GEODESIC DEVIATION EQUATION IN SYMMETRIC TELEPARALLEL GRAVITY

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GEODESIC DEVIATION EQUATION IN SYMMETRIC TELEPARALLEL GRAVITY

ABSTRACT

This dissertation surveys mathematical concepts of pseudo-Riemannian manifolds, reviews several gravity theories and describes the geodesic deviation equation in $f(Q)$ gravity. Symmetric teleparallel gravity was introduced, and its equivalence to general relativity was shown. Moreover, three formalisms of $f(R)$ gravity were explored, which is then followed by the $f(Q)$ gravity theory. By using the notions of one-parameter family of curves, the geodesic deviation equation was formulated. The standard cosmological model called the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology was briefly discussed. In the background of FLRW model, the geodesic deviation equation in $f(Q)$ gravity was presented. Lastly, the two particular cases which are the geodesic deviation for fundamental observers and for past-directed null vector fields were investigated.

Keywords: Geodesic deviation equation, symmetric teleparallel gravity, modified gravity theories, $f(Q)$ gravity, FLRW universe.

PERSAMAAN SISIHAN GEODESIK DALAM GRAVITI TELEPARALLEL SIMETRI

ABSTRAK

Disertasi ini meninjau konsep matematik manifold pseudo-Riemannian, mengkaji beberapa teori graviti dan menerangkan persamaan sisihan geodesik dalam $f(Q)$ graviti. Graviti teleparallel simetri telah diperkenalkan, dan kesetaraannya dengan relativiti am ditunjukkan. Selain itu, tiga formalisme graviti $f(R)$ telah diterokai, yang kemudiannya diikuti oleh teori graviti $f(Q)$. Dengan menggunakan tanggapan keluarga satu parameter lengkung, persamaan sisihan geodesik telah dirumuskan. Model kosmologi piawai yang dipanggil kosmologi Friedmann-Lemaître-Robertson-Walker (FLRW) telah dibincangkan secara ringkas. Dalam latar belakang model FLRW, persamaan sisihan geodesik dalam $f(Q)$ graviti telah dibentangkan. Akhir sekali, dua kes tertentu yang merupakan sisihan geodesik untuk pemerhati asas dan untuk medan vektor nol terarah lalu telah disiasat.

Kata kunci: Persamaan sisihan geodesik, graviti teleselari simetri, teori graviti diubah suai, graviti $f(Q)$, alam semesta FLRW.

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CHAPTER 1: INTRODUCTION

1.1 Background of the Study

One of the most fascinating theories in Physics is the general relativity (GR) proposed by Einsten (1928). It provided a remarkable narrative of the cosmological observational data and created new insights into the concepts of space and time. The mathematical framework of this geometrical theory of gravity is based on pseudo-Riemannian geometry. In short, this theory models spacetime as a 4-dimensional Lorentzian manifold and describes the properties of the gravitational field by using the curvature tensor of the spacetime. Generally, the two fundamental equations in GR are the Einstein's field equations and the geodesic deviation equation. The former equation represents the distribution of matter and energy influences the metric and lead to curvature of spacetime; the latter provides a relationship between the curvature tensor and the relative acceleration of two nearby test particles that depicts the relative motion of free falling particles to bend towards or away from each other, under the influence of gravitational field. However, despite its undeniable success, increasing technological ability in modern observational cosmology posed new questions to GR. It turns out that the validity of GR might only be up to the astrophysical scales not exceeding the Solar system (Brax, 2018; Nojiri et al., 2017).

To resolve the imperfection of GR, one of the approaches is to modify the matter section of the field equations by adding some additional 'dark' components to the energy budget of the universe, and the other one is to modify the gravitational sector. The most common modifications in the latter direction are achieved by generalizing the Einstein-Hilbert action term, precisely by replacing the Ricci scalar R with an arbitrary function of R which produces the so called $f(R)$ gravity theory, first proposed by Buchdahl (1970). Besides that, one of the significances of GR is that the formulation is based on a very special and unique connection, the torsionless and metric-compatible Levi-Civita connection. The Levi-Civita connection ensures that the connection coefficients can be written as a function of the metric tensor only. In fact, there are other gravity theories equivalent to GR, such as the teleparallel gravity and symmetric teleparallel gravity unhindered to this special connection.

The idea of teleparallelism was first employed by Einstein on his failed attempt to unite electromagnetism and gravitation (Einsten, 1928). Subsequently, Moller (1961) with the use of tetrad description of gravitational fields and reformulate teleparallelism as a gravity theory. After that, the very first paper of the symmetric teleparallel gravity was published by Nester and Yo (1999). Since then, several contributions to teleparallel gravity (TG) and symmetric teleparallel gravity (STG) have been made by different authors (Adak, 2006; Aldrovandi & Pereira, 2013; Järv et al., 2018; Jimenez et al., 2018; Nester & Yo, 1999). Unlike GR, where the gravity is depicted by the curvature of spacetime, in both these theories the curvature is set to be zero, and the torsion of the connection represents the gravity in TG and the non-metricity of the connection does so in STG. Inspired by the $f(R)$ theory, the modified $f(T)$ and $f(Q)$ gravity theories were considered where the Ricci scalar R in GR is replaced by the torsion scalar T in TG and the non-metricity scalar Q in STG (Wu & Yu, 2010; Xu et al., 2020, 2019; Zhang et al., 2011). However, both these theories have some drawback in which the consistency of the theory depend on the choice of tetrad in TG and the choice of coordinates in STG (Tamanini & Boehmer, 2012; Zhao, 2021).

The geodesic deviation equation (GDE) is a useful tool for studying timelike, null, and spacelike structure of spacetime. Ellis and Elst (1997) apply the GDE to investigate these structures in the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology. As a natural extension to GR, the GDE was then formulated in $f(R)$ gravity (Guarnizo et al., 2011, 2015), followed by Baffou et al. (2015). Although equivalent to GR, TG is a conceptually different theory. In GR, the motion of particle is described by the curvature of spacetime and the trajectories are subject to geodesics, instead of force equation. On the flip side, in TG, the torsion appeared as a real force, namely, the tidal force. Consequently, there is no geodesic equation in TG, but only force equation. Nevertheless, the teleparallel depiction of the gravitational interaction is totally equivalent to that of GR (Aldrovandi & Pereira, 2013), so it is completely natural to convert the force equation in TG to a geodesic equation in GR. In this way, the corresponding GDE in TG can be obtained. This approach is done by Darabi et al. (2015).

1.2 Objective of the Study

1. To investigate the properties of pseudo-Riemannian manifolds in order to appreciate the geometric description of gravity.
2. To establish the equivalence of general relativity and symmetric teleparallel gravity.
3. To express the geodesic deviation equation for FLRW cosmology in $f(Q)$ gravity theory.

1.3 Scope and Methodology

The scope of the study includes the mathematical prerequisites to begin investigating general relativity and foundations of cosmology, which is then followed by the investigation of symmetric teleparallel gravity and the modified $f(Q)$ gravity. The technical methodology for this research project includes the review of mathematical prerequisites such as smooth manifolds, connections and curvature tensors, which is then followed by a discussion on general relativity. This is then proceeded by deriving the field equations of general relativity, $f(R)$ gravity, symmetric teleparallel gravity and $f(Q)$ gravity. After that, the notion of geodesic deviation equation is introduced, followed by the review of FLRW cosmology included the concepts of observer, Robertson-Walker spacetime, Hubble parameter, and redshift parameter. By using the results obtained, we express the GDE in $f(Q)$ gravity with FLRW model and discuss some of its cosmological applications such as the generalized Raychaudhuri equation and Mattig relation.

1.4 Outline of the Report

This thesis begins with a chapter of mathematical preliminaries. The notion of smooth manifolds is introduced, which is then followed by tangent spaces, vector fields, one-forms, and tensor fields. These are essentially the least differential topology knowledge that needed to pursue a study on differential geometry. After that, we introduce pseudo-Riemannian manifolds with the concept of metric tensors. The notions of affine connections and geodesics were reviewed, and used to describe the curvature tensors. Chapter 2 ends with brief introduction on the Levi-Civita connection. Chapter 3 is to reviews several gravity theories, beginning with the basic implications of general relativity. The uniqueness of this chapter lies in the construction of the symmetric teleparallel gravity and the derivation of the $f(Q)$ gravity field equations. The chapter ends with a discussion on the equivalence between the general relativity and the symmetric teleparallel gravity. Lastly, this dissertation ends with the presentation on the geodesic deviation equation in $f(Q)$ gravity. A short section on FLRW cosmology is provided. The chapter ends with the two important results which is the GDE for fundamental observers and for past-directed null vector fields.

CHAPTER 2: MATHEMATICAL PRELIMINARIES

2.1 Introduction

This chapter is a survey of the basic definitions of differential geometry that will be used throughout this manuscript. The main goal of this chapter is not to exhibit the rigorous formulations, but to fix notations, in local coordinates, as they will be needed for the rest of the chapters and physical applications. The main references used for this chapter include J. Lee (2009); J. M. Lee (1997, 2013); O'neil (1983); W.Tu (2010).

The first four sections will focus on the smooth manifold theory. Smooth manifolds are essentially specific topological spaces with an additional smooth structure and coordinate charts are commonly known as coordinate systems by physicists. We present the tangent space to a manifold at a point, which is a linear approximation for the manifold near the point, and define a tangent vector on a manifold as a derivation at a point so that we can perform calculus on manifolds. A vector field on manifold as a linear map that assigns to each point a tangent vector, whereas one-forms assign to each point a covector which is linear functional on the tangent space at a point. We define the notions of tensors and tensor fields on manifolds by generalizing from linear mappings to multilinear ones. The language of tensors have important implication on physics and will pervade the rest of the thesis.

For the later sections, geometry is introduced into smooth manifold theory. We need an additional structure known as the metric tensor to define geometric concepts like lengths and distances. A smooth manifold with a well defined metric tensor is called a pseudo-Riemannian manifold, whereby the Riemannian and Lorentzian manifolds are the special cases. To study the notion of geodesics and curvature on pseudo-Riemannian manifolds, we need to introduce a new object called a connection. An affine connection allow us to connect tangent spaces at different point of a manifold and define parallel transport of vectors along curves. Geodesics are essentially curves that have parametrizations with zero acceleration. Curvature is defined as a tensor that measure the failure of second covariant derivatives to commute. In the last section, we present two properties: symmetry and compatibility with the metric, that determine a unique connection on every pseudo-Riemannian manifold, called the Levi-Civita connection.

2.2 Smooth Manifolds

Definition 2.1. Let M be a topological space. Then M is a n-dimensional topological manifold if M is a second-countable Hausdorff space such that for every $p \in M$, there is a homeomorphism $\phi : M \supseteq U \to \phi(U) \subseteq \mathbb{R}^n$ which maps the open subset $U \ni p$ of M onto an open subset $\phi(U) \subseteq \mathbb{R}^n$. A *coordinate chart* is a pair (U, ϕ), where U is called a coordinate neighborhood and ϕ is called a coordinate map. Let $p \in U$ and $\phi(p) = (x^1(p), \ldots, x^n(p))$. Then the functions $(x^1, \ldots, x^n)$ are called the *local coordinates* on U.

