac{1}{n}+3}}{(n+1)(3n+1)}\right]$$

$$+\frac{K_2 d}{T^{\frac{1}{n}}}\left[\frac{n\alpha t_d^{\frac{1}{n}+1}}{n+1}-\alpha t_d t_1^{\frac{1}{n}}+\frac{\alpha^2 t_d^2 t_1^{\frac{1}{n}}}{2}-\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2}+\frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right.$$

$$\left.+\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{n+1}-\frac{\alpha^2 t_d t_1^{\frac{1}{n}+1}}{n+1}+\frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)}-\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2(2n+1)}\right]$$

$$+\frac{-K_3 d}{T^{\frac{1}{n}}}\left[T t_1^{\frac{1}{n}}(1-\gamma T)-\frac{t_1^{\frac{1}{n}+1}}{n+1}(1-2\gamma T)-\frac{\gamma t_1^{\frac{1}{n}+2}}{2n+1}\right.$$

$$\left.+\frac{nT^{\frac{1}{n}+1}}{n+1}+\frac{2n^2\gamma T^{\frac{1}{n}+2}}{(2n+1)(n+1)}\right]+\frac{K_4 d}{T^{\frac{1}{n}-1}}\left[\frac{n\gamma T^{\frac{1}{n}+1}}{n+1}+\frac{\gamma t_1^{\frac{1}{n}+1}}{n+1}-\gamma T t_1^{\frac{1}{n}}\right] \tag{6.15}$$

Therefore, we are interested in finding the values of t_1 and T that minimise the function $\psi(t_1,T)$ given in Equation (6.15) in the feasible/attainable region $F(t_1,T){:}\,0 \le t_1 \le T, T > 0$.

6.4 Solution approach

In this section, we find the optimum solution of $(t_1,\ T)$ that minimise the total relevant cost. Take into consideration the partial derivatives of $\psi(t_1,\ T)$ concerning the decision variable t_1 and T such that:

$$\frac{\partial\psi(t_1,T)}{\partial t_1}=0 \qquad and \qquad \frac{\partial\psi(t_1,T)}{\partial T}=0, \tag{6.16}$$

Provided:

$$\left[\frac{\partial^2\psi(t_1,T)}{\partial t_1^2}\right]\left[\frac{\partial^2\psi(t_1,T)}{\partial T^2}\right]-\left[\frac{\partial^2\psi(t_1,T)}{\partial t\partial T}\right]^2>0 \tag{6.17}$$

From Equation (6.16), we get:

$$\frac{\partial\psi(t_1,T)}{\partial t_1}=\frac{K_1 d}{T^{\frac{1}{n}}}\left[\frac{\alpha^2 t_d^3 t_1^{\frac{1}{n}-1}}{3n}-\frac{\alpha t_d^2 t_1^{\frac{1}{n}-1}}{2n}+\frac{t_1^{\frac{1}{n}}(2-\alpha^2 t_d^2)}{2n}+\frac{\alpha t_1^{\frac{1}{n}+1}}{2n}+\frac{\alpha^2 t_1^{\frac{1}{n}+2}}{6n}\right]$$

$$+\frac{K_2 d}{T^{\frac{1}{n}}}\left[\frac{-\alpha t_d t_1^{\frac{1}{n}-1}}{n}+\frac{\alpha^2 t_d^2 t_1^{\frac{1}{n}-1}}{2n}+\frac{\alpha t_1^{\frac{1}{n}}}{n}-\frac{\alpha^2 t_d t_1^{\frac{1}{n}}}{n}+\frac{\alpha^2 t_1^{\frac{1}{n}+1}}{2n}\right]$$

$$-\frac{K_3 d}{T^{\frac{1}{n}}}\left[\frac{Tt_1^{\frac{1}{n}-1}}{n}(1-\gamma T)-\frac{t_1^{\frac{1}{n}}}{n}(1-2\gamma T)-\frac{\gamma t_1^{\frac{1}{n}+1}}{n}\right]$$

$$+\frac{K_4 d}{T^{\frac{1}{n}}}\left[\frac{\gamma t_1^{\frac{1}{n}}}{n}-\frac{\gamma T t_1^{\frac{1}{n}-1}}{n}\right]=0 \tag{6.18}$$

$$\frac{\partial\psi(t_1,T)}{\partial T}=-\frac{A}{T^2}-\frac{K_1 d}{nT^{\frac{1}{n}+1}}\left[\frac{\alpha^2 t_d^3 t_1^{\frac{1}{n}}}{3}-\frac{\alpha t_d^2 t_1^{\frac{1}{n}}}{2}+\frac{t_1^{\frac{1}{n}+1}(2-\alpha^2 t_d^2)}{2(n+1)}+\frac{n\alpha t_d^{\frac{1}{n}+2}}{2n+1}+\frac{\alpha t_1^{\frac{1}{n}+2}}{2(2n+1)}\right.$$

$$\left.+\frac{\alpha^2 t_1^{\frac{1}{n}+3}}{6(3n+1)}-\frac{\alpha^2 n^2 t_d^{\frac{1}{n}+3}}{(n+1)(3n+1)}\right]$$

$$-\frac{K_2 d}{nT^{\frac{1}{n}+1}}\left[\frac{n\alpha t_d^{\frac{1}{n}+1}}{n+1}-\alpha t_d t_1^{\frac{1}{n}}+\frac{\alpha^2 t_d^2 t_1^{\frac{1}{n}}}{2}-\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2}+\frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right.$$

$$\left.+\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{n+1}-\frac{\alpha^2 t_d t_1^{\frac{1}{n}+1}}{n+1}+\frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)}-\frac{\alpha^2 t_d^{\frac{1}{n}+2}}{2(2n+1)}\right]$$

$$+\frac{(1-n)K_3 d t_1^{\frac{1}{n}}}{nT^{\frac{1}{n}}}+\frac{(2-\frac{1}{n})K_3 d\gamma t_1^{\frac{1}{n}}}{T^{\frac{1}{n}-1}}-\frac{K_3 d t_1^{\frac{1}{n}+1}}{n(n+1)T^{\frac{1}{n}+1}}+\frac{K_3 dn}{n+1}+\frac{(1-n)2K_3 d\gamma t_1^{\frac{1}{n}+1}}{(n^2+n)T^{\frac{1}{n}}}$$

$$-\frac{K_3 d\gamma t_1^{\frac{1}{n}+2}}{(2n^2+n)T^{\frac{1}{n}+1}}-\frac{4n^2\gamma T K_3 d}{(2n^2+3n+1)}+\frac{K_4 dn\gamma}{n+1}-\frac{K_4 d\gamma t_1^{\frac{1}{n}+1}}{(n^2+n)T^{\frac{1}{n}+1}}$$

$$+\frac{(1-n)K_4 d\gamma t_1^{\frac{1}{n}}}{nT^{\frac{1}{n}}}=0 \tag{6.19}$$

To obtain an optimal value that minimises the total cost per unit time, we solved the Equation (6.18) and Equation (6.19) simultaneously. It could be observed that the equations are non-linear. Here we use excel solver to find the optimum solution and check for convexity using Equation (6.17).

