CHAPTER 5

SUMMARY AND CONCLUSIONS

5.1 Introduction

The purpose of the study was to explore the understanding of limit of functions among college students, with focus on their views concerning the attainability of the limit. Four research questions were formulated and data were collected to answer these questions.

Subjects of the study comprised 10 college students who had completed a course in Calculus and Analytic Geometry 1, of the School of American University Studies in the summer term of 1999. These students were of mixed abilities based on their performance in the course. The instruments used comprised a questionnaire on beliefs about limits and a task-based interview on the attainability of the limit. Both instruments were adapted from Williams (1991). The questionnaire aimed to obtain information on the students' concept images and concept definitions on the limit of a function. The task-based interview seeks an insight on the students' concept images concerning the attainability of the limit.

The findings of this study are summarised and discussed in the following four sections.

1. Concept images on limit of functions.
2. Concept definitions on limit of functions.

3. Concept images on the attainability of the limit
4. Concept images on the formal view concerning the issue on the attainability of the limit.

5.2 Concept images on limit of functions

Nine out of the 10 students described the limit of a function from an informal point of view. The principal concept images comprised mostly of (a) the dynamic view in which the point (x, f(x)) on a graph, is moving towards the limit, and (b) using limits to find the slope of a tangent at a point on a curve. There was only an average ability student who described the limit of a function from a formal point of view. However, his response indicated that he did not understand the implications of the relationship between the inequality statements $|f(x) - L| < \varepsilon$ and $0 < |x - c| < \delta$. That is, he did not recognise that the statement $0 < |x - c| < \delta$ had to be true in order for the statement $|f(x) - L| < \varepsilon$ to be true. On the whole the findings imply that the formal view on limits is not a dominant aspect of the concept image of these students.

Analysis of the responses to the validity of each statement concerning common beliefs on limits indicated that the students associated limits to a variety of ideas at the same time. These are that the limit

(a) is an approximation that could be made as accurate as one wishes,

(b) a boundary that is impassable,

(c) a number that the y-values of a function can be made arbitrarily close to by restricting x values (formal view),

(d) a point or number that is unreachable, and

(e) a process that involves the notion of getting closer (dynamic view).

Eight out of the 10 students who agreed to the formal view also agreed to the informal views on limits. The students' willingness to accept different descriptions of limits which were at variance with the formal view show they were not aware of the implications of the formal view. These students may assume the phrase "arbitrarily close" in the formal view to mean that a function can never attain its limiting value. This observation is in line with the claim made by Tall and Schwarzenberger (1978) that the word "close" in the phrase "arbitrarily close" is construed as being near but not coincident with the limit. Generally, the results are consistent with that of Williams (1991).

5.3 Concept definitions on limit of functions

The findings indicated that 8 of the 10 students attempted question two of part A which required a definition on the limit of a function. Two other students had forgotten the definition.

Of the 8 students, only 4 students had produced concept definitions. Two of the 4 who were of average ability produced incorrect concept definitions.

The remaining 4 of the 8 students did not produce concept definitions. Two students of above average ability produced concept images relating to the formal definition, which is at variance with the definition. On the other hand, the remaining 2 students who were respectively of average and below average abilities, produced concept images that did not relate to the formal definition. The finding supports the claim made by Vinner (as cited in Tall, 1991) that many people ignore the definition due to the enormous impact that everyday life has on any situation.

In addition to the above findings, a comparison of responses between questions one and two of part A of the questionnaire revealed that all students had no problems in describing the limit of a function, but most of them had difficulty producing a definition of the concept. On the whole, the findings support the general claim that students are rarely able to define limits formally in comparison to describing it(Tall and Vinner, 1981; Davis and Vinner, 1986).

5.4 Concept images on the attainability of the limit

Among the 4 students interviewed, 3 students agreed with the statement that "A limit is a point or number which the function gets close to but never reaches.". The potential conflicting factor in this concept image is that the variable x can never take on the value of the point of interest.

The formation of this concept image is not surprising. The concept of limit gives no importance to the attainability of the point of interest. However, the belief that the limit is unattainable contradicts the notion of continuity. Research has related the formation of the concept image on the limit as something unattainable to various factors. These constitute the view on infinity, influence of language and an understanding of the real line (Tall and Vinner, 1981; Davis and Vinner, 1986; Mamona, 1990; Williams, 1991; Monaghan, 1991).

On the other hand, the only student who believed the limit to be attainable had related attainability to whether or not the function was defined at the point of interest. Subsequently, the student had difficulty in interpreting a function that was discontinuous but defined at the point of interest. This finding that a student's understanding of functions obscured her understanding of limits, was noted by Ferrini and Mundy (1994).

In conclusion, it is noted that the concept images of 3 of the 4 students on the is that the limit is unattainable as the independent variable x is perceived to be infinitesimally close to the point of interest but not coincide with the point of interest. The formation of such a concept image has been attributed to several factors namely, the nature of the limit concept itself which disregards the point of interest, view on infinity, influence of language and understanding of the real line and the concept of continuity.

5.5 Concept Images on the formal view concerning the issue on the attainability of the limit

Only student J of average ability agreed completely with the formal view on limits. He believed that the issue of reaching the limit was irrelevant from the formal viewpoint. On the other hand, the formal viewpoint places importance on the behaviour of the function in the neighbourhood of the point of interest. In addition, his responses implied that he perceived the limit to be attainable for continuous functions.