Definition 2.2. Let M be a topological manifold, and (U_α, ϕ_α), (U_β, ϕ_β) be the charts on M with $U_\alpha \cap U_\beta \neq \emptyset$. If the homeomorphism $\phi_\alpha \circ \phi_\beta^{-1} : \mathbb{R}^n \supseteq \phi_\beta(U_i \cap U_j) \to \phi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$, $(x^1, ..., x^n) \mapsto (y^1, ..., y^n)$, called the *transition map* or *change of coordinates* is smooth, then the charts are said to be smoothly compatible. A *smooth atlas* on M is a collection $\mathcal{A} = \{(U_\Lambda, \phi_\Lambda)\}$ on M such that $M = \bigcup_{(U_\Lambda, \phi_\Lambda) \in \mathcal{A}} U$, and any two charts are smoothly compatible to one another. A smooth atlas on M is maximal if it is not contained in any other larger smooth atlas.

Definition 2.3. A n-dimensional smooth manifold is a pair $(M, \mathcal{A})$, where M is a n-dimensional topological manifold and $\mathcal{A}$ is a maximal smooth atlas on M.

For convention, we usually denote M as a smooth manifold or M^n whenever we want to emphasize the dimension. It is also common to denote $(U, (x^\mu))$ as a coordinate chart or coordinate system, where (x^μ) is the local coordinate on U.

Example 2.1. Let M^m and N^n be smooth manifolds and $M \times N$ be the Cartesian product. Suppose (U, ϕ) is a chart on M and (V, ψ) is a chart on N. Then the map

$$\phi \times \psi : U \times V \to \mathbb{R}^m \times \mathbb{R}^n \tag{2.1}$$

is an atlas on $M \times N$ called the product atlas. With a maximal product atlas, $M \times N$ is called a product manifold.

Definition 2.4. Let $f : M \to N$, (U, ϕ) be a coordinate chart on M^m, with $p \in U$ and (V, ψ) be a coordinate chart on N^n, with $f(p) \in V$. Then, f is *smooth at* p if, $f(U) \subseteq V$ and the composition $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \supseteq \phi(U) \to \psi(V) \subseteq \mathbb{R}^n$ is smooth at $\phi(p)$. The map f is called a *smooth map* if it is smooth at p, $\forall p \in M$.

The smoothness of f is independent to the choice of coordinate charts. A representative map $\hat{f} = \psi \circ f \circ \phi^{-1}$ is defined on open subset of $\mathbb{R}^m$, where m is the dimension of M. Since the dimension of N is n, so $\hat{f} = (\hat{f}^1, \ldots, \hat{f}^n)$ where each $\hat{f}^\mu$ is a m-valued function. If we denote the points in $\mathbb{R}^m$ as $(x^1, \ldots, x^m)$, and $(y^1, \ldots, y^n)$ in $\mathbb{R}^n$, then we may write $y^\mu = \hat{f}^\mu(x^1, \ldots, x^m)$. In common, the hat over the f's are dropped. In particular, we also said a function $f : M \to \mathbb{R}$ is smooth at $p \in M$, if $f \circ \phi^{-1} : \mathbb{R}^n \supseteq \phi(U) \to \mathbb{R}$ is smooth at $\phi(p)$ for some coordinate chart (U, ϕ) with $p \in U$. Consequently, f is a smooth function if it is smooth at every $p \in M$. The set of all smooth maps from M to N is denoted by $C^\infty(M, N)$, while $C^\infty(M)$ for the special case $C^\infty(M, \mathbb{R})$.

Definition 2.5. Let $(U, (x^\mu))$ be a coordinate chart on M^n with $p \in U$. If $f \in C^\infty(M)$, then partial derivative of f on U is defined by

$$\frac{\partial f}{\partial x^\mu}(p) = \frac{\partial (f \circ \phi^{-1})}{\partial u^\mu}(\phi(p)) \tag{2.2}$$

where $(u^1, \ldots, u^n)$ is the standard coordinates on $\mathbb{R}^n$.

2.3 Tangent Spaces

Definition 2.6. A *tangent vector* at $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule

$$v(f \cdot g) = f(p)v(g) + g(p)v(f), \qquad \forall f, g \in C^\infty(M). \tag{2.3}$$

This implies that a tangent vector is a derivation at $p \in M$. At each $p \in M$, we let T_pM denote the set of all tangent vectors to M at p. The set T_pM form a vector space with the operations defined as $(v + w)f = vf + wf$ and $(av)f = a(vf)$, called the tangent space to M at p. Suppose $(U, (x^\mu))$ is a coordinate chart on M^n with $p \in M$, we define the operator $\partial_\mu|_p \equiv \frac{\partial}{\partial x^\mu}\big|_p : C^\infty(M) \to \mathbb{R}$ by

$$\left.\frac{\partial}{\partial x^\mu}\right|_p f = \frac{\partial f}{\partial x^\mu}(p)\,. \tag{2.4}$$

Clearly, $\partial_\mu|_p$ is a derivation at p and so an element of T_pM. It can be shown that the n-tuple $(\partial_1|_p, \ldots, \partial_n|_p)$ form a basis for the tangent space T_pM, called a *coordinate basis*. Therefore, for any $v \in T_pM$, we can write $v = v^\mu \partial_\mu|_p$, where $v^\mu = v(x^\mu)$.

Definition 2.7. Let $f \in C^\infty(M, N)$ be a smooth map. For every $p \in M$, a map

$$df_p : T_pM \to T_{f(p)}N \tag{2.5}$$

which maps v to $df_p(v) \equiv v_f$, is called the *differential of f at p*, where v_f is defined by $v_f(g) = v(g \circ f)$, $\forall v \in T_pM$ and $g \in C^\infty(N)$.

Notice v is linear implies that v_f is linear, and v_f also satisfies the Leibniz rule. Thus, $v_f \in T_{f(p)}N$ and the definition is well-defined. Let $(U, (x^\mu))$ be a chart on M^m at p and $(V, (y^\mu))$ be a chart on N^n at $f(p)$. Then we can show that

$$df_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) = \frac{\partial(y^\mu \circ \phi)}{\partial x^\nu}(p)\left.\frac{\partial}{\partial y^\mu}\right|_{f(p)}. \tag{2.6}$$

The matrix $\left(\frac{\partial(y^\mu \circ \phi)}{\partial x^\nu}(p)\right)_{\mu,\nu}$ is the Jacobian matrix of f with respect to (x^μ) and (y^μ).

Definition 2.8. The *tangent bundle* of M is the disjoint union of tangent spaces at all points in M, denoted as

$$TM = \coprod_{p\in M} T_pM = \{(p, v) \mid p \in M, v \in T_pM\}\,. \tag{2.7}$$

Definition 2.9. A parametrized smooth curve is a smooth map $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^n$, where I is an open interval in $\mathbb{R}$.

Thus, for every $t \in I$, we could write $\gamma(t) = (x^1(t), \ldots, x^n(t)) \in \mathbb{R}^n$, so γ is smooth if and only if each $x^i(t)$ is smooth. Given a smooth curve γ, we can define the tangent vector or the velocity of γ at $t_0 \in I$ as $\dot\gamma(t_0) = (\dot x^1(t), \ldots, \dot x^n(t))$, where dot represents the derivative with respect to t.

Definition 2.10. Let $\gamma : I \to M$ be a smooth curve. The *velocity* of γ at $t_0 \in I$ is

$$\gamma'(t_0) = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) \in T_{\gamma(t_0)}M \tag{2.8}$$

where $d/dt|_{t_0}$ is the coordinate basis vector in $T_{t_0}\mathbb{R}$.

Since $\dot{\gamma}(t_0)$ is the tangent vector at t_0, it can be acts on smooth function f by

$$\dot{\gamma}(t_0)f = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right)f = \left.\frac{d}{dt}\right|_{t_0}(f \circ \gamma) = (f \circ \gamma)'(t_0)\,. \tag{2.9}$$

This implies that $\dot{\gamma}(t_0)$ is simply the derivation at $\gamma(t_0)$ obtained by taking the derivative of a function along γ. Suppose (U, ϕ) is a coordinate chart on M, such that $\gamma(I) \subseteq U$, then for any $t \in I$, we have

$$\phi \circ \gamma(t) = (x^1 \circ \gamma(t), \ldots, x^n \circ \gamma(t)) = (\gamma^1(t), \ldots, \gamma^n(t))\,. \tag{2.10}$$

To simplify the notation, we often express the coordinate representation of γ as $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$, or $\gamma(t) = (x^1(t), \ldots, x^n(t))$ by using the identification between U and $\mathbb{R}^n$. Thus, we have $\dot{\gamma}(t_0) = \dot{\gamma}^i(t_0)\partial_\mu|_{\gamma(t_0)}$.

2.4 Vector Fields and One-Forms

Definition 2.11. A *vector field* X on M is a smooth map $X : M \to TM$, that assigns each point $p \in M$ to a tangent vector $X_p \in T_pM$. The smoothness of X implies that, $\forall f \in C^\infty(M)$, the function $Xf : M \to \mathbb{R}$ defined by $(Xf)(p) = X_p(f)$ is smooth $\forall p \in M$.

We denote $\mathfrak{X}(M)$ as the set of all vector fields on M. Let $X, Y \in \mathfrak{X}(M)$. Then the addition and multiplication of smooth vector fields are defined as $(X + Y)_p = X_p + Y_p$ and $(fX)_p = f(p)X_p$. This implies that $\mathfrak{X}(M)$ is a module over the ring $C^\infty(M)$. Let $(U, (x^\mu))$ be a coordinate chart on M. Then the vector fields defined on U,

$$\frac{\partial}{\partial x^\mu} : p \mapsto \left.\frac{\partial}{\partial x^\mu}\right|_p \tag{2.11}$$

is called the *coordinate vector field*. Thus, for every $X \in \mathfrak{X}(M)$ on U, we can write

$X = X^\mu \partial_\mu = X(x^\mu)\partial_\mu$. In addition, given a vector field X on M, we can view these vector fields like a linear map $X : C^\infty(M) \to C^\infty(M)$ which satisfy the Leibniz rule, $X(fg) = fXg + gXf$, $\forall f, g \in C^\infty(M)$. In other words, X is a derivation on $C^\infty(M)$. In fact, every derivation on $C^\infty(M)$ is a vector field.

Definition 2.12. Let $f \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$. The *Lie bracket* of X and Y is a map $[X, Y] : C^\infty(M) \to C^\infty(M)$ defined by $[X, Y]f = XYf - YXf$.

It is clear that $[X, Y]$ is a vector field on M. Also, we can write $[X, Y]$ at $p \in M$ as $[X, Y]_p f = X_p(Yf) - Y_p(Xf)$. Let $(U, (x^\mu))$ be a chart on U, and $X, Y \in \mathfrak{X}(M)$. Then $X = X^\mu \partial_\mu$ and $Y = Y^\nu \partial_\nu$ and hence

$$[X, Y] = \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial x^\nu} . \tag{2.12}$$

It follows that $[\partial_\mu, \partial_\nu] = 0$.

