MODELING LOSS DATA WITH COMPOSITE MODELS

MASTOUREH MAGHSOUDI

FACULTY OF SCIENCE
UNIVERSITY OF MALAYA
KUALA LUMPUR

2016

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MASTOUREH MAGHSOUDI

THESIS SUBMITTED FOR THE FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

FACULTY OF SCIENCE
UNIVERSITY OF MALAYA
KUALA LUMPUR

2016

UNIVERSITI MALAYA

ORIGINAL LITERARY WORK DECLARATION

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Registration/Matric No: SHB110009

Name of Degree: DOCTOR OF PHILOSOPHY

Title of Project Paper/Research Report/Dissertation/Thesis ("this Work"):

MODELING LOSS DATA WITH COMPOSITE MODELS

Field of Study: ACTUARIAL AND FINANCIAL MATHEMATIC(STATISTICS)

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ABSTRACT

Modeling insurance loss data of a unimodal type with a heavy tail is an interesting topic for actuaries. Recently, the focus for modeling such data has been directed towards composite models following a study by Cooray and Ananda (2005). These models are designed by piecing together two weighted distributions at a certain threshold θ. For simplicity, the distribution up to the threshold θ and the one beyond it are referred to as a head and a tail of the composite model, respectively. Studies have shown that composite models offer some solutions to the problem for covering the whole data sets (Nadarajah and Bakar, 2012).

This study proposed a new approach of finding parameters in terms of mixing weight ϕ and threshold θ parameters. The advantages of this new procedure is that it has a general formula and can be used for all distributions as a head and a tail in constructing composite models. This leads to the composite model being more flexible than the previous models. In previous studies, all parameters in the composite model were estimated in terms of two parameters which are the mixing weight and one of the head distribution parameters.

New composite models with a tail distribution that belongs to the transformed Beta, transformed Gamma, and inverse transformed Gamma families are introduced and fitted to two well-known data sets in actuarial industries. Based on the negative log-likelihood, (NLL), Akaike information criteria, (AIC), and Schwarz's Bayesian criterion, (SBC), the composite Weibull-inverse Paralogistic model and the composite Weibull-inverse transformed Gamma model are found (with the lowest value in all above goodness-of-fit measurements) to have the best fit among all composite models considered. For all new composite models that have been introduced, mixing weight ϕ, threshold θ, and moment generating function are obtained and for the two best fitted composite models, more statistical properties such as mean, variance, skewness, and kurtosis are calculated.

In this study, two real insurance loss data sets are used to illustrate the performance of these new composite models. The first data set is Danish fire insurance loss data which comprises of 2492 losses arising from Copenhagen. The second data set is the allocated loss adjustment expenses (ALAE) data set which consists of 1500 general liability losses recorded in US dollars.

This study also presents the applications of the fitted composite models in risk measurements. Two well-known risk measures are employed, namely, the value-at-risk (VaR) and the conditional tail expectation (CTE) for all composite models in transformed Beta, transformed Gamma, and inverse transformed Gamma families. To test the validity of the value-at-risk and the conditional tail expectation, backtesting method is applied.

All the computations includes constructing the composite model, fitting the distributions and composite models, graphing, measuring the risk measures, and backtesting were performed using the statistical software **R**.

ABSTRAK

Model Permodelan data kerugian insurans berjenis unimodal dengan ekor berat adalah satu topic yang menarik di kalangan ahli aktuari. Terkini, fokus untuk memodelkan data tersebut telah terarah kepada model komposit berikutan kajian oleh Cooray and Ananda (2005). Model-model ini direka dengan mencantumkan bersama dua taburan berpemberat pada suatu ambang θ tertentu. Sebagai keringkasan, taburan sehingga titik ambang θ dan yang selebihnya masing-masing dirujuk sebagai kepala dan ekor model komposit. Kajian telah menunjukkan bahawa model komposit menawarkan beberapa penyelesaian kepada masalah rangkuman keseluruhan data (Nadarajah and Bakar, 2012).

Kajian ini mencadangkan satu pendekatan baru untuk mendapatkan parameter dalam sebutan campuran pemberat, ϕ, dan titik ambang, θ. Kelebihan prosedur baru ini ialah ia mempunyai formula umum dan boleh digunakan untuk semua taburan sebagai kepala dan ekor dalam membina model komposit. Ini menjurus kepada model komposit yang lebih fleksibel daripada model sebelumnya. Dalam kajian lepas, semua parameter dalam model komposit dianggarkan melalui sebutan dua parameter, iaitu, campuran parameter berpemberat dan salah satu parameter daripada taburan kepala.

Model baru komposit dengan taburan ekor yang tergolong dalam keluarga taburan Beta terubah, Gamma terubah dan Gamma songsang terubah diperkenalkan dan disuaikan pada dua set data yang terkenal dalam industri aktuari dan insurans. Berdasarkan kepada log-nilai kebolehjadian negatif, (NLL), kriteria maklumat Akaike, (AIC), dan kriteria Bayesian Schwarz, (SBC), model komposit Weibull-Paralogistic songsang dan model komposit Weibull-Gamma terubah songsang didapati (dengan nilai terendah dalam semua ukuran ketepatan padanan di atas) mempunyai penyuaian terbaik di kalangan semua model komposit yang dikaji. Untuk semua model komposit baru yang telah diperkenalkan, campuran berpemberat ϕ, titik ambang θ, dan fungsi penjana momen diperolehi,

dan bagi dua model komposit dengan penyuaian terbaik, sifat-sifat statistik seperti min, varians, kepencongan, dan kurtosis adalah dihitung.

Dalam kajian ini, dua set data kerugian dalam insurans sebenar digunakan untuk menggambarkan prestasi model komposit baru. Set data pertama adalah data kerugian insurans kebakaran Danish, yang terdiri daripada 2492 kerugian diperoleh dari Copenhagen. Set data kedua ialah perbelanjaan pelarasan kerugian yang diperuntukkan (ALAE), yang terdiri daripada 1500 kerugian liabiliti am yang dicatatkan dalam dolar AS.

Kajian ini juga membentangkan aplikasi model komposit tersuai dalam pengukuran risiko. Dua ukuran risiko yang kerap digunapakai, iaitu, nilai-pada-risiko (VaR) dan ekor jangkaan bersyarat (CTE) untuk semua model komposit dalam keluarga taburan Beta terubah, Gamma terubah, dan Gamma songsang terubah adalah dihitung. Untuk menguji kesahihan nilai-pada-risiko dan ekor jangkaan bersyarat, kaedah ujian tersokong digunakan.

Kesemua pengiraan termasuk pembinaan model komposit, penyuaian taburan dan model komposit, grafik, ukuran risiko dan ujian tersokong dilakukan dengan menggunakan perisian statistik **R**.

ACKNOWLEDGEMENTS

I would like to thank all those who helped me with their sincere accompany:

First and foremost, I am highly thankful to ALLAH who never lets me alone.

I would like to express my heartfelt appreciation to my beloved parents and my dearest sister for their unlimited love, kindness, cooperation, and continuous encouragement in all aspects during the process of completing my studies. Without their endless support, it was not possible for me to start and complete this program.

I would like to extend my special thanks to my supervisors, Prof. Dr. Nor Aishah Binti Hamzah who not only supported the thesis as a supervisor, but also motivated me to go further and Dr. Shaiful Anuar Bin Abu Bakar who had been incredibly generous with his time and for his fruitful ideas as well as enthusiasm to see my success throughout the year of my PhD. Also, I am grateful to Dr. Alireza Bahiraei for all his advices.

Finally, I am indebted to my friends whom their supports helped me stay focused on my graduate study.

I love you all.

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LIST OF SYMBOLS AND ABBREVIATIONS

$E[X]$	Mean for a random variable X
$E[X^2]$	Second moment for a random variable X
$E[X^k]$	k-th moment for a random variable X
$f_X(x)$	Probability density function for a random variable X
$f_X^*(x)$	Truncated probability density function for a random variable X
$f_X^{'}(x)$	First derivation of probability density function for a random variable X
$F_X(x)$	Cumulative distribution function for a random variable X
$F_X^*(x)$	Truncated cumulative distribution function for a random variable X
$F_X^{'}(x)$	First derivation of cumulative distribution function for a random variable X
$M_X(t)$	Moment generating function for a random variable X
$Var(X)$	Variance for a random variable X
ALAE	Allocated Loss Adjustment Expenses
AIC	Akaike Information Criterion
CTE	Conditional Tail Expectation
CTEs	Conditional Tail Expectations
MLE	Maximum Likelihood Estimation
NLL	Negative Log-Likelihood
SBC	Schwarz's Bayesian Criterion
VaR	Value-at-Risk
VaRs	Value-at-Risks

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CHAPTER 1

INTRODUCTION

1.1 General Introduction

In actuarial field, the need to fit a reasonable distribution to insurance loss data and hence to determine the operational quantities involved, such as premiums, is crucial. Over the years, many models have been proposed in the actuarial literature to deal with this issue. Therefore, it is not surprising that the study receives numerous attention considering the practical importance of the field. The univariate distributions which are commonly used in actuarial science for fitting to loss data are the Pareto, Weibull, Lognormal, Gamma, and Exponential distributions.

The insurance loss data normally represents a unimodal structure with some extreme losses. This makes the distribution heavy at the tail. However, many of the earlier attempts do not reasonably capture the nature of heavy tail of the loss data. To account for this, several heavy tailed distributions belonging to three families of distributions which are the transformed Beta, transformed Gamma, and inverse transformed Gamma have been proposed. Each distribution belonging to these families displays some interesting features in the fitting. For instance, Pareto distribution has a longer and thicker upper tail to model the larger loss data and Weibull distribution is a positive skewed, long tail, and flexible distribution (Cooray and Ananda, 2005). All these distributions are defined by a probability density function that consists of several parameters.

More recently, the studies have been devoted to a more complex structure which however mimics the overall shape of the distribution much closely. Among the models are the folded models, skewed models, mixture models, and composite models. In this study, I will focus on composite model which pieces two distributions together. The composite model consists of the first distribution up to a threshold value, θ, and the second distribution for the rest in the range of domain. As will be presented in the forthcoming chapters, composite model provides a better fit than a single distribution.

1.2 Problem Statement

In the context of actuarial science, modeling loss data with univariate distributions is an inevitable problem. Familiar distributions in insurance industries are Exponential distribution, Gamma distribution, Pareto distribution, Lognormal distribution, and Weibull distribution. In some cases, it is seen that these distributions are not sufficient for covering the whole data sets. So, using a new method with combined distributions is logical. These well-known combination methods are examples of mixture distributions and composite models.

In this study, composite models as an alternative to univariate distributions are considered. The most important reasons for choosing composite models are as follows:

- Using a composite model is common among statisticians and actuaries to better fit the whole loss data sets.
- Composite models are used to help find solutions for some parts in insurance such as risk measures, loss estimations, and calculate the future premiums.

Hence, this study introduces two composite models called composite Weibull-inverse Paralogistic model and composite Weibull-inverse transformed Gamma model. It will be discussed that these two composite models are the best composite models by having the best fit to two well-known loss data sets, Danish fire insurance loss and allocated loss adjustment expenses (ALAE), in three distribution families. That is, the composite Weibull-inverse Paralogistic model has the best fit among distributions belongs to the transformed Beta family and the composite Weibull-inverse transformed Gamma model has the best fit among distributions belongs to the transformed Gamma and inverse transformed Gamma families to Danish and ALAE data sets. Also, this study provides various properties of these new composite models.

The application of these new composite models can be extended to risk measures which are related to the insurance industry as predictors of risk. The two well-known risk measures, value-at-risk (VAR) and conditional tail expectation (CTE), are applied in this study.

Another problem is on parameter estimations. That is, in all former composite models, to reduce the number of parameters, two parameters, which are the mixing weight and one parameter in the head part, were estimated in terms of the other parameters. The problem arises when the head distribution is changed. Therefore, the process of finding these two parameters has to be repeated. To overcome this, instead of estimating the parameter of the head, the threshold parameter is estimated. Consequently, the two parameters which are estimated first are the mixing weight and the threshold.

1.3 Research Objectives

The objectives can be listed as below:

1. To construct new composite models as alternatives to univariate distributions. These distributions should satisfy in conditions of (1) continuity and (2) smoothness.

2. To derive the threshold parameter in a composite model.

3. To further explore the properties of composite Weibull-inverse Paralogistic model in the transformed Beta family and the composite Weibull-inverse transformed Gamma model in the inverse transformed Gamma family. These composite models are the best fitted models to two real well-known data sets.

4. To evaluate the Value-at-Risk (VaR) and the conditional tail expectation (CTE), as two famous risk measures in the new composite models which are the composite Weibull-inverse Paralogistic model and the composite Weibull-inverse transformed Gamma model. Backtesting method is applied to verify validity of VaR and CTE.

1.4 Significance of Study

The present study provides several advantages over the previous studies considered in the literature. First, it allows the use of any continuous distribution which can be written in a closed form in the construction of a composite model. Second, by finding two important parameters which are the mixing weight and the threshold in terms of the other parameters, every distribution can be obtained. Third, this study introduces two composite models which are chosen from three well-known families of distributions as the best fitted composite models. Last, this thesis applies some risk measures to the composite models.

1.5 Research Outlines

The construction of thesis is as follows:

Chapter 2 contains litreture review and preliminary concepts such as some basic statistical concepts, brief definitions of maximum likelihood estimation (MLE), goodness-of-fit measures, namely, negative log-likelihood, (NLL), Akaike information criterion, (AIC), and Schwarz's Bayesian criterion, (SBC). At the end of this chapter, several univariate distributions in three well-known distribution families which are transformed Beta family, transformed Gamma family, and inverse transformed Gamma family are described.

In Chapter 3, by presenting a composite model, some criteria, construction, general form, and mathematical properties of the composite model will be discussed.

In Chapter 4, three well-known distribution families are considered. That is, the transformed Beta, the transformed Gamma, and the inverse transformed Gamma. For each family, one composite model is proposed based on Weibull distribution as a head and a distribution that belongs to one of the three families, as a tail. Specifically, a composite Weibull-inverse Paralogistic model and a composite Weibull-inverse transformed Gamma model are described further and their properties are given in details as they tend to give a better fit among the other composite models (see Chapter 5).

In Chapter 5, the results of fitting composite models to two well-known data sets, namely, Danish fire insurance loss data and the allocated loss adjustment expenses (ALAE) data, are presented. As a result, the composite Weibull-inverse Paralogistic model has the best fit among mentioned distributions belong to the transformed Beta family and the composite Weibull-inverse transformed Gamma model has the best fit among mentioned distributions belong to the inverse transformed Gamma family compare to the other models. The final part includes 2 examples of finding some properties of the two best fitted composite models.

In Chapter 6, two well-known risk measures are discussed. These are value-at-risk (VaR) and conditional tail expectation (CTE), and their measures in composite models are evaluated. The final part of the chapter considers the backtesting method to verify the validation of the value-at-risk and conditional tail expectation.

Chapter 7 concludes the summary and some suggestions for further research.

CHAPTER 2

LITERATURE REVIEW AND PRELIMINARY CONCEPTS

2.1 Introduction

This chapter provides literature review and some basic statistical concepts in terminology that will be used in subsequent chapters. For a random variable X, the mean, $E[X]$, the variance, $Var[X]$, the moment generating function, $M_X(t)$, the truncated distribution, the maximum likelihood estimation, (MLE), goodness-of-fit measures such as the negative log-likelihood, (NLL), Akaike information criterion, (AIC), and Schwarz's Bayesian criterion, (SBC), are discussed briefly. Also, several univariate distributions which are belonging to the transformed Beta, transformed Gamma, and inverse transformed Gamma families are presented. Probability density functions and cumulative distribution functions of univariate distributions which are described here will be used to construct a composite model in Chapters 3 and 4.

2.2 Literature Review

In the context of actuarial science, modeling loss data with unimodal distributions is an inevitable problem. In some cases, however, usage of these distributions is insufficient to describe the extreme values of the loss data. Thus, recent methods that create new parametric distributions from existing distributions seem to provide better alternatives in terms of fitting a distribution to the data. These methods include:

- Multiplication by a constant which is also called a change of scale.

- Raising to a power which leads to transformed, inverse, and inverse transformed distributions.

- Exponentiation which allows the distribution to have a heavier tail than the basic distribution.

- Mixing which leads to mixture models which are defined below:

  Suppose a random variable X has a conditional probability density function $f_{X|\Lambda}(x|\lambda)$ and a cumulative distribution function $F_{X|\Lambda}(x|\lambda)$, where λ is a parameter of X and a realization of the random variable Λ with a probability density function $f_{\Lambda}(\lambda)$. Then, the probability density function of X is,

$$f_X(x) = \int f_{X|\Lambda}(x|\lambda) f_{\Lambda}(\lambda) d\lambda,$$

The cumulative distribution function of X is,

$$F_X(x) = \int F_{X|\Lambda}(x|\lambda) f_{\Lambda}(\lambda) d\lambda,$$

The k-th moment of X is,

$$E[X^k] = E[E(X^k|\Lambda)],$$

And the variance of X is,

$$Var[X] = E[Var(X|\Lambda)] + Var[E(X|\Lambda)].$$

It is worthy to note that a special type of mixture distributions is a frailty model which is used in survival analysis to analyze lifetime distributions.

- Splicing which is defined below:

$$f_X(x) = \begin{cases} a_1 f_1(x), & c_0 < x < c_1, \\ a_2 f_2(x), & c_1 < x < c_2, \\ \vdots & \vdots \\ a_k f_k(x), & c_{k-1} < x < c_k, \end{cases} \tag{2.1}$$

where $f_i(x)$ $i = 1,2,...,k$ is the probability density functions, a_i, $i = 1,2,...,k$ denotes mixing weights and $\Sigma_{i=1}^{k} a_i = 1$, and c_i, $i = 1,2,...,k$ represents upper and lower limits of X (Hürlimann, 2010) and (Klugman et al., 2004).

This study focuses on a composite model which is a special case of splicing models. The composite model is defined in terms of two (truncated) probability density functions, mixing weights and range limit of the domain.

Cooray and Ananda (2005) introduced a composite Lognormal-Pareto model and applied this model to the entire range of loss data. The authors explained that a typical insurance payment data are highly positively skewed and distributed with a thicker upper tail. Actuarial industries commonly use Pareto distribution to model larger loss data to provide adequate coverage for higher losses. On the other hand, the Lognormal distribution is used to model smaller data with larger frequencies and/or larger data with lower frequencies. To cover these two tail behaviors, they proposed a composite model. Their composite model consists of a Lognormal density function up to a threshold value, θ, and a Pareto density function for the rest of domain. The result is a density with a shape similar to Lognormal distribution with a larger tail but smaller than the Pareto distribution. Also, they applied maximum likelihood parameter estimation methods for finding parameters of the composite Lognormal-Pareto model. This model was then fitted to Danish data on fire insurance losses as an example which was the best one compare to Lognormal and Pareto distributions.

Scollnik (2007) used the idea of a composite Lognormal-Pareto model which was introduced by Cooray and Ananda (2005). He suggested that for better fitting, having mixing weights that depends on the distribution parameters is preferred to constant coefficients according to Cooray and Ananda (2005). He considered three composite Lognormal-Pareto models and compared them with each other. The first is the composite Lognormal-Pareto model in Cooray and Ananda (2005), the second composite model is the truncated Lognormal and Pareto with a variable mixing weight which is precisely depends on the value of the model parameters, and the third composite model is comprises of Lognormal as a head and a version of generalized Pareto distribution (GDP) above the threshold value, θ, as a tail. For illustration, he applied the composite model to a well-known data set, Danish fire insurance loss data, and concluded that the composite Lognormal-generalized Pareto model showed the best fit.

Teodorescu (2010) considered a truncated composite Lognormal-Pareto model which is used to model insurance payments data. As in Teodorescu and Panaitescu (2009) paper, he explained the advantage of using a truncated distribution such that they can model payments data which appear as a deductible in non-life insurance contracts.

Pigeon and Denuit (2011) mentioned heterogeneity of the threshold parameter situation in a composite Lognormal-Pareto model that can be varied among observations. They considered three composite models. The first is the composite Lognormal-Pareto model in

Cooray and Ananda (2005) with a fixed mixing weight, the second model is the composite Lognormal-generalized Pareto model in Scollnik (2007) with dependent mixing weights on parameters, and the third model is called a mixed composite Lognormal-Pareto model, an extention of the composite Lognormal–Pareto model introduced by Scollnik (2007). In this model, $X_1,...,X_n$ are random samples of size n. Each observation has its own threshold, $\theta_1,...,\theta_n$. These thresholds are assumed as a random variable Θ with a cumulative distribution function G(.). Choosing Gamma and Lognormal distributions for the threshold distribution and computing the stop-loss transform for actuarial data sets led to the following results: using Gamma as a threshold distribution suggests that thresholds change from one contract to another, and using Lognormal threshold does not improve the composite model.

Ciumara (2006) suggested that a composite Weibull-Pareto model which has a similar shape to the Lognormal and the composite Lognormal-Pareto model can be used for the premium to be paid in case of large losses. He mentioned that the advantage of this model is that the parameters can be estimated easily. Similar to the result obtained by Cooray and Ananda (2005), the composite Weibull-Pareto model has a smaller tail than Pareto distribution and larger tail than the Weibull distribution and the restriction of using a fixed mixing weight, θ, is explained as follows: exactly $100\theta\%$ of the observations are from the Weibull model truncated at threshold and exactly 100(1- θ)% of the observations are above the threshold, θ, according to the restricted parameter of the Pareto distribution.

Preda and Ciumara (2006) compared the composite Lognormal-Pareto model which was introduced by Cooray and Ananda (2005) to the composite Weibull-Pareto model which was constructed in Ciumara (2006). These comparisons is include the density functions, cumulative distribution functions, and the k-th moments. Parameter estimations are based on two algorithms: the maximum likelihood estimation and the smooth empirical estimation of percentiles (Klugman et al., 2004). These models are used in insurance for modeling actuarial data, especially when there are some larger loss payments. For two simulated data sets, results show that the shape of these composite densities are similar and have smaller tail than the Pareto distribution and larger tail than the Lognormal and the Weibull distributions.

Cooray (2009) reviewed the construction and properties of the composite Weibull-Pareto and presented it as an alternative to several well-known distributions such as Lognormal, Loglogistic, inverse Gaussian to model the unimodal failure rate data. The composite Weibull-Pareto model has a flexible left tail from the Weibull model and longer and thick upper tail from the Pareto model as well as a closed-form survival and hazard figure that make a reasonable usage of this composite model. Several applications in biostatistical area are illustrated by real data sets of survival times which are survival times of guinea pigs infected with different doses of virulent tubercle bacilli (Bjerkedal, 1960), head and neck cancer data (Efron, 1988), non resectable gastric carcinoma data (Stablien et al., 1981), and nasopharynx cancer survival data (West, 1987).

Teodorescu and Panaitescu (2009) considered a truncated composite Weibull-Pareto model which is an alternative to the composite Lognormal-Pareto model used to model the insurance payments data. They explained that using the truncated distribution is suitable to model payments data which appear in non-life insurance contracts with deductibla. Also, they obtained some properties of the composite Weibull-Pareto model and found the parameter estimations by method of moments and maximum likelihood estimation.

Scollnik and Sun (2012) employed a truncated composite Weibull-Pareto model which allows unrestricted and flexible mixing weight in the model. They compared it with the models in Ciumara (2006) and the truncated composite Weibull-Pareto model. The results showed that the unrestricted mixing weight provides the best fit according to negative log-likelihood, (NLL), Akaike information criterion, (AIC), and Schwarz's Bayesian criterion, (SBC).

Teodorescu and Vernic (2006) introduced a composite Exponential-Pareto model. The result for this composite model is similar to the shape of the Exponential distribution but with a larger tail. This model has a better fit than the Exponential distribution for some heavy tailed insurance claims data with extreme values.

Teodorescu and Vernic (2009) proposed three composite Exponential-Pareto models as an alternative to the composite Lognormal-Pareto model in Scollnik (2007). The first model is the one introduced in Teodorescu and Vernic (2006), the second is same as the first model but with a priori unrestricted mixing weights, and the third is constructed with a Lomax distribution, a version of the generalized Pareto distribution, as a tail. Second and third composite models similar in shape with more parameters but the second composite model can be estimated easier. Hence, it is prefered to the other composite models. They provided a general form for a probability density function and a cumulative distribution function of all types of composite Exponential-Pareto models, considered some properties, and statistical inference.

A more recent paper by Nadarajah and Bakar (2012) is based on the composite Lognormal distribution that serves to improve the fit to Danish data. They introduced a new model which is called a composite Lognormal-Burr model. They estimated the parameters by using maximum likelihood method and compared the model with composite Lognormal-Pareto model to attain a better fit.

The summary of literature reviews is presented in Tables 2.1 and 2.2.

Table 2.1: Summary of the literature review

Name	Year	What Is Done	Result
Klugman et al.	2004	introduced general form of splicing (piecewise) distributions	special case: composite models
Cooray and Ananda	2005	composite Lognormal-Pareto model	model entire range of Danish data
Scollnik	2007	compare 3 composite models: Cooray and Ananda (2005), truncated distributions, and composite Lognormal-GDP model	for better fitting, preferable mixing weights depending on distribution parameters than normalized constant coefficients
Teodorescu	2010	truncated composite Lognormal-Pareto model	insurance payment data as a deductible
Pigeon and Denuit	2011	compare 3 composite models: Cooray and Ananda (2005), Scollnik (2007), and mixed composite Lognormal-Pareto model	heterogeneity of threshold (Gamma / Lognormal) in a composite Lognormal-Pareto model. computing stop-loss
Ciumara	2006	composite Weibull-Pareto model	alternative to Cooray and Ananda (2005) / used for the premium to be paid in case of large losses
Preda and Ciumara	2006	compare 2 composite models: composite Lognormal-Pareto model in Cooray and Ananda (2005) and composite Weibull-Pareto model in Ciumara (2006)	probability density functions, cumulative distribution functions, and k-th initial moments, and MLE. larger loss payments

Table 2.2: Summary of the literature review (continued Table 1.1)

Name	Year	What Is Done	Result
Cooray	2009	review composite Weibull-Pareto model	used real data of survival times and unimodal failure rate
Teodorescu and Panaitescu	2009	truncated composite Weibull-Pareto model	model the insurance payments data
Scollnik and Sun	2012	compare 3 composite Wiebull-Pareto models: Ciumara (2006), truncated composite Wiebull-Pareto model truncated composite Wiebull-generalized Pareto model	flexible mixing weight used to model loss payments
Teodorescu and Vernic	2006	composite Exponential-Pareto model	better fit than Exponential distribution for some heavy tailed insurance claims data with extreme values
Teodorescu and Vernic	2009	on a composite Exponential-Pareto model	general form for a a probability density function and a cumulative distribution function of a composite model
Nadarajah and Bakar	2012	composite Lognormal-Burr model	better fitting to Danish data

2.3 Statistical Concepts And Terminology

For a continuous random variable X with a probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$, some standard statistical functions are given as follows (Abramowitz and Stegun, 1964, p.631-641), (Finan, 2015), and (Patrik, 1980):

- Mean:

  The expectation of X, $E[X]$, also known as mean of X, is defined as:

$$E[X] = \int xf(x)dx. \tag{2.2}$$

  The mean is the most popular measure of central tendency. Since it depends on all values of the data points in a population or a sample, the mean is subjected to influence by outliers that are at the extremes of the data set.

- Variance:

  One of the well-known dispersion measures, variance of X, $Var[X]$, is:

$$Var[X] = E[X^2] - (E[X])^2. \tag{2.3}$$

  where $E[X^2]$ is the second raw moment of X.