We can see from our model that, if $t_d = 0 \quad and \quad \gamma = 0$ then it becomes an inventory model with instantaneous deterioration and complete backlogging where the total relevant cost per unit time is given by

$$\psi(t_1,T) = \frac{A}{T} + \frac{K_1 d}{T^{\frac{1}{n}}}\left[\frac{t_1^{\frac{1}{n}+1}(2-\alpha^2 t_d^2)}{2(n+1)} + \frac{\alpha t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\alpha^2 t_1^{\frac{1}{n}+3}}{6(3n+1)}\right]$$

$$+ \frac{K_2 d}{T^{\frac{1}{n}}}\left[\frac{\alpha^2 t_1^{\frac{1}{n}+2}}{2(2n+1)} + \frac{\alpha t_1^{\frac{1}{n}+1}}{n+1}\right]$$

$$- \frac{K_3 d}{T^{\frac{1}{n}}}\left[T t_1^{\frac{1}{n}} - \frac{t_1^{\frac{1}{n}+1}}{n+1} - \frac{nT^{\frac{1}{n}+1}}{n+1}\right] \tag{6.20}$$

Equation (6.20) is similar to the one obtained by Sicilia et al.(2014). Although in our case, purchasing cost was not taken into consideration.

6.5 Numerical Examples

We elucidate the proposed model with some numerical examples.

Example 1 as found in Sicilia et al. (2014)

$A = 50, d = 100, K_1 = 2, K_2 = 4, K_3 = 12, K_4 = 10, n = 0.5, \quad \alpha = 0.1, t_d = 0.4\gamma = 0.2$ in appropriate unit. The following results was obtained $t_1 = 0.385$ years, $T = 0.440$ years. From Equation (6.15), $\psi(t_1,T) = \$161.78$, $Q = 43.911$ units, $S = 33.663$ units, $P = 10.248$ units

From Equation (6.17) the Hessian is $H(t_1,T) = 11040994$ which is positive, it implies that (t_1,T) is the minimum point.

Example 2.

Repeating the same example 1, but we make n =3. The following was obtain $T = 0.969$ years, $t_1 = 0.831$ years, $S = 92.448$ units, $\psi(t_1,T) = \$97.99$, $P = 4.783$ units, $Q = 97.231$ units.

From Equation (6.17), the Hessian is $H(t_1,\ T) = 1259413.774$, which is positive; it implies that $(t_1,\ T)$ is the minimum point.

Table 6.1: Sensitivity Analysis of the Parameters in the Inventory model.

			Change in:					
P*	V*	C*	T	t_1	TVC	S	P	Q
	120	+20	0.406	0.353	177.29	37.217	11.051	48.268
	110	+10	0.420	0.368	169.71	35.454	10.655	46.109
d	100	0	0.430	0.385	161.78	33.663	10.248	43.911
	90	-10	0.464	0.405	153.44	31.830	9.827	41.667
	80	-20	0.493	0.430	144.59	29.980	9.390	39.371
	60	+20	0.482	0.421	177.19	36.725	11.446	28.171
	55	+10	0.461	0.430	169.63	35.208	10.852	46.060
A	50	0	0.430	0.385	161.78	33.663	10.248	43.911
	45	-10	0.418	0.366	163.52	32.086	9.620	41.716
	40	-20	0.395	0.345	144.77	30.471	8.996	39.467
	2.4	+20	0.413	0.354	173.44	30.292	11.003	41.295
	2.2	+10	0.426	0.368	167.83	31.864	10.631	42.495
K_1	2.0	0	0.440	0.385	161.78	33.663	10.248	43.911
	1.8	-10	0.457	0.404	155.25	35.751	9.855	45.605
	1.6	-20	0.477	0.427	148.16	38.218	9.452	47.669
	4.8	+20	0.439	0.385	161.74	33.697	10.243	43.940
	4.4	+10	0.440	0.385	161.76	33.680	10.246	43.925
K_2	4.0	0	0.440	0.385	161.78	33.663	10.248	43.911
	3.6	-10	0.440	0.385	161.80	33.646	10.250	43.896
	3.2	-20	0.439	0.384	161.83	33.628	10.253	43.881
	14.4	+20	0.430	0.383	164.58	34.186	8.785	42.971
	13.2	+10	0.435	0.384	163.28	33.943	9.460	43.404
K_3	12	0	0.440	0.385	161.78	33.663	10.248	43.911
	10.8	-10	0.446	0.386	160.05	33.334	11.178	44.512
	9.6	-20	0.453	0.387	158.02	32.944	12.194	45.238
	12	+20	0.438	0.385	162.31	33.762	9.968	43.730
	11	+10	0.439	0.385	162.05	33.713	10.106	43.819
K_4	10	0	0.440	0.385	161.78	33.663	10.248	43.911
	9	-10	0.441	0.385	161.51	33.611	10.394	44.005
	8	-20	0.442	0.385	161.23	33.558	10.544	44.102
	0.60	+20	0.479	0.418	154.52	38.182	9.668	47.850
	0.55	+10	0.459	0.401	157.98	35.946	9.934	45.880
n	0.50	0	0.439	0.385	161.78	33.663	10.248	43.911
	0.45	-10	0.420	0.368	165.94	31.338	10.631	41.968
	0.40	-20	0.401	0.352	170.44	28.971	11.115	40.085
	0.48	+20	0.452	0.397	160.77	35.005	10.168	45.172
	0.44	+10	0.445	0.391	161.22	34.281	10.193	44.474
t_d	0.40	0	0.440	0.385	161.78	33.663	10.248	43.911
	0.36	-10	0.435	0.380	162.46	33.148	10.330	43.478
	0.32	-20	0.432	0.376	163.22	32.732	10.437	43.170

Table 6.1, continued.

			Change in:					
P*	V*	C*	T	t_1	TVC	S	P	Q
	0.12	+20	0.442	0.388	161.73	33.403	10.243	43.946
	0.11	+10	0.441	0.387	161.76	33.683	10.244	43.928
α	0.10	0	0.440	0.386	161.78	33.663	10.248	43.911
	0.09	-10	0.439	0.385	161.81	33.642	10.251	43.893
	0.08	-20	0.438	0.384	161.83	33.620	10.254	43.874
	0.24	+20	0.438	0.385	162.28	33.755	9.976	43.732
	0.22	+10	0.439	0.386	162.03	33.710	10.110	43.820
γ	0.20	0	0.440	0.387	161.78	33.663	10.248	43.911
	0.18	-10	0.441	0.388	161.53	33.614	10.389	44.004
	0.16	-20	0.441	0.389	161.26	33.565	10.535	44.099

6. 6 Sensitivity Analysis

Table 6.1 shows the effects of changes of some model parameters on the decision variables base on the first example.

A careful study of the results obtained in the above tables and within the specified range of values of the selected parameters indicate the following observations:

1. As the demand rate d is increasing, the quantity order Q is also increasing, leading to an increase in the total cost TVC. Also, $T and\ t_1$ are decreasing. The economic implication of this is that as the demand is getting higher, the stock will take a short time to finish and so $T and\ t_1$ decreases. Increase in demand rate will produce an improvement in order quantity and the total inventory cost.

2. As the ordering cost A increase, T, t_1, Q, TVC are all increasing. The economic implication of this is that it is advisable to order more quantity when the ordering cost is expensive to prevent the order frequency and damage.

3. T, Q, t_1 decreases with an increase in the holding cost K_1. It is observed that there is an increase in the total inventory cost as the holding cost is increasing. The economic

implication of this is that as the holding cost is rising, it is better to reduce the cycle duration and order quantity to keep the cost of inventory as low as possible.