Definition 2.13. Let $f \in C^\infty(M, N)$, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then vector fields X and Y are said to be *f-related*, if for all $p \in M$, $df(X_p) = Y_{f(p)}$.

Definition 2.14. The *cotangent space* of M at p is the dual space of T_pM, denoted as T_p^*M. The elements of the cotangent space $v^* \in T_p^*M$ are called *covectors*, which are linear maps $v^* : T_pM \to \mathbb{R}$. The *cotangent bundle* of M is the disjoint union of cotangent spaces of M, denoted by

$$TM^* = \coprod_{p \in M} T_p^*M = \{(p, v^*) \mid p \in M, v* \in T_p^*M\} . \tag{2.13}$$

Definition 2.15. A *one-form* θ on M is a smooth map $\theta : M \to TM^*$ assigning each $p \in M$ to a covector $\theta_p \in T_p^*M$. The smoothness of θ implies that, $\forall X \in \mathfrak{X}(M)$, the function $\theta X : M \to \mathbb{R}$ defined by $\theta X(p) = \theta_p(X_p)$ is smooth $\forall p \in M$.

In analogous to vector fields, $\mathfrak{X}^*(M)$ denotes the set of all one-forms on M, which form a module over the ring $C^\infty(M)$.

Definition 2.16. The *differential* of $f \in C^\infty(M)$ is the one-form $df : M \to TM^*$, $p \mapsto df_p$ such that for any tangent vector $v \in T_pM$, $df(v) = v(f)$ implying $(df)_p : T_pM \to \mathbb{R}$.

Since $v_p \in T_pM$ is linear, then $(df)_p : v \mapsto v_p(f)$ is linear. Let $X \in \mathfrak{X}(M)$. Then $df(X) = Xf$ which is known to be smooth. Assume $(U, (x^\mu))$ is a chart on M. Then $(dx^1, \ldots, dx^n)$ are called the coordinate one-forms on U. Note that (dx^μ) are the dual bases to the coordinate vector fields (∂_ν), since for all $p \in U$, we have $dx^\mu(\partial_\nu) = \partial_\nu(dx^\mu) = \delta^\mu_\nu$. Thus, for every $\theta \in \mathfrak{X}^*(M)$ on U, we can write $\theta = \theta_\mu dx^\mu = \theta(\partial_\mu)dx^\mu$. Similar to the case that vector fields were derivations on $C^\infty(M)$, the differentials can also be viewed as a linear map $d : C^\infty(M) \to \mathfrak{X}^*(M)$ that satisfy the Leibniz rule. Thus, d is a derivation on $C^\infty(M)$.

2.5 Tensor Fields

Definition 2.17. Let

$$\mathfrak{X}^*(M)^r = \underbrace{\mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M)}_{r \text{ copies}} \quad \text{and} \quad \mathfrak{X}(M)^s = \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s \text{ copies}}.$$

A *tensor field of type* (r, s) is a $C^\infty(M)$-multilinear function

$$\begin{aligned} \tau &: \mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^s \to C^\infty(M) \\ (\theta^1, \ldots, \theta^r, X_1, \ldots, X_s) &\mapsto f = \tau(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s) \end{aligned} \tag{2.14}$$

where the index r is called the *contravariant type* and s is called the *covariant type*.

Let $\mathcal{T}_s^r(M)$ denotes the set of all (r, s) tensor fields on M which is a module over $C^\infty(M)$. In particular, $\mathcal{T}_0^0(M) = C^\infty(M)$.

Definition 2.18. Let $\tau \in \mathcal{T}_s^r(M)$ and $\tau' \in \mathcal{T}_{s'}^{r'}$. The *tensor product* between tensor fields $\tau \otimes \tau' : \mathfrak{X}^*(M)^{r+r'} \times \mathfrak{X}(M)^{s+s'} \to C^\infty(M)$ is defined as

$$\begin{aligned}\tau \otimes \tau'(&\theta^1, \dots, \theta^{r+r'}, X_1, \dots, X_{s+s'})\\ &= \tau(\theta^1, \dots, \theta^r, X_1, \dots, X_1, \dots, X_s)\tau'(\theta^{r+1}, \dots, \theta^{r+r'}, X_{s+1}, \dots, X_{s+s'}) \end{aligned} \tag{2.15}$$

and $\tau \otimes \tau' \in \mathcal{T}_{s+s'}^{r+r'}(M)$.

It is clearly multilinear and associative. In particular, let $\theta \in \mathfrak{X}^*(M)$. Then we can define a tensor field $\tau_\theta \in \mathcal{T}_1^0(M)$ such that $\tau_\theta : X \mapsto \theta(X)$. In fact, all $(0, 1)$ tensor fields are defined in such a way, which means $\mathcal{T}_1^0(M) = \mathfrak{X}^*(M)$ and so all one-forms are $(0, 1)$ tensor fields. In a similar way, every vector field $X \in \mathfrak{X}(M)$ leads to a tensor field $\tau_X \in \mathcal{T}_0^1(M)$ such that $\tau_X : \theta \mapsto \theta(X)$, and $\mathcal{T}_0^1(M) = \mathfrak{X}(M)$ implying that all vector fields are $(1, 0)$ tensor fields.

Definition 2.19. Tensor fields of type $(0, s)$ are called *covariant fields*, and type $(r, 0)$ are called *contravariant fields*. Consequently, all one-forms are covariant fields, and all vector fields are contravariant fields.

Let $v \in T_pM$ and $v^* \in T_p^*M$. Suppose $\tau \in \mathcal{T}_s^r(M)$, at each point $p \in M$ define $\tau_p : (T_p^*M)^r \times (T_pM)^s \to \mathbb{R}$ by $\tau_p(v^{1*}, \dots, v^{r*}, v_1, \dots, v_s) = \tau(\theta^1, \dots, \theta^r, X_1, \dots, X_s)(p)$ such that $\theta^\mu|_p = v^{\mu*}$ and $X_\nu|_p = v_\nu$, for all $\mu \in \{1, \dots, r\}$ and $\nu \in \{1, \dots, s\}$. It follows that τ_p is $\mathbb{R}$-multilinear, so it is an (r, s) tensor over T_pM. Hence, we have the following definition.

Definition 2.20. A (r, s) *tensor bundle* of M is a disjoint union of (r, s) tensor spaces over T_pM, for all $p \in M$, denoted by

$$\begin{aligned} \mathcal{TM}_s^r &= \coprod_{p \in M} T_s^r(T_pM)) \\ &= \{\tau_p \in L((T_p^*M)^r \times (T_pM)^s; \mathbb{R}) \mid p \in M\}. \end{aligned} \tag{2.16}$$

Thus, every tensor field $\tau \in \mathcal{T}_s^r(M)$ is also a map $\tau : M \to \mathcal{TM}_s^r$ that assigning to each $p \in M$ the tensor τ_p. Recall that if $(U, (x^\mu))$ is a chart on M, then we can write $X \in \mathfrak{X}(M)$ and $\theta \in \mathfrak{X}^*(M)$ as $X = X(x^\mu)\partial_\mu$ and $\theta = \theta(\partial_\mu)dx^\mu$.

Definition 2.21. Let $(U, (x^\mu))$ be a coordinate chart on M. If $\tau \in \mathcal{T}_s^r(M)$, then the *components of* τ *relative to* (x^μ) are

$$\tau_{\nu_1,\ldots,\nu_s}^{\mu_1,\ldots,\mu_r} = \tau(dx^{\mu_1}, \ldots, dx^{\mu_r}, \partial_{\nu_1}, \ldots, \partial_{\nu_s}) \tag{2.17}$$

which is a real-valued functions on U and all the indices run from 1 to n.

Analogous to a vector field or one-form, any tensor has a unique expression on U in terms of its components relative to (x^μ). For $\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s \in \{1, \ldots, n\}$,

$$\partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_s} \tag{2.18}$$

is a (r, s) tensor field on U called the *coordinate tensor field*. Thus, if $\tau \in \mathcal{T}_s^r(M)$, then on U,

$$\tau = \tau_{\nu_1,\ldots,\nu_s}^{\mu_1,\ldots,\mu_r} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_s} \tag{2.19}$$

where each index is summed from 1 to n.

A contraction is a tool to reduce (r, s) tensor to $(r-1, s-1)$ tensor. In tensor fields, it is a unique linear map $C^\mu_\nu : \mathcal{T}^r_s(M) \to \mathcal{T}^{r-1}_{s-1}(M)$ defined by

$$C^\mu_\nu(\theta^1 \otimes \cdots \otimes \theta^\mu \otimes \cdots \otimes \theta^r \otimes X_1 \otimes \cdots \otimes X_\nu \otimes \cdots \otimes X_s) = \theta^\mu X_\nu(\theta^1 \otimes \cdots \otimes \theta^r \otimes X_1 \otimes \cdots \otimes X_s) \tag{2.20}$$

called the (μ, ν) *contraction*. Consequently, if $\tau \in \mathcal{T}^r_s(M)$, which has components $\tau^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}$ with respect to some coordinate chart, then the contracted tensor field $C^\mu_\nu(\tau)$ has components $\tau^{\mu_1,\ldots,\lambda,\ldots,\mu_r}_{\nu_1,\ldots,\lambda,\ldots,\nu_s}$ where λ is inserted at the μ^{th} contravariant index and ν^{th} covariant index.

Definition 2.22. Let $f \in C^\infty(M, N)$ and $\tau \in \mathcal{T}^0_s(N)$. The *pullback* of τ by f denoted by $f^*(\tau)$ is defined as

$$(f^*\tau)(v_1, \ldots, v_s) = \tau(df(v_1), \ldots, df(v_s)) \tag{2.21}$$

$\forall v_\mu \in T_pM$, $p \in M$.

2.6 Metric Tensors

Definition 2.23. Let V be a finite-dimensional vector space. A symmetric bilinear form g on V is a $\mathbb{R}$-bilinear function $g : V \times V \to \mathbb{R}$ such that $g(v, w) = g(w, v)$, for all $v, w \in V$. If $g(v, w) = 0$, for all $w \in V$ implies that $v = 0$, then g is said to be *nondegenerate*. A nondegenerate symmetric bilinear form is called a *scalar product* on V. A *scalar product space* is pair (V, g) where g is a scalar product. If $g(v, v) \geq 0$ ($g(v, v) \leq 0$) for all $v \in V$, and $g(v, v) = 0$ implies $v = 0$, then g is said to be *positive* (negative) *definite*. A positive definite symmetric bilinear form is called an *inner product* and the pair (V, g) is an inner product space.

Definition 2.24. The *index* of a symmetric bilinear form g, denoted as $\operatorname{ind}(g)$, is the dimension of the largest subspace $U \leq V$ such that $g|_U$ is negative definite.

Definition 2.25. Let (V, g) be a scalar product space, U, W be subspaces of V, and $u, v \in V$. Then, u is mutually orthogonal to v, if $g(u, v) = 0$, and U is orthogonal to W, denoted $U \perp W$, if $g(u, w) = 0$, for all $u \in U, w \in W$.