- Moment Generating Function:

  The moment generating function of random variable X, $M_X(t)$, is:

$$M_X(t) = E[e^{tX}] = \int e^{tx} f_X(x)dx, \tag{2.4}$$

It can be shown that:

$$M_X(t) = E[e^{tX}] = E[\sum_{k=0}^{\infty} \frac{tX^k}{k!}] = \sum_{k=0}^{\infty} \frac{t^k}{k!} E[X^k]. \tag{2.5}$$

Proof:

The series expansion of e^{tX} is:

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + ... + \frac{t^kX^k}{k!} + ... \ .$$

Then,

$$\begin{aligned}
M_X(t) &= E[e^{tX}], \\
&= E[1 + tX + \frac{t^2X^2}{2!} + ... + \frac{t^kX^k}{k!} + ...], \\
&= E[\sum_{k=0}^{\infty} \frac{tX^k}{k!}], \\
&= 1 + tE[X] + \frac{t^2E[X^2]}{2!} + ... + \frac{t^kE[X^k]}{k!} + ..., \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} E[X^k].
\end{aligned}$$

□

The derivatives of the moment generating function at point $t = 0$ determine the moments of the random variable X:

$$M_X^{(k)}(0) = E[X^k], \quad k \in \mathbf{N}. \tag{2.6}$$

For example:

$$M_X^{(1)}(0) = E[X], \tag{2.7}$$

$$M_X^{(2)}(0) = E[X^2], \tag{2.8}$$

And

$$Var[X] = M_X^{(2)}(0) - [M_X^{(1)}(0)]^2. \tag{2.9}$$

- k-th Raw Moment:

  If a random variable X possesses a moment generating function $M_X(t)$, then the k-th raw moment of X, $E[X^k]$, exists and is finite for any $k \in \mathbf{N}$:

$$E[X^k] = M_X^{(k)}(0) = \frac{dM_X^{(k)}(t)}{dt}|_{t=0}. \tag{2.10}$$

  It also can be shown that:

$$E[X^k] = \int x^k f_X(x)dx. \tag{2.11}$$

- Skewness:

  For a random variable X, the skewness, γ_1, which refers the lack of symmetry is defined as:

$$\gamma_1 = \frac{E[(X - E[X])^3]}{[\sqrt{Var[X]}]^{\frac{3}{2}}}. \tag{2.12}$$

  The skewness is a measure of asymmetry of a distribution around its mean. The skewness can come in the form of zero skewness, positive skewness, and negative skewness. A perfectly symmetrical distribution has a skewness of 0, while positive (negative) skewness, $\gamma_1 > 0$ ($\gamma_1 < 0$), indicates a distribution with extreme values above (below) the mean and such as an asymmetric tail extends towards more positive (negative) values.

- Kurtosis:

  For a random variable X, the kurtosis, γ_2 is:

$$\gamma_2 = \frac{E[(X - E[X])^4]}{[Var[X]]^2}. \tag{2.13}$$

  The kurtosis is a measure of peakedness or flatness of a distribution. The positive kurtosis indicates a peaked distribution which tends to have a distinct peak, declines rather rapidly, and shorter and thinner tail. The negative kurtosis indicates a flat distribution which tends to have a flat top rather than a sharp peak, and longer and fatter tail.

- Excess Kurtosis:

  For a random variable X, the excess kurtosis, b_2 is:

$$b_2 = \frac{E[(X - E[X])^4]}{[Var[X]]^2} - 3. \tag{2.14}$$

  The excess kurtosis is simply kurtosis-3. The interpretations of excess kurtosis are:

  1. The standard reference is a Normal distribution which has a $\gamma_2 = 3, (b_2 = 0)$. Any distribution same as Normal distribution is called mesokurtic.

  2. A distribution with $\gamma_2 > 3, (b_2 > 0)$ is called leptokurtic.

  3. A distribution with $\gamma_2 < 3, (b_2 < 0)$ is called platykurtic.

- Beta Function:

  For a random variable X where $a > 0$ and $b > 0$, Beta function is:

$$B(a,b) = \int_a^b x^{a-1}(1-x)^{b-1}dx. \tag{2.15}$$

- Gamma Function:

  For a random variable X, definition of Gamma function is:

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx. \tag{2.16}$$

  It can be shown that $\Gamma(s) = (s-1)\Gamma(s-1)$ and for positive integer values, $\Gamma(s) = (s-1)!$.

2.4 Truncated Distribution

For a continuous random variable X with a probability density function $f_X(x)$ and a cumulative distribution function $F_X(x)$, the truncated probability density function which is the conditional distribution of X, $f_X^*(x)$, is:

$$f_X^*(x) = f(x|a < X \leq\ b) = \frac{f_X(x)}{F_X(b) - F_X(a)}, \quad a < X \leq b. \tag{2.17}$$

Truncation trims off the tail of a distribution such that $f_X^*(x)$ is a probability density function. That is, $\int_x f_X^*(s)ds = 1$

The truncated cumulative distribution function, $F_X^*(x)$, is then defined as follows:

$$F_X^*(x) = \begin{cases} 0, & x \leq\ a, \\ \frac{F_X(x)-F_X(a)}{F_X(b)-F_X(a)}, & a < x \leq\ b, \\ 1, & x > b. \end{cases} \tag{2.18}$$

2.5 Maximum Likelihood Estimation, (MLE)

By having a probability density function $f_X(x)$, parameter estimations can be obtained by using the maximum likelihood estimation (MLE) method which was introduced by Gauss (1821) and extended by Fisher (1935).

1. Likelihood Function, $L(\theta)$:

   Let $X_1, X_2, ..., X_n$ have a joint density function $f(X_1, X_2, ..., X_n|\theta)$. Given $X_1 = x_1, X_2 = x_2, ..., X_n = x_n$ is observed, the likelihood function $L(\theta)$, the function of parameters θ, is defined by:

   $$L(\theta) = L(\theta|x_1, ..., x_n) = \prod_{i=1}^{n} f(x_i|\theta). \tag{2.19}$$

   where

   n: the number of observations, or equivalently, the sample size,

   θ: parameters of the model.

2. Log-Likelihood Function, $l(\theta)$:

   Since the logarithm is a monotonically increasing function, maximizing the log-likelihood function, $l(\theta)$, is equivalent to maximizing the likelihood function, $L(\theta)$, hence, $l(\theta)$ is prefer as it is more convenient to work with. It is shown by:

   $$l(\theta) = \ln L(\theta) = \ln L(\theta|x_1, ..., x_n) = \sum_{i=1}^{n} \ln f(x_i|\theta). \tag{2.20}$$

   where

   n: the sample size,

   $x_1, ..., x_n$: observed data,

   θ: parameters of the model.

3. MLE of Parameters:

   The procedure to find maximum likelihood estimators of parameters starts by taking the derivation of a log-likelihood function with respect to each parameter and equating to zero.

$$\frac{\partial l(\theta)}{\partial \theta} = 0. \tag{2.21}$$

   Solving above equation, leads to find the ML estimator of θ, $\hat{\theta}_{ML}$ (Edwards, 1974).

2.6 Goodness-of-Fit Measures

Several goodness-of-fit measures are used to measure the appropriateness of the fitted models. They also assess the validity of the models considered. The measures which are considered in this study are negative log-likelihood, (NLL), Akaike information criterion, (AIC), and Schwarz's Bayesian criterion, (SBC). A lower value of measures gives an indication of the preferred model (Calderín-Ojeda and Kwok, 2015) and (Williams, 2013).

2.6.1 Negative Log-Likelihood, (NLL)

Negative log-likelihood, (NLL), is the negative value of the log-likelihood function. Optimizers usually work by minimizing the result of a function in statistical topics. The logarithm is a monotonic function, so optimizing a function is same as optimizing the logarithm of it.

Let $l(\theta)$ denotes the log-likelihood function, Equation (2.20), then the negative log-likelihood is defined by:

$$NLL = -l(\theta). \tag{2.22}$$

NLL is only applicable for comparing models with the same number of parameters. A lower value indicates a better fit of the model to the data set.

2.6.2 Akaike Information Criterion, (AIC)

Akaike information criterion, (AIC), was developed by Akaike (1974), under the name of An Information Criterion. It measures the relative goodness-of-fit of a statistical model. That is, given a collection of models for the data, AIC estimates the quality of each model, relative to each of the other models. Hence, AIC provides a tool for a model selection. It has some advantages of comparing both the nested and non-nested models. (Two models are called as a nested model when one of them contains the other as a particular case). However, the disadvantages of AIC is that it won't give any warning of the quality of models, if all the candidate models fit poorly.

Akaike information criterion is defined by:

$$AIC = 2k - 2l(\theta), \tag{2.23}$$

where k is the number of parameters of the fitted model. That is, k is the number of degrees of freedom and $l(\theta)$ is defined in Equation (2.20). Given a set of candidate models for the data, the preferred model is the one with the minimum AIC value.

Compare to the NLL, AIC not only rewards goodness-of-fit which is assessed by the likelihood function, but also includes a penalty which is adding $2k$ that is an increasing function of the number of parameters. It means the penalty discourages over-fitting.

2.6.3 Schwarz's Bayesian Criterion, (SBC)

Schwarz's Bayesian criterion, (SBC), or Bayesian information criterion, (BIC), or Schwarz-Bayesian information criterion, (SBIC), was introduced by Schwarz (1978). It allows the comparison of both nested and non-nested models.

Schwarz's Bayesian criterion, (SBC), is defined by:

$$SBC = k\ln(n) - 2l(\theta), \tag{2.24}$$

where k is the number of parameters of the fitted model, n is the number of observations, and $l(\theta)$ is defined in Equation (2.20).

The difference between AIC and BIC is that AIC uses the constant 2, whereas SBC uses $\ln(n)$ and hence, SBC penalizes model more than does the AIC. That is, the penalty term is larger in Schwarz's Bayesian criterion, than in Akaike information criterion. Trott et al. (2004) indicates that whenever two or more models are nested, the AIC may fail to choose the model that has the fewest parameters which is called the most parsimonious. In another case, if all the models are non-nested and only one is well specified, the AIC chooses the well-specified model asymptotically; Because this model has the largest value of the log-likelihood function. The SBC avoids the problem discussed above by replacing 2 in the AIC function with the $\ln(n)$ term. As $n \to \infty$, asymptotically, BIC would pick the more parsimonious model than AIC might suggest. It is clear that SBC decreases with likelihood and increases with the number of parameters, k. A smaller value of the SBC indicates a better fit among compared models.

2.7 Families of Distributions

There are several ways to arrange distributions into groups. Since there are some relationships among distributions, a change in parameter values may lead to another distribution such as when the parameters tend to their limiting values of zero or infinity. In risk models, it is necessary for all variables to have a support on the positive real line and be positively skewed. According to these conditions, three families can be considered: transformed Beta family, transformed Gamma family, and inverse transformed Gamma family. In these families, it is shown that most of distributions are special cases of each other.

In this section, distributions are presented via their probability density functions and cumulative distribution functions. Their usage will be in the following chapters. Distributions belonging to the transformed Beta family are the Burr distribution, the inverse Burr distribution, the Pareto distribution, the inverse Pareto distribution, the Loglogistic distribution, the Paralogistic distribution, the inverse Paralogistic distribution, and the generalized Pareto distribution. For the transformed Gamma and the inverse transformed Gamma families, the Weibull distribution, the inverse Weibull distribution, the Gamma distribution, the inverse Gamma distribution, the Exponential distribution, the inverse Exponential distribution, the transformed Gamma distribution, and the inverse transformed Gamma distribution are considered. Several basic properties for probability density functions can be found in the following papers (Abdelkader and Al-Marzouq, 2010), (Abramowitz and Stegun, 1964), (Dutang et al., 2008), (Johnson et al., 1995, p.210-275), (Kleiber and Kotz, 2003), (Venter, 1983), and (Walck, 2007).

2.7.1 Transformed Beta Family

In the transformed Beta family, several univariate distributions are obtained from the transformed Beta distribution. To construct a transformed Beta distribution, we need a random variable X which has a Beta distribution (a,b) as the underlying distribution. So, for $0 < x < 1$, $a > 0$, and $b > 0$, the probability density function and the cumulative distribution function are presented, respectively, as follows:

$$f_X(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1. \tag{2.25}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{B(x;a,b)}{B(a,b)}, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases} \tag{2.26}$$

where $B(.,.)$ is defined in Equation (2.15) and $B(x;.,.)$ is an incomplete lower Beta function and is defined as:

$$B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt. \tag{2.27}$$

For $0 < x < 1$, $a > 0$, and $b > 0$, random variable $Z = \frac{X}{1-X}$ with Beta distribution (type II) has a probability density function as follows:

$$f_Z(z) = \frac{z^{a-1}}{B(a,b)(1+z)^{a+b}}. \tag{2.28}$$

where $B(.,.)$ is defined in Equation (2.15).

Replacing $a = \beta$ and $b = \tau$, leads to a new random variable $Y = \lambda X^{\frac{1}{\eta}} \quad \lambda > 0, \eta > 0$ which has a transformed Beta distribution. For $0 < y < 1$, $\beta > 0$, and $\tau > 0$, the probability density function and the cumulative distribution function are as follows, respectively:

$$f_Y(y) = \frac{1}{B(\beta,\tau)} \frac{\eta(\frac{y}{\lambda})^{\tau\eta}}{y(1+(\frac{y}{\lambda})^{\eta})^{\beta+\tau}}, \quad 0 < y < 1. \tag{2.29}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{B(\frac{(\frac{y}{\lambda})^{\eta}}{1+(\frac{y}{\lambda})^{\eta}};\beta,\tau)}{B(\beta,\tau)}, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases} \tag{2.30}$$

where $B(.,.)$ and $B(y;.,.)$ are defined in Equations (2.15) and (2.27), respectively.

In order to find a specific distribution in this family, by choosing different values for parameters, we can obtain the following univariate distributions such as Burr distribution, inverse Burr distribution, Pareto distribution, inverse Pareto distribution, Loglogistic distribution, Paralogistic distribution, inverse Paralogistic distribution, and generalized Pareto distribution (Abdelkader and Al-Marzouq, 2010), (Dutang et al., 2008), (Kleiber and Kotz, 2003, p.183-234), and (Venter, 1983).

2.6.1.1 Univariate Distribution in Transformed Beta Family

Here, several univariate distributions which belongs to the transformed Beta family are presented. In addition, some properties are given.

In order to obtain the Burr distribution and the inverse Burr distribution, let $\tau = 1$ and $\beta = 1$ in transformed Beta distribution, respectively.

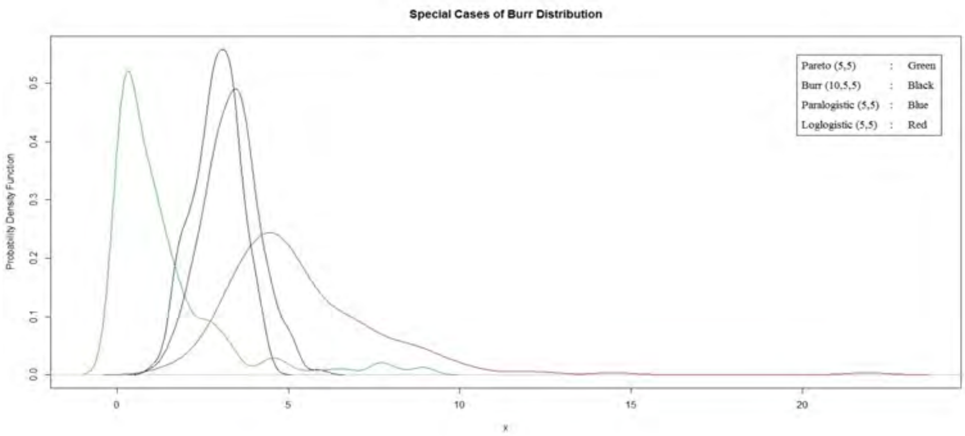

Figure 2.1: Special Cases of Burr Distribution

- Burr Distribution:

A random variable X has a Burr distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, shape parameters $\beta(\beta > 0)$ and $\eta(\eta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\beta\eta(\frac{x}{\lambda})^{\eta}}{x[1+(\frac{x}{\lambda})^{\eta}]^{\beta+1}}, \quad x > 0, \tag{2.31}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1-[1+(\frac{x}{\lambda})^{\eta}]^{-\beta}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.32}$$

Figure 2.1 presents special cases of the Burr distribution which are obtained as follows:

1. When $\beta = 1$, this is a Loglogistic distribution with parameters (η, λ).
2. When $\eta = 1$, this is a Pareto distribution with parameters (β, λ).
3. When $\beta = \eta$, this is a Paralogistic distribution with parameters (β, λ).

When X is a random variable with a Beta distribution with parameters $(1, \beta)$, then

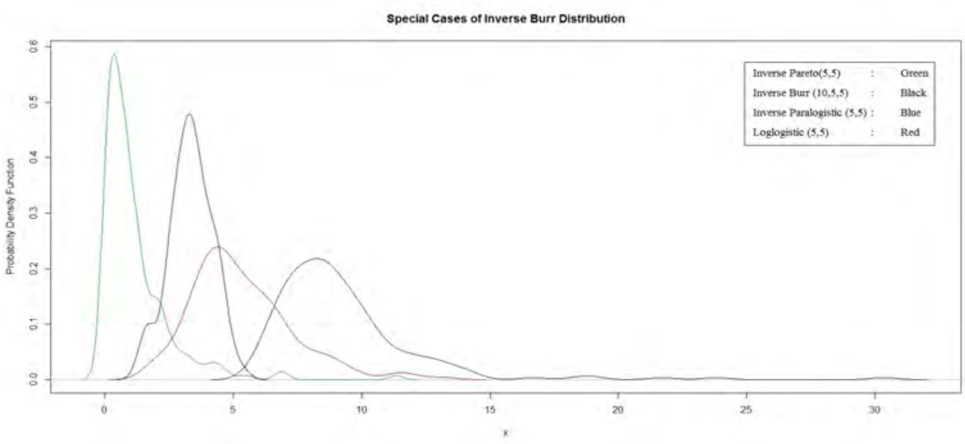

Figure 2.2: Special Cases of Inverse Burr Distribution

a random variable $Y = \lambda(\frac{X}{1-X})^{\frac{1}{\beta}}$ has a Burr distribution with parameters (β, η, λ).

- Inverse Burr Distribution:

  A random variable X has an inverse Burr distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, shape parameters $\beta(\beta > 0)$ and $\eta(\eta) > 0$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\beta\eta(\frac{x}{\lambda})^{\beta\eta}}{x[1+(\frac{x}{\lambda})^{\eta}]^{\beta+1}}, \quad x > 0, \tag{2.33}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{(\frac{x}{\lambda})^{\beta\eta}}{[1+(\frac{x}{\lambda})^{\eta}]^{\beta}}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.34}$$

  Figure 2.2 presents special cases of the inverse Burr distribution which are obtained as follows:

  1. When $\beta = 1$, this is a Loglogistic distribution with parameters (η, λ).
  2. When $\eta = 1$, this is an inverse Pareto distribution with parameters (β, λ).
  3. When $\beta = \eta$, this is an inverse Paralogistic distribution with parameters (β, λ).

When X is a random variable with a Beta distribution with parameters $(\beta, 1)$, then a random variable $Y = \lambda(\frac{X}{1-X})^{\frac{1}{\beta}}$ has an inverse Burr distribution with parameters (β, η, λ).

- Pareto Distribution:

A random variable X has a Pareto distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, a shape parameter $\beta(\beta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\beta \lambda^{\beta}}{(x+\lambda)^{\beta+1}}, \quad x > 0, \tag{2.35}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - (\frac{\lambda}{x+\lambda})^{\beta}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.36}$$

This distribution is also known as Lomax distribution which is used for reliability modeling and life testing (Hassan and Al-Ghamdi, 2009), applied to income and wealth distribution data (Atkinson and Harrison, 1978, p.12). In addition, it has found application in the biological sciences and also, for modeling the distribution of sizes of computer files on servers (Holland et al., 2006). Bryson (1974) have suggested that this distribution can be as an alternative to the Exponential distribution when the data is heavy-tailed.

- Inverse Pareto Distribution:

  A random variable X has an inverse Pareto distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, a shape parameter $\beta(\beta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\beta\lambda x^{\beta-1}}{(x+\lambda)^{\beta+1}}, \quad x > 0, \tag{2.37}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ (\frac{x}{x+\lambda})^{\beta}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.38}$$

- Loglogistic Distribution:

  A random variable X has a Loglogistic distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, a shape parameter $\beta(\beta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\beta(\frac{x}{\lambda})^{\beta}}{x[1+(\frac{x}{\lambda})^{\beta}]^2}, \quad x > 0, \tag{2.39}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{(\frac{x}{\lambda})^{\beta}}{1+(\frac{x}{\lambda})^{\beta}}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.40}$$

- Paralogistic Distribution:

  A random variable X has a Paralogistic distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, a shape parameter $\beta(\beta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\beta^2(\frac{x}{\lambda})^{\beta}}{x[1+(\frac{x}{\lambda})^{\beta}]^{\beta+1}}, \quad x > 0, \tag{2.41}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - [\frac{1}{1+(\frac{x}{\lambda})^{\beta}}]^{\beta}, & 0 \le x < \infty, \\ 1, & x \ge \infty. \end{cases} \tag{2.42}$$

- Inverse Paralogistic Distribution:

  A random variable X has an inverse Paralogistic distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, a shape parameter $\beta(\beta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\beta^2(\frac{x}{\lambda})^{\beta^2}}{x[1+(\frac{x}{\lambda})^{\beta}]^{\beta+1}}, \quad x > 0, \tag{2.43}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{(\frac{x}{\lambda})^{\beta^2}}{[1+(\frac{x}{\lambda})^{\beta}]^{\beta}}, & 0 \le x < \infty, \\ 1, & x \ge \infty. \end{cases} \tag{2.44}$$

- Generalized Pareto Distribution:

  A random variable X has a generalized Pareto distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, shape parameters $\beta(\beta > 0)$ and $\eta(\eta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\Gamma(\beta+\eta)\lambda^{\beta}x^{\eta-1}}{\Gamma(\beta)\Gamma(\eta)(x+\lambda)^{\beta+\eta}}, \quad x > 0, \tag{2.45}$$

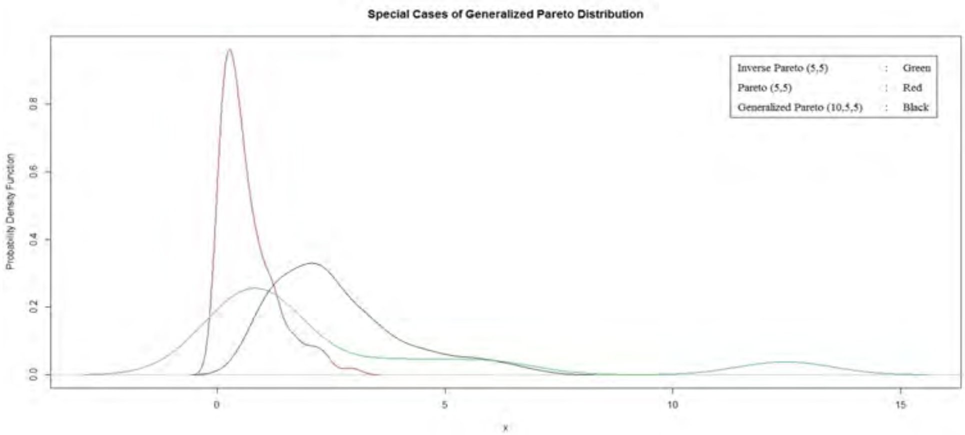

Figure 2.3: Special Cases of Generalized Pareto Distribution

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \mathbf{B}(\frac{x}{x+\lambda}; \eta, \beta), & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.46}$$

Figure 2.3 presents special cases of the generalized Pareto distribution which are obtained as follows:

1. When $\eta = 1$, this is a Pareto distribution with parameters (β, λ).

2. When $\beta = 1$, this is an inverse Pareto distribution with parameters (η, λ).

When X is a random variable with a Beta distribution with parameters (β, η), then a random variable $Y = \lambda(\frac{X}{1-X})$ is a generalized Pareto distribution with parameters (β, η, λ).

2.7.2 Transformed Gamma and Inverse Transformed Gamma Families

In the inverse transformed Gamma family, several univariate distributions are obtained from the inverse transformed Gamma distribution. To construct the inverse transformed Gamma distribution, we need a random variable X which has a Gamma distribution $(\beta, 1)$ as the underlying distribution. So, for $x > 0$ and $\beta > 0$, a probability density function and a cumulative distribution function are presented, respectively, as follows:

$$f_X(x) = \frac{1}{\Gamma(\beta)} x^{\beta-1} e^{-x}, \quad x > o. \tag{2.47}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{\gamma(\beta,x)}{\Gamma(\beta)}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.48}$$

where $\Gamma(.)$ is defined in Equation (2.16) and $\gamma(.,x)$ is the lower incomplete Gamma function which is defined as follows (Gautschi, 1979):

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt. \tag{2.49}$$

A random variable $Z = \frac{X^{\frac{1}{\tau}}}{\lambda}$ with transformed Gamma distribution can be defined and has a probability density function and a cumulative distribution function, respectively for $z > 0$, $\beta > 0$, $\tau > 0$, and $\lambda > 0$ as follows:

$$f_Z(z) = \frac{\lambda^{\beta\tau}}{\Gamma(\beta)} \tau z^{\beta\tau-1} e^{-(\lambda z)^{\tau}}, z > 0. \tag{2.50}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{\gamma(\beta,(\lambda z)^{\tau})}{\Gamma(\beta)}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.51}$$

where $\Gamma(.)$ and $\gamma(.,z)$ are defined in Equations (2.16) and (2.49), respectively.

For $\tau < 0$, replacing $\tau^* = -\tau$, leads to a new random variable $Y = \frac{1}{\lambda X^{\frac{1}{\tau^*}}}$ which has an inverse transformed Gamma distribution. For $y > 0$, $\beta > 0$, $\tau > 0$, and $\lambda > 0$, the probability density function and the cumulative distribution function are presented respectively as follows:

$$f_Y(y) = \frac{\tau^* e^{-(\lambda y)^{-\tau^*}}}{\lambda^{\beta\tau^*} y^{\beta\tau^*} \Gamma(\beta)}, y > 0. \tag{2.52}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{1-\gamma(\beta,(\lambda y)^{-\tau^*})}{\Gamma(\beta)}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.53}$$

where $\Gamma(.)$ and $\gamma(.,y)$ are defined in Equations (2.16) and (2.49), respectively.

As a result, a general form of inverse transformed Gamma family is calculated. In order to find a specific distribution in this family, we can choose different values for parameters and hence obtain several univariate distributions (Abdelkader and Al-Marzouq, 2010), (Dutang et al., 2008), (Kleiber and Kotz, 2003, p.147-182), and (Venter, 1983).

2.6.2.1 Univariate Distribution in Transformed Gamma and Inverse Transformed Gamma Families

Here, some univariate distributions which belongs to the transformed Gamma family such as the Weibull distribution, the Gamma distribution, the Exponential distribution, the transformed Gamma distribution and several univariate distributions which belongs to the inverse transformed Gamma family such as the inverse Weibull distribution, the inverse Gamma distribution, the inverse Exponential distribution, and the inverse transformed Gamma distribution are presented. Moreover, some properties are mentioned.