4. As the deteriorating cost K_2 is increasing, there is an increase in T, t_1, Q, which lead to a slight decrease in the total inventory cost. It implies that when the deteriorating price is higher, there is a need to order more quantity and increase the cycle period to take the opportunity of reduced deteriorating cost.

5. As $K_1 and\ K_2$ are increasing, there is decease in $T, t_1 and\ Q$, which result in an increase in the inventory total cost TVC.

6. It is discovered that as the backlogging rate γ increases, the parameters $T, t_1 and\ Q$, reduces while there is a slight increase in the total inventory cost.

7. Increase in the parameters $n\ and\ t_d$ lead to an increase in $T, t_1, and\ Q$ while the total inventory cost reduces.

6.7 Concluding Observations

In this chapter, an EOQ Inventory model for delayed deteriorating items with power demand, considering shortages and lost sales is presented. It extended a similar model carried out by Sicilia et al.(2014). We incorporate delay deterioration and lost sales. The effect of demand rate, constant rate of deterioration and partial backlogging rate on order quantity and total inventory cost per unit time are reported. Numerical examples are given, and sensitivity analysis carried out to show how the optimal decision is affected by changes in different parameters in the model. The following are our concluding remarks:

1. When the demand rate d is increasing, the inventory total cost is rising.
2. When the ordering cost A is increasing, the inventory total cost is also growing.

3. As the rate of deterioration α is increasing, the inventory total cost is decreasing leading to the increase in the inventory level S.
4. When the backlogging rate γ is expanding, the inventory total cost is increasing.
5. As the length of time t_d for the stock to deplete to zero is rising, the Inventory total cost is decreasing.
6. When the power index n is growing, the inventory total cost is decreasing.

From the above observations, the effect of demand rate, deteriorating rate and backlogging rate on optimal replenishment policy cannot be easily neglected. The propounded model can be utilised in inventory control of some delayed deteriorating item such as food items, vegetables, milk, fish so on. The model investigated in this chapter provides a basis for various feasible extension. In this direction, future research can enrich the model by incorporating time and stock demand dependent, preservation technologies, and so on.

CHAPTER 7: SUMMARY, CONCLUSION AND RECOMMENDATIONS

7.1 Summary

The primary purpose of this study is to fill the gaps in the academic literature as enumerated in Section 2.1.8 and to advance mathematical models for deteriorating items with power pattern demand rate. The aim has been achieved through four main objectives. These objectives were accomplished by developing four models which are deterministic in nature.

The first model developed is an inventory model for deteriorating items with power demand pattern and time-dependent holding cost without shortages. It is observed that in many economic order quantity models, holding cost is always considered as a constant function of time. In this model, the holding cost is assumed to be time-dependent, and the deterioration rate is supposed to be constant. The aim is to minimise the total average cost by finding the optimal cycle time and ordering quantity, and shortages are not considered in this model.

A high non-linear differential equation was arrived at after the development of the model which was solved by excel solver and backup by maple software 2018, which was used to evaluate the optimal cycle time and optimal ordering quantity. Numerical example has been presented to illustrate the applicability of the developed model. From the numerical example, the study of the outcome of various several in some feasible combination of the parameters in the model on decision variables are also carried out, and the results are depicted graphically.

In the second model, an inventory model for linearly time-dependent deteriorating rate and time-varying demand with shortages partially backlogged is proposed. The model extends the first model by adding shortages and time-dependent deteriorating rate. Based on the assumptions, the initial inventory height is M units at time $t = 0$. During the period $t = 0$ and $t = t_1$, the inventory level diminishes, due to deterioration and demand until it gets to zero level at $t = t_1$. Throughout the interval $[t_1, T]$ the system experiences shortages which are backlogged to the close of the cycle. At the interval $t = T$, the inventory get to a maximum shortages level m to get rid of the backlogged and the inventory level increases again to level M.

Numerical illustrations are given at the end of the developed model, and sensitivity analysis administered base on one of the example. The total cost function concerning the decision variables t_1 and T is plotted, and we obtained a strictly convex graph to establish the result. We also represented the sensitivity analysis graphically.

The third model is based on all the attributes and assumption of the other three models proposed. In this model, a linearly deteriorating inventory policy with power demand pattern and variable holding cost considering shortages which are partially backlogged is developed. When $t_d = 0$ and the deterioration rate is constant, we have similar work considered by (Mishra (2016b), Sicilia et al. (2014)). The main purpose of the model is to evaluate the optimal replenishment cycle length to minimise the total variable cost per unit time. Numerical illustration is made available to establish the application of the developed model, and the example was used to investigate the consequences of various alterations in some possible combination of the parameters in the model on the decision variables in the policy. The convexity graph of the total function concerning the decision variables is also plotted which is a strictly convex function. Also, we make the sensitivity analysis clearer by plotting the optimum cycle time T^*, the optimal total cost U^* and the optimum ordering quantity G^* against the inventory input parameters and make our observations.

The fourth model developed is an inventory model for delayed deteriorating items with power demand considering shortages and lost sales. In this model, a lot size Q unit enters the system at the inception of each cycle. The deterioration takes places after time t_d and attains zero inventory level at time t_1. The shortages happen in the interval and partially backlogged and lost sales at the end of the cycle time.

The main focus is to minimise the total average cost per unit time by calculating the optimum time value t at which the inventory comes down to zero level and the schedule period $T.$ The model extends the paper of Sicilia et al. (2014) without considering purchase cost. A highly non-linear differential equations are derived and solved by excel solver and maple. Numerical illustrations are made available at the end of the developed model and sensitivity analysis carried out.

7.2 Conclusion

In this thesis, different deterministic inventory models are investigated. Models are established for items with time-dependent and delayed deteriorating rate, time-varying holding cost, and power demand pattern rate. At the end of each developed inventory model, some specific conclusions are given. Nevertheless, general findings will now be drawn across the whole thesis and summarised as follows:

1. An inventory model for deteriorating items is proposed in chapter 3. The model considers a fixed rate of deterioration and the holding cost is linear time-dependent. When holding cost is assumed as time-dependent, it represents a real-life situation and is valid in the storage of some deteriorating food products such as meats, cake, wheat, flours, vegetables etc. The outcomes unveil that the effect of demand rate parameter r and holding cost parameter h on total inventory cost is significant.

Sensitivity analysis results of the decision variables as against the changes in the model parameters indicate that T, TC and Q are exorbitantly sensitive to changes in r, K, a while they are moderately sensitive in the parameter α, C_1, h and less insensitive to change in parameter b and n.

The economic implication of this is that optimum ordering quantity along with total cost per unit time are sensitive to power demand parameter, ordering cost and the holding cost. Moreover, it can be said they are less insensitive to unit purchasing cost and parameter b.

2. In chapter 4, a deterministic inventory model for linearly time-dependent deterioration rate is proposed. Demand rate varying with time and shortages are partially backlogged. The effect of power-dependent demand rate, backlogging cost, holding cost, deteriorating cost and purchasing cost on the optimal replenishment policy is high and hence should not be left out in developing this type of inventory model. The developed model can be useful in controlling the inventory of particular products that deteriorate with the advance of time and demand also varies with time. Examples of such products are android phone, computer chips, fashion, electronics etc.