Definition 2.26. Let (V, g) be a scalar product space and $v \in V$. The *norm* or *length* of v is defined as $\|v\| = \sqrt{|g(v, v)|}$. If $\|v\| = 1$, that is, $g(v, v) = \pm 1$, then v is called a unit vector.

Definition 2.27. Let (V, g) be a scalar product space with $\dim(V) = n$. Then, $(e_1, \ldots, e_n)$ is called an orthonormal basis, if $g(e_\mu, e_\nu) = \epsilon_\mu \delta_{\mu\nu}$, for all $\mu, \nu \in \{1, \ldots, n\}$, where $\epsilon_\mu = g(e_\mu, e_\mu) = \pm 1$. The list $(\epsilon_1, \ldots, \epsilon_n)$ is called the *signature* of g.

Definition 2.28. Let (V_1, g_1) and (V_2, g_2) be scalar product spaces. Then, an isomorphism $\phi : V_1 \longrightarrow V_2$ is called a linear isometry if $g_1(v, w) = g_2(\phi v, \phi w), \forall v, w \in V_1$.

An isometry preserves the scalar product and so g_1 and g_2 have the same signature. Furthermore, we often denote $g(v, w)$ as $\langle v, w \rangle$ without confusion. Next, we extend the metric tensors to smooth manifolds.

Definition 2.29. Let M be a smooth manifold. A *Riemannian metric tensor* g is a $(0, 2)$ tensor field $g \in \mathcal{T}_2^0(M)$ that is symmetric and positive definite at T_pM, for all $p \in M$. A *Riemannian manifold* is a pair (M, g), where g is a Riemannian metric.

Thus, $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$, for all $p \in M$. Moreover, it is obvious that g has a constant index, that is, g_p has the same index for every $p \in M$, since $\operatorname{ind}(g) = 0$.

Definition 2.30. A pseudo-Riemannian (or semi-Riemannian) metric tensor g is a $(0,2)$ tensor field $g \in \mathcal{T}_2^0(M)$ that is nondegenerate, symmetric and has a constant index on M. A *pseudo-Riemannian* (or *semi-Riemannian*) *manifold* is a pair (M, g), where g is a pseudo-Riemannian metric.

Definition 2.31. A *Lorentzian manifold* is a pseudo-Riemannian manifold with signature $(-1, +1, +1, \dots)$, provided $\dim(M) \geq 2$.

An example of a pseudo-Riemannian metric is $\mathbb{R}^n$ with the metric g defined as

$$\langle x, y\rangle_k = -\sum_{i=1}^{k} x^i y^i + \sum_{j=k+1}^{n} x^j y^j \tag{2.22}$$

where $k = \text{ind}(g)$, we denote such a scalar product space as $\mathbb{R}^n_k$, called a pseudo-Euclidean space. Therefore, Riemannian and Lorentzian manifolds are simply the special case of pseudo-Riemannian manifolds. Note that if $X, Y \in \mathfrak{X}(M)$, then $g(X, Y) = \langle X, Y\rangle$ is a smooth function on M so that $\langle X, Y\rangle(p) = \langle X_p, Y_p\rangle$, where $X_p, Y_p \in T_pM$.

Consider a coordinate chart $(U, (x^\mu))$ on M, then

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{2.23}$$

where $g_{\mu\nu} = g(\partial_\mu, \partial_\nu) = \langle \partial_\mu, \partial_\nu\rangle$. Hence, if $X = X^\mu \partial_\mu$ and $Y = Y^\nu \partial_\nu$ on U, then $\langle X, Y\rangle = g_{\mu\nu} X^\mu Y^\nu$. At point $p \in U$, the matrix $[g_{\mu\nu}(p)]$ is clearly nondegenerate, that is, the $\det[g_{\mu\nu}(p)] \neq 0$, so $g_{\mu\nu}(p)$ is invertible and the inverse is denoted as $g^{\mu\nu}(p)$. Furthermore, at each point $p \in M$, let $q(X_p) = \langle X_p, X_p\rangle$. Then q provides the associated quadratic form of the metric tensor at p. Frequently, q is called the *line element* of M, denoted by ds^2. In terms of a coordinate chart, we have

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{2.24}$$

In addition, with the notion of metric, we can now define a local smooth frame $(e_1, \ldots, e_n)$ for M on an open subset $U \subseteq M$ as a local smooth orthonormal frame if $(e_1|_p, \ldots, e_n|_p)$ form an orthonormal basis for T_pM, for all $p \in U$, or equivalently, $\langle e_\mu, e_\nu \rangle = \epsilon_\mu \delta_{\mu\nu}$. One can easily show that there always exists a local smooth orthonormal frame at every point of a pseudo-Riemannian manifold. Note that, however, it does not guarantee that there are local coordinates on a neighborhood of p for which the coordinate frame is orthonormal.

Definition 2.32. Let (M, g) be a Lorentzian manifold. A tangent vector $v \in T_pM$ is called *spacelike* if $\langle v, v \rangle > 0$ or $v = 0$, *null* if $\langle v, v \rangle = 0$ and $v \neq 0$, *timelike* if $\langle v, v \rangle < 0$.

The category into which a given tangent vector falls indicate the *casual character*. The set of all null vectors in T_pM is called the *nullcone* at $p \in M$. The set $C(v) = \{w \in T_pM : \langle v, w \rangle < 0\}$ is called the *timecone* at $p \in M$. This implies that there are exactly two timecones at each $p \in M$ and their union is the set of all timelike vectors at point p.

Definition 2.33. A smooth curve $\gamma : I \to M$ is called *spacelike*, *null*, or *timelike* if $\dot{\gamma}(t) \in T_{\gamma(t)}M$ is spacelike, null, or timelike respectively, for all $t \in I$. A vector field $X \in \mathfrak{X}(M)$ is called *spacelike*, *null*, or *timelike* if X_p is spacelike, null, or timelike respectively, for all $p \in M$.

Definition 2.34. A Lorenztian manifold (M, g) is said to be *time-orientable* if there exists a timelike vector field $X \in \mathfrak{X}(M)$. A *time orientation* of M is a smooth function that assigns each $p \in M$ to a timecone $C^+(p) \in T_pM$ such that there exists a timelike $X \in \mathfrak{X}(M)$ with $X_p \in C^+(p)$ for each p. The timecone $C^+(p)$ is called the *future timecone* at p, while the other timecone denoted as $C^-(p)$ is called the *past timecone* at p. Timelike vectors in the future timecone are said to be *future directed* or *future pointing* and those in the past timecone are *past directed* or *past pointing*.

In this way, the meaning of a future directed timelike curve is well defined.

Definition 2.35. Let g and g' be pseudo-Riemannian metrics on M. Then g and g' are said to be *conformally related* to each other if there exists a positive function $f \in C^\infty(M)$ such that $g' = fg$.

Definition 2.36. Let (M, g) and (M', g') be pseudo-Riemannian manifolds. If there is a diffeomorphism $\psi : M \to M'$ such that the the pullback of g' is conformally related to g, that is, $\psi^* g' = fg$ for some positive $f \in C^\infty(M)$, then (M, g) and (M', g') are said to be *conformally equivalent*.

Definition 2.37. Let (M, g) be a pseudo-Riemannian manifold. Then (M, g) is said to be *conformally flat* if every point of M has a neighborhood that is conformally equivalent to an open subset in $(\mathbb{R}^n, \bar{g})$, where $\bar{g}$ denotes the pseudo-Euclidean metric.

2.7 Affine Connections and Geodesics

Definition 2.38. Let M be a smooth manifold. An *affine connection* on M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ which maps (X, Y) to $\nabla_X Y$ and satisfies

1. $\nabla_{(fX+gY)} Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X (fY) = f\nabla_X Y + (Xf)Y$

for all $f, g \in C^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. Given $X \in \mathfrak{X}(M)$, the map $\nabla_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $Y \mapsto \nabla_X Y$ is called the *covariant derivative of* Y *in the direction* X.

Definition 2.39. Let $(U, (x^\mu))$ be a coordinate chart and ∇ be an affine connection on M. The *connection coefficients* of ∇ with respect to (x^μ) are the smooth functions $\Gamma^\lambda_{\mu\nu} : U \to \mathbb{R}$ such that

$$\nabla_{\partial_\mu}\partial_\nu = \Gamma^\lambda_{\mu\nu}\partial_\lambda \,. \tag{2.25}$$

Let $X, Y \in \mathfrak{X}(M)$, in a coordinate chart $(U, (x^\mu))$. We can write $X = X^\mu\partial_\mu$ and $Y = Y^\nu\partial_\nu$. It follows that,

$$\begin{aligned}\nabla_X Y &= X^\mu(\partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu}Y^\nu)\partial_\lambda \\ &= X^\mu(\nabla_\mu Y^\lambda)\partial_\lambda\end{aligned} \tag{2.26}$$

where $\nabla_\mu Y^\lambda = \partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu}Y^\nu$ is called the *components* of $\nabla_{\partial_\mu}Y$. This can be easily extend to tensor fields. Example, for $\tau \in \mathcal{T}^1_1(M)$, the components of $\nabla_{\partial_\mu}\tau$ is given by

$$\nabla_\mu\tau^\lambda_\nu = \partial_\mu\tau^\lambda_\nu + \Gamma^\lambda_{\mu\alpha}\tau^\alpha_\nu - \Gamma^\alpha_{\nu\mu}\tau^\lambda_\alpha \,. \tag{2.27}$$

Note that connection ∇ is not a tensor, so the connection coefficients do not obey the usual transformation rule under change of coordinates. However, a simple calculation claim that the difference between two connection is a tensor.

Proposition 2.40. If ∇ and $\hat{\nabla}$ are affine connections on M, then a map $S : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$S(X, Y) = \hat{\nabla}_X Y - \nabla_X Y \tag{2.28}$$

is a $(1, 2)$-tensor field.

Definition 2.41. Suppose $\gamma : I \to M$ is a smooth curve. A *vector field along a curve* γ is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$, for all $t \in I$.

The set of all vector field along γ denoted by $\mathfrak{X}(\gamma)$ is a module over the ring $C^\infty(I)$. An example of $V \in \mathfrak{X}(\gamma)$ is the velocity vector $\dot{\gamma}(t) \in T_{\gamma(t)}M$, for all $t \in I$.

Definition 2.42. A vector field $V \in \mathfrak{X}(\gamma)$ is extendible if there exists a $\tilde{V} \in \mathfrak{X}(M)$ on a neigborhood of the image $\gamma(I)$ such that V is induced by $\tilde{V}$ in the sense that $V(t) = \tilde{V}_{\gamma(t)}, \forall t \in I$.

Theorem 2.43. *Let ∇ be an affine connection on M and $\gamma : I \to M$ be a smooth curve. Then, there exist a unique operator $D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ which satisfy*

1. $D_t(V + W) = D_t(V) + D_t(W)$
2. $D_t(fV) = f'V + fD_t(V)$
3. *if V is extendible and induced $\tilde{V} \in \mathfrak{X}(M)$ such that $V(t) = \tilde{V}_{\gamma(t)}$, then*

$$D_t(V) = \nabla_{\gamma'(t)}\tilde{V} \tag{2.29}$$

$\forall V, W \in \mathfrak{X}(\gamma)$, $f \in C^\infty(I)$ and D_tV is called the covariant derivative of V along γ.

