APPENDIX A

CURRICULUM CONTENT ON LIMITS OF FUNCTIONS

1. Rationale for the study of limits of functions discussed - the problem of finding the slope of a tangent to the curve.
2. Informal definition of limits: $\lim x \to c\ f(x) = L$ is interpreted as y-coordinate of a point (x, f(x)) on the graph of f approaches the single number L as x approaches c from each side of c.
3. Evaluating limits by using graphical and numerical investigations (table of values used to predict the limit of the function).
4. Emphasis by examples, that the method of substituting the point of interest into the function does not always give a correct value for the limit of a function. Hence the need for graphical or numerical investigations for predicting the value of the limit.
5. Criteria for the existence on the limit of a function is dependent on the function being defined in the neighbourhood of the point of interest. What happens at the point of interest is irrelevant.
6. Discussion of functions that approach the limit in various ways, to emphasise that function need not approach its limit monotonically.
7. Continuos functions: $\lim x \to a\ f(x) = f(a)$. This is further explained by using functions discussed in (6).
8. Non-existence of a limit: f(x) does not approach a single number as $x \to a$.

9. One-sided limits.
10. Non-existence of a limit: Left-hand limit is not equal to the right-hand limit.
11. Limits that involve infinity
12. Horizontal and vertical asymptotes.
13. Return to numerical investigations in predicting the limit. Limitations of this method discussed. The method can be misleading in the value of the limit projected. Hence special care must be taken on the nature of the values near the point of interest. For e.g. lim $x \to 0$ Sin π/x by numerical investigation proposes that the limit be zero with values of x such as 0.1, 0.01, 0.001, when in actual fact the limit does not exist. Another reason for being misled on the limit value is that the use of values to be substituted in x, may not be close enough to the point of interest. For example, lim $x \to 0$ (x^3 + (Cos 500)/10000) can be thought to be zero, using values of x such as 0.1, 0.05, 0.01. With a little bit more of perseverance, it can be shown that values of x smaller than those used previously leads to the correct limit, 0.0001.
14. Since numerical investigations and graphs are not always reliable in predicting the value of the limit, theorems on calculating limits are proposed. Amongst these is the Sandwich Theorem.
15. Rationale for the study of a more formal definition on limits of functions: Informal definition is imprecise because of words like "approaches". For example, $\lim_{x \to 0} x^2 = 0$. But x^2 also approaches -1, -2, etc, as $x \to 0$. This leads to a discussion on how close x should be to a point of interest, before L can be said to be the limit of a

function.

The example used is
$$f(x) = \begin{cases} 2x - 1, & x \neq 3 \\ 6, & x = 3 \end{cases}$$

16. The symbol "δ" is defined as the distance between x and the point of interest. A scenario of challenges (values of ε) and responses (values of δ) are presented. The symbol "ε" is not introduced as yet. Instead the distance between f (x) and L is fixed using successive values that get closer to zero, for example: 0.1, 0.01, 10^{-5}. A corresponding "δ" is found for each of these values. Finally the symbol "ε" is introduced as some positive number, no matter how small, and the corresponding δ, which is in terms of ε is found. Therefore the formal definition gives precision to the magnitude of closeness between x and the point of interest, and f (x) and the limit.

17. Presentation of the formal definition of limit of functions. Explanations on the meaning of expressions like " $0 < |x - a| < \delta$" is given. For continuos functions, the restriction of $|x - a| > 0$ need not be there because if a ∈ Domain of f, then f is defined at a; and if lim x → a f(x) = f(a), then $|f(a) - f(a)| = 0 < \varepsilon$, when x = a.

18. Implications of the formal definition:

- Order of ε-δ is important: ε determined first, then a corresponding δ is found.
- Definition must work for every ε >0, no matter how small.

- δ is not unique. For if a specific δ can be found, then any smaller positive number, δ_1, will also satisfy the requirements.

All of the above implications were discussed in class, except for the following:

- In the definition, $|f(x) - L| < \varepsilon$. There is no restriction that f(x) - L be greater than zero. Hence f(x) can equal L.

The instructor omits this point, as she would like to determine if students are aware of this subtle implication in her research study.