3. The third developed model deals with inventory model for delayed deteriorating items. The inventory demand rate is in power pattern form considering shortages and lost sales which are partially backlogged. It is discovered from the results of Table 4 that the impacts of changing the parameters of the model $d, A, K_1, K_2, K_3, K_4, n, t_d, \alpha, \gamma$ on the replenishment policy disclose the following:

- At the time the demand rate d is rising, the inventory total cost is increasing.
- During the time the holding cost K_1 is rising, the inventory total cost is increasing

When the deteriorating cost K_2 is growing, the total cost is increasing

- When the shortage cost K_3 is increasing, the inventory total cost is increasing
- When the value of lost sales K_4 is rising, the inventory total cost is increasing
- As the deterioration rate α is expanding, the total cost is decreasing
- When the backlogging parameter increase, there is also a corresponding increase in the total inventory cost.

The model is essential because, in a product life cycle, demand is increasing with time during growth phrase, then after reaching its zeniths, the demand attains stability for a limited period otherwise known as maturity stage. After that, the demand begins to decrease with time and eventually tending to 0.

4. The fourth model deals with a linear deterioration inventory policy for products with power demand pattern and variable holding cost considering shortages. The model found deteriorating rate as time-dependent. The application of linear time-dependent holding cost is an accurate representation of any real-life situation and correct for the storage of some perishable and decaying items such as food products. The results obtained shows that the effect of demand rate d, backlogging parameter α and purchasing cost Z on total inventory cost U^* is quite enormous.

The results of the sensitivity analysis of the decision variables concerning the alterations in the model parameters indicate the following results:

- U^* and G^* are highly sensitive to overestimation and underestimation of the model parameters d and γ.
- U^* and G^* are less insensitive to the overestimation and underestimation of the model parameters h and θ.

- U^* is highly sensitive to overestimation and underestimation of the model parameters Z.
- U^* is moderately susceptible to overestimation and underestimation of the model parameters A, K and S and somewhat sensitive to the overestimation of the parameters n.
- U^* is less insensitive to both overestimate and underestimate of the model parameters β and less impervious to underestimate of the parameter n.
- G^* is highly sensitive to overestimation and underestimation of the model parameters A and K and highly responsive to overestimation of the parameter n.
- G^* is lowly sensitive to overestimation and underestimation of the model parameters Z and B and also lowly sensitive to underestimation of the parameter n.
- G^* is less insensitive to overestimation and underestimation of the model parameter S and less insensitive to the underestimation of the parameter n.

The contributions of this research include the following:

1. A model has been developed of an inventory policy for deteriorating items with power pattern and time-dependent holding cost without shortages. This model provides a simple to understand solution as against the general belief that holding cost is a constant function as found in Singh (2017). From the results obtained, it has been shown that enlargement in the total cost will lead to an enlargement in the holding cost. The model is an extension of the first contribution, where holding cost is time-dependent, along with the demand rate is in power form function of time. A graphical representation of convexity of total cost concerning the decision variable is provided. No existing model is known to have considered such an inventory model ahead of this one.

2. An attempt or effort has been made to propound and extend the model of Rajeswari & Indrani (2015), who proposed an Eoq inventory model for deteriorating products with linearly time-dependent deterioration rate. In the consideration of this model, the schedule period is presumed to be fixed, and the decision variables are the time at which inventory level drops to zero. Here in this model, the scheduled time is not fixed and is considered as one of the decision variables which accurately represents an inventory model. We provided the graphical description of the convexity of the total cost with respect to the decision variables to establish the reliability of the model. It is yet to be discovered in works of literature that such a model has been investigated elsewhere.

3. A model has been presented of an inventory system for delayed deteriorating items with power demand pattern considering shortages and lost sales. Here we broaden the work of Sicilia et al. (2014) by adding lost sales that are partially backlogged. This model contributes to the existing ones in that the effect of demand rate, deteriorating rate and backlogging rate parameter on optimal replenishment policy is quite significant and cannot be ignored in inventory modelling development. No published work to the best of our understanding has been done similar to this model.

4. A model has been developed for a linear time-dependent deterioration rate along with time-varying demand considering shortages which are partially backlogged. The model is a reality of the behaviours of the customers in the time of the stock-out period such that some consumers will be ready to be patient for the arrival of new stocks, while others will look elsewhere as a result of intolerant. We assumed that the holding cost is a linear function of time to reflect the truth that holding cost increase linearly with time and also deteriorating rate is taken into consideration as time dependent to show that deterioration increases continuously as time progresses. The work of (Adaraniwon & Omar (2019), Mishra (2016b)) is made more realistic. Time-varying demand, linear deterioration rate along with time-

dependent holding cost combined has not been so prevalent in literature.

7.3 Recommendations

Since inventory modelling deals mainly on reducing the inventory costs, it will be commendable for those operating inventory management of deteriorating items to exploit the findings of this research work in their decision-making procedures or process.

It is therefore recommended for future research work that attentions can be geared toward the following areas:

1. Investigation can be carried out on probabilistic re-order point, and order-level policy for deteriorating/degenerating items.
2. Explore a model in inventory for deteriorating items for linear and other different types of power demand pattern.
3. Extension of all the four inventory models to financial analysis although it is non-inventory. It is necessary because there is high global inflations and high-interest rates, which can help the decision-makers for planning purposes in banking and financial markets.
4. Research can be carried out on probabilistic inventory models. Only a few research works have been investigated in this direction.
5. Research can be carried out to include time-value of money, trade credits, discount rate, salvage values etc.
6. The models can also be developed in a fuzzy environment.

REFERENCES

Abad, P. (1996). Optimal pricing and lot-sizing under conditions of perishability and partial backordering. *Management Science*, *42*(8), 1093–1104.

Abad, P. L. (2003). Optimal pricing and lot-sizing under conditions of perishability, finite production and partial backordering and lost sale. *European Journal of Operational Research*, *144*(3), 677–685.

Abdul-Jalbar, B., Gutiérrez, J. M., & Sicilia, J. (2009). At two-echelon inventory/distribution system with power demand pattern and backorders. *International Journal of Production Economics*, *122*(2), 519–524.

Adaraniwon, A. O., & Mohd, B. O. (2019). An inventory model for delayed deteriorating items with power demand considering shortages and lost sales. *Journal of Intelligent & Fuzzy Systems*, *36*(6), 5397–5408.

Alfares, H. K. (2007). Inventory model with stock-level dependent demand rate and variable holding cost. *International Journal of Production Economics*, *108*(1), 259–265.

Alfares, H. K., & Ghaithan, A. M. (2016). Inventory and pricing model with price-dependent demand, time-varying holding cost, and quantity discounts. *Computers and Industrial Engineering*, *94*, 170–177.

Alfares, H. K., & Ghaithan, A. M. (2019). Eoq and epq production-inventory models with variable holding cost: State-of-the-art review. *Arabian Journal for Science and Engineering*, *44*(3), 1737–1755.

Amutha, R., & Chandrasekaran, E. (2013). An eoq model for deteriorating items with quadratic demand and the dependent holding cost. *International Journal of Emerging Science and Engineering*, *1*(5), 5–6.

Anchal, A., Isha, S., & Smita, R. (2016). A partial backlogging inventory model for non-instantaneous decaying items under trade credit financing facility. *Indian Journal of Science and Technology*, *9*(34), 1-11.

Aouam, T., Geryl, K ., Kumar, K., & Brahimi, N. (2018). Production planning with order acceptance and demand uncertainty. *Computers and Operations Research*, *91*, 145–159.

Baker, R. A., & Urban, T. L. (1988). A deterministic inventory system with an inventory-level-dependent demand rate. *Journal of the Operational Research Society*, *39*(9), 823–831.