Suppose $(U, (x^\mu))$ is a chart on M such that $\gamma(I) \subseteq U$. Then $V \in \mathfrak{X}(\gamma)$ can be written $V(t) = V^\nu(t)\partial_\nu$. Since ∂_ν is extendible, we obtain

$$D_tV = \left(\dot{V}^\lambda + \dot{\gamma}^\mu V^\nu \Gamma^\lambda_{\mu\nu}(\gamma)\right)\partial_\lambda \,. \tag{2.30}$$

Definition 2.44. Let $\gamma : I \to M$ be a smooth curve. The *acceleration* of γ is the vector field $D_t\dot{\gamma}$ along γ. In particular, a curve is said to be a *geodesic* with respect to ∇ if $D_t\dot{\gamma} = 0$.

By writing the components of γ with respect to a chart as $\gamma(t) = (x^1(t), \ldots, x^n(t))$, we have

$$D_t\dot{\gamma} = \left(\ddot{x}^\lambda + \dot{x}^\mu \dot{x}^\nu \Gamma^\lambda_{\mu\nu}(x)\right)\partial_\lambda \tag{2.31}$$

so γ is a geodesic provided that

$$\ddot{x}^\lambda + \dot{x}^\mu \dot{x}^\nu \Gamma^\lambda_{\mu\nu}(x) = 0 \tag{2.32}$$

which is called the *geodesic equation*.

Definition 2.45. Let γ be a smooth curve on M. Then $V \in \mathfrak{X}(\gamma)$ is *parallel transported along* γ with respect to ∇ if $D_t V = 0$.

Hence, it is clear that the velocity vector field of a curve is parallel along the curve.

2.8 Curvature Tensors

Definition 2.46. Let (M, g) be a pseudo-Riemannian manifold with affine connection ∇. The *curvature tensor* is a map $Rm : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$Rm(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.33}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

It is clear that Rm is a $(1,3)$ tensor field. It measures the failure of ∇_X and ∇_Y to commute. In addition, Rm could also viewed as a map $Rm(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ for each pair of $X, Y \in \mathfrak{X}(M)$. Let $(U, (x^\mu))$ be a coordinate chart on M. Then the curvature tensor can be written as

$$Rm = R^{\rho}{}_{\sigma\mu\nu}\, \partial_\rho \otimes dx^\sigma \otimes dx^\mu \otimes dx^\nu \tag{2.34}$$

where the components $R^{\rho}{}_{\sigma\mu\nu}$ are defined by

$$R(\partial_\mu, \partial_\nu)\partial_\sigma = R^{\rho}{}_{\sigma\mu\nu}\partial_\rho \,. \tag{2.35}$$

A straightforward calculation shows that

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \,. \tag{2.36}$$

Note that the curvature tensor here has only one obvious symmetry, which is the antisymmetric in the last two indices, $R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}$.

Definition 2.47. A pseudo-Riemannian manifold (M, g) is said to be *flat* if the curvature tensor is identically zero, that is $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z$.

Theorem 2.48. *Let (M, g) be a pseudo-Riemannian manifold. If $X \in \mathfrak{X}(M)$, and let $\theta_X \in \mathfrak{X}^*(M)$ such that $\theta_X(Y) = \langle X, Y\rangle$, for any $Y \in \mathfrak{X}(M)$, then the map $\phi : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ that maps $X \mapsto \theta_X$ is a $C^\infty(M)$-linear isomorphism.*

This shows that every vector field can be transformed to a unique one-form, due to the isomorphism, which implies $\mathcal{T}_0{}^1(M) \simeq \mathcal{T}_1{}^0(M)$. This approach can be extend to arbitrary dimensions of tensor fields. Let $\tau \in \mathcal{T}_s{}^r(M)$. The corresponding lowered index tensor field $\tau_* \in \mathcal{T}^{r-1}_{s+1}(M)$ is defined as

$$
\begin{aligned}
&(\tau_*)(\theta^1, ..., \theta^{r-1}, X_1, ..., X_{s+1}) \\
&\qquad = \tau(\theta^1, ..., \theta^{\alpha-1}, X^*_\beta, \theta^{\alpha+1}, ..., \theta^{r-1}, X_1, ..., X_{\beta-1}, X_{\beta+1}, ..., X_s) \quad (2.37)
\end{aligned}
$$

where $\alpha \in \{1, ..., r\}$ and $\beta \in \{1, ..., s\}$. Then the function $\phi(\tau) = \tau_*$ transform a vector field in the β^{th} position to a one-form, in the α^{th} position. Since ϕ is an isomorphism, so ϕ^{-1} exists, and it follows that $\phi^{-1}(\tau) = \tau^*$ where $\tau^* \in \mathcal{T}^{r+1}_{s-1}$, which represents the raising index operation. Recall that $dx^\mu(\partial_\nu) = \partial_\nu(dx^\mu) = \delta^\mu_\nu$ with respect to a chart $(U, (x^\mu))$. Therefore, for any $Y \in \mathfrak{X}(M)$, we have $\langle \partial_\lambda, Y\rangle = g_{\lambda\nu} dx^\nu(Y)$, so the unique isomorphism can be defined as $\partial_\lambda \mapsto g_{\lambda\nu} dx^\nu$ with the corresponding inverse $dx^\lambda \mapsto g^{\lambda\nu}\partial_\nu$. Thus, any vector field $X \in \mathfrak{X}(M)$ can be transformed by $\phi(X) = \phi(X^\mu\partial_\mu) = X^\mu\phi(\partial_\mu) = X^\mu g_{\mu\nu} dx^\nu$ which clearly determines a unique one-form in $\mathfrak{X}^*(M)$. Similar argument holds for one-form.

With these notions, we could also have a curvature tensor $Rm_* : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ defined by $Rm_*(X, Y, Z, W) = \langle Rm(X, Y)Z, W\rangle$. It is clear that Rm_* is the lowered index of the curvature tensor, which is a $(0, 4)$ tensor field. In terms of local coordinates, we have

$$
\begin{aligned}
Rm_* &= (R^\rho{}_{\sigma\mu\nu}\, g_{\rho\lambda} dx^\lambda) \otimes dx^\mu \otimes dx^\nu \otimes dx^\sigma \\
&= (R^\lambda{}_{\sigma\mu\nu}\, g_{\rho\lambda}) dx^\rho \otimes dx^\mu \otimes dx^\nu \otimes dx^\sigma \\
&= R_{\rho\sigma\mu\nu}\, dx^\rho \otimes dx^\mu \otimes dx^\nu \otimes dx^\sigma \qquad (2.38)
\end{aligned}
$$

where $R_{\rho\sigma\mu\nu} = R^\lambda{}_{\sigma\mu\nu}\, g_{\rho\lambda}$ is the components of Rm_*.

Definition 2.49. Let Rm be a curvature tensor on M. The *Ricci tensor Rc* is the contraction

$C_3^1(Rm) \in \mathcal{T}_2^0(M)$. The *Ricci scalar* R is the contraction $C_{12}(Rc) \in C^\infty(M)$.

In a coordinate chart $(U, (x^\mu))$, the Ricci tensor Rc are given as

$$Rc = R_{\mu\nu}\, dx^\mu \otimes dx^\nu \tag{2.39}$$

where

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\nu\mu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\nu\mu} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\lambda\mu} \tag{2.40}$$

and the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}\,. \tag{2.41}$$

Note that the Ricci tensor is defined from a contraction of the curvature tensor without using the metric and the Ricci scalar is uniquely defined.

Definition 2.50. Let (M^n, g) be a pseudo-Riemannian manifold, Rm be a curvature tensor, Rc be the Ricci tensor and R be the Ricci scalar. Suppose $(U, (x^\mu))$ is a coordinate chart on M. The *Weyl tensor of g* is a (0,4) tensor field defined by

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} &+ \frac{1}{n-2}(R_{\alpha\delta}g_{\beta\gamma} - R_{\alpha\gamma}g_{\beta\delta} + R_{\beta\gamma}g_{\alpha\delta} - R_{\beta\delta}g_{\alpha\gamma}) \\ &+ \frac{1}{(n-1)(n-2)}R(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})\,. \end{aligned} \tag{2.42}$$

An important consequent of the formula of Weyl tensor is that if g is conformally flat provided $n \geq 4$, then its Weyl tensor vanishes identically, that is, $C_{\alpha\beta\gamma\delta} = 0$.

2.9 Levi-Civita Connection

Definition 2.51. Let (M, g) be a pseudo-Riemannian manifold with affine connection ∇. Then ∇ is said to be a *metric connection* or to be *compatible with* g, if for any $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \,. \tag{2.43}$$

From the definition of the covariant derivative, we can write

$$(\nabla g)(X, Y, Z) = (\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \,. \tag{2.44}$$

It follows that $\nabla g = 0$ if and only if (2.43) is satisfied for all X, Y, Z. Thus, it is equivalent to said that ∇ is compatible with g if and only if $\nabla g = 0$. A metric connection is also said to be have vanishing *non-metricity*.

Definition 2.52. Let (M, g) be a pseudo-Riemannian manifold with affine connection ∇. The *torsion* of ∇ is a map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.45}$$

for all $X, Y \in \mathfrak{X}(M)$.

The torsion T is a $(1, 2)$ tensor field, called the torsion tensor. If the torsion vanishes, that is, $[X, Y] = \nabla_X Y - \nabla_Y X$, then the connection ∇ is said to be *symmetric* or *torsion-free*. In any coordinate chart, the connection is symmetric if and only if $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$.

Theorem 2.53. *Let (M, g) be a pseudo-Riemannian manifold. There exists a unique connection $\mathring{\nabla}$ on M that is symmetric and compatible with g, called the Levi-Civita* *connection.*

The complete proof can be found in J. M. Lee (1997). The theorem ensures that in any coordinate chart on M, the connection coefficients of the Levi-Civita connection $\mathring{\nabla}$, often called the *Christoffel symbols* are determined by

$$\mathring{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \,. \tag{2.46}$$

Consequently, we also express the curvature tensor defined by $\mathring{\nabla}$ as

$$\mathring{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \mathring{\Gamma}^{\rho}_{\nu\sigma} - \partial_\nu \mathring{\Gamma}^{\rho}_{\mu\sigma} + \mathring{\Gamma}^{\rho}_{\mu\lambda} \mathring{\Gamma}^{\lambda}_{\nu\sigma} - \mathring{\Gamma}^{\rho}_{\nu\lambda} \mathring{\Gamma}^{\lambda}_{\mu\sigma} \,. \tag{2.47}$$

so as the Ricci tensor $\mathring{R}_{\mu\nu} = \mathring{R}^{\lambda}{}_{\mu\lambda\nu}$ and the Ricci scalar $\mathring{R} = g^{\mu\nu} \mathring{R}_{\mu\nu}$. Due to the symmetric property of $\mathring{\nabla}$, we have $\mathring{R}_{\rho\sigma\mu\nu} = -\mathring{R}_{\sigma\rho\mu\nu} = -\mathring{R}_{\rho\sigma\nu\mu} = \mathring{R}_{\mu\nu\rho\sigma}$.