- Weibull Distribution:

  A random variable X has a Weibull distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, a shape parameter $\alpha(\alpha > 0)$, and a scale parameter $\sigma(\sigma > 0)$ as follows:

$$f_X(x) = \frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1}e^{-(\frac{x}{\sigma})^{\alpha}}, \quad x > 0, \tag{2.54}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-(\frac{x}{\sigma})^{\alpha}}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.55}$$

- Inverse Weibull Distribution:

  A random variable X has an inverse Weibull distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, a shape parameter $\beta(\beta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\beta(\frac{\lambda}{x})^{\beta}e^{-(\frac{\lambda}{x})^{\beta}}}{x}, \quad x > 0, \tag{2.56}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ e^{-(\frac{\lambda}{x})^{\beta}}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.57}$$

  This distribution is also known as Log-Gompertz distribution.

  When $\beta = 1$, this is an inverse Exponential distribution with parameter (λ).

- Gamma Distribution:

  A random variable X has a Gamma distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, a shape parameter $\beta(\beta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{(\frac{x}{\lambda})^{\beta} e^{-(\frac{x}{\lambda})}}{x\Gamma(\beta)}, \quad x > 0, \tag{2.58}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{\gamma(\beta, \frac{x}{\lambda})}{\Gamma(\beta)}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.59}$$

  When $\beta = 1$, this is an Exponential distribution with parameter (λ).

- Inverse Gamma Distribution:

  A random variable X has an inverse Gamma distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, a shape parameter $\beta(\beta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{(\frac{\lambda}{x})^{\beta} e^{-(\frac{\lambda}{x})}}{x\Gamma(\beta)}, \quad x > 0, \tag{2.60}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - \frac{\gamma(\beta, \frac{\lambda}{x})}{\Gamma(\beta)}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.61}$$

  When $\beta = 1$, this is an inverse Exponential distribution with parameter (λ).

  When X is a random variable with a Gamma distribution with parameters (β, λ), then a random variable $Y = \frac{1}{X}$ is an inverse Gamma distribution with parameters (β, λ).

When X is a random variable with an inverse Gamma distribution with parameters $(\frac{1}{2}, \frac{\sigma}{2})$, then a random variable $Y = X + \mu$ has a Levy distribution with parameters (μ, σ).

- Exponential Distribution:

A random variable X has an Exponential distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{e^{-(\frac{x}{\lambda})}}{\lambda}, \quad x > 0, \tag{2.62}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-(\frac{x}{\lambda})}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.63}$$

- Inverse Exponential Distribution:

A random variable X has an inverse Exponential distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\lambda e^{-(\frac{\lambda}{x})}}{x^2}, \quad x > 0, \tag{2.64}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ e^{-(\frac{\lambda}{x})}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.65}$$

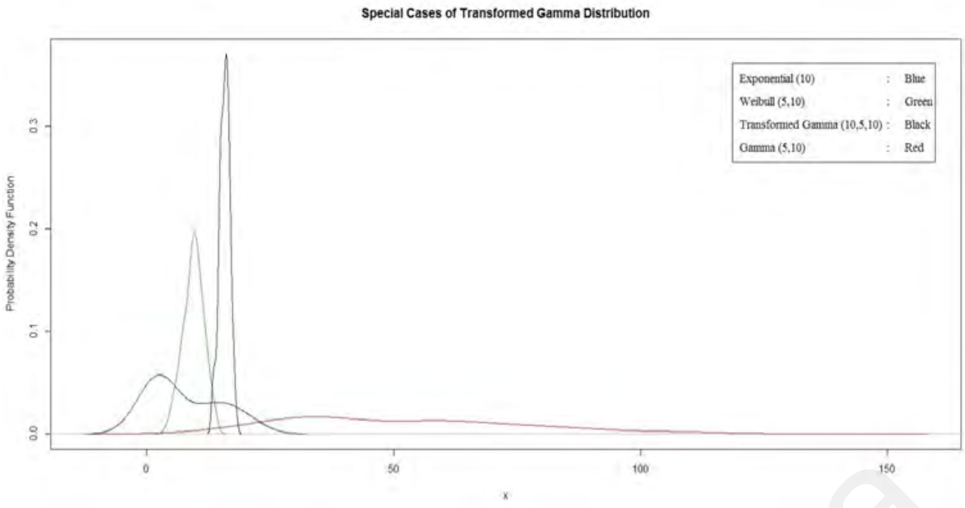

Figure 2.4: Special Cases of Transformed Gamma Distribution

- Transformed Gamma Distribution:

A random variable X has a transformed Gamma distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, shape parameters $\beta(\beta > 0)$ and $\eta(\eta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\eta(\frac{x}{\lambda})^{\beta\eta} e^{-(\frac{x}{\lambda})^{\eta}}}{x\Gamma(\beta)}, \quad x > 0, \tag{2.66}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{\gamma(\beta,(\frac{x}{\lambda})^{\eta})}{\Gamma(\beta)}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.67}$$

Figure 2.4 presents special cases of the transformed Gamma distribution which are obtained as follows:

1. When $\beta = 1$, this is a Weibull distribution with parameters (η, λ).
2. When $\eta = 1$, this is a Gamma distribution with parameters (β, λ).
3. When $\beta = \eta = 1$, this is an Exponential distribution with parameters (λ).

When X is a random variable with a Gamma distribution with parameters $(\beta, 1)$,

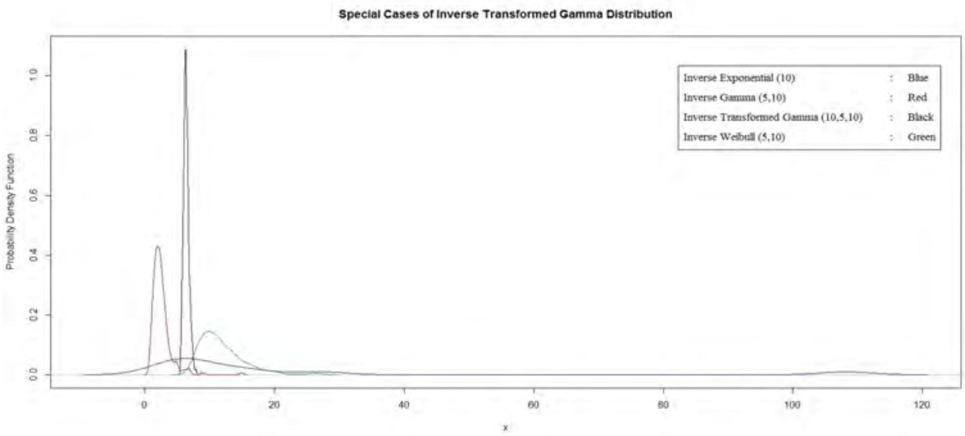

Figure 2.5: Special Cases of Inverse Transformed Gamma Distribution

then a random variable $Y = \lambda X^{\frac{1}{\eta}}$ is a transformed Gamma distribution with parameters (β, η, λ).

- Inverse Transformed Gamma Distribution:

  A random variable X has an inverse transformed Gamma distribution with a probability density function $f_X(x)$, a cumulative distribution function $F_X(x)$, shape parameters $\beta(\beta > 0)$ and $\eta(\eta > 0)$, and a scale parameter $\lambda(\lambda > 0)$ as follows:

$$f_X(x) = \frac{\eta(\frac{\lambda}{x})^{\beta\eta} e^{-(\frac{\lambda}{x})^{\eta}}}{x\Gamma(\beta)}, \quad x > 0, \tag{2.68}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - \frac{\gamma(\beta, (\frac{\lambda}{x})^{\eta})}{\Gamma(\beta)}, & 0 \leq x < \infty, \\ 1, & x \geq \infty. \end{cases} \tag{2.69}$$

Figure 2.5 presents special cases of the inverse transformed Gamma distribution which are obtained as follows:

1. When $\beta = 1$, this is an inverse Weibull distribution with parameters (η, λ).

2. When $\eta = 1$, this is an inverse Gamma distribution with parameters (β, λ).

3. When $\beta = \eta = 1$, this is an inverse Exponential distribution with parameters (λ).

When X is a random variable with a Gamma distribution with parameters $(\beta, 1)$, then a random variable $Y = \lambda X^{\frac{-1}{\eta}}$ is an inverse transformed Gamma distribution with parameters (β, η, λ).

This distribution is also known as inverse generalized Gamma distribution. More details can be found in the following papers: (Dutang et al., 2008), and (Kleiber and Kotz, 2003).

2.8 Summary

In this chapter, the litreture review and some preliminaries for used in subsequent chapters are discussed briefly. These include some statistical properties such as mean, variance, moment generating function, skewness, kurtosis, maximum likelihood estimation (MLE), some goodness-of-fit measures such as negative log-likelihood, (NLL), Akaike information criterion, (AIC), and Schwarz's Bayesian criterion, (SBC). In the last section, which is 2.6, univariate distributions which belongs to three families of distributions, the transformed Beta, the transformed Gamma, and the inverse transformed Gamma, are described for later uses.

CHAPTER 3

NEW APPROACH ON THE CONSTRUCTION OF COMPOSITE MODELS

3.1 Introduction

To construct a composite model, this chapter begins with the introduction of a composite model. First, the most two important criteria for a composite model which are continuity and differentiability are considered. Then, according to these conditions, the construction of a composite model and a general form of it are described, respectively. In this study, an important part in the construction of a composite model is a new approach to find the threshold parameter θ instead of finding one of the parameters of the head which is done in the previous issues. Hence, in this study, two parameters which are the mixing weight, ϕ, and the threshold, θ, can be estimated in terms of other parameters and makes it easy to use any distribution as a head and as a tail in a composite model. This chapter will be ended by expressing some mathematical properties for the general form of a composite model.

3.2 Composite Model

As discussed in the literature review, one way of combining distributions in order to improve the coverage of data sets is through composite models. General form of a truncated spliced distribution was shown by Nadarajah and Bakar (2012) as follows:

$$f_X(x) = \begin{cases} a_1 f_1^*(x), & c_0 < x < c_1, \\ a_2 f_2^*(x), & c_1 < x < c_2, \\ \vdots \\ a_k f_k^*(x), & c_{k-1} < x < c_k, \end{cases} \tag{3.1}$$

where,

$$f_i^*(x) = \frac{f_i(x)}{\int_{c_{i-1}}^{c_i} f_i(x)dx}, \quad i = 1,2,...,k, \tag{3.2}$$

denotes a truncated probability density function. $f_i(x)$, denotes a probability density function, a_i, is the i-th mixing weight, and c_i shows range limit of domain.

A composite model is defined in terms of two probability density functions. Let $f_1^*(x), \quad -\infty < x < \theta$, be the truncated probability density function up to the threshold point θ and supposed as a head and $f_2^*(x), \quad \theta < x < +\infty$, be the truncated probability density function and considered as a tail which can be written as a mathematical form as follows:

$$f_X(x) = \begin{cases} a_1 f_1^*(x), & -\infty < x < \theta, \\ a_2 f_2^*(x), & \theta < x < +\infty. \end{cases} \tag{3.3}$$

where $a_i \quad i = 1,2$ is the i-th mixing weight and θ is a threshold point.

3.2.1 Criteria for the Composite Model

This section describes continuity and differentiability conditions for a composite model. To obtain a smooth composite probability density function, it is necessary to have these two following conditions at the threshold θ:

3.2.1.1 Continuity

In general, a composite model is not continuous. So, to allow for continuity at a threshold point θ, that is, to join the head of distribution to the tail at the threshold θ, we have:

$$\lim_{x\to\theta^-} f_X(x) = \lim_{x\to\theta^+} f_X(x), \tag{3.4}$$

or, equivalently,

$$f_X(\theta-) = f_X(\theta+), \tag{3.5}$$

where the threshold point θ is regarded as a model parameter.

3.2.1.2 Differentiability

For a composite model, the condition of differentiability which provides smoothness in probability density function at the threshold point θ leads to:

$$\lim_{x\to\theta^-} \frac{df_X(x)}{dx} = \lim_{x\to\theta^+} \frac{df_X(x)}{dx}, \tag{3.6}$$

or, equivalently,

$$f'_X(\theta-) = f'_X(\theta+). \tag{3.7}$$

The initial probability density function, $f_X(x)$, must be continuous at the threshold θ (Nadarajah and Bakar, 2012).

3.2.2 Construction of a Composite Model

To describe the methodological procedure of constructing a composite model with any two pieces of distributions, it is worthy to mention the previous methods which was introduced and developed by Cooray and Ananda (2005), Scollnik and Sun (2012), and

Nadarajah and Bakar (2012). Improvements among these models is related to two parameters, namely, the mixing weights and one of the parameters of the head. In a first paper, the mixing weight is a normalized constant. In the last two papers, the mixing weight is consider as a dependent component on distribution parameters and the authors found the mixing weight and one of the head distribution parameter in terms of other parameters. This helps to reduce the total number of parameters to be estimated.

For the first time, Cooray and Ananda (2005) introduced a composite model as follows:

$$f_X(x) = \begin{cases} cf_1(x), & 0 < x \leq \theta, \\ cf_2(x), & \theta \leq x < +\infty, \end{cases} \tag{3.8}$$

where $f_1(x)$ is a Lognormal density function with two parameters μ and σ, $f_2(x)$ is a Pareto density function with two parameters α and θ, c is a normalizing constant, and θ is an unknown parameter such that $\theta > 0$. Here, $f_1(x)$ and $f_2(x)$ are shown as follows:

$$f_1(x) = \frac{(2\pi)^{\frac{-1}{2}}}{x\sigma} e^{\frac{1}{2}(\frac{\ln x - \mu}{\sigma})^2}, \quad x > 0, \tag{3.9}$$

and

$$f_2(x) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}, \quad x > 0, \tag{3.10}$$

where μ, σ, and α are unknown parameters such that $\mu \in \mathbf{R}$, $\sigma > 0$, and $\alpha > 0$.

For a composite Lognormal-Pareto model, the authors presented probability density function, cumulative distribution function, some properties, and maximum likelihood estimations as follows:

$$f_X(x) = \begin{cases} \frac{\alpha\theta^x}{(1+\Phi(k))x^{\alpha+1}} e^{-\frac{\alpha^2}{2k^2}\ln^2(\frac{x}{\theta})}, & 0 < x \leq \theta, \\ \frac{\alpha\theta^x}{(1+\Phi(k))x^{\alpha+1}}, & \theta \leq x < +\infty. \end{cases} \tag{3.11}$$

is the probability density function where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution and $k = 0.372$ is a known constant which is defined as $e^{-k^2} = 2\pi k^2$. Also, we have $\alpha\sigma = k$, $c = \frac{1}{1+\Phi(k)}$, $\ln\theta - \mu = \alpha\sigma^2$, and $e^{-\alpha^2\sigma^2} = 2\pi\alpha^2\sigma^2$. Therefore, a composite Lognormal-Pareto model has only two parameters: $\theta > 0$ and $\alpha > 0$.

For this composite Lognormal-Pareto model, the cumulative density function $F_X(x)$ is:

$$F_X(x) = \begin{cases} \frac{1}{(1+\Phi(k))}\Phi((\frac{\alpha}{k})\ln(\frac{x}{\theta}) + k), & 0 < x \leq \theta, \\ 1 - \frac{1}{(1+\Phi(k))}(\frac{\theta}{x})^{\alpha}, & \theta \leq x < +\infty. \end{cases} \tag{3.12}$$

The corresponding t-th moment of X is:

$$E(X^t) = \frac{\theta^t}{1+\Phi(k)}\{\Phi(k - \frac{kt}{\alpha})e^{[\frac{1}{2}(\frac{k}{\alpha})^2(t^2-2t\alpha)] + \frac{\alpha}{(\alpha-t)}}\}, \quad t < \alpha. \tag{3.13}$$

In order to obtain the parameters estimations, the maximum likelihood estimation (MLE) of parameters α and θ are:

$$\hat{\alpha}_{ML} = \frac{nk}{\sqrt{n\sum_{i=1}^{n}(\ln x_i)^2 - (\sum_{i=1}^{n}\ln x_i)^2}}. \tag{3.14}$$

and

$$\hat{\theta}_{ML} = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}} e^{\frac{k^2}{\hat{\alpha}_{ML}}}. \tag{3.15}$$

Scollnik and Sun (2012) proposed a composite model with mixing weight r which can be expressed as follows:

$$f_X(x) = \begin{cases} r f_1^*(x), & 0 < x \leq \theta, \\ (1-r) f_2^*(x), & \theta \leq x < +\infty, \end{cases} \tag{3.16}$$

where the truncated Weibull probability density function $f_1^*(x)$ and the truncated generalized Pareto density function $f_2^*(x)$ are respectively as follows:

$$f_1^*(x) = \frac{f_1(x)}{F_1(\theta)}, \tag{3.17}$$

and

$$f_2^*(x) = \frac{f_2(x)}{1 - F_2(\theta)}. \tag{3.18}$$

Thus, the probability density functions are:

$$f_1(x) = \frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}, \quad x > 0, \tag{3.19}$$

and

$$f_2(x) = \frac{\tau(\lambda+\theta)^{\tau}}{(\lambda+x)^{\tau+1}}, \quad x > 0. \tag{3.20}$$

where

$\alpha > 0$, $\sigma > 0$, $\tau > 0$, $\theta > 0$, $\lambda > -\theta$, and a mixing weight $0 \leq r \leq 1$. $F_1(\theta)$ and $F_2(\theta)$ are the cumulative distribution functions of Weibull and generalized Pareto at θ, respectively.

By applying the continuty condition, Equation (3.5), a closed form for the mixing weight r is given as follows:

$$r = \frac{\frac{\tau}{\alpha}}{\frac{\lambda+\theta}{\theta}\frac{(\frac{\theta}{\sigma})^{\alpha}}{e^{(\frac{\theta}{\sigma})^{\alpha}}-1}+\frac{\tau}{\alpha}}. \tag{3.21}$$

By applying the differentiability condition, Equation (3.7), we have:

$$\tau = \frac{\theta^2 - (\alpha(\lambda+\theta)+\theta)(1-\alpha)(\lambda+\theta)}{\theta(1-\alpha)(\lambda+\theta)-\theta^2}. \tag{3.22}$$

According to Scollnik (2007), the mixing weight r can be expressed as a function of the distributional parameters involved but it should be noted that this function may not always have an explicit form.

Following Scollnik and Sun (2012), Nadarajah and Bakar (2012) applied the continuity condition, Equation (3.5), for mixing weight r that leads to:

$$r = \frac{f_1(\theta)f_2(\theta)}{f_1(\theta)[1-F_2(\theta)]+f_2(\theta)F_1(\theta)}, \tag{3.23}$$

where $0 \leq r \leq 1$.

Also, by choosing the mixing weight r in a logistic form, that is:

$$r = \frac{1}{1+\phi}, \tag{3.24}$$

and

$$1-r = \frac{\phi}{1+\phi}. \tag{3.25}$$

The mixing weight $\phi \geq 0$ is defined as:

$$\phi = -\frac{\frac{d}{d\theta}\ln F_1(\theta)}{\frac{d}{d\theta}\ln[1-F_2(\theta)]}, \tag{3.26}$$

or, equivalently,

$$\phi = \frac{f_1(\theta)[1-F_2(\theta)]}{f_2(\theta)F_1(\theta)}. \tag{3.27}$$

Proof:

The procedure of using Equations (3.24) and (3.25) to obtain Equation (3.27) is explained by applying the continuity condition of Equation (3.5) as follows:

$$f_X(\theta-) = f_X(\theta+),$$
$$\frac{1}{1+\phi}\frac{f_1(\theta)}{F_1(\theta)} = \frac{\phi}{1+\phi}\frac{f_2(\theta)}{[1-F_2(\theta)]},$$

Then, ϕ is given as:

$$\phi = \frac{f_1(\theta)[1-F_2(\theta)]}{f_2(\theta)F_1(\theta)}.$$

□

In order to serve as a legitimate probability density function, it is necessary that the summation of the mixing weights equals to 1.

According to the new definition for the mixing weight ϕ, Equations (3.24) and (3.25), Nadarajah and Bakar (2012) defined a composite model as:

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} f_1^*(x), & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} f_2^*(x), & \theta \leq x < +\infty. \end{cases} \tag{3.28}$$

According to Equations (3.17) and (3.18), $f_1^*(x)$ and $f_2^*(x)$ are truncated density functions of the head and the tail, respectively. They assumed the Lognormal probability density function as a head and the Burr probability density function as a tail for a composite Lognormal-Burr model. The Lognormal probability density function, $f_1(x)$, and the Burr probability density function, $f_2(x)$, are respectively as follows:

$$f_1(x) = \frac{1}{x\sigma}\Psi(\frac{lnx-\mu}{\sigma}), \quad x > 0, \tag{3.29}$$

and

$$f_2(x) = \frac{\alpha\beta(\frac{x}{s})^{\beta}}{x[1+(\frac{x}{s})^{\beta}]^{\alpha+1}}, \quad x > 0. \tag{3.30}$$

where $\Psi(.)$ is the standard probability density function, $\mu \in \mathbf{R}, \sigma > 0, \alpha > 0, \beta > 0$, and $s > 0$.

By applying the continuty condition, Equation (3.5), we have:

$$\phi = \frac{(\theta^{\beta}+s^{\beta})\Psi(\frac{lnx-\mu}{\sigma})}{\sigma\alpha\beta\theta^{\beta}\Phi(\frac{lnx-\mu}{\sigma})}, \tag{3.31}$$

where $\mu \in \mathbf{R}, \sigma > 0, \alpha > 0, \beta > 0$, $s > 0$, $\theta > 0$, $\Psi(.)$ denotes the standard Normal probability density function, and $\Phi(.)$ denotes the standard Normal cumulative distribution function.

By applying the differentiability condition, Equation (3.7), we have:

$$\mu = \ln\theta - \sigma^2\{\frac{(\alpha+1)\beta\theta^{\beta}}{\theta^{\beta}+s^{\beta}} - \beta\}. \tag{3.32}$$

Therefore, the composite Lognormal-Burr model has five unknown parameters $\theta > 0, \sigma > 0, \alpha > 0, \beta > 0$, and $s > 0$.

3.3 A New Approach in Constructing a Composite Model

In this study, a new approach to develop in constructing a composite model is proposed. This approach is related to the threshold parameter θ that can be found in terms of the other parameters. The advantage of using this method is that by having the mixing weight ϕ and the threshold θ and choosing any distribution for both parts of a composite model, the head and the tail, composite model can be constructed easily (Abu Bakar et al., 2015).

For this new approach, methodological procedure of constructing a composite model is described as follows:

1. Any two pieces of distributions can be considered as a head and a tail of a composite model as in Equation (3.28). So, we have

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} f_1^*(x), & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} f_2^*(x), & \theta \leq x < +\infty. \end{cases}$$

   According to Equations (3.17) and (3.18), $f_1^*(x) = \frac{f_1(x)}{F_1(\theta)}$ and $f_2^*(x) = \frac{f_2(x)}{1-F_2(\theta)}$ are truncated probability density functions, respectively, ϕ is a mixing weight, and θ is a threshold point.

2. By applying the continuity condition, Equation (3.5), the head is joined to the tail of the distribution at a threshold θ.

3. By applying the differentiability condition, Equation (3.7), the smoothness of the distribution is ensured.

4. Mixing Weight ϕ:

As it is shown in Equation (3.27), the mixing weight $\phi \geq 0$ is defined as:

$$\phi = \frac{f_1(\theta)[1 - F_2(\theta)])}{f_2(\theta)F_1(\theta)}.$$

5. Threshold θ:

The parameter θ can be obtained through the following equation by solving in terms of θ:

$$\frac{d}{d\theta} \ln \frac{f_1(\theta)}{f_2(\theta)} = 0. \qquad (3.33)$$

Although a closed form of θ may not exist, it may be solved by using numerical methods.

Proof:

Equation (3.33) is obtained from the differentiability condition, Equation (3.7) as follows:

$$f'_X(\theta-) = f'_X(\theta+),$$

$$\frac{d}{d\theta}[\frac{1}{1+\phi}\frac{f_1(\theta)}{F_1(\theta)}] = \frac{d}{d\theta}[\frac{\phi}{1+\phi}\frac{f_2(\theta)}{1-F_2(\theta)}],$$

$$\frac{1}{1+\phi}\frac{\frac{d}{d\theta}f_1(\theta)}{F_1(\theta)} = \frac{\phi}{1+\phi}\frac{\frac{d}{d\theta}f_2(\theta)}{1-F_2(\theta)}.$$

By replacing Equation (3.27), we have:

$$\frac{d}{d\theta}f_1(\theta) = \frac{f_1(\theta)}{f_2(\theta)}\frac{d}{d\theta}f_2(\theta),$$

$$\frac{\frac{d}{d\theta}f_1(\theta)}{f_1(\theta)} - \frac{\frac{d}{d\theta}f_2(\theta)}{f_2(\theta)} = 0,$$

$$\frac{d}{d\theta}\ln f_1(\theta) - \frac{d}{d\theta}\ln f_2(\theta) = 0,$$

Then,

$$\frac{d}{d\theta}\ln\frac{f_1(\theta)}{f_2(\theta)} = 0.$$

□

3.3.1 General Form of a Composite Model

As a result of previous discussion, general closed forms of the probability density function $f_X(x)$ and the cumulative distribution function $F_X(x)$ of a composite model can be written as follows:

$$f_X(x) = \begin{cases} \frac{1}{1+\phi}\frac{f_1(x)}{F_1(\theta)}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi}\frac{f_2(x)}{1-F_2(\theta)}, & \theta \leq x < +\infty. \end{cases} \tag{3.34}$$

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{1+\phi}\frac{F_1(x)}{F_1(\theta)}, & 0 \leq x < \theta, \\ \frac{1}{1+\phi} + \frac{\phi}{1+\phi}\frac{F_2(x)-F_2(\theta)}{1-F_2(\theta)}, & \theta \leq x < +\infty, \\ 1, & x \geq +\infty. \end{cases} \tag{3.35}$$

where $f_1(x)$ and $f_2(x)$ are probability density functions of a head and a tail, respectively, $F_1(\theta)$ and $F_2(\theta)$ are the corresponding cumulative distribution functions. The mixing weight ϕ has a form of Equation (3.27) and the threshold θ can be found by Equation (3.33) (Teodorescu and Vernic, 2009).

In general, by having a probability density function $f_X(x)$ and a cumulative probability function $F_X(x)$ of a composite model, a p-th quantile in point x_p is obtained such that $F_X(x_p) = p$.

$$Q_p = \begin{cases} F_1^{-1}[p(1+\phi)F_1(\theta)], & 0 < p \leq \frac{1}{1+\phi}, \\ F_2^{-1}[\frac{p(1+\phi)-1}{\phi}[1-F_2(\theta)]+F_2(\theta)], & \frac{1}{1+\phi} \leq p < 1. \end{cases} \tag{3.36}$$

where $F_1(\theta)$ and $F_2(\theta)$ are the cumulative distribution functions of a head and a tail, respectively, $F_1^{-1}(.)$ and $F_2^{-1}(.)$ are the corresponding inverse cumulative distribution functions. The mixing weight ϕ has a form of Equation (3.27) and the threshold θ can be obtained by Equation (3.33) (Nadarajah and Bakar, 2013).

3.4 Summary

In this chapter, the construction of a composite model is explained and the general forms of probability density function and cumulative distribution function are presented. A new method of finding parameters in terms of two well-known parameters, the mixing weight ϕ and the threshold θ parameters, is presented. As a result, the advantage of using the new method can be explained as follows: in the previous methods, applying continuity and differentiability conditions led to finding the mixing weight ϕ and one of the parameters of the head but in the new approach, in each composite model regardless of distributions used to describe as a head and a tail, one can easily estimate the mixing weight ϕ and the threshold θ in terms of other parameters in the model. This chapter concludes by presenting a general form of the probability density function, the cumulative distribution function, and quantiles of a composite model.