Bakker, M., Riezebos, J., & Teunter, R. H. (2012). Review of inventory systems with deterioration since 2001. *European Journal of Operational Research*, *221*(2), 275–284.

Balkhi, Z .T., & Benkherouf, L. (1996). A production lot size inventory model for deteriorating items and arbitrary production and demand rates. *European Journal of Operational Research*, *92*(2), 302–309.

Banerjee, S., & Agrawal, S. (2017). Inventory model for deteriorating items with freshness and price dependent demand: Optimal discounting and ordering policies. *Applied Mathematical Modelling*, *52*, 53–64.

Bard, J., & Moore, J. (1990). Production planning with variable demand. *Omega*, *18*(1), 35–42.

Benkherouf, L., Skouri, K., & Konstantaras, I. (2017). Inventory decisions for a finite horizon problem with product substitution options and time varying demand. *Applied Mathematical Modelling*, *51*, 669–685.

Bhunia, K., & Maiti, M. (1998). Deterministic inventory model for deteriorating items with finite rate of replenishment dependent on inventory level. *Computers and Operations Research*, *25*(11), 997–1006.

Bukhari, F. A., & El-Gohary, A. (2012). Optimal control of a production maintenance system with deteriorating items. with deteriorating items. *Journal of King Saud University-Science*, *24*(4), 351–357.

Chakrabarti, D., & Chaudhuri. K. S. (1997). An eoq model for deteriorating items with a linear trend in demand and shortages in all cycles. *International Journal of Production Economics*, *49*(3), 205–213.

Chakraborty, D., Jana, D. K., & Roy, T. K . (2018). Two-warehouse partial backlogging inventory model with ramp type demand rate,three-parameter weibull distribution deterioration under inflation and permissible delay in payments. Computers and Industrial Engineering, *123*, 157–179.

Chang, C. T., Cheng, M. C., & Ouyang, L. Y. (2015). Optimal pricing and ordering policies for non-instantaneously deteriorating items under order-size-dependent delay in payments. *Applied Mathematical Modelling*, *39*(2), 747–763.

Chen, C. K., Hung, T. W., & Weng, T. C. (2007). Optimal replenishment policies with allowable shortages for a product life cycle. *Computers and Mathematics with Applications*, *53*(10), 1582–1594.

Chen, S. C., Teng, J. T., & Skouri, K. (2014). Economic production quantity models for deteriorating items with up-stream full trade credit and down-stream partial trade credit. *International Journal of Production Economics*, *155*, 302–309.

Chowdhury, M. R., & Chaudhuri, K. (1983). An order-level inventory model for deteriorating items with finite rate of replenishment. *OPSEARCH*, *20*(2), 99–106.

Chung, K. J. (2009). A complete proof on the solution procedure for non-instantaneous deteriorating items with permissible delay in payment. *Computers and Industrial*

Engineering, *56*(1), 267–273.

Chung, K. J., & Tsai, S. F. (1997). An algorithm to determine the eoq for deteriorating items with shortage and a linear trend in demand. *International Journal of Production Economics*, *51*(3), 215–221.

Covert, R. P., & Philip, G. C. (1973). An eoq model for items with weibull distribution deterioration. *AIIE Transactions*, *5*(4), 323–326.

Datta, T., & Pal, A. K. (1988). Order level inventory system with power demand pattern for items with variable rate of deterioration. *Indian Journal of Pure and Applied Mathematics*, *19*(11), 1043–1053.

Datta, T. K., & Pal, A. K. (1991). Effects of inflation and time-value of money on an inventory model with linear time-dependent demand rate and shortages. *European Journal of Operational Research*, *52*(3), 326–333.

Dave, U. (1979). On a discrete-in-time order-level inventory model for deteriorating items. *Journal of the Operational Research Society*, *30*(4), 349–354.

Dave, U., & Patel, L. K. (1981). (t, s i) policy inventory model for deteriorating items with time proportional demand. *Journal of the Operational Research Society*, *32*(2), 137–142.

Deng, P. S., Lin, H. J., & Chu, P. (2007). A note on the inventory models for deteriorating items with ramp type demand rate. *European Journal of Operational Research*, *178*(1), 112–120.

Donaldson, W. (1977). Inventory replenishment policy for a linear trend in demand: An analytical solution. *Journal of the Operational Research Society*, *28*(3), 663–670.

Duan, Y., Cao, Y., & Huo, J. (2018). Optimal pricing, production, and inventory for deteriorating items under demand uncertainty: The finite horizon case. *Applied Mathematical Modelling*, *58*, 331–348.

Dye, C. Y. (2013). The effect of preservation technology investment on a non-instantaneous deteriorating inventory model. *Omega*, *41*(5), 872–880.

Elsayed, E., & Teresi, C. (1983). Analysis of inventory systems with deteriorating items. *The International Journal of Production Research*, *21*(4), 449–460.

Escuin, D., Polo, L., & Ciprés, D. (2017). On the comparison of inventory replenishment policies with time-varying stochastic demand for the paper industry. *Journal of Computational and Applied Mathematics*, *309*, 424–434.

Ferguson, M., Jayaraman, V., & Souza, G. C. (2007). Note:An application of the eoq model with nonlinear holding cost to inventory management of perishables. *European Journal of Operational Research*, *180*(1), 485–490.

Geetha, K., & Uthayakumar, R. (2010). Economic design of an inventory policy or non-instantaneous deteriorating items under permissible delay in payments. *Journal of Computational and Applied Mathematics*, *233*(10), 2492–2505.

Ghare, P. (1963). A model for an exponentially decaying inventory. *Journal of Industrial Engineering*, *14*, 238–243.

Ghasemi, N., & Afshar Nadjafi, B. (2013). Eoq models with varying holding cost. *Journal of Industrial Mathematics*, *2*(13), 1-8.

Giri, B., & Chaudhuri, K. (1998). Deterministic models of perishable inventory with stock-dependent demand rate and nonlinear holding cost. *European Journal of Operational Research*, *105*(3), 467–474.

Giri, B., Pal, S., Goswami, A., & Chaudhuri, K. S. (1996). An inventory model for deteriorating items with stock-dependent demand rate. *European Journal of Operational Research*, *95*(3), 604–610.

Goh, M. (1994). Eoq models with general demand and holding cost functions. *European Journal of Operational Research*, *73*(1), 50–54.

Goswami, K., & Chaudhuri, K. (1991). Eoq model for an inventory with a linear trend in demand and finite rate of replenishment considering shortages. *International Journal of Systems Science*, *22*(1), 181–187.

Goswami, A., & Chaudhuri, K. (1992). Variations of order-level inventory models for deteriorating items. *International Journal of Production Economics*, *27*(2), 111–117.

Goyal, S., & Gunasekaran, A. (1995). An integrated production inventory marketing model for deteriorating items. *Computers and Industrial Engineering*, *28*(4), 755–762.

Goyal, S. K., & Giri, B. C. (2001). Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research*, *134*(1), 1–16.

Goyal, S. K., & Giri, B. C. (2003). A simple rule for determining replenishment intervals of an inventory item with linear decreasing demand rate. *International Journal of Production Economics*, *83*(2), 139–142.

Gupta, P. N., & Jauhari, R. (1995). An eoq model for deteriorating items with power demand pattern subject to permissible delay in payments. *Ganita Sandesh*, *9*(2), 65–71.