19. The formal definition is used to verify the limits in a few examples.

APPENDIX B

QUESTIONNAIRE ON BELIEFS ABOUT LIMITS

The purpose of this questionnaire is to gather information on the understanding of limits of functions among college students. It is **not an examination** and the results will be used strictly for the purpose of research.

Instructions: There are two parts to this questionnaire. The first part is part A, in which you are to write your responses in the space provided below the question. Part B will be given to you once you have completed part A.

Please read the questions **carefully** and answer **all** questions.

Your cooperation in this project is very much appreciated.

Name : ________________________________

(Your name will be kept in strict confidence in the report of the findings)

PART A

1. Suppose a friend of yours asks you to explain to him or her what a limit of a function is. How would you describe to your friend what **you** understand, a limit of a function to be?

2. If possible, please write down a formal definition (ε-δ definition) of limit of functions. You may write the definition using your own words.

PART B

NAME: ____________

(1) Please mark the following six statements about limits as being true or false:

(a) A limit describes how a function moves as x moves toward a certain point. T F

(b) A limit is a number or point which a function cannot go past. T F

(c) A limit is a number that the y-values of a function can be made arbitrarily close (as close as we please) to by restricting x-values. (by making x sufficiently close to the point of interest). T F

(d) A limit is a number or point the function gets close to but never reaches. (What is meant by "never reaches" here is that the function never takes on the value of its limit.) T F

(e) A limit is an approximation that can be made as accurate as you wish. T F

(f) A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached. ("Reached" here means that the function takes on the value of its limit.) T F

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(2) Which of the above statements **best** describes a limit as you understand it? (Circle one)

(a) (b) (c) (d) (e) (f) None

APPENDIX C

INTERVIEW: Attainability of the Limit

TASK 1

I would like you to read this statement. (Presents the subject with a card on which the statement is written).

Statement: A Limit is a point or number which the function gets close to but never reaches.

If "reaches" means that the function can take on the value of its limit;

(a) Do you agree with this statement?

(b) What is your rationale?

TASK 2

I would like you to read two students' viewpoints on limits. I am going to ask you some questions about what these students say. Please read this student's (student S_1) viewpoint on limits).

Student S_1

(1) A limit is a number or point that the function gets close to but never reaches (attains or takes on) the value of its limit.

(2) If you take the limit of f(x) as $x \rightarrow a$, x is never really equal to a, it just gets close. Same with f(x); f(x) never really equals the limit, but it gets close.

(3) In fact, you can make f(x) as close as you want to the limit, but if f(x) ever equals the limit, you don't really have a limit.

(4) For example, if $f(x) = 2x + 3$, and you take the limit as $x \rightarrow 1$, any number you substitute into x, that is close to 1 but not equal to 1 will give you a number close to 5 but not equal to 5.

(5) The function doesn't ever reach (take on the value) 5 unless x exactly equals 1, and that never happens in limits.

Can you explain to me what student S_1 is saying?

Now, I'd like you to read this student's (student S_2) viewpoint on limits.

Student S_2

It isn't really correct to say that a function never reaches (takes on) the value of its limit.

(1) What it means when we say " a function never reaches its limit" is that when you are talking about the limit of a function f as $x \rightarrow a$, you are only concerned with numbers *close to a* and you don't care whether x ever *equals* a.

(2) But certainly f could *equal* the limit. If f is a constant function, for instance, $f(x) = 7$, the limit as $x \rightarrow$ *any number* is 7, and f(x) "reaches" it because f is exactly equal to 7 for any number you substitute into x.

(3) I'm not saying that you can always get the limit by substituting the value $x = a$ into the function; the point is that reaching the limit or not reaching the limit is irrelevant.

(4) The function could take on the value of the limit as many times as it wants.

(5) What matters is what happens to f(x) in the neighbourhood of the number a.

(6) To say that f(x) never gets to the limit clouds the issue.

Can you explain to me what student S_2 is saying?

Further questions pertaining to the viewpoints about limits

1. Can you describe in your own words the difference between the two viewpoints?

Probes: a) What do they say about the issue on reaching the limit?

b) What is important concerning limits according to student S_2?

2. Which is most like the way you think about limits?

3. Why?

4. What is your rationale for not selecting the other viewpoint?