CHAPTER 4

NEWLY DEVELOPED COMPOSITE MODELS

4.1 Introduction

Reasons for choosing the Weibull distribution as a head and a distribution belonging to one of the three famous distribution families, which are transformed Beta, transformed Gamma, and inverse transformed Gamma, as a tail are stated at the begining of this chapter. Several composite models with a Weibull distribution as a head and other distributions as a tail are proposed. Among composite models belonging to mentioned families, the composite Weibull-inverse Paralogistic and the composite Weibull-inverse transformed Gamma are chosen as the best fitting models to the real data sets described in Chapter 5 and hence, these two composite models explained more in details by finding their parameter estimations via MLE method and the mathematical properties. For other composite models, the k-th raw moments and moment generating functions are obtained in APPENDIX A.

4.2 Weibull Distribution as a Head for a Composite Model

By introducing a new approach to develop a composite model, any type of distributions can be chosen as a head and a tail. In this study, a Weibull distribution is considered as a head and different distributions belonging to the transformed Beta, transformed Gamma, and inverse transformed Gamma families as a tail (Bourguignon et al., 2014), (Meyers, 2005), (Teodorescu and Panaitescu, 2009), and (Venter, 1983).

Since this study is related to an actuarial science and the Weibull distribution has a more usage in insurance, the reasons of using the Weibull distribution are as follows:

- Since the Weibull distribution is a positively skewed distribution with a long tail, it is suitable to use for claim distributions.
- The Weibull distribution is a very flexible distribution which can be used as a model for losses in insurance.
- The Weibull distribution is widely used in reliability and life data analysis due to its versatility (JMP and Proust, 2013).
- Depending on the values of the parameters, the Weibull distribution can be used to model a variety of life behaviors (JMP and Proust, 2013).
- In general insurance (non-life insurance), the Weibull distribution has been used to model the size of reinsurance claims (Teodorescu and Panaitescu, 2009).

In actuary, insurance claims seem to be skewed to the right. So, the logical way to model the loss data sets is using a proper distribution with a long tail. Among different families of distributions, three well-known families are chosen, namely, the transformed Beta, the transformed Gamma, and the inverse transformed Gamma. Generally, the transformed Beta family is a good candidate for casualty loss severity distributions with a heavy tailed distribution. Also, this family offers the possibility of improved fits. The transformed Gamma and the inverse transformed Gamma families used in casualty loss severity distributions where the tail is long. These families have been found to be accurate and efficient when they model claims which are positive and usually positively skewed with variances often proportional to the mean squared (Venter, 1983).

4.3 Composite Models in the Transformed Beta Family

In this section, for constructing composite models, a Weibull distribution is employed as a head and some univariate distributions from transformed Beta family which are popular in modeling loss severity and considered in 2.7.1.1, are chosen as a tail of composite models. These univariate distributions includes the Burr distribution, the inverse Burr distribution, the Pareto distribution, the inverse Pareto distribution, the Loglogistic distribution, the Paralogistic distribution, the inverse Paralogistic distribution, and the generalized Pareto distribution. To describe the procedure of the new method, the composite Weibull-inverse Paralogistic is considered and is explained in details.

4.3.1 Composite Weibull-Inverse Paralogistic Model

A composite Weibull-inverse Paralogistic model introduced contains the Weibull distribution as the head and the inverse Paralogistic distribution as the tail. According to Equation (3.28), for a random variable $X > 0$, we have:

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} f_1^*(x), & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} f_2^*(x), & \theta \leq x < +\infty. \end{cases}$$

where $f_1^*(x)$ is the truncated Weibull probability density function and $f_2^*(x)$ is the truncated inverse Paralogistic probability density function.

By considering Equations (2.43), (2.44), (2.54), and (2.55), $f(x)$ can be rewritten as a close form as follows:

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\beta^2(\frac{x}{\lambda})^{\beta^2}[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}}{x[1+(\frac{x}{\lambda})^{\beta}]^{\beta+1}[[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}-(\frac{\theta}{\lambda})^{\beta^2}]}, & \theta \leq x < +\infty. \end{cases} \tag{4.1}$$

where $\alpha > 0, \sigma > 0, \beta > 0$, $\lambda > 0$, $\theta > 0$, and $\phi > 0$.

Applying Equation (3.27) at the threshold θ leads to find ϕ:

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1}e^{-(\frac{\theta}{\sigma})^{\alpha}}\theta[1+(\frac{\theta}{\lambda})^{\beta}][[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}-(\frac{\theta}{\lambda})^{\beta^2}]}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\beta^2(\frac{\theta}{\lambda})^{\beta^2}}. \tag{4.2}$$

and applying Equation (3.33) and solving Equation (4.3) in terms of θ by using numerical methods, leads to find the parameter θ:

$$(\alpha-\beta^2)\frac{1}{\theta}-\frac{\alpha}{\sigma^{\alpha}}\theta^{\alpha-1}+(\beta+1)\frac{\beta\theta^{\beta-1}}{\lambda^{\beta}+\theta^{\beta}}=0. \tag{4.3}$$

Hence, the composite Weibull-inverse Paralogistic model has four unknown parameters $\alpha>0, \sigma>0, \beta>0$, and $\lambda>0$.

4.3.1.1 Parameter Estimation

By having the probability density function for the composite Weibull-inverse Paralogistic model as Equation (4.1), the maximum likelihood estimation, Equation (2.20), can be used as:

$$\begin{aligned}
l(\theta) &= \ln L(\theta|x_1,\dots,x_n),\\
&= \sum_{i=1}^{n}\ln f(x_i|\theta),\\
&= \sum_{i=1}^{n}\ln[\frac{1}{1+\phi}\frac{f_1(x_i)}{F_1(\theta)}I_{(0,\theta]}(x_i)+\sum_{i=1}^{n}\frac{\phi}{1+\phi}\frac{f_2(x_i)}{1-F_2(\theta)}I_{[\theta,+\infty)}(x_i)],\\
&= \sum_{i=1}^{n}\ln[\frac{1}{1+\phi}\frac{\frac{\alpha}{\sigma}(\frac{x_i}{\sigma})^{\alpha-1}e^{-(\frac{x_i}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}I_{(0,\theta]}(x_i),\\
&\quad+\frac{\phi}{1+\phi}\frac{\beta^2(\frac{x_i}{\lambda})^{\beta^2}[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}}{x_i[1+(\frac{x_i}{\lambda})^{\beta}]^{\beta+1}[[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}-(\frac{\theta}{\lambda})^{\beta^2}]}I_{[\theta,+\infty)}(x_i)].
\end{aligned} \tag{4.4}$$

The estimation of parameters can be found by differentiating Equation (4.4) in terms of each parameter as presented in Equation (2.20) (Hassan and Al-Ghamdi, 2009).

To estimate all parameters in the composite model, except the mixing weight ϕ and the threshold θ, we have to solve the simultaneous equation, Equation (4.5), as a likelihood system in terms of all parameters. Thus, $\hat{\alpha}_{ML}$, $\hat{\sigma}_{ML}$, $\hat{\beta}_{ML}$, and $\hat{\lambda}_{ML}$ will be obtained. In Chapter 5, these parameter estimators are obtained by using software **R**.

$$\left.\begin{aligned}
\frac{\partial l(\theta)}{\partial \alpha} &= \frac{n}{\alpha} + n(\alpha-1)^{n-1}\textstyle\sum_{i=1}^{n} \ln(\frac{x_i}{\sigma}) - \sum_{i=1}^{n}(\frac{x_i}{\sigma})^{\alpha}\ln(\frac{x_i}{\sigma}) - \frac{n\frac{\theta}{\sigma}\ln(\frac{-\theta}{\sigma})e^{-(\frac{\theta}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}} = 0, \\
\frac{\partial l(\theta)}{\partial \sigma} &= \frac{-n\alpha}{\sigma} - (\alpha-1)^{n}\frac{n}{\sigma} + \textstyle\sum_{i=1}^{n}\frac{\alpha x_i^{\alpha}}{\sigma^{\alpha+1}} + \frac{n\alpha\theta^{\alpha}e^{-(\frac{\theta}{\sigma})}}{\sigma^{\alpha+1}[1-e^{-(\frac{\theta}{\sigma})}]} = 0, \\
\frac{\partial l(\theta)}{\partial \beta} &= \frac{2n}{\beta} + 2n\beta^{2n-1}\Sigma_{i=1}^{n}\ln(\frac{x_i}{\lambda}) + n\ln[1+(\frac{\theta}{\lambda})^{\beta}] + n\beta\frac{(\frac{\theta}{\lambda})^{\beta}\ln(\frac{\theta}{\lambda})}{1+(\frac{\theta}{\lambda})^{\beta}} \\
&\quad -(\beta+1)^{n}\frac{d}{d\beta}[\Sigma_{i=1}^{n}\ln[1+(\frac{x_i}{\lambda})^{\beta}]^{\beta+1} - n\ln[[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta} - (\frac{\theta}{\lambda})^{\beta^2}]] = 0, \\
\frac{\partial l(\theta)}{\partial \lambda} &= -\frac{n\beta^2 n}{\lambda} - \frac{n\beta^2\theta^{\beta}}{\lambda^{\beta+1}[1+(\frac{x_i}{\lambda})^{\beta}]} + (\beta+1)^{n}\Sigma_{i=1}^{n}\frac{\beta(\beta+1)x_i^{\beta}}{\lambda^{\beta+1}[1+(\frac{x_i}{\lambda})^{\beta}]} \\
&\quad - \frac{n[\frac{\theta^{\beta}\beta^2[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta+1}}{\lambda^{\beta+1}} + \frac{\beta^2\theta^{\beta^2}}{\lambda^{\beta^2+1}}]}{[[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta} - (\frac{\theta}{\lambda})^{\beta^2}]} = 0.
\end{aligned}\right\} \tag{4.5}$$

4.3.1.2 Properties

By having the probability density function for the composite Weibull-inverse Paralogistic model as Equation (4.1), the mean of X, $E[X]$, the second raw moment of X, $E[X^2]$, the variance of X, $Var[X]$, the k-th raw moment of X, $E[X^k]$, the moment generating function of X, $M_X(t)$, the skewness γ_1, and the excess kurtosis b_2 can be calculated below.

The mean of X, $E[X]$, is obtained according to Equation (2.1) as follows:

$$
\begin{aligned}
E[X] &= \int x f_X(x)dx, \\
&= \int_0^{\theta} x \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}} dx \\
&\quad + \int_{\theta}^{\infty} x \frac{\phi}{1+\phi} \frac{\beta^2 (\frac{x}{\lambda})^{\beta^2} [1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}}{x[1+(\frac{x}{\lambda})^{\beta}]^{\beta+1} [[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta} - (\frac{\theta}{\lambda})^{\beta^2}]} dx, \\
&= \frac{1}{1+\phi} \frac{\alpha}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}} \int_0^{\theta} (\frac{x}{\sigma})^{\alpha} e^{-(\frac{x}{\sigma})^{\alpha}} dx \\
&\quad + \frac{\phi}{1+\phi} \frac{\beta^2 [1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}}{[[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta} - (\frac{\theta}{\lambda})^{\beta^2}]} \int_{\theta}^{\infty} \frac{(\frac{x}{\lambda})^{\beta^2}}{[1+(\frac{x}{\lambda})^{\beta}]^{\beta+1}} dx,
\end{aligned}
$$

Let:

$$
\begin{aligned}
&(\frac{x}{\sigma})^{\alpha} = u, \quad \Rightarrow \quad x = \sigma u^{\frac{1}{\alpha}}, \quad dx = (u^{\frac{1}{\alpha}-1}) \frac{\sigma du}{\alpha}. \\
&1+(\frac{x}{\lambda})^{\beta} = \frac{1}{u}, \quad \Rightarrow \quad x = \lambda [\frac{1-u}{u}]^{\frac{1}{\beta}}, \quad dx = \frac{-\lambda}{\beta} \frac{du}{u^{1+\frac{1}{\beta}} (1-u)^{1-\frac{1}{\beta}}}.
\end{aligned}
$$

Then, $E[X]$ can be rewritten as follows:

$$
\begin{aligned}
E[X] &= \frac{1}{1+\phi} \frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}} \int_0^{(\frac{\theta}{\sigma})^{\alpha}} u^{\frac{1}{\alpha}} e^{-u} du \\
&\quad - \frac{\phi}{1+\phi} \frac{\beta\lambda [1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}}{[[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta} - (\frac{\theta}{\lambda})^{\beta^2}]} \int_0^{[1+(\frac{\theta}{\lambda})^{\beta}]^{-1}} u^{-\frac{1}{\beta}} (1-u)^{\beta+\frac{1}{\beta}-1} du,
\end{aligned}
$$

By applying Equations (2.27) and (2.49), we will have:

$$
\begin{aligned}
E[X] &= \frac{1}{1+\phi} \frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}} \gamma(\frac{1}{\alpha}+1, (\frac{\theta}{\sigma})^{\alpha}) \\
&\quad - \beta\lambda \frac{\phi}{1+\phi} \frac{[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}}{[[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta} - (\frac{\theta}{\lambda})^{\beta^2}]} B([1+(\frac{\theta}{\lambda})^{\beta}]^{-1}; 1-\frac{1}{\beta}, \beta+\frac{1}{\beta}).
\end{aligned}
\tag{4.6}
$$

The second raw moment of X, $E[X^2]$, is obtained according to Equation (2.11) for $k=2$ as follows:

$$\begin{aligned}
E[X^2] &= \int x^2 f(x)dx. \\
&= \int_0^\theta x^2 \frac{1}{1+\phi}\frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1}e^{-(\frac{x}{\sigma})^\alpha}}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}dx \\
&\quad + \int_\theta^\infty x^2 \frac{\phi}{1+\phi}\frac{\beta^2(\frac{x}{\lambda})^{\beta^2}[1+(\frac{\theta}{\lambda})^{\beta^2}]^\beta}{x[1+(\frac{x}{\lambda})^\beta]^{\beta+1}[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}dx, \\
&= \frac{1}{1+\phi}\frac{\alpha}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\int_0^\theta x(\frac{x}{\sigma})^\alpha e^{-(\frac{x}{\sigma})^\alpha}dx \\
&\quad + \frac{\phi}{1+\phi}\frac{\beta^2[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}\int_\theta^\infty x\frac{(\frac{x}{\lambda})^{\beta^2}}{[1+(\frac{x}{\lambda})^\beta]^{\beta+1}}dx,
\end{aligned}$$

Let:

$$\begin{aligned}
&(\frac{x}{\sigma})^\alpha = u, \quad \Rightarrow \quad x=\sigma u^{\frac{1}{\alpha}}, \quad dx=(u^{\frac{1}{\alpha}-1})\frac{\sigma du}{\alpha}. \\
&1+(\frac{x}{\lambda})^\beta = \frac{1}{u}, \quad \Rightarrow \quad x=\lambda[\frac{1-u}{u}]^{\frac{1}{\beta}}, \quad dx=\frac{-\lambda}{\beta}\frac{du}{u^{1+\frac{1}{\beta}}(1-u)^{1-\frac{1}{\beta}}}.
\end{aligned}$$

Then, $E[X^2]$ can be rewritten as follows:

$$\begin{aligned}
E[X^2] &= \frac{1}{1+\phi}\frac{\sigma^2}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\int_0^{(\frac{\theta}{\sigma})^\alpha} u^{\frac{2}{\alpha}}e^{-u}du \\
&\quad - \frac{\phi}{1+\phi}\frac{\beta\lambda^2[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}\int_0^{[1+(\frac{\theta}{\lambda})^\beta]^{-1}} u^{-\frac{2}{\beta}}(1-u)^{\beta+\frac{2}{\beta}-1}du,
\end{aligned}$$

By applying Equations (2.27) and (2.49), we have:

$$\begin{aligned}
E[X^2] &= \frac{1}{1+\phi}\frac{\sigma^2}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{2}{\alpha}+1,(\frac{\theta}{\sigma}^\alpha)) \\
&\quad -\beta\lambda^2\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{2}{\beta},\beta+\frac{2}{\beta}). \qquad (4.7)
\end{aligned}$$

The variance of X, $Var[X]$, is obtained according to Equation (2.3) and using Equations (4.6) and (4.7) as follows:

$$
\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2. \\
&= \frac{1}{1+\phi}\frac{\sigma^2}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{2}{\alpha}+1,(\frac{\theta}{\sigma}^{\alpha})) \\
&\quad -\beta\lambda\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{2}{\beta},\beta+\frac{2}{\beta}) \\
&\quad -[\frac{1}{1+\phi}\frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma}^{\alpha})) \\
&\quad -\beta\lambda\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{1}{\beta},\beta+\frac{1}{\beta})]^2. \qquad (4.8)
\end{aligned}
$$

The k-th raw moment of X, $E[X^k]$, is obtained according to Equation (2.11) as follows:

$$
\begin{aligned}
E[X^k] &= \int x^k f(x)dx. \\
&= \int_0^\theta x^k \frac{1}{1+\phi}\frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1}e^{-(\frac{x}{\sigma})^\alpha}}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}dx \\
&\quad +\int_\theta^\infty x^k\frac{\phi}{1+\phi}\frac{\beta^2(\frac{x}{\lambda})^{\beta^2}[1+(\frac{\theta}{\lambda})^{\beta^2}]^\beta}{x[1+(\frac{x}{\lambda})^\beta]^{\beta+1}[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}dx, \\
&= \frac{1}{1+\phi}\frac{\alpha}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\int_0^\theta x^{k-1}(\frac{x}{\sigma})^\alpha e^{-(\frac{x}{\sigma})^\alpha}dx \\
&\quad +\frac{\phi}{1+\phi}\frac{\beta^2[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}\int_\theta^\infty x^{k-1}\frac{(\frac{x}{\lambda})^{\beta^2}}{[1+(\frac{x}{\lambda})^\beta]^{\beta+1}}dx, \qquad (4.9)
\end{aligned}
$$

Let:

$$
\begin{aligned}
&(\frac{x}{\sigma})^\alpha = u, \quad \Rightarrow \quad x=\sigma u^{\frac{1}{\alpha}}, \quad dx=(u^{\frac{1}{\alpha}-1})\frac{\sigma du}{\alpha}. \\
&1+(\frac{x}{\lambda})^\beta=\frac{1}{u}, \quad \Rightarrow \quad x=\lambda[\frac{1-u}{u}]^{\frac{1}{\beta}}, \quad dx=\frac{-\lambda}{\beta}\frac{du}{u^{1+\frac{1}{\beta}}(1-u)^{1-\frac{1}{\beta}}}.
\end{aligned}
$$

Then, $E[X^k]$ can be rewritten as follows:

$$E[X^k] = \frac{1}{1+\phi}\frac{\sigma^k}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\int_0^{(\frac{\theta}{\sigma})^\alpha} u^{\frac{k}{\alpha}}e^{-u}du$$
$$-\frac{\phi}{1+\phi}\frac{\beta\lambda^k[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}\int_0^{[1+(\frac{\theta}{\lambda})^\beta]^{-1}} u^{-\frac{k}{\beta}}(1-u)^{\beta+\frac{k}{\beta}-1}du,$$

By applying Equations (2.27) and (2.49), we will have:

$$E[X^k] = \frac{1}{1+\phi}\frac{\sigma^k}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{k}{\alpha}+1,(\frac{\theta}{\sigma}^\alpha))$$
$$-\beta\lambda^k\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{k}{\beta},\beta+\frac{k}{\beta}). \quad (4.10)$$

The moment generating function of X, $M_X(t)$, is obtained according to Equation (2.4) as follows:

$$M_X(t) = E[e^{tX}],$$
$$= \int_0^\infty e^{tx}f_X(x)dx,$$
$$= \int_0^\theta e^{tx}\frac{1}{1+\phi}\frac{f_1(x)}{F_1(\theta)}dx + \int_\theta^\infty e^{tx}\frac{\phi}{1+\phi}\frac{f_2(x)}{1-F_2(\theta)}dx,$$
$$= \frac{1}{1+\phi}\frac{1}{F_1(\theta)}\int_0^\theta e^{tx}f_1(x)dx + \frac{\phi}{1+\phi}\frac{1}{1-F_2(\theta)}\int_\theta^\infty e^{tx}f_2(x)dx.$$

According to Equations (2.5) and (4.10), we have:

$$M_X(t) = \frac{1}{1+\phi}\frac{1}{[1-e^{-(\frac{\theta}{\sigma})^\alpha}]}\sum_{k=0}^\infty\frac{t^k}{k!}\sigma^k\gamma(\frac{k}{\alpha}+1,(\frac{\theta}{\sigma})^\alpha)$$
$$-\frac{\phi}{1+\phi}\sum_{k=0}^\infty\frac{t^k}{k!}\beta\lambda^k\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{k}{\beta},\beta+\frac{k}{\beta}).$$

(4.11)

The skewness of X, γ_1, can be found by using Equation (2.12) as follows:

$$
\begin{aligned}
\gamma_1 &= \frac{E[(X-E[X])^3]}{[Var[X]]^{\frac{3}{2}}},\\
&= E[(X - \frac{1}{1+\phi}\frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma}^\alpha))\\
&\quad + \beta\lambda\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{1}{\beta},\beta+\frac{1}{\beta}))^3]/\\
&[\frac{1}{1+\phi}\frac{\sigma^2}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{2}{\alpha}+1,(\frac{\theta}{\sigma}^\alpha))\\
&\quad - \beta\lambda\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{2}{\beta},\beta+\frac{2}{\beta})\\
&\quad - [\frac{1}{1+\phi}\frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma}^\alpha))\\
&\quad - \beta\lambda\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{1}{\beta},\beta+\frac{1}{\beta})]^2]^{\frac{3}{2}}. \qquad (4.12)
\end{aligned}
$$

The excess kurtosis of X, b_2, can be found by using Equation (2.14) as follows:

$$
\begin{aligned}
b_2 &= \frac{E[(X-E[X])^4]}{[Var[X]]^2}-3,\\
&= [E[(X - \frac{1}{1+\phi}\frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma}^\alpha))\\
&\quad + \beta\lambda\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{1}{\beta},\beta+\frac{1}{\beta}))^4]/\\
&[\frac{1}{1+\phi}\frac{\sigma^2}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{2}{\alpha}+1,(\frac{\theta}{\sigma}^\alpha))\\
&\quad - \beta\lambda\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{2}{\beta},\beta+\frac{2}{\beta})\\
&\quad - [\frac{1}{1+\phi}\frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma}^\alpha)\\
&\quad - \beta\lambda\frac{\phi}{1+\phi}\frac{[1+(\frac{\theta}{\lambda})^\beta]^\beta}{[[1+(\frac{\theta}{\lambda})^\beta]^\beta-(\frac{\theta}{\lambda})^{\beta^2}]}B([1+(\frac{\theta}{\lambda})^\beta]^{-1};1-\frac{1}{\beta},\beta+\frac{1}{\beta})]^2]^2-3. \qquad (4.13)
\end{aligned}
$$

Several composite models based on the Weibull distribution as the head are presented below. By following the procedure as described above to find $\phi > 0$ and $\theta > 0$, expressions for these new composite models are given. It is worthy to note that the tail distributions belong to the transformed Beta family which are presented in subsubsection 2.7.1.1.

4.3.2 Composite Weibull-Burr Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0, \eta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.31), (2.32), (2.54), and (2.55):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\beta\eta(\frac{x}{\lambda})^{\eta}[1+(\frac{\theta}{\lambda})^{\eta}]^{\beta+1}}{x[1+(\frac{x}{\lambda})^{\eta}]^{\beta+1}}, & \theta \leq x < +\infty. \end{cases} \tag{4.14}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}}[1+(\frac{\theta}{\lambda})^{\eta}]}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\beta\eta(\frac{\theta}{\lambda})^{\eta}}. \tag{4.15}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha - \eta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + (\beta+1)\frac{\eta\theta^{\eta-1}}{\lambda^{\eta}}[(1+\frac{\theta}{\lambda})^{\eta}]^{-1} = 0. \tag{4.16}$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$\begin{aligned} M_X(t) = & \frac{1}{1+\phi}\frac{1}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]}\sum_{k=0}^{\infty}\frac{t^k}{k!}\sigma^k\gamma(\frac{k}{\alpha}+1,(\frac{\theta}{\sigma})^{\alpha}) \\ & - \frac{\phi}{1+\phi}[\frac{1}{1+(\frac{\theta}{\lambda})^{\eta}}]^{\beta+1}\sum_{k=0}^{\infty}\frac{t^k}{k!}\beta\lambda^k B([1+(\frac{\theta}{\lambda})^{\eta}]^{-1}, -\beta-\frac{k}{\eta}, \frac{k}{\eta}+1). \end{aligned} \tag{4.17}$$

4.3.3 Composite Weibull-Inverse Burr Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0, \eta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.33), (2.34), (2.54), and (2.55):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\beta\eta(\frac{x}{\lambda})^{\beta\eta}[1+(\frac{\theta}{\lambda})^{\eta}]^{\beta}}{x[1+(\frac{x}{\lambda})^{\eta}]^{\beta+1}[[1+(\frac{\theta}{\lambda})^{\eta}]^{\beta}-(\frac{\theta}{\lambda})^{\beta\eta}]}, & \theta \leq x < +\infty. \end{cases} \tag{4.18}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} [[1+(\frac{\theta}{\lambda})^{\eta}]^{\beta} - (\frac{\theta}{\lambda})^{\beta\eta}]\theta[1+(\frac{\theta}{\lambda})^{\eta}]}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\beta\eta(\frac{\theta}{\lambda})^{\beta\eta}}. \tag{4.19}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha - \beta\eta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + (\beta+1)\frac{\eta\theta^{\eta-1}}{\lambda^{\eta}}[(1+\frac{\theta}{\lambda})^{\eta}]^{-1} = 0. \tag{4.20}$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$\begin{aligned} M_X(t) = & \frac{1}{1+\phi} \frac{1}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sigma^k \gamma(\frac{k}{\alpha}+1, (\frac{\theta}{\sigma})^{\alpha}) \\ & - \frac{\phi}{1+\phi}[\frac{1}{1+(\frac{\theta}{\lambda})^{\eta}}]^{\beta+1} \sum_{k=0}^{\infty} \frac{t^k}{k!} \beta\lambda^k B([1+(\frac{\theta}{\lambda})^{\eta}]^{-1}, \beta - \frac{k}{\eta}, \frac{k}{\eta}+1). \end{aligned} \tag{4.21}$$

4.3.4 Composite Weibull-Pareto Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.35), (2.36), (2.54), and (2.55):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\beta(\theta+\lambda)^{\beta}}{(x+\lambda)^{\beta+1}}, & \theta \leq x < +\infty. \end{cases} \tag{4.22}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}}(\theta+\lambda)}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\beta}. \tag{4.23}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha-1)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + (\beta+1)\frac{1}{\theta+\lambda} = 0. \tag{4.24}$$

The moment generating function, $M_X(t)$, cannot be calculated in a closed form.

4.3.5 Composite Weibull-Inverse Pareto Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.37), (2.38), (2.54), and (2.55):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\beta \lambda x^{\beta-1}(\theta+\lambda)^{\beta}}{(x+\lambda)^{\beta+1}[(\theta+\lambda)^{\beta}-\theta^{\beta}]}, & \theta \leq x < +\infty. \end{cases} \tag{4.25}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} (\theta+\lambda)[(\theta+\lambda)^{\beta} - \theta^{\beta}]}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\beta\lambda\theta^{\beta-1}}. \tag{4.26}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha-\beta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + (\beta+1)\frac{1}{\theta+\lambda} = 0. \tag{4.27}$$

The moment generating function, $M_X(t)$, cannot be calculated in a closed form.