Hariga, M. (1993). The inventory replenishment problem with a linear trend in demand. *Computers and Industrial Engineering*, *24*(2), 143–150.

Harris, F. W. (1913). How many parts to make at once. *The Magazine of Management*, *10*(2), 135–136.

Hollier, R., & Mak, K. (1983). Inventory replenishment policies for deteriorating items in a declining market. *International Journal of Production Research*, *21*(6), 813–836.

Jaggi, C. K., Goel, S. K., & Tiwari, S. (2018). Two-warehouse inventory model for non-instantaneous deteriorating items under different dispatch policies. *Investigación Operacional*, *38*(4), 343–365.

Jain, S. (2016). An eoq inventory model for items with ramp type demand, three-parameter weibull distribution deterioration and starting with shortage. *Yugoslav Journal of Operations Research*, *20*(2), 249-259.

Jalali, H., Carmen, R., Van Nieuwenhuyse, I., & Boute, R. (2019). Quality and pricing decisions in production/inventory systems. *European Journal of Operational Research*, *272*(1), 195–206.

Jalan, A., Giri, R., & Chaudhuri, K. (1996). Eoq model for items with weibull distribution deterioration, shortages and trended demand. *International Journal of Systems Science*, *27*(9), 851–855.

Janssen, L., Claus, T., & Sauer, J. (2016). Literature review of deteriorating inventory models by key topics from 2012 to 2015. *International Journal of Production Economics*, *182*, 86–112.

Jiafu, T., Zhendong, P., Gong Jun, G., & Shixin, L. (2008). Combined heuristics for determining order quantity under time-varying demands. *Journal of Systems Engineering and Electronics*, *19*(1), 99–111.

Kaliraman, N. K. (2019). Inventory model for three–parameter weibull deterioration and partial backlogging. *Journal of Advances in Mathematics and Computer Science*, *33*(5), 1–9.

Khanra, S. (2016). A production-inventory model for a deteriorating item with shortage and time dependent demand. *Yugoslav Journal of Operations Research*, *21*(1), 29-45.

Krishnamoorthi, C., & Sivashankari, C. (2016). Production inventory models for deteriorative items with three levels of production and shortages. *Yugoslav Journal of Operations Research*, *27*(4), 499–519.

Krishnaraj, R. B., & Ramasamy, K. (2012). An inventory model with power demand pattern, weibull distribution deterioration and without shortage. *The Bulletin of Society for Mathematical Services and Standards*, *2*, 33–37.

Kumar, P. (2019). An inventory planning problem for time-varying linear demand and parabolic holding cost with salvage value. *Croatian Operational Research Review*, *10*, 187–199.

Li, G., He, X., Zhou, J., & Wu, H. (2019). Pricing, replenishment and preservation technology investment decisions for non-instantaneous deteriorating items. *Omega*, *84*, 114–126.

Li, R., Lan, H., & Mawhinney, J. (2010). A review on deteriorating inventory study. *Journal of Service Science and Management*, *3*(01), 117-129.

Lo, W. Y., Tsai, C. H., & Li, R. K. (2002). Exact solution of inventory replenishment policy for a linear trend in demand–two-equation model. *International Journal of Production Economics*, *76*(2), 111–120.

Luo, W. (1998). An integrated inventory system for perishable goods with backordering. *Computers and Industrial Engineering*, *34*(3), 685–693.

Mahmoudinezhad, M., Ghoreishi, M., Mirzazadeh, A., & Ghodratnama, A. (2014). An inventory model for non-instantaneous deteriorating items with imperfect quality,delay in payments and time value money. J*ournal of Industrial Engineering and Management Studies*, *1*(1), 58–71.

Maihami, R., & Abadi, I. N. K. (2012). Joint control of inventory and its pricing for non-instantaneously deteriorating items under permissible delay in payments and partial backlogging. *Mathematical and Computer Modelling*, *55*(5-6), 1722–1733.

Maihami, R., & Kamalabadi, I. N. (2012). Joint pricing and inventory control for non-instantaneous deteriorating items with partial backlogging and time and price dependent demand. *International Journal of Production Economics*, *136*(1), 116–122.

Maihami, R., & Karimi, B. (2014). Optimizing the pricing and replenishment policy for non-instantaneous deteriorating items with stochastic demand and promotional efforts. *Computers and Operations Research*, *51*, 302–312.

Mak, M. K. (1982). A production lot size inventory model for deteriorating items. *Computers and Industrial Engineering*, *6*(4), 309–317.

Malik, A., Shekhar, C.,Vashisth, V., Chaudhary, A. K., & Singh, S. (2016). Sensitivity analysis of an inventory model with non-instantaneous and time-varying deteriorating items. In *American Institute of Physics, Conference Proceedings, 1715*, 20-59.

Mishra, S., Raju, L. K., Misra,U. K., & Misra, G. (2012). A study of eoq model with power demand of deteriorating items under the influence of inflation. *General Mathematics Note*, *10*(1), 41–50.

Mishra, U. (2016a). An eoq model with time dependent weibull deterioration, quadratic demand and partial backlogging. *International Journal of Applied and Computational Mathematics*, *2*(4), 545–563.

Mishra, U. (2016b). A waiting time deterministic inventory model for perishable items in stock and time dependent demand. *International Journal of System Assurance Engineering and Management*, *7*(1), 294–304.

Mishra, V. K. (2012). Production inventory model for deteriorating items with shortages and salvage value under reverse logistics. *International Journal of Mathematical*

Modelling and Computations, *02*(02), 99–110.

Mishra, V. K., Singh, L. S., & Kumar, R. (2013). An inventory model for deteriorating items with time-dependent demand and time-varying holding cost under partial backlogging. *Journal of Industrial Engineering International*, *9*(1), 41-50.

Misra, R. B. (1975). Optimum production lot size model for a system with deteriorating inventory. *The International Journal of Production Research*, *13*(5), 495–505.

Mitra, A., Cox, J. F., & Jesse Jr, R. R. (1984). A note on determining order quantities with a linear trend in demand. *Journal of the Operational Research Society*, *35*(2), 141–144.

Mokhtari, H., Naimi-Sadigh, A., & Salmasnia, A. (2017). A computational approach to economic production quantity model for perishable products with backordering shortage and stock-dependent demand. *Scientia Iranica*, *24*(4), 2138–2151.

Muckstadt, J. A. (2010). Principle of inventory management. *Springer-Verlag*.

Muhlemann, A., & Valtis-Spanopoulos, N. (1980). A variable holding cost rate eoq model. *European Journal of Operational Research*, *4*(2), 132–135.

Mukhopadhyay, S., Mukherjee, R., & Chaudhuri, K. (2005). An eoq model with two-parameter weibull distribution deterioration and price-dependent demand. *International Journal of Mathematical Education in Science and Technology*. *36*(1), 25–33.

Naddor, E. (1966). Inventory systems. *Technical Reports, 4,* Article#55.

Nahmias, S. (1982). Perishable inventory theory: A review. *Operations Research, 30*(4), 680–708.

Omar, M., & Yeo, I. (2009). A model for a production–repair system under a time-varying demand process. *International Journal of Production Economics*, *119*(1), 17–23.

Ouyang, L. Y., Wu, K. S., & Yang, C. T. (2006). A study on an inventory model for non-instantaneous deteriorating items with permissible delay in payments. *Computers and Industrial Engineering*, *51*(4), 637–651.