Lastly, we emphasize that connections are defined independently to the metric. Roughly speaking, the metric encodes distances and angles, while the connection alone defines covariant derivatives and parallel transport. Note that curvature, torsion, and non-metricity are generally all properties of the connection. Riemann-Cartan geometry is obtained by vanishing non-metricity, teleparallel geometry is obtained by vanishing curvature, and torsion-free geometry is obtained by vanishing torsion. We could also impose double conditions on the connection. Vanishing torsion and non-metricity gives us the Levi-Civita connection and so pseudo-Riemannian geometry. The Weitzenböck connection is based on the assumption that curvature and non-metricity are both zero. Taking curvature and torsion vanish, resulting symmetric teleparallel geometry. Finally, setting all three to zero yields Minkowski geometry.

CHAPTER 3: GRAVITY THEORIES

3.1 Introduction

This chapter begins with a brief overview of the fundamentals of general relativity based on Hawking and Ellis (1973); Heller (1992); Wald (1984). The main concerns of the general relativity are the two postulates, called the equivalence principle and the local conservation of energy and momentum principle. The former said that freely falling objects are geodesics in spacetime, while the latter suggested the form of the Einstein field equations. As a mean to overcome certain limitations of GR, we introduce the concept of modified gravity theories and their significances. In particular, we present a quick review on the $f(R)$ gravity theory in section 3.3. A complete review of the $f(R)$ gravity can be found in Sotiriou (2007); Sotiriou and Faraoni (2010).

Next, we consider a relatively new gravity theory, called the symmetric teleparallel gravity (Jimenez et al., 2018; Lu et al., 2019). The full construction of the symmetric teleparallel gravity is presented in section 3.4. The result is then applied to section 3.5, to show the modified gravity theory, namely, $f(Q)$ gravity. We end this chapter with the comment on the equivalence between the general relativity and the symmetric teleparallel gravity.

3.2 General Relativity

In Einstein's theory of general relativity (GR), the mathematical model of the universe is a spacetime manifold, defined as a triple $(M, g, \mathring{\nabla})$, where M is a time-oriented connected 4-dimensional smooth manifold, g is a Lorentz metric with signature $(-1, +1, +1, +1)$ on M and $\mathring{\nabla}$ is the Levi-Civita connection. The points of spacetime M are called *events*. An *observer* is a timelike future-directed curve $\gamma : I \to M$ such that $|\gamma'(\tau)| = 1$ for all $\tau \in I$, where τ is called the *proper time* of the observer. Otherwise, if $\langle \gamma', \gamma' \rangle = -m^2$, then γ is called a *particle* with the rest mass m. A *lightlike particle* such as photon or light beam is a null future-directed geodesic. For any kind of particle, the image $\gamma(I) \subseteq M$ of the curve $\gamma : I \to M$ is a one-dimensional submanifold of M called the *worldline* of observer or particle.

The first postulate of general relativity is very depend on the principles of special relativity. In short, the *equivalence principle* said that an observer cannot discriminate between reciprocal cases of spacetime accelerating through him/her, or his/her own acceleration through spacetime which implies that a gravitational field cannot be distinguished from a appropriately chosen accelerated reference frame. Thus, the equivalence between inertial mass and gravitational mass. Given that there is no difference between a test particle at rest in a gravitational field and a test particle accelerated by a force equal to gravity, freely falling objects under the influence of gravity can be classified as objects moving along geodesics on a spacetime manifold. Suppose γ is a freely falling observer, that means γ is a geodesic and so must satisfies the geodesic equation (2.32). Let $(x^\mu) = (x^0, x^1, x^2, x^3)$ be a coordinate system. Then the geodesic equation can be written as

$$\frac{d^2x^\lambda}{d\tau^2} = -\frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.1}$$

where the Christoffel symbols $\mathring{\Gamma}^\lambda_{\mu\nu}$ are written out explicitly. The non-vanishing of $\mathring{\Gamma}^\lambda_{\mu\nu}$ indicates the presence of inertial forces due to the non inertial reference frame. Nevertheless, it is always possible to choose coordinates at an event $p \in M$ such that $\mathring{\Gamma}^\lambda_{\mu\nu}$ vanishes at that point. This implies that at that point, gravity and relative acceleration are precisely balanced. But in general, there is no coordinate system in which $\mathring{\Gamma}^\lambda_{\mu\nu}$ vanish in a coordinate chart, except if the metric is locally flat.

The second postulate, often called the *local conservation of energy and momentum* *principle* stated that the curvature of spacetime is caused by energy-momentum tensor T, provided the divergence of the tensor T is identically zero. This gives the intuition for Einstein to relate geometric quantities such as the metric tensor, Ricci tensor, and Ricci scalar to the the energy-momentum tensor T, which is a physical quantity. The tensor T is to be determined based on physical considerations, as its physical meaning is to manage matter fields which describes the distribution of matter and energy. In local coordinates, the *Einstein field equations* (EFE) can be written as

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.2}$$

where $\kappa = 8\pi$. In addition, we can define the Einstein tensor G as the symmetric $(0, 2)$ tensor field describing the left hand side of the EFE where $\mathring{G}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu}$. Due to the divergence property of the second postulate, we have the conservation laws

$$\mathring{\nabla}_{\nu} T^{\mu\nu} = 0\,. \tag{3.3}$$

The Einstein tensor follows equally to have

$$\mathring{\nabla}_{\nu} \mathring{G}^{\mu\nu} = 0\,. \tag{3.4}$$

Consequently, the EFE provide only six non-linear partial differential equations in the metric and its first two derivatives.

Alternatively, the EFE can be derived from the Einstein-Hilbert (EH) action through the principle of least action. The action is given as

$$
\begin{aligned}
S = \int \mathcal{L} d^{(4)}V &= \int (\mathcal{L}_{EH} + \mathcal{L}_m) d^{(4)}V \\
&= \frac{1}{2\kappa} \int d^4x \sqrt{-g} \mathring{R} + \int d^4x \sqrt{-g} \mathcal{L}_m \\
&= S_{EH} + S_m
\end{aligned} \tag{3.5}
$$

where $g = \det([g_{\mu\nu}])$ is the determinant of the metric tensor matrix, $\mathcal{L}$ is the Lagrange density and the integral is take over the whole spacetime. In particular, $\mathcal{L}_m = 0$ in the vacuum case. The principle of least action requires that $\delta S = 0$, where the variation is with respect to $g_{\mu\nu}$. Varying S_{EH} with respect to $g^{\mu\nu}$, we obtain

$$
\delta S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} (\mathring{R}_{\mu\nu} - \frac{1}{2} \mathring{R} g_{\mu\nu}) \delta g^{\mu\nu} . \tag{3.6}
$$

Then, by applying the definition of the energy-momentum tensor

$$
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \tag{3.7}
$$

with $\delta S = 0$, we obtain the EFE

$$
\mathring{R}_{\mu\nu} - \frac{1}{2} \mathring{R} g_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.8}
$$

and so the trace

$$
\mathring{R} = -\kappa T \tag{3.9}
$$

where $T = g^{\mu\nu}T_{\mu\nu}$. To include the cosmological constant Λ, the action can be modified as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(\mathring{R} - 2\Lambda) + \int d^4x\sqrt{-g}\mathcal{L}_m \,. \tag{3.10}$$

It follows that the EFE with cosmological constant is

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \,. \tag{3.11}$$

3.3 $f(R)$ Gravity

One of the most basic attempts to modify the GR is adding higher order invariants to the Einstein-Hilbert action, resulting in so called higher-order gravity theories. The $f(R)$ gravity theory is under one of this class which is to replace the Ricci scalar in (3.5) with an arbitrary function of R. There are essentially three formalisms to calculate the variation, each yielding a different field equations. Variation with respect to the metric gives *metric $f(R)$ gravity*, variation with respect to the metric and the connection while the matter action is independent of the connection gives *Palatini $f(R)$ gravity* and variation with respect to the metric and the connection while the matter Lagrangian depends on the connection gives *metric-affine $f(R)$ gravity*.

The action for metric $f(R)$ gravity is given by

$$\begin{aligned} S &= S_{met} + S_m \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(\mathring{R}) + S_m \,. \end{aligned} \tag{3.12}$$

The variation of the action in (3.12) with respect to the inverse metric $g^{\mu\nu}$ gives

$$\delta S_{met} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(f'(\mathring{R})\mathring{R}_{\mu\nu} - \frac{1}{2}f(\mathring{R})g_{\mu\nu} - \mathring{\nabla}_\mu\mathring{\nabla}_\nu f'(\mathring{R}) + g_{\mu\nu}\Box f'(\mathring{R})\right)\delta g^{\mu\nu}$$

(3.13)

where $\mathring{\nabla}$ is the Levi-Civita connection and $\Box$ is the d'Alembert operator defined as $\Box = \mathring{\nabla}^{\mu}\mathring{\nabla}_{\mu} = g^{\mu\nu}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}$. Applying the same energy-momentum tensor stated in (3.7) and using the least action principle, we obtain the fourth order field equations

$$f'(\mathring{R})\mathring{R}_{\mu\nu} - \frac{1}{2}f(\mathring{R})g_{\mu\nu} - \mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}f'(\mathring{R}) + g_{\mu\nu}\Box f'(\mathring{R}) = \kappa T_{\mu\nu} \tag{3.14}$$

which has trace

$$f'(\mathring{R})\mathring{R} - 2f(\mathring{R}) + 3\Box f'(\mathring{R}) = \kappa T\ . \tag{3.15}$$

Note that when $f(\mathring{R}) = \mathring{R}$, the field equations reduce to EFE as expected. Moreover, in contrast to (3.9), $T = 0$ does not imply $\mathring{R} = 0$ in (3.15). This suggests that the field equations of $f(R)$ gravity will allow for a wider range of solutions than GR. The conservation of energy-momentum is satisfied since we still have $\mathring{\nabla}_{\nu}T^{\mu\nu} = 0$. Furthermore, if we rearrange (3.14) as

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} = \frac{\kappa T_{\mu\nu}}{f'(\mathring{R})} - \frac{\mathring{R}f'(\mathring{R}) - f(\mathring{R})}{2f'(\mathring{R})}g_{\mu\nu} + \frac{\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}f'(\mathring{R}) - g_{\mu\nu}\Box f'(\mathring{R})}{f'(\mathring{R})} \tag{3.16}$$

and define the effective energy-momentum tensor

$$T^{(e)}_{\mu\nu} = \frac{1}{\kappa}\left(\frac{f(\mathring{R}) - \mathring{R}f'(\mathring{R})}{2}g_{\mu\nu} + \mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}f'(\mathring{R}) - g_{\mu\nu}\Box f'(\mathring{R})\right) \tag{3.17}$$

then the field equations of $f(R)$ gravity can be rewritten in the form of EFE as

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} = \kappa\left(T_{\mu\nu} + T^{(e)}_{\mu\nu}\right)\ . \tag{3.18}$$

This simplification is very useful in calculation and determining suitable $f(R)$ models.