4.3.6 Composite Weibull-Loglogistic Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.39), (2.40), (2.54), and (2.55):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\beta(\frac{x}{\lambda})^{\beta}[1+(\frac{\theta}{\lambda})^{\beta}]}{x[1+(\frac{x}{\lambda})^{\beta}]^2}, & \theta \leq x < +\infty. \end{cases} \tag{4.28}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} \theta[1+(\frac{\theta}{\lambda})^{\beta}]}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\beta(\frac{\theta}{\lambda})^{\beta}}. \tag{4.29}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha-\beta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + \frac{2\beta\theta^{\beta-1}}{\lambda^{\beta}}[(1+\frac{\theta}{\lambda})^{\beta}]^{-1} = 0. \tag{4.30}$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$\begin{aligned} M_X(t) = & \frac{1}{1+\phi} \frac{1}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sigma^k \gamma(\frac{k}{\alpha}+1, (\frac{\theta}{\sigma})^{\alpha}) \\ & - \frac{\phi}{1+\phi}[1+(\frac{\theta}{\lambda})^{\beta}] \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k B([1+(\frac{\theta}{\lambda})^{\beta}]^{-1}, 2-\frac{k}{\beta}, \frac{k}{\beta}+1). \end{aligned} \tag{4.31}$$

4.3.7 Composite Weibull-Paralogistic Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.41), (2.42), (2.54), and (2.55):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\beta^2(\frac{x}{\lambda})^{\beta}[1+(\frac{\theta}{\lambda})^{\beta}]^{\beta}}{x[1+(\frac{x}{\lambda})^{\beta}]^{\beta+1}}, & \theta \leq x < +\infty. \end{cases} \tag{4.32}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} \theta[1+(\frac{\theta}{\lambda})^{\beta}]}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\beta^2(\frac{\theta}{\lambda})^{\beta}}. \tag{4.33}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha-\beta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + (\beta+1)\frac{\beta\theta^{\beta-1}}{\lambda^{\beta}}[(1+\frac{\theta}{\lambda})^{\beta}]^{-1} = 0. \tag{4.34}$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$\begin{aligned} M_X(t) = & \frac{1}{1+\phi} \frac{1}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sigma^k \gamma(\frac{k}{\alpha}+1, (\frac{\theta}{\sigma})^{\alpha}) \\ & - \frac{\phi}{1+\phi} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k \beta^2 [1+(\frac{\theta}{\lambda})^{\beta}]^{\beta} B([1+(\frac{\theta}{\lambda})^{\beta}]^{-1}; 1-\frac{k}{\beta}, \beta+\frac{k}{\beta}). \end{aligned} \tag{4.35}$$

4.3.8 Composite Weibull-Generalized Pareto Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0, \eta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.45), (2.46), (2.54), and (2.55):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^\alpha}}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\Gamma(\eta+\beta)}{\Gamma(\eta)\Gamma(\beta)} \frac{\lambda^\eta x^{\beta-1}}{(x+\lambda)^{\beta+1}[1-\mathbf{B}(\beta,\eta;\frac{\theta}{\theta+\lambda})]}, & \theta \leq x < +\infty. \end{cases} \tag{4.36}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^\alpha}[1-\mathbf{B}(\eta,\beta;\frac{\theta}{\theta+\lambda})]\Gamma(\beta)\Gamma(\eta)(\theta+\lambda)^{\beta+\eta}}{[1-e^{-(\frac{\theta}{\sigma})^\alpha}]\Gamma(\beta+\eta)\lambda^\beta\theta^{\eta-1}}. \tag{4.37}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha-\eta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^\alpha} - (\beta+\eta)\frac{1}{\theta+\lambda} = 0. \tag{4.38}$$

The moment generating function, $M_X(t)$, cannot be calculated in a closed form.

4.4 Composite Models in the Transformed Gamma and the Inverse Transformed Gamma Families

This section follows the same method as explained in 4.3. That is, for constructing composite models, the Weibull distribution is considered as a head and among distributions in the transformed Gamma and the inverse transformed Gamma families, some univariate distributions such as the Weibull distribution, the inverse Weibull distribution, the Gamma distribution, the inverse Gamma distribution, the Exponential distribution, the inverse Exponential distribution, the transformed Gamma distribution, and the inverse transformed Gamma distribution which are mentioned in 2.7.2.1 are chosen as the tail for the composite model.

To describe the procedure of the new method, the composite Weibull-inverse transformed Gamma is considered and explained in details.

4.4.1 Composite Weibull-Weibull Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.54) and (2.55):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\frac{\beta}{\lambda}(\frac{x}{\lambda})^{\beta-1} e^{-(\frac{x}{\lambda})^{\beta}}}{e^{-(\frac{\theta}{\lambda})^{\beta}}}, & \theta \leq x < +\infty. \end{cases} \tag{4.39}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}}}{[1 - e^{-(\frac{\theta}{\sigma})^{\alpha}}]\frac{\beta}{\lambda}(\frac{\theta}{\lambda})^{\beta-1}}. \tag{4.40}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha - \beta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + \frac{\beta\theta^{\beta-1}}{\lambda^{\beta}} = 0. \tag{4.41}$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$M_X(t) = \frac{1}{1+\phi}\frac{1}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]}\sum_{k=0}^{\infty}\frac{t^k}{k!}\sigma^k\gamma(\frac{k}{\alpha}+1,(\frac{\theta}{\sigma})^{\alpha})$$
$$+\frac{\phi}{1+\phi}\frac{1}{[e^{-(\frac{\theta}{\lambda})^{\beta}}]}\sum_{k=0}^{\infty}\frac{t^k}{k!}\lambda^k\gamma(\frac{k}{\beta}+1,(\frac{\theta}{\lambda})^{\beta}). \quad (4.42)$$

4.4.2 Composite Weibull-Inverse Weibull Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.54), (2.55), (2.56), and (2.57):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi}\frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1}e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi}\frac{\beta(\frac{\lambda}{x})^{\beta}e^{-(\frac{\lambda}{x})^{\beta}}}{x[1-e^{-(\frac{\theta}{\lambda})^{\beta}}]}, & \theta \leq x < +\infty. \end{cases} \quad (4.43)$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1}e^{-(\frac{\theta}{\sigma})^{\alpha}}[1-e^{-(\frac{\lambda}{\theta})^{\beta}}]}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\frac{\beta}{\theta}(\frac{\lambda}{\theta})^{\beta}e^{-(\frac{\lambda}{\theta})^{\beta}}}. \quad (4.44)$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha-\beta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + \frac{\beta\lambda^{\beta}}{\theta^{\beta+1}} = 0. \quad (4.45)$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$M_X(t) = \frac{1}{1+\phi}\frac{1}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]}\sum_{k=0}^{\infty}\frac{t^k}{k!}\sigma^k\gamma(\frac{k}{\alpha}+1,(\frac{\theta}{\sigma})^{\alpha})$$
$$-\frac{\phi}{1+\phi}\frac{1}{[1-e^{-(\frac{\lambda}{\theta})^{\beta}}]}\sum_{k=0}^{\infty}\frac{t^k}{k!}\lambda^k\gamma(1-\frac{k}{\alpha},(\frac{\lambda}{\theta})^{\beta}). \quad (4.46)$$

4.4.3 Composite Weibull-Gamma Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.54), (2.55), (2.58), and (2.59):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\frac{x}{\lambda}^{\beta} e^{-\frac{x}{\lambda}}}{x[\Gamma(\beta)-\gamma(\beta,\frac{\theta}{\lambda})]}, & \theta \leq x < +\infty. \end{cases} \tag{4.47}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} \theta[\Gamma(\beta) - \gamma(\beta, \frac{\theta}{\lambda})]}{[1 - e^{-(\frac{\theta}{\sigma})^{\alpha}}](\frac{\theta}{\lambda})^{\beta} e^{-\frac{\theta}{\lambda}}}. \tag{4.48}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha - \beta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} - \frac{1}{\lambda} = 0. \tag{4.49}$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$M_X(t) = \frac{1}{1+\phi} \frac{1}{[1 - e^{-(\frac{\theta}{\sigma})^{\alpha}}]} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sigma^k \gamma(\frac{k}{\alpha} + 1, (\frac{\theta}{\sigma})^{\alpha}) + \frac{\phi}{1+\phi} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\lambda^k}{[\Gamma(\beta) - \gamma(\beta, \frac{\theta}{\lambda})]} \gamma(\beta + k + 2; \frac{\lambda}{\theta}). \tag{4.50}$$

4.4.4 Composite Weibull-Inverse Gamma Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.54), (2.55), (2.60), and (2.61):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\frac{\lambda}{x}^{\beta} e^{-\frac{\lambda}{x}}}{x\gamma(\beta, \frac{\lambda}{\theta})}, & \theta \leq x < +\infty. \end{cases} \quad (4.51)$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} \theta\gamma(\beta, \frac{\lambda}{\theta})}{[1 - e^{-(\frac{\theta}{\sigma})^{\alpha}}](\frac{\lambda}{\theta})^{\beta} e^{-\frac{\lambda}{\theta}}}. \quad (4.52)$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha + \beta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + \frac{\lambda}{\theta^2} = 0. \quad (4.53)$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$\begin{aligned} M_X(t) = & \frac{1}{1+\phi} \frac{1}{[1 - e^{-(\frac{\theta}{\sigma})^{\alpha}}]} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sigma^k \gamma(\frac{k}{\alpha} + 1, (\frac{\theta}{\sigma})^{\alpha}) \\ & - \frac{\phi}{1+\phi} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\lambda^k}{\gamma(\beta, \frac{\lambda}{\theta})} \gamma(\beta - k; \frac{\lambda}{\theta}). \end{aligned} \quad (4.54)$$

4.4.5 Composite Weibull-Exponential Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.54), (2.55), (2.62), and (2.63):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{e^{\frac{\theta-x}{\lambda}}}{\lambda}, & \theta \leq x < +\infty. \end{cases} \quad (4.55)$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} \lambda}{[1 - e^{-(\frac{\theta}{\sigma})^{\alpha}}]}. \quad (4.56)$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha - 1)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} + \frac{1}{\lambda} = 0. \quad (4.57)$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$M_X(t) = \frac{1}{1+\phi} \frac{1}{[1 - e^{-(\frac{\theta}{\sigma})^{\alpha}}]} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sigma^k \gamma(\frac{k}{\alpha} + 1, (\frac{\theta}{\sigma})^{\alpha}) - \frac{\phi}{1+\phi} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k e^{\frac{\theta}{\lambda}} \gamma(3 + k; \frac{\theta}{\lambda}). \quad (4.58)$$

4.4.6 Composite Weibull-Inverse Exponential Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.54), (2.55), (2.64), and (2.65):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\lambda e^{-\frac{\lambda}{x}}}{x^2[1-e^{-\frac{\lambda}{\theta}}]}, & \theta \leq x < +\infty. \end{cases} \tag{4.59}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} \theta^2 [1 - e^{-\frac{\lambda}{\theta}}]}{1 - e^{-(\frac{\theta}{\sigma})^{\alpha}} \lambda e^{-\frac{\lambda}{\theta}}}. \tag{4.60}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha + 1)\frac{1}{\theta} - \frac{\alpha \theta^{\alpha-1}}{\sigma^{\alpha}} - \frac{\lambda}{\theta^2} = 0. \tag{4.61}$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$\begin{aligned} M_X(t) = & \frac{1}{1+\phi} \frac{1}{[1 - e^{-(\frac{\theta}{\sigma})^{\alpha}}]} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sigma^k \gamma(\frac{k}{\alpha} + 1, (\frac{\theta}{\sigma})^{\alpha}) \\ & - \frac{\phi}{1+\phi} \sum_{k=0}^{\infty} \frac{t^k}{k!} [1 - e^{-(\frac{\lambda}{\theta})}]^{-1} \lambda^k \gamma(1-k, \frac{\lambda}{\theta}). \end{aligned} \tag{4.62}$$

4.4.7 Composite Weibull-Transformed Gamma Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0, \eta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is presented by using Equations (2.54), (2.55), (2.66), and (2.67):

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\eta(\frac{x}{\lambda})^{\beta\eta} e^{-(\frac{x}{\lambda})^{\eta}}}{x[\Gamma(\beta)-\Gamma(\beta,(\frac{\theta}{\lambda})^{\eta})]}, & \theta \leq x < +\infty. \end{cases} \tag{4.63}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} \theta[\Gamma(\beta) - \Gamma(\beta,(\frac{\lambda}{\theta})^{\eta})]}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\eta(\frac{\theta}{\lambda})^{\eta\beta} e^{-(\frac{\theta}{\lambda})^{\eta}}}. \tag{4.64}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha - \beta\eta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} - \frac{\eta\theta^{\eta-1}}{\lambda^{\eta}} = 0. \tag{4.65}$$

According to Equation (2.4), the moment generating function, $M_X(t)$, can be obtained as follows:

$$\begin{aligned} M_X(t) = & \frac{1}{1+\phi} \frac{1}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sigma^k \gamma(\frac{k}{\alpha}+1,(\frac{\theta}{\sigma})^{\alpha}) \\ & - \frac{\phi}{1+\phi}[\frac{1}{[\Gamma(\beta)-\Gamma(\beta,(\frac{\theta}{\lambda})^{\eta})]} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k \gamma(\frac{k}{\eta}+\beta+2,(\frac{\lambda}{\theta})^{\eta}). \end{aligned} \tag{4.66}$$

4.4.8 Composite Weibull-Inverse Transformed Gamma Model

For a random variable $X > 0$ and parameters $\alpha > 0, \sigma > 0, \beta > 0, \eta > 0$, and $\lambda > 0$, the probability density function $f_X(x)$ is defined by using Equations (2.54), (2.55), (2.68), and (2.69) as follows:

$$f_X(x) = \begin{cases} \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}, & 0 < x \leq \theta, \\ \frac{\phi}{1+\phi} \frac{\eta(\frac{\lambda}{x})^{\beta\eta} e^{-(\frac{\lambda}{x})^{\eta}}}{x\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})}, & \theta \leq x < +\infty. \end{cases} \tag{4.67}$$

The mixing weight ϕ is calculated by using Equation (3.27):

$$\phi = \frac{\frac{\alpha}{\sigma}(\frac{\theta}{\sigma})^{\alpha-1} e^{-(\frac{\theta}{\sigma})^{\alpha}} \theta\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]\eta(\frac{\lambda}{\theta})^{\eta\beta} e^{-(\frac{\lambda}{\theta})^{\eta}}}. \tag{4.68}$$

θ can be found By applying Equation (3.33) and differentiating in terms of θ, using numerical methods, and solving the following equation:

$$(\alpha + \beta\eta)\frac{1}{\theta} - \frac{\alpha\theta^{\alpha-1}}{\sigma^{\alpha}} - \frac{\eta\lambda^{\eta}}{\theta^{\eta}+1} = 0. \tag{4.69}$$

4.4.8.1 Parameter Estimation

By having the probability density function of the composite Weibull-inverse transformed gamma model as Equation (4.67), the maximum likelihood estimation, Equation (2.20), can be used as:

$$l(\theta) = \ln L(\theta|x_1,...,x_n),$$
$$= \sum_{i=1}^{n} \ln f(x_i|\theta),$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \ln[\frac{1}{1+\phi}\frac{f_1(x_i)}{F_1(\theta)} I_{(0,\theta]}(x_i) + \sum_{i=1}^{n} \frac{\phi}{1+\phi}\frac{f_2(x_i)}{1-F_2(\theta)} I_{[\theta,+\infty)}(x_i)], \\
&= \sum_{i=1}^{n} \ln[\frac{1}{1+\phi}\frac{\frac{\alpha}{\sigma}(\frac{x_i}{\sigma})^{\alpha-1} e^{-(\frac{x_i}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}} I_{(0,\theta]}(x_i) + \frac{\phi}{1+\phi}\frac{\eta(\frac{\lambda}{x})^{\beta\eta} e^{-(\frac{\lambda}{x})^{\eta}}}{x\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})} I_{[\theta,+\infty)}(x_i)].
\end{aligned} \tag{4.70}
$$

The estimation of parameters can be found by differentiating Equation (4.67) in terms of each parameter as presented in Equation (2.21) (Hassan and Al-Ghamdi, 2009).

To estimate all parameters in the composite model, except the mixing weight ϕ and the threshold θ, we have to solve the simultaneous equation, Equation (4.71), as a likelihood system in terms of all parameters. Thus, $\hat{\alpha}_{ML}$, $\hat{\sigma}_{ML}$, $\hat{\beta}_{ML}$, $\hat{\eta}_{ML}$, and $\hat{\lambda}_{ML}$ will be obtained. In Chapter 5, these parameter estimators are obtained by using software **R**.

$$
\left.
\begin{aligned}
&\frac{\partial l(\theta)}{\partial \alpha} = \frac{n}{\alpha} + n(\alpha-1)^{n-1}\textstyle\sum_{i=1}^{n}\ln(\frac{x_i}{\sigma}) - \textstyle\sum_{i=1}^{n}(\frac{x_i}{\sigma})^{\alpha}\ln(\frac{x_i}{\sigma}) - \frac{n\frac{\theta}{\sigma}\ln\frac{-\theta}{\sigma}e^{-(\frac{\theta}{\sigma})^{\alpha}}}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}} = 0, \\
&\frac{\partial l(\theta)}{\partial \sigma} = \frac{-n\alpha}{\sigma} - (\alpha-1)^{n}\frac{n}{\sigma} + \textstyle\sum_{i=1}^{n}\frac{\alpha x_i^{\alpha}}{\sigma^{\alpha+1}} + \frac{n\alpha\theta^{\alpha}e^{-(\frac{\theta}{\sigma})}}{\sigma^{\alpha+1}[1-e^{-(\frac{\theta}{\sigma})}]} = 0, \\
&\frac{\partial l(\theta)}{\partial \beta} = n\eta^{n}\beta^{n-1}\Sigma_{i=1}^{n}\ln(\frac{\lambda}{x_i}) - n\frac{d}{d\beta}\ln[\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})] = 0, \\
&\frac{\partial l(\theta)}{\partial \eta} = \frac{n}{\eta} + n\eta^{n-1}\beta^{n}\Sigma_{i=1}^{n}\ln(\frac{\lambda}{x_i}) + \Sigma_{i=1}^{n}(\frac{\lambda}{x_i})^{\eta}\ln[\Sigma_{i=1}^{n}(\frac{\lambda}{x_i})^{\eta}] - n\frac{d}{d\eta}\ln[\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})] = 0. \\
&\frac{\partial l(\theta)}{\partial \lambda} = \frac{n\beta^{n}\eta^{n}}{\lambda} - \Sigma_{i=1}^{n}\frac{\eta\lambda^{\eta-1}}{x_i} - \frac{n(\frac{\lambda}{\theta})^{\eta(\beta-1)}e^{-(\frac{\lambda}{\theta})^{\eta}}}{\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})} = 0.
\end{aligned}
\right\}
\tag{4.71}
$$

It is worthy to note that the derivation of lower incomplete gamma function, Equation (2.49), is (Abramowitz and Stegun, 1964, p.262):

$$
\frac{\partial \gamma(a,x)}{\partial x} = x^{a-1}e^{-x}. \tag{4.72}
$$

4.4.8.2 Properties

By having the probability density function of the composite Weibull-inverse transformed Gamma model as Equation (4.67), the mean of X, $E[X]$, the second raw moment of X, $E[X^2]$, the variance of X, $Var[X]$, the k-th raw moment of X, $E[X^k]$, the moment generating function of X, $M_X(t)$, the skewness γ_1, and the excess kurtosis b_2 can be calculated as follows:

The mean of X, $E[X]$, is obtained according to Equation (2.2) as follows:

$$\begin{aligned}
E[X] &= \int x f_X(x)dx. \\
&= \int_0^\theta x \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^\alpha}}{1-e^{-(\frac{\theta}{\sigma})^\alpha}} dx + \int_\theta^\infty x \frac{\phi}{1+\phi} \frac{\eta(\frac{\lambda}{x})^{\beta\eta} e^{-(\frac{\lambda}{x})^\eta}}{x\gamma(\beta,(\frac{\lambda}{\theta})^\eta)} dx, \\
&= \frac{1}{1+\phi} \frac{\alpha}{1-e^{-(\frac{\theta}{\sigma})^\alpha}} \int_0^\theta (\frac{x}{\sigma})^\alpha e^{-(\frac{x}{\sigma})^\alpha} dx + \frac{\phi}{1+\phi} \frac{\eta}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)} \int_\theta^\infty (\frac{\lambda}{x})^{\beta\eta} e^{-(\frac{\lambda}{x})^\eta} dx.
\end{aligned}$$

Let:

$$\begin{aligned}
(\frac{x}{\sigma})^\alpha = u, \quad &\Rightarrow \quad x = \sigma u^{\frac{1}{\alpha}}, \quad dx = (u^{\frac{1}{\alpha}-1})\frac{\sigma du}{\alpha}. \\
(\frac{\lambda}{x})^\eta = u, \quad &\Rightarrow \quad x = \frac{\lambda}{u^{\frac{1}{\eta}}}, \quad dx = \frac{-\lambda du}{\eta u^{\frac{1+\eta}{\eta}}}.
\end{aligned}$$

Then, $E[X]$ can be rewritten as follows:

$$E[X] = \frac{1}{1+\phi} \frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^\alpha}} \int_0^{(\frac{\theta}{\sigma})^\alpha} u^{\frac{1}{\alpha}} e^{-u} du - \frac{\phi}{1+\phi} \frac{\eta}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)} \int_0^{(\frac{\lambda}{\theta})^\eta} u^{\beta-\frac{1}{\eta}-1} e^{-u} du.$$

By applying Equation (2.49), we will have:

$$E[X] = \frac{1}{1+\phi} \frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^\alpha}} \gamma(\frac{1}{\alpha}+1, (\frac{\theta}{\sigma})^\alpha) - \frac{\phi}{1+\phi} \frac{\lambda}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)} \gamma(\beta - \frac{1}{\eta}, (\frac{\lambda}{\theta})^\eta). \quad (4.73)$$

The second raw moment of X, $E[X^2]$, is obtained according to Equation (2.11) for $k=2$ as follows:

$$
\begin{aligned}
E[X^2] &= \int x^2 f(x)dx. \\
&= \int_0^\theta x^2 \frac{1}{1+\phi} \frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^\alpha}}{1-e^{-(\frac{\theta}{\sigma})^\alpha}} dx + \int_\theta^\infty x^2 \frac{\phi}{1+\phi} \frac{\eta(\frac{\lambda}{x})^{\beta\eta} e^{-(\frac{\lambda}{x})^\eta}}{x\gamma(\beta,(\frac{\lambda}{\theta})^\eta)} dx, \\
&= \frac{1}{1+\phi} \frac{\alpha}{1-e^{-(\frac{\theta}{\sigma})^\alpha}} \int_0^\theta x(\frac{x}{\sigma})^\alpha e^{-(\frac{x}{\sigma})^\alpha} dx \\
&\quad + \frac{\phi}{1+\phi} \frac{\eta}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)} \int_\theta^\infty x(\frac{\lambda}{x})^{\beta\eta} e^{-(\frac{\lambda}{x})^\eta} dx.
\end{aligned}
$$

Let:

$$
\begin{aligned}
&(\frac{x}{\sigma})^\alpha = u, \quad \Rightarrow \quad x = \sigma u^{\frac{1}{\alpha}}, \quad dx = (u^{\frac{1}{\alpha}-1})\frac{\sigma du}{\alpha}. \\
&(\frac{\lambda}{x})^\eta = u, \quad \Rightarrow \quad x = \frac{\lambda}{u^{\frac{1}{\eta}}}, \quad dx = \frac{-\lambda du}{\eta u^{\frac{1+\eta}{\eta}}}.
\end{aligned}
$$

Then, $E[X^2]$ can be rewritten as follows:

$$
\begin{aligned}
E[X^2] = \frac{1}{1+\phi} \frac{\sigma^2}{1-e^{-(\frac{\theta}{\sigma})^\alpha}} \int_0^{(\frac{\theta}{\sigma})^\alpha} u^{\frac{2}{\alpha}} e^{-u} du \\
- \frac{\phi}{1+\phi} \frac{\lambda^2}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)} \int_0^{(\frac{\lambda}{\theta})^\eta} u^{\beta-\frac{2}{\eta}-1} e^{-u} du,
\end{aligned}
$$

By applying Equation (2.49), we will have:

$$
\begin{aligned}
E[X^2] = \frac{1}{1+\phi} \frac{\sigma^2}{1-e^{-(\frac{\theta}{\sigma})^\alpha}} \gamma(\frac{2}{\alpha}+1,(\frac{\theta}{\sigma})^\alpha) \\
- \frac{\phi}{1+\phi} \frac{\lambda^2}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)} \gamma(\beta-\frac{2}{\eta},(\frac{\lambda}{\theta})^\eta).
\end{aligned}
\tag{4.74}
$$

The variance of X, $Var[X]$, is obtained according to Equation (2.3) and using Equations (4.73) and (4.74) as follows:

$$\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2, \\
&= \frac{1}{1+\phi}\frac{\sigma^2}{[1-e^{-(\frac{\theta}{\sigma})^\alpha}]}\gamma(\frac{2}{\alpha}+1,(\frac{\theta}{\sigma})^\alpha) \\
&\quad - \frac{\phi}{1+\phi}\frac{\lambda^2}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)}\gamma(\beta-\frac{2}{\eta},(\frac{\lambda}{\theta})^\eta) \\
&\quad - [\frac{1}{1+\phi}\frac{\sigma}{[1-e^{-(\frac{\theta}{\sigma})^\alpha}]}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma})^\alpha) \\
&\quad - \frac{\phi}{1+\phi}\frac{\lambda}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)}\gamma(\beta-\frac{1}{\eta},(\frac{\lambda}{\theta})^\eta)]^2. \qquad (4.75)
\end{aligned}$$

The k-th raw moment of X, $E[X^k]$, is obtained according to Equation (2.11) as follows:

$$\begin{aligned}
E[X^k] &= \int x^k f(x)dx. \\
&= \int_0^\theta x^k \frac{1}{1+\phi}\frac{\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{\alpha-1}e^{-(\frac{x}{\sigma})^\alpha}}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}dx + \int_\theta^\infty x^k \frac{\phi}{1+\phi}\frac{\eta(\frac{\lambda}{x})^{\beta\eta}e^{-(\frac{\lambda}{x})^\eta}}{x\gamma(\beta,(\frac{\lambda}{\theta})^\eta)}dx, \\
&= \frac{1}{1+\phi}\frac{\alpha}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\int_0^\theta x^{k-1}(\frac{x}{\sigma})^\alpha e^{-(\frac{x}{\sigma})^\alpha}dx \\
&\quad + \frac{\phi}{1+\phi}\frac{\eta}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)}\int_\theta^\infty x^{k-1}(\frac{\lambda}{x})^{\beta\eta}e^{-(\frac{\lambda}{x})^\eta}dx.
\end{aligned}$$

Let:

$$\begin{aligned}
&(\frac{x}{\sigma})^\alpha = u, \quad \Rightarrow \quad x = \sigma u^{\frac{1}{\alpha}}, \quad dx = (u^{\frac{1}{\alpha}-1})\frac{\sigma du}{\alpha}. \\
&(\frac{\lambda}{x})^\eta = u, \quad \Rightarrow \quad x = \frac{\lambda}{u^{\frac{1}{\eta}}}, \quad dx = \frac{-\lambda du}{\eta u^{\frac{1+\eta}{\eta}}}.
\end{aligned}$$