Pal, S., Mahapatra, G., & Samanta, G. (2015). A production inventory model for deteriorating item with ramp type demand allowing inflation andshortages under fuzziness. *Economic Modelling*, *46*, 334–345.

Palanivel, M., & Uthayakumar, R. (2014). An eoq model for non-instantaneous deteriorating items with power demand, time dependent holding cost, partial backlogging and permissible delay in payments. *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, *8*(8), 1127–1137.

Pando, V., Garcıa, J., San-José, L. A., & Sicilia, J. (2012). Maximizing profits in an inventory model with both demand rate and holding cost per unit time dependent on the stock level. *Computers and Industrial Engineering*, *62*(2), 599–608.

Pando, V., San-José, L. A., Garcia-Laguna, J., & Sicilia, J. (2013). An economic lot-size model with non-linear holding cost hinging on time and quantity. *International Journal of Production Economics*, *145*(1), 294–303.

Pervin, M., Roy, S. K., & Weber, G. W. (2018). Analysis of inventory contromodel with shortage under time-dependent demand and time-varying holding cost including stochastic deterioration. *Annals of Operations Research*, *260*(2), 437–460.

Philip, G. C. (1974). A generalized eoq model for items with weibull distribution deterioration. *America Institute of Industrial Engineers Transactions*, *6(*2), 159–162.

Pradhan, S., Shial, R., & Tripathy, P. (2016). An inventory model with power demand pattern under inflation. *Investigación Operacional*, *37*(3), 278–288.

Pramanik, P., & Maiti, M. K. (2019). An inventory model for deteriorating items with inflation induced variable demand under two level partial trade credit: A hybrid abc-ga approach. *Engineering Applications of Artificial Intelligence*, *85*, 194–207.

Raafat, F. (1991). Survey of literature on continuously deteriorating inventory models. *Journal of the Operational Research Society*, *42*(1), 27–37.

Raafat, F. F., Wolfe, P. M., & Eldin, H. K. (1991). An inventory model for deteriorating items. C*omputers and Industrial Engineering*, *20*(1), 89–94.

Rajeswari, N., & Indrani, K. (2015). Eoq policies for linearly time dependentdeteriorating items with power demand and partial backlogging. *International Journal of Mathematical Archive EISSN 2229-5046*, *6*(2), 122–130.

Rajeswari, N., Indrani, K., & Vanjkkodi, T. (2015). Fuzzy inventory model for constantly deteriorating items with power demand and partial backlogging. *IOSR Journal of Mathematics*, *11*(6), 44–53.

Rajeswari, N., & Vanjikkodi, T. (2012). An inventory model for items with two parameter weibull distribution deterioration and backlogging. *American Journal of Operations Research*, *2*(02), 247-252.

Rangarajan, K., & Karthikeyan, K. (2017). A partially backlogged inventory model for non-instantaneous deteriorating items. *International Journal of Pure and Applied Mathematics*, *116*(24), 143–154.

Rasel, S. (2017). A deterministic inventory system with power distribution Deterioration and fully backlogged shortage. *Asian Journal of Current Research*, *2*(3), 81–88.

Ritchie, E. (1984). The eoq for linear increasing demand: A simple optimal solution. *Journal of the Operational Research Society*, *35*(10), 949–952.

Ritha, W., & Vinoline, I. A. (2018). An economic order quantity model for weibull distribution deterioration with selling price sensitive demand and varying hold cost. *International Journal of Pure and Applied Mathematics*, *118*(6), 499–507.

Samanta, G. (2016). A production inventory model with deteriorating items and shortages. *Yugoslav Journal of Operations Research*, *14*(2), 219-230.

Sana, S., Goyal, S. K., & Chaudhuri, K. (2004). A production–inventory model for a deteriorating item with trended demand and shortages. *European Journal of Operational Research*, *157*(2), 357–371.

San-José, L. A., Sicilia, J., Cárdenas-Barrón, L. E., & Gutiérrez, J. M. (2019). Optimal price and quantity under power demand pattern and non-linear holding cost. *Computers and Industrial Engineering*, *129*, 426–434.

San-José, L. A., Sicilia, J., & Juan Garcia-Laguna, J. (2009). A general model for eoq inventory systems with partial backlogging and linear shortage costs. *International Journal of Systems Science*, *40*(1), 59–71.

San-José, I. A., Sicilia, J., González-De-la Rosa, M., & Febles-Acosta, J. (2017a). Optimal inventory policy under power demand pattern and partial backlogging. *Applied Mathematical Modelling*, *46*, 618–630.

San-José, I. A., Sicilia, J., González-De-la Rosa, M., & Febles-Acosta, J. (2017b) Optimal policy for deteriorating items with power demand and shortages. *Journal of Engineering and Science Research*, *1*(2), 110–117.

Sanni, S. S., & Chukwu, W. (2013). An economic order quantity model for items with three-parameter weibull distribution deterioration, ramp-type demandand shortages. *Applied Mathematical Modelling*, *37*(23), 9698–9706.

Santhi, G., & Karthikeyan, K. (2017). An eoq model for weibull distribution deterioration with time-dependent cubic demand and backlogging. *In Materials Science and Engineering Conference Series*, *263*(4), 1-19.

Sarbjit, S., & Shivraj, S. (2011). Deterministic and probabilistic eoq models for products having power demand pattern. *In Proceedings of the World Congress on Engineering*, *1*, 1-6.

Sarkar, S., Saren, S., & Wee, H. M. (2013). An inventory model with variable demand, component cost and selling price for deteriorating items. *Economic Modelling*, *30*, 306–310.

Sazvar, Z., Rekik, Y., Jokar, M. A., Baboli, A., & Al-E-Hashem, S. M. (2013). A new up-to level inventory model for deteriorating products with non-linear holding cost. *IFAC Proceedings*, *46*(9), 1702–1707.

Shah, N. H., Soni, H. N., & Patel, K. A. (2013). Optimizing inventory and marketing policy for non-instantaneous deteriorating items with generalized type deterioration and holding cost rates. *Omega*, *41*(2), 421–430.

Shah, N. H., & Vaghela, C. R. (2018). An epq moodel for deteriorating items with price dependent demand and two level trade credit financing. *Investigación Operacional*, *39*(2), 170–180.

Shaikh, A. A., Cárdenas-Barrón, I. E., Bhunia, A. K., & Tiwari, S. (2019). An inventory model of a three parameter weibull distributed deteriorating item with variable demand dependent on price and frequency of advertisement under trade credit. *RAIRO-Operations Research*, *53*(3), 903–916.

Shao, Y. E., Fowler, J. W., & Runger, G. C. (2000). Determining the optimal target for a process with multiple markets and variable holding costs. *International Journal of Production Economics*, *65*(3), 229–242.

Sharma, S., Yadav, A. S., & Swami, A. (2016). An optimal ordering policy for non-instantaneous deteriorating items with conditionally permissible delay in payment under two storage management. *International Journal of Computer Applications*, *140*(4), 16-25.

Sharmila, D., & Uthayakumar, R. (2018). A two warehouse deterministic inventory model for deteriorating items with power demand, time varying holding costs and trade credit in a supply chain system. *Uncertain Supply Chain Management*, *6(*2), 195–212.

Shukla, H. S., Tripathi, R. P., & Sang, N. (2017). Eoq model with stock-level dependent demand and different holding cost functions. *International Journal of Operations Research and Information Systems (IJORIS)*, *8*(4), 59–75.