For Palatini formalism, the connection is independent to the metric, denoted by $\bar{\nabla}$. For clarity, we denote the Ricci tensor and Ricci scalar constructed with $\bar{\nabla}$ as $\bar{R}_{\mu\nu}$ and $\bar{R}$ respectively. However, we assert that the independent connection $\bar{\nabla}$ does not define covariant derivative and the geometry is really pseudo-Riemannian. In other words, the covariant derivative is still defined by the Levi-Civita connection of the metric. Thus, the action now takes the form

$$S_{pal} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(\bar{R})\,. \tag{3.19}$$

By using the equation

$$\delta\bar{R}_{\mu\nu} = \bar{\nabla}_\lambda\delta\bar{\Gamma}^\lambda_{\mu\nu} - \bar{\nabla}_\nu\delta\bar{\Gamma}^\lambda_{\mu\lambda} \tag{3.20}$$

and varying the action (3.19) independently with respect to the metric and the connection yields

$$f'(\bar{R})\bar{R}_{(\mu\nu)} - \frac{1}{2}f(\bar{R})g_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.21}$$

$$-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\bar{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\bar{R})g^{\sigma(\mu}\right)\delta^{\nu)}_\lambda = 0 \tag{3.22}$$

where $T_{\mu\nu}$ is defined as stated in (3.7). Taking the trace of (3.22), we can easily shown that

$$\bar{\nabla}_\sigma\left(\sqrt{-g}f'(\bar{R})g^{\sigma\mu}\right) = 0\,. \tag{3.23}$$

Hence, the field equations reduce to

$$f'(\bar{R})\bar{R}_{(\mu\nu)} - \frac{1}{2}f(\bar{R})g_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.24}$$

$$\bar{\nabla}_{\lambda}\left(\sqrt{-g}f'(\bar{R})g^{\mu\nu}\right) = 0. \tag{3.25}$$

In the case $f(\bar{R}) = \bar{R}$ and so $f'(\bar{R}) = 1$, (3.25) turns into the definition of the Levi-Civita connection for the independent connection $\bar{\nabla}$. Therefore, $\bar{R}_{\mu\nu} = \mathring{R}_{\mu\nu}$, $\bar{R} = \mathring{R}$ and (3.24) reproduces Einstein field equations. The energy-momentum tensor is conserved by the covariant derivative defined with Levi-Civita connection, that is, $\mathring{\nabla}_{\nu}T^{\mu\nu} = 0$, but $\bar{\nabla}_{\nu}T^{\mu\nu} \neq 0$. It is now clear that generalizing the action in the Palatini formalism to be an arbitrary function of $\bar{R}$ is just as natural as generalizing the Einstein-Hilbert action in the metric formalism.

The metric-affine formalism is too tedious to be discuss here. In short, besides considering the connection is independent to the metric, we also drop the assumption that the connection is symmetric and metric compatible. In addition, this connection defines the covariant derivative and parallel transport, unlike Palatini formalism which remains to the Levi-Civita connection. Therefore, the geometry is completely different to pseudo-Riemannian and is usually called the Einstein-Cartan-Weyl geometry (Adak, 2006). In fact, this forces the matter action of the metric-affine theory to be depend on the connection, that is, $S_m = S_m\left(g_{\mu\nu}, \bar{\Gamma}^{\lambda}_{\mu\nu}, \psi\right)$, where ψ represents the matter fields. In this way, we have

$$\delta S_m = \int d^4x \frac{\delta S_m}{\delta g^{\mu\nu}}\delta g^{\mu\nu} + \int d^4x \frac{\delta S_m}{\delta \bar{\Gamma}^{\lambda}_{\mu\nu}}\delta \bar{\Gamma}^{\lambda}_{\mu\nu} \tag{3.26}$$

where we also define a new tensor called the hypermomentum as

$$\Delta_{\lambda}{}^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta \bar{\Gamma}^{\lambda}_{\mu\nu}} \,. \tag{3.27}$$

The vanishing of $\Delta_{\lambda}{}^{\mu\nu}$ actually imply independence of the matter action from the connection. The full explanation and the field equations can be refer to Sotiriou and Liverati (2007).

3.4 Symmetric Teleparallel Gravity

Because general relativity is essentially a geometric theory expressed in pseudo-Riemannian space, another interesting route for to generalized gravity theories is to look for more universal geometric structures that can represent the gravitational field. Weitzenböck's study, which established what is currently known as teleparallelism, resulted in an approach that did find significant physical applications. The core idea underlying the teleparallel approach the gravity is to replace the fundamental variable describing gravitational features, the metric $g_{\mu\nu}$ of spacetime, with a set of tetrad fields. Therefore, the torsion produced by the tetrad fields can be utilised to fully describe gravitational effects, with the curvature being replaced by the torsion. As a result, we obtain the so called teleparallel equivalent of general relativity (Hayashi & Shirafuji, 1979), and then leads to the $f(T)$ gravity theory. Thus, in teleparallel theory, torsion totally compensates curvature, resulting in the spacetime becoming flat. The field equations of the $f(T)$ gravity theory are of second order, as opposed to fourth order field equations in the metric $f(R)$ gravity. Also, $f(T)$ theory has been widely applied to the study of cosmology, where it is used to explain the late time accelerating expansion of the Universe, without the necessity of introducing dark energy.

General relativity can be described in two equivalent geometric representations, according to the above presentation: the curvature representation in which torsion and non-metricity vanish, and the teleparallel representation in which curvature and non-metricity vanish. Nevertheless, a third comparable representation in which the fundamental geometric variable characterizing the features of the gravitational interaction is represented by the non-metricity, which geometrically describes the variation of the length of a vector in parallel transport, is also conceivable. This approach is called the symmetric teleparallel gravity and it has the advantages that calculation can be done in local coordinates instead of tetrad fields. To begin, let us clarify that STG is differs from metric-affine gravity theory which extends GR by allowing connection to have torsion and non-metricity in addition to curvature, whereas such additional geometric structures often require specific types or properties of matter to excite and investigate. By enforcing vanishing curvature and torsion, the GR gravitational action is reinterpreted in terms of non-metricity in STG, and all gravitational effects associated to curvature in GR are now equivalently attributed to non-metricity. Hence, in STG, the matter content can remain unchanged, and non-metricity is generated by the usual matter energy-momentum, which is analogous to Einstein's equations.

Let ∇ be a flat and torsion-free affine connection on M and $\mathring{\nabla}$ be the Levi-Civita connection on M. Define a map $L : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $L(X,Y) = \nabla_X Y - \mathring{\nabla}_X Y$, for all $X, Y \in \mathfrak{X}(M)$. Suppose $(U, (x^\mu))$ is a coordinate chart on M. Then

$$L(\partial_\mu, \partial_\nu) = (\Gamma^\lambda_{\mu\nu} - \mathring{\Gamma}^\lambda_{\mu\nu})\partial_\lambda \,. \tag{3.28}$$

Hence, with respect to $(U, (x^\mu))$, we can write

$$L = L^\lambda{}_{\mu\nu}\partial_\lambda \otimes dx^\mu \otimes dx^\nu \tag{3.29}$$

where $L^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \mathring{\Gamma}^\lambda_{\mu\nu}$ are the components of L. It is clear that L is a $(1,2)$-tensor field which is symmetric with respect to the second and third indices. Consequently, L can be expressed as

$$L(X,Y) = \frac{1}{2}\big(L(X,Y) - L(Y,X)\big) + \frac{1}{2}\big(L(X,Y) + L(Y,X)\big)$$
$$= A(X,Y) + S(X,Y) \tag{3.30}$$

where $A(X,Y) = \frac{1}{2}\big(L(X,Y) - L(Y,X)\big)$ is antisymmetric and $S(X,Y) = \frac{1}{2}\big(L(X,Y) + L(Y,X)\big)$ is symmetric parts of L. Thus, with respect to $(U, (x^{\mu}))$, the components of A are

$$A^{\lambda}{}_{\mu\nu} = \frac{1}{2}(\Gamma^{\lambda}_{\mu\nu} - \mathring{\Gamma}^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} + \mathring{\Gamma}^{\lambda}_{\nu\mu}) = \frac{1}{2}(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}) = 0 \tag{3.31}$$

and the components of S are

$$S^{\lambda}{}_{\mu\nu} = \frac{1}{2}(\Gamma^{\lambda}_{\mu\nu} - \mathring{\Gamma}^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\mu} - \mathring{\Gamma}^{\lambda}_{\nu\mu}) = \frac{1}{2}(\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\mu}) - \mathring{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \mathring{\Gamma}^{\lambda}_{\mu\nu} \tag{3.32}$$

because $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$ for torsion-free connection. Thus, the components of L become

$$L^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \mathring{\Gamma}^{\lambda}_{\mu\nu} \tag{3.33}$$

and L is called the *disformation tensor*. By using the formula for the Levi-Civita connection

(2.46), we obtain

$$
\begin{aligned}
L^{\lambda}{}_{\mu\nu} &= \Gamma^{\lambda}_{\mu\nu} - \frac{1}{2}g^{\lambda\alpha}(\partial_{\mu}g_{\nu\alpha} + \partial_{\nu}g_{\mu\alpha} - \partial_{\alpha}g_{\mu\nu}) \\
&= \Gamma^{\lambda}_{\mu\nu} - \frac{1}{2}g^{\lambda\alpha}\big(\nabla_{\mu}g_{\nu\alpha} + g_{\rho\alpha}\Gamma^{\rho}_{\nu\mu} + g_{\nu\rho}\Gamma^{\rho}_{\alpha\mu} \\
&\qquad\qquad + \nabla_{\nu}g_{\mu\alpha} + g_{\rho\alpha}\Gamma^{\rho}_{\mu\nu} + g_{\mu\rho}\Gamma^{\rho}_{\alpha\nu} \\
&\qquad\qquad - \nabla_{\alpha}g_{\mu\nu} - g_{\rho\nu}\Gamma^{\rho}_{\mu\alpha} - g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha}\big) \\
&= \Gamma^{\lambda}_{\mu\nu} - \frac{1}{2}g^{\lambda\alpha}(\nabla_{\mu}g_{\nu\alpha} + \nabla_{\nu}g_{\mu\alpha} - \nabla_{\alpha}g_{\mu\nu}) - \delta^{\lambda}_{\rho}\Gamma^{\rho}_{\mu\nu} \\
&= -\frac{1}{2}g^{\lambda\alpha}(\nabla_{\mu}g_{\nu\alpha} + \nabla_{\nu}g_{\mu\alpha} - \nabla_{\alpha}g_{\mu\nu})\,.
\end{aligned} \tag{3.34}
$$