Then, $E[X^k]$ can be rewritten as follows:

$$E[X^k] = \frac{1}{1+\phi}\frac{\sigma^k}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\int_0^{(\frac{\theta}{\sigma})^\alpha} u^{\frac{k}{\alpha}} e^{-u} du$$
$$- \frac{\phi}{1+\phi}\frac{\lambda^k}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)}\int_0^{(\frac{\lambda}{\theta})^\eta} u^{\beta-\frac{k}{\eta}-1} e^{-u} du,$$

By applying Equation (2.49), we have:

$$E[X^k] = \frac{1}{1+\phi}\frac{\sigma^k}{1-e^{-(\frac{\theta}{\sigma})^\alpha}}\gamma(\frac{k}{\alpha}+1,(\frac{\theta}{\sigma})^\alpha)$$
$$- \frac{\phi}{1+\phi}\frac{\lambda^k}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)}\gamma(\beta-\frac{k}{\eta},(\frac{\lambda}{\theta})^\eta). \quad (4.76)$$

The moment generating function of X, $M_X(t)$, is obtained according to Equation (2.4) as follows:

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx} f_X(x)dx, \\ &= \int_0^\theta e^{tx}\frac{1}{1+\phi}\frac{f_1(x)}{F_1(\theta)}dx + \int_\theta^\infty e^{tx}\frac{\phi}{1+\phi}\frac{f_2(x)}{1-F_2(\theta)}dx, \\ &= \frac{1}{1+\phi}\frac{1}{F_1(\theta)}\int_0^\theta e^{tx} f_1(x)dx + \frac{\phi}{1+\phi}\frac{1}{1-F_2(\theta)}\int_\theta^\infty e^{tx} f_2(x)dx. \end{aligned}$$

According to Equations (2.5) and (4.76), we have:

$$M_X(t) = \frac{1}{1+\phi}\frac{1}{[1-e^{-(\frac{\theta}{\sigma})^\alpha}]}\sum_{k=0}^{\infty}\frac{t^k}{k!}\sigma^k\gamma(\frac{k}{\alpha}+1,(\frac{\theta}{\sigma})^\alpha)$$
$$+ \frac{\phi}{1+\phi}\frac{1}{\gamma(\beta,(\frac{\lambda}{\theta})^\eta)}\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{\lambda^k}{\Gamma(\beta)}\gamma(\beta-\frac{k}{\lambda},(\frac{\lambda}{\theta})^\eta). \quad (4.77)$$

The skewness of X, γ_1, can be found by using Equation (2.12) as follows:

$$
\begin{aligned}
\gamma_1 &= \frac{E[(X-E[X])^3]}{[Var[X]]^{\frac{3}{2}}}, \\
&= E[(X - \frac{1}{1+\phi}\frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma}^{\alpha})) - \frac{\phi}{1+\phi}\frac{\lambda}{\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})}\gamma(\beta-\frac{1}{\eta},(\frac{\lambda}{\theta})^{\eta}))^3]/ \\
&[\frac{1}{1+\phi}\frac{\sigma^2}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]}\gamma(\frac{2}{\alpha}+1,(\frac{\theta}{\sigma})^{\alpha}) - \frac{\phi}{1+\phi}\frac{\lambda^2}{\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})}\gamma(\beta-\frac{2}{\eta},(\frac{\lambda}{\theta})^{\eta}) \\
&\quad - [\frac{1}{1+\phi}\frac{\sigma}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma})^{\alpha}) - \frac{\phi}{1+\phi}\frac{\lambda}{\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})}\gamma(\beta-\frac{1}{\eta},(\frac{\lambda}{\theta})^{\eta})]^2]^{\frac{3}{2}}.
\end{aligned}
\tag{4.78}
$$

The excess kurtosis of X, b_2, can be found by using Equation (2.14) as follows:

$$
\begin{aligned}
b_2 &= \frac{E[(X-E[X])^4]}{[Var[X]]^2} - 3, \\
&= E[(X - \frac{1}{1+\phi}\frac{\sigma}{1-e^{-(\frac{\theta}{\sigma})^{\alpha}}}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma}^{\alpha})) - \frac{\phi}{1+\phi}\frac{\lambda}{\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})}\gamma(\beta-\frac{1}{\eta},(\frac{\lambda}{\theta})^{\eta}))^4]/ \\
&[\frac{1}{1+\phi}\frac{\sigma^2}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]}\gamma(\frac{2}{\alpha}+1,(\frac{\theta}{\sigma})^{\alpha}) - \frac{\phi}{1+\phi}\frac{\lambda^2}{\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})}\gamma(\beta-\frac{2}{\eta},(\frac{\lambda}{\theta})^{\eta}) \\
&\quad - [\frac{1}{1+\phi}\frac{\sigma}{[1-e^{-(\frac{\theta}{\sigma})^{\alpha}}]}\gamma(\frac{1}{\alpha}+1,(\frac{\theta}{\sigma})^{\alpha}) - \frac{\phi}{1+\phi}\frac{\lambda}{\gamma(\beta,(\frac{\lambda}{\theta})^{\eta})}\gamma(\beta-\frac{1}{\eta},(\frac{\lambda}{\theta})^{\eta})]^2]^2 - 3.
\end{aligned}
\tag{4.79}
$$

4.5 Summary

In this chapter, several composite models are constructed by choosing the Weibull distribution as the head and distributions that belonging to the transformed Beta, transformed Gamma, and inverse transformed Gamma families as the tail. For each composite model, the probability density function $f_X(x)$, the mixing weight ϕ, the threshold θ, and the moment generating function $M_X(t)$ are presented. For the composite Weibull-inverse Paralogistic model and the composite Weibull-inverse transformed Gamma model, additive parts are parameter estimations which is done by MLE method and some derivatives of moment generating function such as mean, variance, skewness, and excess kurtosis.

CHAPTER 5

APPLICATION OF THE NEW COMPOSITE MODELS IN REAL DATA SETS

5.1 Introduction

In this chapter, two well-known loss data sets related to insurance, Danish fire loss data and allocated loss adjustment expenses data, are employed to illustrate the goodness-of-fit of new composite models.

The computations are performed by using the statistical software **R** and the "actuar" package which is an add-on by Dutang et al. (2008). The **R** software for statistical calculation and figures can be found in www.r-project.org and cran.r-project.org (Team et al., 2012) and (Ricci, 2005). The estimation of parameters for each model is done based on maximum likelihood estimation (MLE) method through the help of "nlm" function which is non-linear minimization and in some cases, "optim" function which is optimization based on Nelder–Mead, Quasi-Newton, and conjugate-gradient algorithms. The negative log-likelihood (NLL) as defined in Equation (2.22) is suitable for comparing models with the same number of parameters. However, when comparing across models with the different number of parameters, the goodness-of-fit can be measured using Akaike information criterion (AIC) in Equation (2.23), and also, Schwarz's Bayesian criterion (SBC) in Equation (2.24). Smaller values of NLL, AIC, and SBC indicate a better fit of the model to data sets(Akaike, 1992) and (Schwarz et al., 1978).

According to the goodness-of-fit measures in section (5.3), the results show that in the transformed Beta family, the composite Weibull-inverse Paralogistic model and in the inverse transformed Gamma family, the composite Weibull-inverse transformed Gamma model are the best fitted models among the considered composite models.

5.2 Data sets

In this study, two data sets are used and fitted to composite models which are constructed with the Weibull distribution as the head and the tail chosen from these families: the transformed Beta, the transformed Gamma, and the inverse transformed Gamma. The two data sets are Danish fire insurance loss data and allocated loss adjustment expenses (ALAE) data which are described in details below. (Achieng et al., 2010) and (Beirlant et al., 2004).

5.2.1 Danish Fire Insurance Loss Data

Danish fire insurance loss data consists of 2,492 losses supplied by Mette Rytgaard in Copenhagen. These data describe fire insurance claims in Denmark from 1980 until 1990. Danish fire insurance loss data set obtained from R package "SMPracticals". McNeil et al. (1997) analyzed part of the data. Several authors used this data, which is highly skewed, to fit their models. This data set has been applied by Cooray and Ananda (2005) to fit composite Lognormal-Pareto model, Eling (2012) to fit skewed-Normal and skewed-Student distributions, Nadarajah and Bakar (2012) to fit composite Lognormal-Burr model and Scollnik (2007) to fit composite Lognormal-generalized Pareto model, and Scollnik and Sun (2012) to fit truncated composite Weibull-generalized Pareto model. Also, Bernardi et al. (2012) proposed Bayesian approach for skew mixture models. Analysis of this data can be found in Resnick (1997).

To analysis the Danish fire insurance loss data set, descriptive statistics are presented in Table 5.1, Figures 5.1 and 5.2 as follows:

Table 5.1: Summary Statistics of Danish Fire Insurance Loss Data

$No.of$ $Observation$	$Mean$	$Minimum$	$1st quartile$	$Median$	$3rd quartile$	$Maximum$
2492	3.063	0.313	1.157	1.634	2.646	263.250

Table 5.1 contains summary statistics for Danish data set. According to this Table, Danish fire insurance loss data set has a mean of 3.063. The five-number summary that provides information about the five most important percentiles which are the minimum, the first quartile, the median, the third quartile, and the maximum are 0.313, 1.157, 1.634, 2.646, and 263.250, respectively. Apart from the mean being greater than the median, the maximum data is also very far from the 3*rd* quartile. Hence, this data set is skewed to the right.

Figure 5.1 shows the histograms of Danish fire insurance loss data set. Histogram gives a quick visual summary of the data and it is a common tool to show centering, dispersion (spread), and shape (relative frequency) of the data. From Figure 5.1, it can be seen that data set spreads with a right skewness.

Figure 5.2 shows the box plot of Danish fire insurance loss data set. A box plot provides five number summaries which are considered in Table 5.1. From Figure 5.2, it is obvious that data set is skewed to the right.

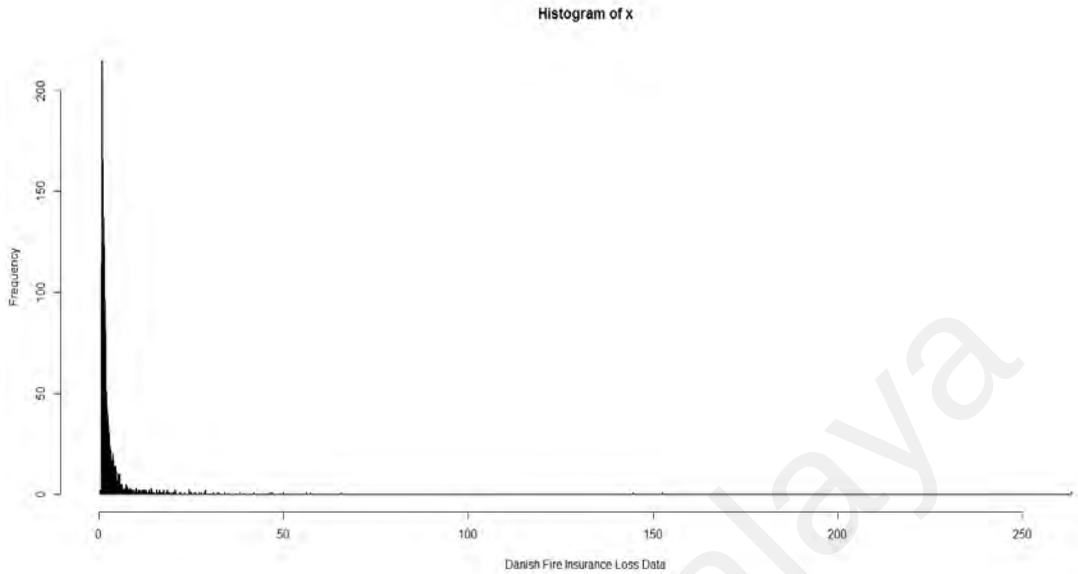

Figure 5.1: Histogram of Danish Fire Insurance Loss Data

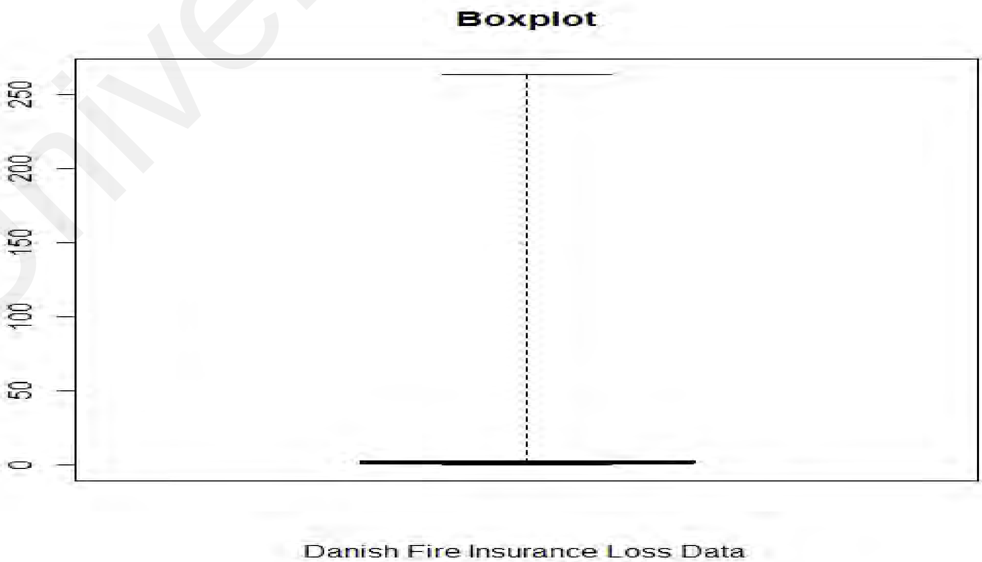

Figure 5.2: Box Plot of Danish Fire Insurance Loss Data

5.2.2 Allocated Loss Adjustment Expenses Data (ALAE)

The second data set is allocated loss adjustment expenses data which consists of 1,500 general liability claims recorded in US dollars. An insurer's estimate of the money which will pay out in claims, as well as expenses associated with processing those claims are presented as allocated loss adjustment expenses (ALAE) plus an unallocated loss adjustment expenses (ULAE). So, allocated loss adjustment expenses are those that are linked directly to the processing of a specific claim. Examples include: legal fees, adjusting fees, court costs, medical costs containment expenses, services required by law or insurance regulation. Unallocated loss adjustment expenses pertains to handling claims that cannot be specifically attributable to a claim. As a result, allocated loss adjustment expenses are charges related by adjusting claims whereas ULAE is the cost to the insurance company of managing its claims department to adjust and resolve claims.

Allocated loss adjustment expenses (ALAE) data set can be found in the **R** package copula and "evd". ALAE has been mentioned in various actuarial studies such as Frees and Valdez (1998) and Klugman and Parsa (1999). For scaling purposes and simplicity of the calculation, the loss data are divided by 1000.

To analysis the allocated loss adjustment expenses data set, descriptive statistics are presented in Table 5.2, Figures 5.3 and 5.4 as follows:

Table 5.2: Descriptive Statistics of Allocated Loss Adjustment Expenses Data

$No.of$ $Observation$	$Mean$	$Minimum$	$1stquartile$	$Median$	$3rdquartile$	$Maximum$
1500	12.588	0.015	2.333	5.471	12.572	501.863

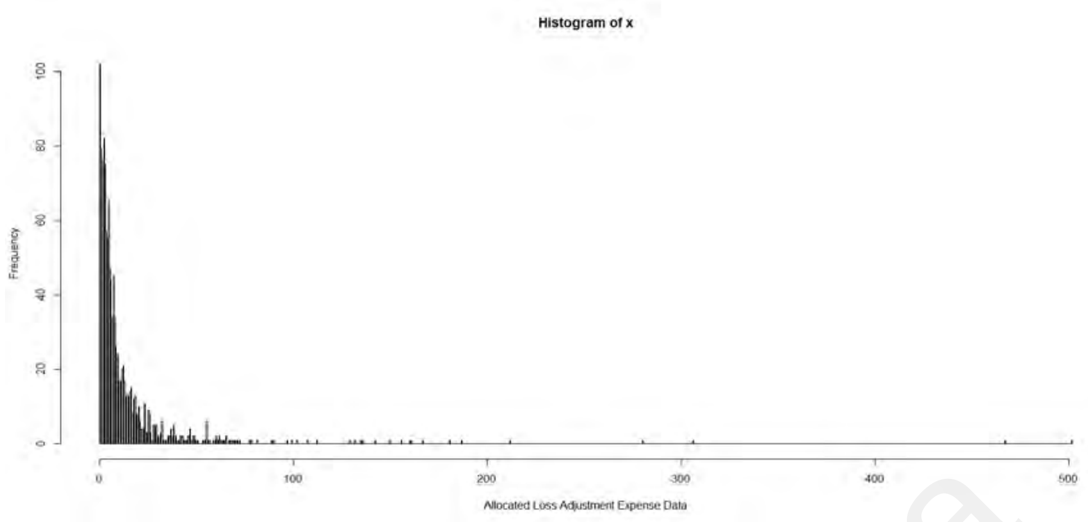

Figure 5.3: Histogram of Allocated Loss Adjustment Expenses Data

Table 5.2 contains summary statistics for ALAE data. According to this Table, allocated loss adjustment expense data set has a mean of 12.588. The five-number summary that provides information about the five most important percentiles which are the minimum, the first quartile, the median, the third quartile, and the maximum are 0.015, 2.333 ,5.471, 12.572 , and 501.863, respectively. Apart from the mean being greater than the median, the maximum data is also very far from the 3*rd* quartile. Hence, this data set is skewed to the right.

Figure 5.3 shows the histograms of allocated loss adjustment expenses data set. Histogram gives a quick visual summary of the data and it is a common tool to show centering, dispersion (spread), and shape (relative frequency) of the data. From Figure 5.3, it can be seen that data set spreads with a right skewness.

Figure 5.4 shows the box plot of allocated loss adjustment expenses data set. A box plot provides five number summaries which are considered in Table 5.2. From Figure 5.4, it is obvious that data set is skewed to the right.

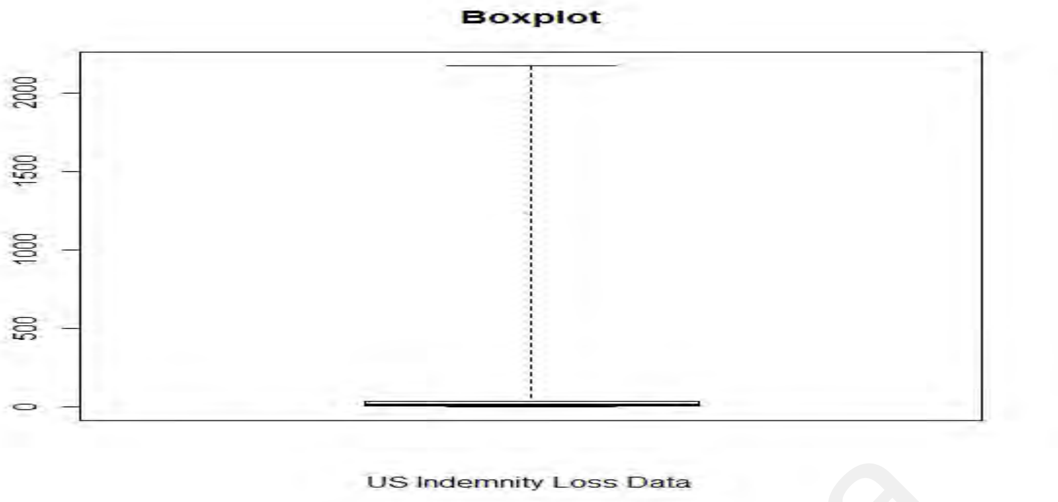

Figure 5.4: Box Plot of Allocated Loss Adjustment Expenses Data

5.3 Fitted Composite Models to Data Sets

For comparison purposes, the results of fitting composite models to the data sets are produced. These models are the composite Weibull-Burr model, the Weibull-inverse Burr model, the composite Weibull-Pareto model, the composite Weibull-inverse Pareto model, the composite Weibull-Loglogistic model, the composite Weibull-Paralogistic model, the composite Weibull-inverse Paralogistic model, and the composite Weibull-generalized Pareto model for the transformed Beta family and the composite Weibull-Weibull model, the composite Weibull-inverse Weibull model, the composite Weibull-Gamma model, the composite Weibull-inverse Gamma model, the composite Weibull-Exponential model, the composite Weibull-inverse Exponential model, the composite Weibull-transformed Gamma model, and the composite Weibull-inverse transformed Gamma model for the transformed Gamma and the inverse transformed Gamma families. Tables 5.3 to 5.10 give the parameter estimations based on maximum likelihood estimation (MLE) method and goodness-of-fit measures which are NLL, AIC, and SBC for Danish fire insurance loss data and allocated loss adjustment expenses (ALAE) data in all three transformed Beta, transformed Gamma, and inverse transformed Gamma families, respectively. The best model provides the lowest value for NLL, AIC, and SBC. These values are bolded for the best model in the Tables.

5.3.1 Fitted Composite Models in Transformed Beta Family

Tables 5.3 to 5.6 provide the estimations as well as the goodness-of-fit measures for both data sets. Tables 5.3 and 5.4 display fitted models to Danish fire insurance loss data in Beta family and Tables 5.5 and 5.6 display fitted models to allocated loss adjustment expenses data in the transformed Beta family, respectively.

As a result, among composite models in transformed Beta family, the composite Weibull-inverse Paralogistic model is prefered and shows the lowest NLL, AIC, and SBC values.

From Table 5.3, it is obvious that among the univariate distributions belongs to the transformed Beta family, the Burr distribution is prefered as it provides the lowest NLL, AIC, and SBC values.

From Table 5.4, it is obvious that among the composite models from the transformed Beta family, the composite Weibull-Burr model shows the lowest NLL and AIC values and the composite Weibull-inverse Paralogistic model shows the lowest SBC values. The composite Weibull-inverse Paralogistic model is preferable because of the smaller number of parameters. That is, the composite Weibull-Burr model has 7 parameters while the composite Weibull-inverse Paralogistic model has 6 parameters. AIC and SBC are used instead of NLL due to different number of model parameters. Also, compared to AIC, since SBC penalizes model more than does the AIC, so rely on SBC is logical.

Table 5.3: Parameter estimates of univariate distributions for Danish fire insurance loss data in transformed Beta family

Distribution	Estimated Parameters	NLL	AIC	SBC
Weibull	$\alpha = 0.9476$ $\sigma = 2.9525$	5270.471	10544.942	10556.584
Burr	$\beta = 0.0878$ $\eta = 14.927$ $\lambda = 0.9209$	**3835.119**	**7676.238**	**7693.701**
Inverse Burr	$\beta = 1.0220$ $\eta = 1.0670$ $\lambda = 11.497$	3967.904	7941.808	7959.271
Pareto	$\beta = 5.1693$ $\lambda = 11.900$	5051.907	10107.814	10119.456
Inverse Pareto	$\beta = 1.1790$ $\lambda = 9.3110$	4647.717	9299.434	9311.076
Loglogistic	$\beta = 2.6526$ $\lambda = 1.7703$	4280.587	8565.174	8576.816
Paralogistic	$\beta = 1.8457$ $\lambda = 2.8066$	4514.882	9033.764	9045.406
Inverse Paralogistic	$\beta = 2.4130$ $\lambda = 1.1008$	4093.318	8190.636	8202.278
Generalized Pareto	$\beta = 1.1800$ $\eta = 1.0810$ $\lambda = 13.497$	4100.322	8206.644	8224.107

Table 5.4: Parameter estimates of composite models for Danish fire insurance loss data in transformed Beta family

Distribution	**Estimated Parameters**	**NLL**	**AIC**	**SBC**
Weibull-Burr	$\alpha = 16.203$ $\sigma = 0.9487$ $\beta = 0.3945$ $\eta = 3.6464$ $\lambda = 0.8457$	**3817.570**	**7645.140**	7674.244
Weibull-Inverse Burr	$\alpha = 15.555$ $\sigma = 0.9630$ $\beta = 1.0450$ $\eta = 1.5680$ $\lambda = 0.6680$	3821.202	7652.404	7681.508
Weibull-Pareto	$\alpha = 15.343$ $\sigma = 0.9689$ $\beta = 1.6526$ $\lambda = 0.5604$	3823.698	7655.396	7678.679
Weibull-Inverse Pareto	$\alpha = 15.629$ $\sigma = 0.9630$ $\beta = 27.383$ $\lambda = 0.7651$	3894.884	7797.768	7821.051
Weibull-Loglogistic	$\alpha = 15.652$ $\sigma = 0.9623$ $\beta = 1.5678$ $\lambda = 0.6799$	3821.229	7650.458	7673.741
Weibull-Paralogistic	$\alpha = 15.512$ $\sigma = 0.9655$ $\beta = 1.2666$ $\lambda = 0.6224$	3822.441	7652.882	7676.165
Weibull-Inverse Paralogistic	$\alpha = 15.806$ $\sigma = 0.9600$ $\beta = 1.5670$ $\lambda = 0.5630$	3820.935	7649.870	**7673.153**
Weibull-Generalized Pareto	$\alpha = 15.847$ $\sigma = 0.9630$ $\beta = 1.6440$ $\eta = 4.9490$ $\lambda = 0.1910$	3822.752	7655.504	7684.608

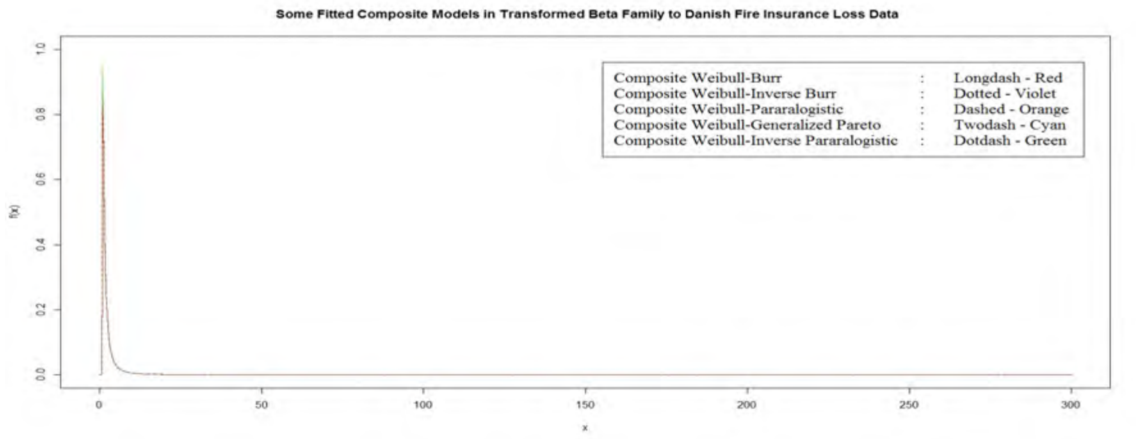

Figure 5.5: Some Fitted Composite Models to the Danish Fire Insurance Loss Data Set in Transformed Beta Family

In Figure 5.5, according to Tables 5.3 and 5.4 for the Danish fire insurance loss data set, some similar fitted composite models to the composite Weibull-inverse Paralogistic model in the transformed Beta family are plotted. The rest univariate distributions and composite models are not such clear that can be plotted in Figure 5.5.