Sicilia, J., González-De-la Rosa, J., Febles-Acosta, J., & Alcaide-López-de Pablo, D. (2014). An inventory model for deteriorating items with shortages and time-varying demand. *International Journal of Production Economics*, *155*, 155–162.

Silver, E. A. (1979). A simple inventory replenishment decision rule for a linear trend in demand. *Journal of the Operational Research Society*, *30*(1), 71–75. .

Singh, S. (2017). Optimal ordering policy for deteriorating items having constant demand and deterioration rate. *Theoretical Economics Letters, 7*(7), 2152-2158.

Singh, S. R., Kumar, S., & Gaur, N. K. (2009). Inventory model with power demand, partial backlogging, and incremental holding cost under inflation. *International Transactions in Applied Sciences*, *1*(2), 93–102.

Singh, T., Pattanayak, H., Nayak, A. K., & Sethy, N. N. (2018). An optimal policy with three-parameter weibull distribution deterioration, quadratic demand, and salvage value under partial backlogging. *International Journal of Rough Sets and Data Analysis (IJRSDA)*, *5*(1), 79–98.

Soni, H. N. (2013). Optimal replenishment policies for non-instantaneous deteriorating items with price and stock sensitive demand under permissible delay in payment. *International Journal of Production Economics*, *146*(1), 259–268.

Tai, A. H. (2013). Economic production quantity models for deteriorating/imperfect products and service with rework. *Computers and Industrial Engineering*, *66*(4), 879–888.

Taleizadeh, A. A., Mohammadi, B., Cárdenas-Barrón, I. E., & Samimi, H. (2013). An eoq model for perishable product with special sale and shortage. *International Journal of Production Economics*, *145*(1), 318–338.

Taleizadeh, A. A., Noori-daryan, M., & Cárdenas-Barrón, L. E. (2015). Joint optimization of price, replenishment frequency, replenishment cycle and production rate in vendor managed inventory system with deteriorating items. *International Journal of Production Economics*, *159*, 285–295.

Taleizadeh, A. A., Pentico, D. W., Jabalameli, M .S., & Aryanezhad, M. (2013). An eoq model with partial delayed payment and partial backordering. *Omega*, *41*(2), 354–368.

Taleizadeh, A. A., Soleymanfar, V. R., & Govindan, K. (2018). Sustainable economic production quantity models for inventory systems with shortage. *Journal of Cleaner Production*, *174*, 1011–1020.

Teng, J. T. (1996). A deterministic inventory replenishment model with a linear trend in demand. *Operations Research Letters*, *19*(1), 33–41.

Teng, J. T., Ouyang, L. Y., & Chang, C. T. (2005). Deterministic economic production quantity models with time-varying demand and cost. *Applied Mathematical Modelling*, *29*(10), 987–1003.

Tiwari, S., Cárdenas-Barrón, L. E., Khanna, A., & Jaggi, C. K. (2016). Impact of trade credit and inflation on retailer's ordering policies for non-instantaneous deteriorating items in a two-warehouse environment. *International Journal of Production Economics*, *176*, 154–169.

Tripathi, R. (2013). Inventory model with different demand rate and different holding cost. *International Journal of Industrial Engineering Computations*, *4*(3), 437–446.

Tuan, H. W., Lin, S. W., & Julian, P. (2017). Improvement for amelioration inventory model with weibull distribution. *Mathematical Problem in Engineering*, *2017*, 1-8.

Udayakumar, R., & Geetha, K. (2018). An eoq model for non-instantaneous deteriorating tems with two levels of storage under trade credit policy. *Journal of Industrial Engineering International*, *14*(2), 343–365.

Uthayakumar, R., & Karuppasamy, S. (2017). An inventory model for variable deteriorating pharmaceutical items with time dependent demand and time dependent holding cost under trade credit in healthcare industries. *Communications in Applied Analysis*, *21*(4), 533–549.

Viji, G., & Karthikeyan, K. (2018). An economic production quantity model for three levels of production with weibull distribution deterioration and shortage. *Ain Shams Engineering Journal*, *9*(4), 1481–1487.

Weiss, H. J. (1982). Economic order quantity models with non-linear holding costs. *European Journal of Operational Research*, 9(1), 56–60.

Whitin, T. M. (1957). *Theory of Inventory Management*, Princeton University Press.

Widyadana, G. A., & Wee, H. M. (2011). Optimal deteriorating items production inventory models with random machine breakdown and stochastic repair time. *Applied Mathematical Modelling*, *35*(7), 3495–3508.

Wu, J., Skouri, K., Teng, J. T., & Ouyang, L.Y. (2014). A note on an optimal replenishment policies for non-instantaneous deteriorating items with price and stock sensitive demand under permissible delay in payment. *International Journal of Production Economics*, *155*, 324–329.

Wu, J., Ouyang, L. Y., & Yang, C. T. (2006). An optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging. *International Journal of Production Economics*, *101*(2), 369–384.

Wu, K. S. (2001). An eoq inventory model for items with weibull distribution deterioration, ramp type demand rate and partial backlogging. *Production Planning and Control*, *12*(8), 787–793.

Yang, H. L. (2004). Two-warehouse inventory models for deteriorating items with shortages under inflation. *European Journal of Operational Research*, *157*(2), 344–356.

Yang, J., Zhao, G., & Rand, G. (2004). An eclectic approach for replenishment with non-linear decreasing demand. *International Journal of Production Economics*, *92*(2), 125–131.

Zhang, J., Wang, Y., Lu, I., & Tang, W. (2015). Optimal dynamic pricing and replenishment cycle for non-instantaneous deterioration items with inventory-level-dependent demand. *International Journal of Production Economics*, *170*, 136–145.

Zhao, G., Yang, J., & Rand, G. (2001). Heuristics for replenishment with linear decreasing demand. *International Journal of Production Economics*, *69*(3), 339–345.

Zhao, S. T., Wu, K., & Yuan, X. M. (2016). Optimal production-inventory policy for an integrated multi-stage supply chain with time-varying demand. *European Journal of Operational Research*, *255*(2), 364–379.

Zhou, Y. W. (2003). A multi-warehouse inventory model for items with time-varying demand and shortages. *Computers and Operations Research*, *30(14)*, 2115–2134.

Zhou, Y. W., Lau, H. S., & Yang, S. I. (2004). A finite horizon lot-sizing problem with time-varying deterministic demand and waiting-time-dependent partial backlogging. *International Journal of Production Economics*, *91*(2), 109–119.

LIST OF PUBLICATIONS AND PAPERS PRESENTED

List of publications

1. **Adaraniwon, A. O.,** & Mohd, B. O. (2019). An inventory model for delayed deteriorating items with power demand considering shortages and lost sales. *Journal of Intelligent & Fuzzy Systems*, *36*(6), 5397-5408.

2. **Adaraniwon, A. O.,** & Mohd, B. O. (2020). An inventory model for linearly time-dependent deteriorating rate and time-varying demand with shortages partially backlogged. *Journal of Intelligent & Fuzzy Systems,* (Preprint), 1-13.

Papers Presented

1. **Adaraniwon, A. O.,** & Mohd, B. O. (2018). An inventory model for linearly time-dependent deteriorating items with power pattern, shortages and time-varying demand. Paper presented at the 2nd Asia AIMC International Multidisciplinary Conference, 12-13 May 2018, Universiti Teknology, Johor Bahru, Malaysia.