Student F of above average ability did not agree with the formal view that the issue of attaining the limit was irrelevant. She initially perceived attainability to be the rationale for finding the limit. Hence the attainability of the limit was important, but this contradicted the formal viewpoint. Student F readily agreed that constant functions attained their limits at the point of interest and after some probing concluded that this applied to continuous functions as well.

While students Y and M also agreed with the formal view that "the point of interest is immaterial in the concept of limits", they believed that the limit could never be attained. This was because the variable x could only get arbitrarily close to the point of interest, but could never coincide with the point itself. Student Y believed that constant functions were an exception to the rule. On the other hand, student M disagreed on this as his concept image on the attainability of the limit pertained strictly to the point of interest. Hence, although constant functions equalled their limits, the attainability of the limit was specifically associated with a particular point, and since x never got to this point, constant functions could not be said to have attained their limits. Although student M was aware that continuous functions took on the value of its limit at the point of interest, he separated the process of approaching the limit from the situation in which the function took on its limiting value.

In conclusion, the potential conflicting factor in the concept images of the 2 students who believed the limit to be unattainable was that the independent variable x never attained the point of interest. It was also observed that constant functions were readily accepted as functions that attained their limits because the value of the function is the same as its limit, irrespective of the value of the independent variable x. There was also a tendency to view the process of approaching the limit separately from the situation in which the function took on the value of its limit.

5.6 Implications and suggestions

In general, the findings indicate that these students were not aware of the inconsistent and conflicting aspects of their concept image and its variance with the formal view. This further supports the finding that students rarely understood the formal definition. The lack of formal understanding may be due to the difficulty of correctly interpreting quantifiers like "for all" and "there exists", and also to the lack of appreciation for the need to understand the formal definition.

Such observations warrant teachers to take alternative pedagogical approaches in teaching the topic. For example, students can be given the opportunity to explore the meanings of certain words and phrases used in limits, namely, "approaches", "tends to", "getting closer" and "arbitrarily close". The discussion should centre on the differences of its interpretation in the non-mathematical and mathematical contexts. The imprecision of the phrase "arbitrarily close" can be further exemplified through problems faced by mathematicians in interpreting it. This would hopefully encourage students to take an interest in understanding the formal definition of limits as constructed by Karl Weierstrass .

The students generally understood that the limit did not concern the point of interest, as this was normally emphasised in instruction. Therefore, it was not surprising to find that they believed that the limit is unattainable. Another observation made was that the process of approaching the limit was perceived to be unrelated to the attainment of the limit. This suggests that the issue on the attainability of the limit should be addressed when discussing continuity. The attainability of the limit could be used to link the notion of limits to that of continuity so that students do not perceive the two concepts to be unrelated.

5.7 Recommendations for further research

A similar study can be extended to a larger sample and varied ability groups to validate the findings of this study. Perhaps, further analysis can be conducted on comparisons of concept images and concept definitions between different ability level groups. If there is a significant difference in the understanding of the limit concept among varying ability levels, then careful consideration would have to be given to the approach in teaching the limit concept.

The second phase of the study can be extended to include interviews on other informal views on limits such as the dynamic view and the view of the limit as a boundary. A study similar to that of Williams (1991) can be conducted to identify the extent to which students are willing to alter their concept images, once they are made aware of the inadequacies of their concept images. This could be done by presenting them with unusual limit problems, that are designed to cause a conflict with the existing concept image and hence to encourage students to adopt a more formal view. Findings could include factors affecting changes or no changes in the concept images of the students.

A study can also be considered on the effectiveness of existing computer software in promoting a formal understanding of the limit concept. The findings of the study can generate information on the limitations of the software and provide suggestions for improvements.

5.8 Conclusion

This study had investigated the understanding of college students on the concept of limit of functions. The following findings were obtained:

(i) Only one student described limits from the formal viewpoint. Most of the students' principal concept images related to the movement of the point (x, f(x)) on a graph towards the limit and the use of limits in determining the slope of a tangent to a curve.

(ii) The students exhibited a variety of concept images on limits, which are at variance with the formal view.

(iii) Only 2 out of 10 students produced correct concept definitions on limits. This implied that most of the students did not understand the definition.

(iv) Three out of 4 students perceived the limit to be unattainable as the variable x is perceived as never attaining the point of interest.

(v) Contrary to the formal viewpoint that does not address the issue on the attainability of the limit, 2 students believed the limit to be unattainable. The potential conflicting factor was that the variable x never attained the point of interest. However, one student believed that constant functions were an exception to the rule. On the other hand, another student of above average ability believed that since limits is associated to a point of interest, the constant function could not attain its limit. In addition he viewed the process of approaching the limit separate from the situation in which the function attained its limit.

In conclusion, this study has definitely provided some insight, at least to the researcher, on the varied concept images that students have on the limit of functions and the difficulties in producing concept definitions that are not at variance with the formal definition. In particular, the complexities involved in the understanding of limits as shown by the students warrant teachers and educators to reconsider careful planning in their instructions. Last, but not least, this study hopes to serve as stimulus for more research in mathematics education at college level.