From Table 5.5, it is obvious that among the univariate distributions belongs to the transformed Beta family, the inverse Burr distribution is prefered as it provides the lowest NLL, AIC, and SBC values.

From Table 5.6, it is obvious that among the composite models with tail belonging to the transformed Beta family, the composite Weibull-inverse Paralogistic model is prefered as it provides the lowest NLL, AIC, and SBC values.

Table 5.5: Parameter estimates of univariate distributions for allocated loss adjustment expenses data in transformed Beta family

Distribution	**Estimated Parameters**	**NLL**	**AIC**	**SBC**
Weibull	$\alpha = 0.7417$ $\sigma = 9.9829$	5133.528	10271.056	10281.682
Burr	$\beta = 1.7048$ $\eta = 1.0991$ $\lambda = 10.334$	5048.334	10102.754	10118.693
Inverse Burr	$\beta = 0.6398$ $\eta = 1.5391$ $\lambda = 8.6018$	**5047.755**	**10101.51**	**10117.449**
Pareto	$\beta = 2.2229$ $\lambda = 16.133$	5051.816	10107.632	10118.258
Inverse Pareto	$\beta = 1.3947$ $\lambda = 3.3636$	5099.633	10203.266	10217.892
Loglogistic	$\beta = 1.2749$ $\lambda = 5.3184$	5061.091	10126.182	10136.808
Paralogistic	$\beta = 1.2158$ $\lambda = 6.7065$	5053.235	10110.47	10121.096
Inverse Paralogistic	$\beta = 1.1709$ $\lambda = 4.3667$	5071.857	10147.714	10158.340
Generalized Pareto	$\beta = 2.0249$ $\eta = 1.1241$ $\lambda = 11.505$	5049.057	10104.114	10134.680

Table 5.6: Parameter estimates of composite models for allocated loss adjustment expenses data in transformed Beta family

Distribution	**Estimated Parameters**	**NLL**	**AIC**	**SBC**
Weibull-Burr	$\alpha = 71.5160$ $\sigma = 750.529$ $\beta = 3.18400$ $\eta = 0.99900$ $\lambda = 37.4120$	1861.202	3732.405	3758.971
Weibull-Inverse Burr	$\alpha = 85.9400$ $\sigma = 129.075$ $\beta = 0.2070$ $\eta = 0.4920$ $\lambda = 93.152$	1767.376	3544.752	3571.318
Weibull-Pareto	$\alpha = 71.5160$ $\sigma = 750.529$ $\beta = 3.18400$ $\lambda = 37.4120$	1861.724	3731.448	3752.701
Weibull-Inverse Pareto	$\alpha = 28.4700$ $\sigma = 179.068$ $\beta = 0.00000$ $\lambda = 57.6890$	4083.808	8175.617	8196.869
Weibull-Loglogistic	$\alpha = 72.0150$ $\sigma = 698.964$ $\beta = 1.19800$ $\lambda = 3.84900$	1777.565	3563.131	3584.384
Weibull-Paralogistic	$\alpha = 66.9570$ $\sigma = 1537.19$ $\beta = 1.13400$ $\lambda = 5.37800$	1976.765	3961.530	3982.783
Weibull-Inverse Paralogistic	$\alpha = 76.883$ $\sigma = 360.051$ $\beta = 1.1740$ $\lambda = 2.8230$	**1601.291**	**3210.583**	**3231.836**
Weibull-Generalized Pareto	$\alpha = 77.0200$ $\sigma = 353.738$ $\beta = 8.67600$ $\eta = 1.53800$ $\lambda = 105.256$	2226.836	4463.672	4490.238

5.3.2 Fitted Composite Models in Transformed Gamma and Inverse Transformed Gamma Families

Tables 5.7 to 5.10 provide the estimations as well as the goodness-of-fit measures for both data sets.

Tables 5.7 and 5.8 display the parameter estimates and goodness-of-fit measures which are NLL, AIC, and SBC for the fitted models to Danish fire insurance loss data in the transformed Gamma and the inverse transformed Gamma families, respectively. Tables 5.9 and 5.10 give the parameter estimates and goodness-of-fit measures which are NLL, AIC, and SBC for the fitted models to the allocated loss adjustment expenses (ALAE) data in the transformed Gamma and inverse transformed Gamma families, respectively.

Among the composite models in the transformed Gamma and the inverse transformed Gamma families, the composite Weibull-inverse transformed Gamma model is prefered as it provides the lowest NLL, AIC, and SBC values.

From Table 5.7, it is obvious that among the univariate distributions that belongs to the transformed Gamma and the inverse transformed Gamma families, the inverse transformed Gamma distribution is prefered as it provides the lowest NLL, AIC, and SBC values.

From Table 5.8, it is obvious that among the composite models with tail belonging to the transformed Gamma and the inverse transformed Gamma families, the composite Weibull-inverse transformed Gamma model shows the lowest NLL and AIC values and the composite Weibull-inverse Weibull model shows the lowest SBC values.

Table 5.7: Parameter estimates of univariate distributions for Danish fire insurance loss data in transformed Gamma and inverse transformed Gamma families

Distribution	**Estimated Parameters**	**NLL**	**AIC**	**SBC**
Weibull	$\alpha = 0.9476$ $\sigma = 2.9525$	5270.47	10544.94	10556.582
Inverse Weibull	$\beta = 2.0103$ $\lambda = 1.4395$	3966.83	7937.66	7949.302
Gamma	$\beta = 1.2579$ $\lambda = 2.4346$	5243.027	10490.054	10501.696
Inverse Gamma	$\beta = 2.7533$ $\lambda = 4.4469$	4097.877	8199.754	8211.396
Exponential	$\lambda = 0.3265$	5281.287	10564.574	10570.395
Inverse Exponential	$\lambda = 0.6192$	4645.854	9293.708	9299.529
Transformed Gamma	$\beta = 15.689$ $\eta = 0.3062$ $\lambda = 0.0026$	4620.971	9247.942	9265.405
Inverse Transformed Gamma	$\beta = 0.5553$ $\eta = 2.9217$ $\lambda = 1.0889$	**3931.374**	**7868.748**	**7886.211**

Table 5.8: Parameter estimates of composite models for Danish fire insurance loss data in transformed Gamma and inverse transformed Gamma families

Distribution	**Estimated Parameters**	**NLL**	**AIC**	**SBC**
Weibull-Weibull	$\alpha = 15.971$ $\sigma = 0.9597$ $\beta = 0.2367$ $\lambda = 0.0018$	3835.84	7679.68	7702.963
Weibull-Inverse Weibull	$\alpha = 16.094$ $\sigma = 0.9550$ $\beta = 1.5553$ $\lambda = 0.9075$	3820.01	7648.02	**7674.303**
Weibull-Gamma	$\alpha = 15.048$ $\sigma = 0.9366$ $\beta = 0.0003$ $\lambda = 4.9655$	4102.975	8213.95	8237.233
Weibull-Inverse Gamma	$\alpha = 15.575$ $\sigma = 0.9639$ $\beta = 1.6350$ $\lambda = 1.1213$	3822.126	7652.252	7675.535
Weibull-Exponential	$\alpha = 15.276$ $\sigma = 0.9105$ $\lambda = 0.4438$	4590.169	9186.338	9203.801
Weibull-Inverse Exponential	$\alpha = 15.619$ $\sigma = 0.9630$ $\lambda = 1.2687$	3894.885	7795.770	7813.233
Weibull-Transformed Gamma	$\alpha = 13.366$ $\sigma = 0.9905$ $\beta = 0.3945$ $\eta = 0.2094$ $\lambda = 0.0006$	3832.343	7674.686	7703.790
Weibull-Inverse Transformed Gamma	$\alpha = 16.173$ $\sigma = 0.9490$ $\beta = 0.4836$ $\eta = 2.9885$ $\lambda = 0.8500$	**3817.939**	**7645.878**	7674.982

Table 5.9: Parameter estimates of univariate distributions for allocated loss adjustment expenses data in transformed Gamma and inverse transformed Gamma families

Distribution	Estimated Parameters	NLL	AIC	SBC
Weibull	$\alpha = 0.7417$ $\sigma = 9.9829$	5133.528	10271.056	10281.682
Inverse Weibull	$\beta = 0.6044$ $\lambda = 2.3820$	5353.240	42825.920	10721.106
Gamma	$\beta = 0.6630$ $\lambda = 18.987$	5200.042	10404.084	10414.710
Inverse Gamma	$\beta = 0.4592$ $\lambda = 0.5678$	5596.994	11197.988	11208.614
Exponential	$\lambda = 0.0794$	5299.135	10600.270	10605.583
Inverse Exponential	$\lambda = 0.0002$	6024.144	12050.288	12055.601
Transformed Gamma	$\beta = 6.3376$ $\eta = 0.2889$ $\lambda = 0.0111$	**5064.080**	**10134.160**	**10150.099**
Inverse Transformed Gamma	$\beta = 7.8181$ $\eta = 0.2433$ $\lambda = 176.41$	5151.374	10308.748	10324.688

From Table 5.9, it is obvious that among the univariate distributions that belongs to the transformed Gamma and inverse transformed Gamma families, the transformed Gamma distribution is prefered as it provides the lowest NLL, AIC, and SBC values.

From Table 5.10, it is obvious that among composite models with tail belonging to the transformed Gamma and inverse transformed Gamma families, the composite Weibull-inverse transformed Gamma model is prefered as it provides the lowest NLL, AIC, and SBC values.

Table 5.10: Parameter estimates of composite models for allocated loss adjustment expenses data in transformed Gamma and inverse transformed Gamma families

Distribution	Estimated Parameters	NLL	AIC	SBC
Weibull-Weibull	$\alpha = 1.0246$ $\sigma = 6.6396$ $\beta = 0.2645$ $\lambda = 0.0689$	5047.663	10103.326	10124.579
Weibull-Inverse Weibull	$\alpha = 15.073$ $\sigma = 0.6451$ $\beta = 0.9249$ $\lambda = 2.0432$	6620.716	13249.432	13270.685
Weibull-Gamma	$\alpha = 1.3704$ $\sigma = 2.4899$ $\beta = 0.2773$ $\lambda = 25.259$	5124.477	10256.954	10278.207
Weibull-Inverse Gamma	$\alpha = 14.971$ $\sigma = 0.6432$ $\beta = 1.3448$ $\lambda = 2.0385$	6935.158	13878.316	13899.569
Weibull-Exponential	$\alpha = 14.974$ $\sigma = 0.6433$ $\lambda = 0.0915$	6777.478	13560.956	13576.896
Weibull-Inverse Exponential	$\alpha = 14.984$ $\sigma = 0.6435$ $\lambda = 0.3862$	6614.502	13235.004	13250.944
Weibull-Transformed Gamma	$\alpha = 1.1097$ $\sigma = 4.8283$ $\beta = 0.0018$ $\eta = 0.2787$ $\lambda = 0.4441$	5063.995	10137.99	10164.556
Weibull-Inverse Transformed Gamma	$\alpha = 1.0290$ $\sigma = 6.1102$ $\beta = 17.413$ $\eta = 0.0702$ $\lambda = 11.022$	**4862.754**	**9735.508**	**9762.074**

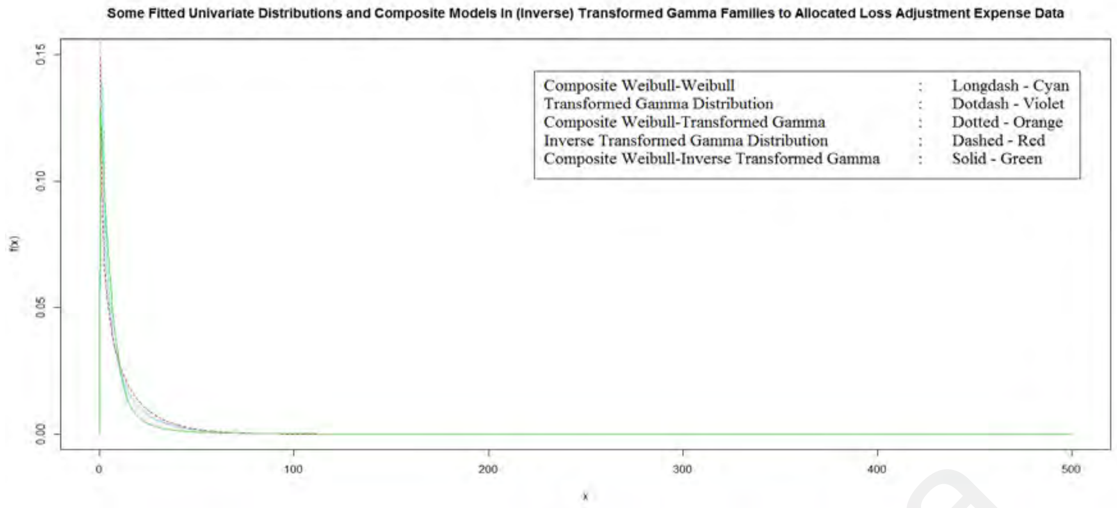

Figure 5.6: Some Fitted Univariate Distributions and Composite Models to the Allocated Loss Adjustment Expenses Data Set in (Inverse) Transformed Gamma Family

In Figure 5.6, according to Tables 5.9 and 5.10 for the allocated loss adjustment expenses data set, some similar fitted univariate distributions and composite models in the transformed Gamma and inverse transformed Gamma families are plotted. The rest univariate distributions and composite models are not clear such that can be plotted in Figure 5.6.

5.4 Statistical Properties of some Fitted Composite Models

According to the results from section 5.3, the composite Weibull-inverse Paralogistic model from the transformed Beta family and the composite Weibull-inverse transformed Gamma model from the transformed Gamma and inverse transformed Gamma families are chosen as the best fitted composite models to Danish fire insurance loss data and allocated loss adjustment expenses (ALAE) data. As a practical example, some statistical properties which are obtained in subsubsections 4.3.1.2 and 4.4.8.2 from section 2.3 and also, plots of fitted composite models to the two data sets are presented.

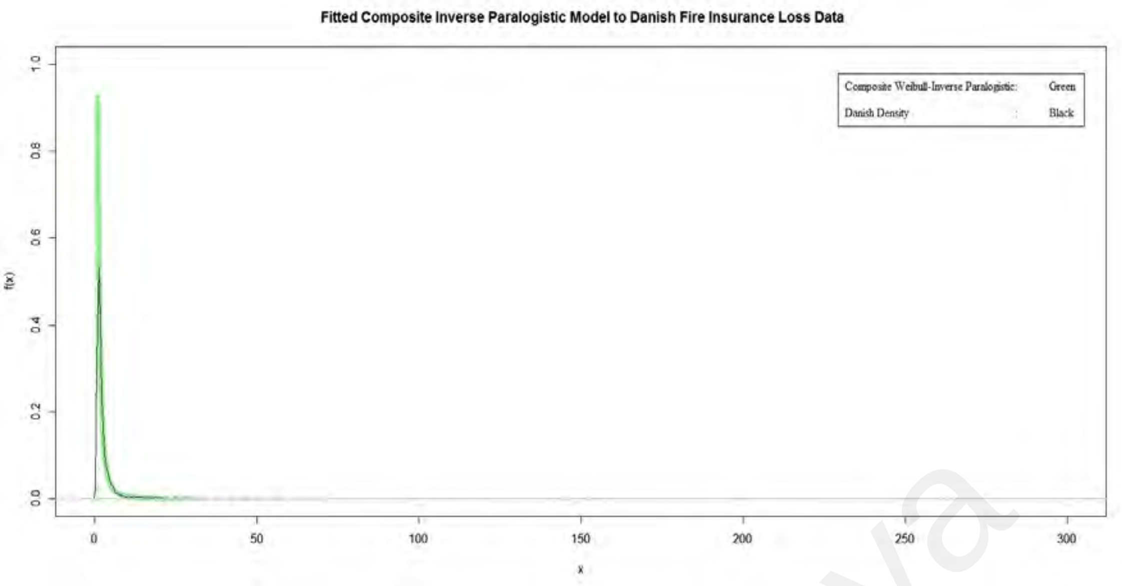

Figure 5.7: Fitted Composite Inverse Paralogistic Model to the Danish Fire Insurance Loss Data

5.4.1 Example 1: Fitted composite Weibull-inverse Paralogistic model to the Danish data set

Since Table 5.4 displays estimated parameters of the fitted composite Weibull-inverse Paralogistic model to the Danish fire insurance loss data set, then some statistical properties which are obtained in subsubsection 4.3.1.2 can be calculated and presented in Table 5.11. It can be seen that the mean, the variance, the skewness, and the excess kurtosis are so close to the empirical values for the Danish data set.

Figure 5.7 shows the probability density function of the fitted composite Weibull-inverse Paralogistic model to the Danish data set and also, the density of this data set. It is seen that this composite model is fitted so close to this data set. So, it is confirmed that the composite Weibull-inverse Paralogistic model is the best fitted model to this data set.

Table 5.11: Statistical properties of composite Weibull-inverse Paralogistic model for Danish fire insurance loss data

	Fitted Model	Data Set
$f_X(x)$	$\begin{cases} (4.369)x^{14.806}e^{-(\frac{x}{0.96})^{15.806}}, & 0 < x \leq 0.958, \\ \frac{(21.193)x^{2.456}}{x[1+(\frac{x}{0.563})^{1.567}]^{2.567}}, & 0.958 \leq x < +\infty. \end{cases}$	-
$F_X(x)$	$\begin{cases} 0, & x < 0, \\ (0.145)[1-e^{-(\frac{x}{0.96})^{15.806}}], & 0 < x \leq 0.958, \\ 0.09 + \frac{(8.632)x^{2.456}-(1.196)[1+(\frac{x}{0.563})^{1.567}]^{1.567}}{[1+(\frac{x}{0.563})^{1.567}]^{1.567}}, & 0.958 \leq x < +\infty, \\ 1, & x \geq +\infty. \end{cases}$	-
Q_p	$\begin{cases} F_1^{-1}[(6.558)p], & 0 < p \leq 0.09, \\ F_2^{-1}[\frac{(0.293)[(10.5779)p-1]}{[1+(\frac{x}{0.563})^{1.567}]^{1.567}} + \frac{3.6887}{[1+(\frac{x}{0.563})^{1.567}]^{1.567}}], & 0.09 \leq p < 1. \end{cases}$	-
$E[X]$	3.084	3.063
$Var[X]$	63.628	66.236
$M_X(t)$	$(0.145)\sum_{k=0}^{\infty}\frac{t^k}{k!}(0.96)^k\gamma(\frac{k}{15.806}+1, 0.968)$ $-(3.3)\sum_{k=0}^{\infty}\frac{t^k}{k!}(0.563)^k B([0.303; 1-\frac{k}{1.567}, 1.567+\frac{k}{1.567}).$	-
γ_1	16.560	19.884
b_2	421.997	546.133

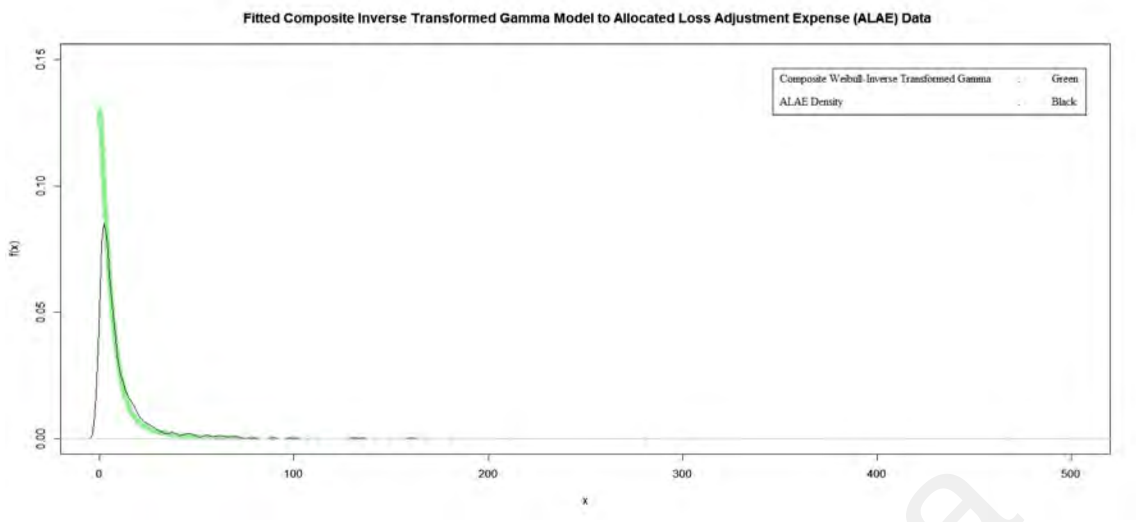

Figure 5.8: Fitted Composite Inverse Transformed Gamma Model to the Allocated Loss Adjustment Expense (ALAE) Data

5.4.2 Example 2: Fitted composite Weibull-inverse transformed Gamma model to the allocated loss adjustment expenses data set

Since Table 5.10 displays estimated parameters of the fitted composite Weibull-inverse transformed Gamma model to the allocated loss adjustment expenses (ALAE) data set, then some statistical properties which are obtained in Subsubsection 4.4.8.2 can be calculated and presented in Table 5.12. It can be seen that the mean, variance, skewness, and kurtosis are close to the empirical values for the ALAE data set.

Figure 5.8 shows the probability density function of the fitted composite Weibull-inverse transformed Gamma model to the allocated loss adjustment expenses (ALAE) data set and also, the density of this data set. It is seen that this composite model is fitted so close to this data set. So, it is confirmed that the composite Weibull-inverse transformed Gamma model is the best fitted model to this data set.

Table 5.12: Statistical properties of composite Weibull-inverse transformed Gamma model for allocated loss adjustment expenses (ALAE) data

	Fitted Model	Data Set
$f_X(x)$	$\begin{cases} (0.146)x^{0.029}e^{-(\frac{x}{6.1102})^{1.029}}, & 0 < x \leq 12.682, \\ 128.585(\frac{1}{x})^{2.222}e^{-(\frac{11.022}{x})^{0.0702}}, & 12.682 \leq x < +\infty. \end{cases}$	-
$F_X(x)$	$\begin{cases} 0, & x < 0, \\ (0.909)[1-e^{-(\frac{x}{6.1102})^{1.029}}], & 0 < x \leq 0.2575, \\ 1-(10.582)\gamma(17.413,(\frac{11.029}{x})^{0.0702}), & 0.2575 \leq x < +\infty, \\ 1, & x \geq +\infty. \end{cases}$	-
Q_p	$\begin{cases} F_1^{-1}[1.107p], & 0 < p \leq 0.8, \\ F_2^{-1}[1+(1.379*10^{-15})p], & 0.8 \leq p < 1. \end{cases}$	-
$E[X]$	20.213	12.588
$Var[X]$	535.707	792.177
$M_X(t)$	$(0.909)\sum_{k=0}^{\infty}\frac{t^k}{k!}(6.1102)^k\gamma(\frac{k}{1.029}+1,2.12)$ $+(1.58*10^{-13})\sum_{k=0}^{\infty}\frac{t^k}{k!}(11.022)^k\gamma(17.413-\frac{k}{11.022},1.302).$	-
γ_1	13.716	9.241
b_2	193.087	124.546

5.5 Summary

In this chapter, two famous data sets are considered which are Danish fire insurance loss data and allocated loss adjustment expenses (ALAE) data. Each data set is described by their summary statistics and illustrated via histogram, and box plot. Several univariate distributions and composite models in the transformed Beta, transformed Gamma, and the inverse transformed Gamma families are fitted to each data set and the results are presented in Tables 5.3 to 5.10. For both data sets, the composite Weibull-inverse Paralogistic model from the transformed Beta family and the composite Weibull-inverse transformed Gamma model from the inverse transformed Gamma family provide the lowest value in all NLL, AIC, and SBC and hence, these two models are considered as the best fitted models. For two data sets, some fitted univariate distributions and composite models in all three families are plotted. The summaries of the properties and figures of the fitted composite Weibull-inverse Paralogistic model to Danish fire insurance loss data and the fitted composite Weibull-inverse transformed Gamma model to the allocated loss adjustment expenses (ALAE) data are presented in section 5.4. The estimations obtained through the fitted composite models are provided good estimations for the considered statistics.

CHAPTER 6

RISK MEASURES OF THE NEW COMPOSITE MODELS

6.1 Introduction

Loss distributions for insurance policies are commonly used in actuarial industry. Characteristics of loss distributions are applied for risk management, pricing, and reserving. Risk measure is one of the most significant tools which maps a loss distribution to a value. Also, it is used to determine the amount of an asset or set of assets to be kept in reserve to make the risks taken by financial institutions such as banks and insurance companies acceptable to the moderator. In actuarial science, the first use of risk measures is in premium principles which are described in Bühlmann (1970, p.85), Cizek et al. (2005, p.408), Gerber (1979), Laeven and Goovaerts (2008), Minkova (2010), and many more. By having a loss distribution, premium principles can be used to determine an appropriate premium to charge for the risk. Some of them are mentioned below.

6.2 Premium Principles

Let X denotes a non-negative random variable describing the loss (risk) with a probability density function $f_X(t)$ and a cumulative distribution function $F_X(t)$. Moreover, assume that the expected value $E[X]$, the variance $Var[X]$ and the moment generating function $M_X(T) = E[e^{tX}]$ exist. If the premium is denoted by P, we have:

- Pure risk principle:

  This type of premium is the simplest, which equals to the expectation of a loss variable. That is, the premium is $P = E[X]$. This premium is often applied in life and some in non-life insurance.

- Expected value principle:

  This type of premium is with safety (security) loading and equals to $P = (1+\alpha)E[X], \quad$ for some $\alpha > 0$ where α and $\alpha E[X]$ are the relative and total safety loadings, respectively. This premium is an increasing linear function of α and it equals to the pure risk premium for $\alpha = 0$. This premium is sufficient for a risk neutral insurer only.

- Variance principle:

  This premium equals to $P = E[X] + \alpha Var[X], \quad$ for some $\alpha > 0$. In this case, the premium depends not only on the expectation but also on the variance of the loss. This premium is an increasing linear function of α and it is obvious that for $\alpha = 0$, it equals to the pure risk premium.

- Standard deviation principle:

  This premium equals to $P = E[X] + \alpha\sqrt{Var[X]}, \quad$ for some $\alpha > 0$. In this case, the premium depends on the expectation and also on the standard deviation of the loss. This premium is an increasing linear function of α and clearly for $\alpha = 0$, it reduces to the pure risk premium.

- Zero utility principle:

  This principle states that the premium P for a loss (risk) X and utility function u should be calculated such that $E[u(P-X)] \geq u(0) = 0$. That is, the expected utility is (at least) equal to the zero utility. This principle yields a minimum premium, P_L, in the sense that the loss (risk) X should not be accepted at a premium below P.

It is obvious that except the pure risk principle, other measures generate a premium which is greater than the expected loss.

In terms of practical aspect, by using real data sets and applying descriptive statistics which are mentioned in section 2.3, above premium principles can be computed.

6.3 Risk Measure

A risk measure is a function which is mapping a loss (risk) distribution to a real number. Thus, for a loss random variable X and a functional risk measure H, we have: $H : X \rightarrow \mathbf{R}$, where $\mathbf{R}$ is a real number.

Risk measures are divided to two types: individual losses and aggregate losses where each has its own measurements. One way to estimate the distribution function for individual and aggregate losses approximately is the one which is designed primarily to calculate percentiles of the right tail for skewed distributions. Risk measurements which belong to the individual losses and described here are value-at-risk, (VaR), and conditional tail expectation, (CTE), (Artzner, 1999), (Artzner et al., 2009), (Abu Bakar et al., 2015), (Hardy, 2006), (Heath et al., 1999), and (Philippe, 2001).

6.3.1 Value-at-Risk, (VaR)

Value-at-risk, (VaR), has been widely adopted for measuring market risk in trading profiles during the 1990's but its origins goes further back around 1922 at which time the New York Stock Exchange (NYSE) imposed requirements on member firms. Value-at-risk provides measures to three variables: a loss amount, a probability of loss amount, which is confidence level, and a time frame. Value-at-risk is the maximum loss not exceeding a given probability defined as the confidence level, over a given period of time. This measure is used by risk managers in order to measure and control the level of risk which the firm undertakes. Risk managers should ensure that risks are not taken beyond a level at which the firm can absorb the losses of a probable worst outcome. Generally, it reduces all loss information to a value. For example, a financial firm may determine that it has a 5% one month value-at-risk of $100 million, which translate to a 5% chance that the firm could lose more than $100 million in any given month. Therefore, a $100 million loss should be expected to occur once every 20 months. Also, losses greater than the value-at-risk may appear only with a specified small probability (Acerbi and Tasche, 2002), (Albrecht et al., 1996), (Carl et al., 2010), (Hyndman and Fan, 1996), (Käärik and Zegulova, 2012), and (Linsmeier and Pearson, 1996).

There are several reasons to why VaR have become a popular risk measure. Some of them are listed below:

- VaR has a simple interpretation. It is expressed as money-at-risk which makes it very easy to communicate to non-economists as well.
- VaR is probabilistic which gives the useful information about the probability associated with different loss values.

- VaR can be applied to any type of a portfolio which allows us to compare the risk between each other.

Although, VaR is the most popular risk measure, some of its drawbacks and disadvantages should be considered such as:

- VaR is sensitive to incorrect assumptions about the underlying distribution. They may cause to an overestimate or an underestimate values for VaR.
- VaR is silent about the size or value if a loss greater than VaR occurs.
- VaR is not coherent.

The latter one is the most crucial drawback and need some further explanation: As Delbaen (2000) mentioned, for a risk random variable X, a coherent risk measure, ρ, is a function $\rho : \mathbf{L} \to \mathbf{R}$, where $\mathbf{R}$ is a real number and $\mathbf{L}$ is a sets of risks, such that satisfies with the below axioms:

1. Monotonicity: If $X,Y \in \mathbf{L}$ and $X \leq Y$, Then $\rho(X) \leq \rho(Y)$.
2. Positive homogeneity: If $X \in \mathbf{L}$ and $\tau \geq 0$, Then $\rho(\tau X) = \tau\rho(X)$.
3. Translation invariance: $\rho(X+k) = \rho(X)+k, \quad \forall k \in \mathbf{R}$
4. Sub-additivity: If $X,Y \in \mathbf{L}$, Then $\rho(X+Y) \leq \rho(X)+\rho(Y)$.

In simple words, the above axioms means whenever a portfolio is risker than the others, it will have a higher risk value as long as the risk measure is coherent (Artzner et al., 2002), (Finan, 2015), (Meyers, 2005), and (Mitra and Ji, 2010). Sub-additivity is the one which is not satisfied by VaR. In fact, sub-additivity of a risk measure is a mathematical way which leads to less risk (Ias Letmark, 2010).

In this study, two ways of calculation of VaRs are considered which are the empirical and the theoretical methods. The closest theoretical value to the corresponding empirical value is determines the best composite model (Nadarajah and Bakar, 2013).

6.3.1.1 Empirical Value-at-Risk

Let X be a random variable with a probability density function $f_X(x)$ and a cumulative distribution function $F_X(x)$. Definition of the empirical value-at-risk, V_α, which is the percentile of distribution at α confidence level where $0 < \alpha < 1$, is defined as:

$$F_X(V_\alpha) = \alpha, \Rightarrow \quad V_\alpha = \hat{F}_X^{-1}(\alpha). \tag{6.1}$$

where $\hat{F}_X^{-1}(.)$ denotes the inverse empirical cumulative distribution function. $\hat{F}_X^{-1}(.)$ exists, since we are dealing with continuous and strictly increasing distributions.

6.3.1.2 Theoretical Value-at-Risk

The theoretical value-at-risk, VaR_α, for a composite model can be expressed as follows:

$$VaR_\alpha = \begin{cases} F_1^{-1}(\alpha(1+\phi)F_1(\theta)), & 0 < \alpha \leq \frac{1}{1+\phi}, \\ F_2^{-1}(F_2(\theta) + (\alpha(1+\phi)-1)(1-F_2(\theta))/\phi), & \frac{1}{1+\phi} \leq \alpha < 1. \end{cases} \tag{6.2}$$

where F_1^{-1} and F_2^{-1} denote the inverse cumulative distribution functions of the head and the tail, respectively, ϕ is a mixing weight, and θ is a threshold.

6.3.2 Conditional Tail Expectation, (CTE)

One problem with the value-at-risk, (VaR), is that it does not give any information about the severity of loss by which it is exceeded. That is, it is caused by the loss distribution above the quantile that does not affect the risk measure. To solve this problem, conditional tail expectation, (CTE), is used (Hardy, 2006), (Necir et al., 2009), and Sarykalin et al. (2008). Another point is that CTE assesses how bad things can get if the VaR loss is exceeded. That is, CTE is the expected loss given that the VaR loss is exceeded (Mitra and Ji, 2010). Conditional Tail Expectation, (CTE), has a number of names such as tail Value-at-Risk (TVaR) which is introduced by Desmedt and Jean-Francois (2008), tail conditional expectation (TCE) and conditional Value-at-Risk (CVaR) which are introduced by Rockafellar and Uryasev (2002), and expected shortfall introduced by Acerbi and Tasche (2002). CTE satisfies the coherent risk measure axioms as in section 6.3.1 and consequently, is more robust than the VaR (Ias Letmark, 2010).

In this study, two ways of calculation of CTEs are considered which are the empirical and the theoretical methods. The closest theoretical value to the corresponding empirical value determines the best composite model.

6.3.2.1 Empirical Conditional Tail Expectation

Let X be a random variable with a probability density function $f_X(x)$ and a cumulative distribution function $F_X(x)$. Definition of the empirical conditional tail expectation, C_α, which is the percentile of distribution at α confidence level where $0 < \alpha < 1$, is:

$$C_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 \hat{F}_s^{-1} ds. \tag{6.3}$$

where $\hat{F}_{\cdot}^{-1}$ denotes the inverse empirical cumulative distribution function.

6.3.2.2 Theoretical Conditional Tail Expectation

Theoretical conditional tail expectation, CTE_α, for a composite model can be expressed as follows:

$$CTE_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_s ds. \tag{6.4}$$

where α is a confidence level and VaR_s is the theoretical value-at-risk.

6.3.3 Applications

In this section, VaRs and CTEs are calculated for both data sets, Danish fire insurance loss data and allocated loss adjustment expenses (ALAE) data at 99% confidence level.

6.3.3.1 VaR for Composite Models in Transformed Beta Family

Table 6.1 displays the computed VaRs for both Danish and ALAE data sets, respectively with estimated parameters presented in Tables 5.4 and 5.6. The calculated value-at-risk at 99% confidence level for different composite models with the Weibull distribution as the head and the tail distribution, which is made up from the transformed Beta family, are shown in Table 6.1.

Table 6.1: VaR at 99% Confidence Level for Danish Fire Insurance Loss Data and Allocated Loss Adjustment Expenses data

Composite Models	VaR **Danish Data**	VaR **ALAE Data**
Emprical	**24.613**	**131.7090**
Weibull-Burr	**25.182**	446.8320
Weibull-Inverse Burr	22.680	170235.0
Weibull-Pareto	22.648	**121.9470**
Weibull-Inverse Pareto	86.780	873.6330
Weibull-Loglogistic	22.698	180.2070
Weibull-Paralogistic	22.609	191.4890
Weibull-Inverse Paralogistic	22.640	163.8170
Weibull-Generalized Pareto	22.489	95.21200

Table 6.1 shows that the composite Weibull-Burr model provides the best fit to the Danish data set and the composite Weibull-Pareto model is the closest to the empirical VaR value for the ALAE data set.

As a result, we may say these two composite models are reasonable fits for both data sets.

6.3.3.2 VaR for Composite Models in Transformed Gamma and Inverse Transformed Gamma Families

Table 6.2 displays the computed VaRs for both Danish and ALAE data sets, respectively with estimated parameters presented in Tables 5.8 and 5.10. The calculated value-at-risk at 99% confidence level for different composite models with the Weibull distribution as the head and the tail distribution, which is made up from the transformed Gamma and the inverse transformed Gamma families, are shown in Table 6.2.

Table 6.2: VaR at 99% Confidence Level for Danish Fire Insurance Loss Data and Allocated Loss Adjustment Expenses Data

Composite Models	VaR **Danish Data**	VaR **ALAE Data**
Emprical	**24.614**	**131.709**
Weibull-Weibull	18.625	117.686
Weibull-Inverse Weibull	22.765	308.104
Weibull-Gamma	14.719	74.6767
Weibull-Inverse Gamma	22.431	55.9928
Weibull-Exponential	11.186	50.8822
Weibull-Inverse Exponential	86.776	260.196
Weibull-Transformed Gamma	19.944	149.354
Weibull-Inverse Transformed Gamma	**25.046**	**140.846**

Table 6.2 shows that the composite Weibull-inverse transformed Gamma model provides the best fit to both data sets, Danish and ALAE.

As a result, we may say this composite model is the reasonable fits for both data sets.

6.3.3.3 CTE for Composite Models in Transformed Beta Family

Table 6.3 displays the computed CTEs for both Danish and ALAE data sets, respectively with estimated parameters presented in Tables 5.4 and 5.6. The calculated conditional tail expectations at 99% confidence level for different composite models with the Weibull distribution as the head and the tail distribution, which is made up from the transformed Beta family, are shown in Table 6.3.

Table 6.3: CTE at 99% Confidence Level for Danish Fire Insurance Loss Data and Allocated Loss Adjustment Expenses Data

Composite Models	CTE **Danish Data**	CTE **ALAE Data**
Emprical	**54.604**	**222.680**
Weibull-Burr	82.609	1273.04
Weibull-Inverse Burr	62.740	33.000
Weibull-Pareto	**58.210**	**189.410**
Weibull-Inverse Pareto	NA[1]	4651.21
Weibull-Loglogistic	62.796	633.520
Weibull-Paralogistic	60.353	597.010
Weibull-Inverse Paralogistic	62.650	598.450
Weibull-Generalized Pareto	58.149	121.560

[1] NA shows that a conditional tail expectation value could not be computed due to convergence issues.

Table 6.3 shows that the composite Weibull-Pareto model provides the best fit to both data sets, Danish and ALAE. As a result, we may say this composite model is the reasonable fit for both data sets.

6.3.3.4 CTE for Composite Model in Transformed Gamma and Inverse Transformed Gamma Families

Table 6.4 displays the computed CTEs for both Danish and ALAE data sets, respectively with estimated parameters presented in Tables 5.8 and 5.10. The calculated conditional tail expectations at 99% confidence level for different composite models with the Weibull distribution as the head and the tail distribution, which is made up from the transformed Gamma and the inverse transformed Gamma families, are shown in Table 6.4.

Table 6.4: CTE at 99% Confidence Level for Danish Fire Insurance Loss Data and Allocated Loss Adjustment Expenses Data

Composite Models	CTE **Danish Data**	CTE **ALAE Data**
Emprical	**54.604**	**222.680**
Weibull-Weibull	31.546	210.575
Weibull-Inverse Weibull	63.863	NA[1]
Weibull-Gamma	18.762	96.400
Weibull-Inverse Gamma	**58.431**	**220.907**
Weibull-Exponential	13.439	61.808
Weibull-Inverse Exponential	NA[1]	NA[1]
Weibull-Transformed Gamma	35.711	295.793
Weibull-Inverse Transformed Gamma	81.301	505.382

[1] NA shows that a conditional tail expectation value could not be computed due to convergence issues.

Table 6.4 shows that the composite Weibull-inverse Gamma model provides the best fit to both data sets, Danish and ALAE.

As a result, we may say this composite model is a reasonable fit for both data sets.

6.4 Validity of Risk Measures: Backtesting

Backtesting is a statistical method to investigate the validity of risk measures. To check that the results from value-at-risk, (VaR), calculations are reliable, the models should be backtested. This is called reality checks by Jorion (2001, 18-25). Backtesting is a method where actual profits and losses are compared to projected VaR estimates. If VaR estimates are not precise, the models should be re-examined for misspecified models, incorrect parameters, or inaccurate assumptions. Since VaR measures the expected loss just under normal assumption, using backtesting can provide more interests in good estimating (Nieppola et al., 2009).

In this section, backtesting procedure as a test for validity of VaR and conditional tail expectation (CTE) is calculated for both data sets, Danish fire insurance loss data and allocated loss adjustment expenses (ALAE) data. The backtesting examines whether the proportion of violations and the empirical mean of violations obtained using the theoretical estimates of VaR and CTE are compatible with the expected confidence levels of two risk measures. This can be verified through a Binomial test comparing the number of violations observed with probability of VaR, %1, which was proposed by Kupiec (1995) as an unconditional backtesting and through a t-test comparing the CTE of each model with the average of the values that empirically exceeded the VaR calculated using each model. Based on the p-value of the t-test, the null hypothesis that the model is appropriate, will be accepted or rejected (Abu Bakar et al., 2015).

6.4.1 Backtesting of VaR for Composite Models in Transformed Beta Family

For both Danish and ALAE data sets with estimated parameters presented in Tables 5.4 and 5.6, the computed backtesting of VaRs are displayed in Tables 6.5 and 6.6 for different composite models with the Weibull distribution as the head and the tail distribution, which is made up from the transformed Beta family.

Table 6.5: Backtesting of VaR for Danish Fire Insurance Loss Data

Composite Models	VaR	Viol.	Prop. Viol.	Conf. Int.	p-value
Weibull-Burr	25.182	24	0.010	0.006-0.014	**1.000**
Weibull-Inverse Burr	22.680	28	0.011	0.007-0.016	**0.544**
Weibull-Pareto	22.648	28	0.011	0.007-0.016	**0.544**
Weibull-Inverse Pareto	86.780	3	0.001	0.000-0.004	0.000
Weibull-Loglogistic	22.698	28	0.011	0.007-0.016	**0.544**
Weibull-Paralogistic	22.609	28	0.011	0.007-0.016	**0.544**
Weibull-Inverse Paralogistic	22.640	28	0.011	0.007-0.016	**0.544**
Weibull-Generalized Pareto	22.489	28	0.011	0.007-0.016	**0.544**

From Table 6.5, based on the p-values of the Binomial test, all composite models except the composite Weibull-inverse Pareto model, do not reject the null hypothesis and thus the estimates of VaR under these models are reasonable for Danish data.

Table 6.6: Backtesting of VaR for Allocated Loss Adjustment Expenses Data

Composite Models	VaR	Viol.	Prop. Viol.	Conf. Int.	p-value
Weibull-Burr	446.832	2	0.001	0.000-0.005	0.000
Weibull-Inverse Burr	170,235.013	0	0.000	0.000-0.002	0.000
Weibull-Pareto	121.947	17	0.011	0.007-0.018	**0.602**
Weibull-Inverse Pareto	873.633	0	0.000	0.000-0.002	0.000
Weibull-Loglogistic	180.207	7	0.005	0.002-0.010	0.037
Weibull-Paralogistic	191.489	5	0.003	0.001-0.008	0.006
Weibull-Inverse Paralogistic	163.817	8	0.005	0.002-0.011	**0.069**
Weibull-Generalized Pareto	95.212	22	0.015	0.009-0.022	**0.089**

From Table 6.6, based on the p-values of the Binomial test, the composite Weibull-Pareto model, the composite Weibull-inverse Paralogistic model, and the composite Weibull-generalized Pareto model do not reject the null hypothesis and thus the estimates of VaR under these models are reasonable for ALAE data.

6.4.2 Backtesting of VaR for Composite Model in Transformed Gamma and Inverse Transformed Gamma Families

For both Danish and ALAE data sets with estimated parameters presented in Tables 5.8 and 5.10, the computed backtesting of VaRs are displayed in Tables 6.7 and 6.8 for different composite models with the Weibull distribution as the head and the tail distribution, which is made up from the transformed Gamma and inverse transformed Gamma families.

Table 6.7: Backtesting of VaR for Danish Fire Insurance Loss Data

Composite Models	VaR	Viol.	Prop. Viol.	Conf. Int.	p-value
Weibull-Weibull	18.626	44	0.018	0.013-0.023	0.000
Weibull-Inverse Weibull	22.765	28	0.011	0.008-0.016	**0.060**
Weibull-Gamma	14.719	61	0.025	0.019-0.032	0.000
Weibull-Inverse Gamma	22.431	28	0.011	0.008-0.016	**0.060**
Weibull-Exponential	11.186	93	0.037	0.030-0.046	0.000
Weibull-Inverse Exponential	86.776	3	0.001	0.000-0.004	0.000
Weibull-Transformed Gamma	19.944	36	0.014	0.010-0.020	0.033
Weibull-Inverse Transformed Gamma	25.046	24	0.010	0.006-0.015	**0.933**

From Table 6.7, based on the p-values of the Binomial test, the composite Weibull-inverse Weibull model, the composite Weibull-inverse Gamma model, and the composite Weibull-inverse transformed Gamma model do not reject the null hypothesis and thus the estimates of VaR under these models are reasonable for Danish data.

Table 6.8: Backtesting of VaR for Allocated Loss Adjustment Expenses Data

Composite Models	VaR	Viol.	Prop. Viol.	Conf. Int.	p-value
Weibull-Weibull	117.686	17	0.011	0.007-0.019	**0.697**
Weibull-Inverse Weibull	308.104	2	0.001	0.000-0.005	0.001
Weibull-Gamma	74.677	28	0.019	0.013-0.027	0.001
Weibull-Inverse Gamma	55.993	52	0.035	0.026-0.046	0.000
Weibull-Exponential	50.882	61	0.041	0.032-0.052	0.000
Weibull-Inverse Exponential	260.196	4	0.003	0.001-0.007	0.006
Weibull-Transformed Gamma	149.354	12	0.008	0.004-0.014	**0.517**
Weibull-Inverse Transformed Gamma	140.846	13	0.009	0.005-0.015	**0.697**

From Table 6.8, based on the p-values of the binomial test, the composite Weibull-Weibull model, the composite Weibull-transformed Gamma model, and the composite Weibull-inverse transformed Gamma model do not reject the null hypothesis and thus the estimates of VaR under these models are reasonable for ALAE data.

6.4.3 Backtesting of CTE for Composite Models in Transformed Beta Family

For both Danish and ALAE data sets with estimated parameters presented in Tables 5.4 and 5.6, the computed backtesting of CTEs are displayed in Tables 6.9 and 6.10 for different composite models with the Weibull distribution as the head and the tail distribution, which is made up from the transformed Beta family.

Table 6.9: Backtesting of CTE for Danish Fire Insurance Loss Data

Composite Models	CTE	Mean. Viol.	Conf. Int.	p-value
Weibull-Burr	82.609	55.839	32.5-79.2	0.027
Weibull-Inverse Burr	62.740	51.340	31.1-71.6	**0.259**
Weibull-Pareto	58.210	51.340	31.1-71.6	**0.493**
Weibull-Inverse Pareto	NA[1]	186.774	NA-NA[1]	NA[1]
Weibull-Loglogistic	62.796	51.340	31.1-71.6	**0.257**
Weibull-Paralogistic	60.357	51.340	31.1-71.6	**0.370**
Weibull-Inverse Paralogistic	62.650	51.340	31.1-71.6	**0.263**
Weibull-Generalized Pareto	58.149	51.340	31.1-71.6	**0.497**

[1] NA means either that the theoretical estimate of conditional tail expectation was missing or that the number of observations needed for testing was not sufficient.

From Table 6.9, based on the p-values of the t-test, all composite models except the composite Weibull-Burr model and the composite Weibull-inverse Pareto model, do not reject the null hypothesis and thus the estimates of CTE under these models are reasonable for Danish data.

Table 6.10: Backtesting of CTE for Allocated Loss Adjustment Expenses Data

Composite Models	CTE	Mean. Viol.	Conf. Int.	p-value
Weibull-Burr	1273.04	484.554	265-704	0.014
Weibull-Inverse Burr	33.0000	NA[1]	NA-NA[1]	NA[1]
Weibull-Pareto	189.409	211.842	153-271	**0.430**
Weibull-Inverse Pareto	4651.21	NA[1]	NA-NA[1]	NA[1]
Weibull-Loglogistic	633.524	304.892	183-427	0.001
Weibull-Paralogistic	597.010	353.391	198-509	0.012
Weibull-Inverse Paralogistic	598.452	287.642	178-397	0.000
Weibull-Generalized Pareto	121.557	187.212	138-236	0.011

[1] NA means either that the theoretical estimate of conditional tail expectation was missing or that the number of observations needed for testing was not sufficient.

From Table 6.10, based on the p-values of the t-test, the composite Weibull-Pareto model does not reject the null hypothesis and thus the estimate of CTE under this model is reasonable for ALAE data.

6.4.4 Backtesting of CTE for Composite Model in Transformed Gamma and Inverse Transformed Gamma Families

For both Danish and ALAE data sets with estimated parameters presented in Tables 5.8 and 5.10, the computed backtesting of CTEs are displayed in Tables 6.11 and 6.12 for different composite models with the Weibull distribution as the head and the tail distribution, which is made up from the transformed Gamma and the inverse transformed Gamma families.

Table 6.11: Backtesting of CTE for Danish Fire Insurance Loss Data

Composite Models	CTE	Mean. Viol.	Conf. Int.	p-value
Weibull-Weibull	31.546	39.989	26.563-53.415	**0.212**
Weibull-Inverse Weibull	63.863	51.339	31.052-71.628	**0.216**
Weibull-Gamma	18.762	33.522	23.577-43.468	0.004
Weibull-Inverse Gamma	58.431	51.339	31.052-71.628	**0.479**
Weibull-Exponential	13.439	26.419	19.647-33.190	0.000
Weibull-Inverse Exponential	NA[1]	186.774	NA[1]	NA[1]
Weibull-Transformed Gamma	35.711	44.639	28.507-60.773	**0.269**
Weibull-Inverse Transformed Gamma	81.301	55.839	32.459-79.219	0.034

[1] NA means either that the theoretical estimate of conditional tail expectation was missing or that the number of observations needed for testing was not sufficient.

From Table 6.11, based on the p-values of the t-test, the composite Weibull-Weibull model, the composite Weibull-inverse Weibull model, the composite Weibull-inverse Gamma model, and the composite Weibull-transformed Gamma do not reject the null hypothesis and thus the estimates of CTE under these models are reasonable for Danish data.

From Table 6.12, based on the p-values of the t-test, the composite Weibull-Weibull model and the composite Weibull-transformed Gamma model do not reject the null hypothesis and thus the estimates of CTE under these models are reasonable for ALAE data.

Table 6.12: Backtesting of CTE Allocated Loss Adjustment Expenses Data

Composite Models	CTE	Mean. Viol.	Conf. Int.	p-value
Weibull-Weibull	210.575	211.842	153.134-270.551	**0.964**
Weibull-Inverse Weibull	NA[1]	484.555	NA[1]	NA[1]
Weibull-Gamma	96.400	165.11	123.946-206.292	0.002
Weibull-Inverse Gamma	220.907	118.514	92.752-144.275	0.000
Weibull-Exponential	61.808	109.099	86.482-131.717	0.000
Weibull-Inverse Exponential	NA[1]	388.846	NA[1]	NA[1]
Weibull-Transformed Gamma	295.793	243.965	165.815-322.115	**0.172**
Weibull-Inverse Transformed Gamma	505.382	236.139	162.961-309.317	3.68e-06

[1] NA means either that the theoretical estimate of conditional tail expectation was missing or that the number of observations needed for testing was not sufficient.

6.5 Summary

In this chapter, two well-known risk measures in actuary, value-at-risk, (VaR), and conditional tail expectation, (CTE), are described and applications of composite models in these risk measures are considered. For each data set that are mentioned in section 5.2, VaR and CTE are computed and the best composite model which has the closest value to the corresponding empirical value is chosen. The results of the backtesting procedure to verify the validity of the computed VaRs and CTEs are also discussed.

CHAPTER 7

CONCLUSION

7.1 Introduction

One important task of actuaries is modeling the claims size distributions for predicting future claim costs in order to calculated premiums, probabilities of ruin, assess financial situations, and other risk evaluation purposes. Thus, finding a pattern for claims which is usually done by fitting a parametric probability distribution to these claims is an important aspect in actuarial science.

One way to find the parametric probability distribution is using a composite model. With recent development in parametric models, fitting insurance loss data distributions such as Pareto, Lognormal, Weibull, and Gamma distributions has become less attractive for modeling highly skewed loss data.

In this study, the most important objective is to find the best model which has the best fit to data sets. So, by considering three families of distributions, two best models are introduced. These two chosen composite models, the composite Weibull-inverse Paralogistic model and the composite Weibull-inverse transformed Gamma model, are constructed by a new approach which is easier to apply and is more appeal in the sense that it increases applicability in the insurance industry. That is, in the previous studies, the first two parameters that should be estimated in the composite model were the mixing weight and one of the head parameters. By applying the new approach, which is mentioned in section 3.3, these two parameters are changed to the mixing weight ϕ and the threshold θ. Regardless of which two distributions are selected for the head and the tail of a composite model, these two parameters can be estimated.

7.2 Summary of Contribution

In summary, the conceptual working ideas have been considered in Chapter 1 while some preliminary statistical concepts have been discussed in Chapter 2. The structure of composite models has been described in Chapter 3 and a new approach in estimating the parameters of the composite model, where the tail belongs to the heavy tailed distribution families, has been presented. Chapter 4 discussed several composite models that were proposed. In Chapter 5, the usages of two well-known data sets for the mentioned composite models are presented and the best fitted composite models are determined. Some risk measures are used in Chapter 6 to show the applications of the new composite models in the actuarial area.

Several composite models are examined with the Weibull distribution as the head and different distributions belonging to the transformed Beta, the transformed Gamma, and the inverse transformed Gamma families as a tail for composite models. These models were developed based on the new approach which leads to reduce the number of estimated parameters. For both insurance loss data sets, in the transformed Beta family, the composite Weibull-inverse Paralogistic model and in the transformed Gamma and the inverse transformed Gamma families, the composite Weibull-inverse transformed Gamma model show consistent results. That is, these two composite models give the lowest NLL, AIC, and SBC.

As a conclusion, the result of this study shows that the composite Weibull-inverse Paralogistic model and the composite Weibull-inverse transformed Gamma model are superior in modeling highly skewed data than do any other models considered in this study. According to the composite Weibull-inverse transformed Gamma model, having a higher number of parameters seems to increase the goodness-of-fit for the data, as expected .

7.3 Future Work

The following findings are highlighted:

- It is interesting to fit obtained composite models with other real insurance data sets to justify the results provided in this study. That is, to show if these composite models be able to capture the whole loss data sets and also, the best composite models which are introduced here still remain as the best composite models among the others.

- One of the most important motivations of finding the loss probability density function and the loss cumulative distribution function in this study is to calculate the premium. By having real data sets and sufficient information about the types of insurance contracts, actual values for different types of premiums can be calculated.

- Besides the practical aspect, another purpose for using composite models is to improve the statistical and actuarial models. They are much more useful in important areas such as finance, medical, health care and especially in the insurance industry, compared to the univariate distributions used in previous studies.

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