Since ∇ is a non-metric connection, that is, $\nabla g \neq 0$, so we define the non-metricity tensor

$$
Q_{\lambda\mu\nu} = \nabla_{\lambda}g_{\mu\nu}\,. \tag{3.35}
$$

It follows that Q is symmetric with respect to its second and third indices, or equivalently, $Q_{\lambda\mu\nu} = Q_{\lambda(\mu\nu)}$. Hence, (3.34) can be rewritten as

$$
\begin{aligned}
L^{\lambda}{}_{\mu\nu} &= -\frac{1}{2}g^{\lambda\alpha}(Q_{\mu\alpha\nu} + Q_{\nu\alpha\mu} - Q_{\alpha\mu\nu}) \\
&= \frac{1}{2}(Q^{\lambda}{}_{\mu\nu} - Q_{\mu}{}^{\lambda}{}_{\nu} - Q_{\nu}{}^{\lambda}{}_{\mu})\,.
\end{aligned} \tag{3.36}
$$

This implies that L is fully determined by the non-metricity tensor. Moreover, we denote the trace of Q on first two and last two pair of indices by

$$
\tilde{Q}_{\mu} \equiv g^{\nu\lambda}Q_{\nu\lambda\mu} = Q_{\nu}{}^{\nu}{}_{\mu}\,, \qquad Q_{\mu} \equiv g^{\nu\lambda}Q_{\mu\nu\lambda} = Q_{\mu}{}^{\nu}{}_{\nu}\,. \tag{3.37}
$$

Consequently, we also have

$$L_{\nu}{}^{\nu}{}_{\mu} = -\frac{1}{2}Q_{\mu}\,, \qquad L_{\mu}{}^{\nu}{}_{\nu} = \frac{1}{2}Q_{\mu} - \tilde{Q}_{\mu}\,. \tag{3.38}$$

Furthermore, in terms of components, we know that the curvature tensor is given by

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,. \tag{3.39}$$

It follows from (3.33), the two curvature tensors can be related by

$$R^{\rho}{}_{\sigma\mu\nu} = \mathring{R}^{\rho}{}_{\sigma\mu\nu} + \mathring{\nabla}_{\mu}L^{\rho}{}_{\nu\sigma} - \mathring{\nabla}_{\nu}L^{\rho}{}_{\mu\sigma} + L^{\rho}{}_{\mu\lambda}L^{\lambda}{}_{\nu\sigma} - L^{\rho}{}_{\nu\lambda}L^{\lambda}{}_{\mu\sigma}\,. \tag{3.40}$$

But since ∇ is flat, that is, $R^{\rho}{}_{\sigma\mu\nu} = 0$, so we have

$$\mathring{R}^{\rho}{}_{\sigma\mu\nu} = -\mathring{\nabla}_{\mu}L^{\rho}{}_{\nu\sigma} + \mathring{\nabla}_{\nu}L^{\rho}{}_{\mu\sigma} - L^{\rho}{}_{\mu\lambda}L^{\lambda}{}_{\nu\sigma} + L^{\rho}{}_{\nu\lambda}L^{\lambda}{}_{\mu\sigma}\,. \tag{3.41}$$

By taking the contraction of the curvature tensor on its first and third indices, we obtain the Ricci tensor

$$\mathring{R}_{\sigma\nu} = -\frac{1}{2}\mathring{\nabla}_{\nu}Q_{\sigma} - \mathring{\nabla}_{\rho}L^{\rho}{}_{\nu\sigma} + \frac{1}{2}Q_{\lambda}L^{\lambda}{}_{\nu\sigma} + L^{\rho}{}_{\nu\lambda}L^{\lambda}{}_{\rho\sigma} \tag{3.42}$$

and thereby the Ricci scalar

$$\mathring{R} = \mathring{\nabla}_{\lambda}\tilde{Q}^{\lambda} - \mathring{\nabla}_{\lambda}Q^{\lambda} + \frac{1}{4}Q_{\lambda}Q^{\lambda} - \frac{1}{2}Q_{\lambda}\tilde{Q}^{\lambda} + L_{\rho\nu\lambda}L^{\lambda\rho\nu}\,. \tag{3.43}$$

To obtain the field equations in symmetric teleparallel gravity, we construct an invariant, called the non-metricity scalar

$$Q = g^{\mu\nu}(L^{\alpha}{}_{\beta\mu}L^{\beta}{}_{\nu\alpha} - L^{\alpha}{}_{\beta\alpha}L^{\beta}{}_{\mu\nu}) \tag{3.44}$$

and the non-metricity conjugate (or superpotential)

$$P^{\lambda}{}_{\mu\nu} = -\frac{1}{2}L^{\lambda}_{\mu\nu} + \frac{1}{4}(Q^{\lambda} - \tilde{Q}^{\lambda})g_{\mu\nu} - \frac{1}{4}\delta^{\lambda}_{(\mu}Q_{\nu)} \,. \tag{3.45}$$

By doing so, we obtain

$$Q = Q_{\lambda\mu\nu}P^{\lambda\mu\nu} = -\frac{1}{2}Q_{\lambda\mu\nu}L^{\lambda\mu\nu} + \frac{1}{4}Q_{\lambda}Q^{\lambda} - \frac{1}{2}Q_{\lambda}\tilde{Q}^{\lambda} \,. \tag{3.46}$$

Then, the action for symmetric teleparallel gravity is defined by

$$\begin{aligned} S &= S_{STG} + S_m \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,Q + S_m \,. \end{aligned} \tag{3.47}$$

Note that from (3.43) and (3.46), one finds that

$$Q = \mathring{R} + \mathring{\nabla}_{\lambda}(Q^{\lambda} - \tilde{Q}^{\lambda}) \tag{3.48}$$

and so

$$S_{STG} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,[\mathring{R} + \mathring{\nabla}_{\lambda}(Q^{\lambda} - \tilde{Q}^{\lambda})] \tag{3.49}$$

This shows that the action is identical to the Einstein-Hilbert action in the GR up to a boundary term. Thus, the symmetric teleparallel gravity theory is equivalent to general relativity. Before taking the variation of the action, we have to assume that $\Gamma^{\lambda}_{\mu\nu} = 0$, so all the covariant derivatives reduce to the partial derivatives, $\nabla_{\lambda} = \partial_{\lambda}$ because the variation δ and the covariant derivative ∇_{λ} is not commute. However, a flat and torsion-free connection only implies that there exists some coordinate systems (y^{μ}) in which $\Gamma^{\lambda}_{\mu\nu} = 0$. This special coordinate system is called the *coincident gauge* based on Lin and Zhai (2021). As shown in Zhao (2021), in any other coordinate systems (x^{μ}) in which the connection does not vanish, the connection coefficients will take the form,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial y^{\alpha}}\partial_{\mu}\partial_{\nu}y^{\alpha} \tag{3.50}$$

which is purely inertial. These term only devote into the boundary term in the action (3.49) and has no effect on the equation of motion. Thus, we can always assume the coincident gauge and so the metric is the only fundamental variable. Then, varying S_{STG} with respect to the inverse metric $g^{\mu\nu}$ gives

$$\begin{aligned}&\delta S_{STG}\\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\frac{2}{\sqrt{-g}}\partial_{\lambda}(\sqrt{-g}P^{\lambda}{}_{\mu\nu}) - \frac{1}{2}Qg_{\mu\nu} + (Q_{\mu\rho\sigma}P_{\nu}{}^{\rho\sigma} - 2Q_{\rho\sigma\nu}P^{\rho\sigma}_{\mu})\right]\delta g^{\mu\nu}.\end{aligned} \tag{3.51}$$

Once again define the energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{3.52}$$

and using the least action principle, $\delta S = 0$, we obtain the second-order field equations

$$\frac{2}{\sqrt{-g}}\partial_{\lambda}(\sqrt{-g}P^{\lambda}{}_{\mu\nu}) - \frac{1}{2}Qg_{\mu\nu} + (Q_{\mu\rho\sigma}P_{\nu}{}^{\rho\sigma} - 2Q_{\rho\sigma\nu}P_{\mu}{}^{\rho\sigma}) = \kappa T_{\mu\nu}\,. \tag{3.53}$$

3.5 $f(Q)$ Gravity

A straightforward attempt to modify the symmetric teleparallel gravity is to replace the non-metricity scalar Q in the action (3.47) with an arbitrary function of Q, called the $f(Q)$ gravity theory. Thus, the action of $f(Q)$ gravity is

$$S_{f(Q)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, f(Q)\,. \tag{3.54}$$

In this case, the action can no longer be written in (3.49), so both the connection and the metric are dynamic variable. If we assume the coincident gauge, that is, $\Gamma^\lambda_{\mu\nu} = 0$ and varying $S_{f(Q)}$ with respect to $g^{\mu\nu}$, then we have the field equations

$$\frac{2}{\sqrt{-g}} \nabla_\lambda (\sqrt{-g} f_Q P^\lambda{}_{\mu\nu}) - \frac{1}{2} f g_{\mu\nu} + f_Q (Q_{\mu\rho\sigma} P_\nu{}^{\rho\sigma} - 2Q_{\rho\sigma\nu} P_\mu{}^{\rho\sigma}) = \kappa T_{\mu\nu} \tag{3.55}$$

where $f_Q = \partial f / \partial Q$. We call this field equations as the type-I $f(Q)$ field equations. The complete derivation of this field equations is presented in Appendices. Since we have applied the coincident gauge, the type-I field equations are only valid in some coordinate system in which $\Gamma^\lambda_{\mu\nu} = 0$. This also suggested by the fact that the first term of the left hand side in (3.55) is not in tensor form. To avoid this problem and for a better understanding of the field equations, we transform it into a form similar to EFE. According to Appendices, the type-I field equations in (3.55) can be rewritten as

$$f_Q \left(\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R} \right) + \frac{1}{2} g_{\mu\nu} (Q f_Q - f) + 2 f_{QQ} \nabla_\lambda Q P^\lambda{}_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.56}$$

where $f_{QQ} = \partial f_Q / \partial Q$. This is called the type-II $f(Q)$ field equations. Notice that (3.56) is in tensor form which implies that it is actually coordinate independent. As a result, the type-II field equations will be applied in later sections, since it is valid in all coordinate systems. Alternatively, the $f(Q)$ gravity theory can be formulated in a covariant way. This approach is presented in Zhao (2021), but in our case, the coincident gauge condition will be sufficient.

3.6 Discussion

At the classical level, gravitation has a fairly peculiar property in which given the same initial conditions and follow the same path, particles with distinct masses experience it acquire the same acceleration. The equivalence principle reflects this phenomena, which is known as universality of free fall. It is gravity's most unique and strange property, as no other fundamental interaction in nature display it. Non-gravitational effects, on the other hand, have been recognized for a long time and are felt equally by all particles. They are called the inertial effects such as the Coriolis and centrifugal forces on Earth. The inspiration of Einstein for developing general relativity was based on the universality of both gravitational and inertial effects.

Another point that inspire Einstein was the notion of field, since each of the known forces can be described mathematically as a field. If gravitation is also to be represented by a field, then the field must be universal and felt by all particles equally. Thus, it is believed that gravitation changes spacetime itself and the most straightforward method to do so is to change the metric, which appears to be the most fundamental field. In short, the existence of a gravitational field is indicated by a change in the metric of spacetime. Nevertheless, the metric tensor does not define curvature or non-metricity on its own. In fact, curvature and non-metricity both demand a connection to be defined. On a given spacetime, several different connections can be defined, each with a different curvature and non-metricity tensor. Therefore, determining the appropriate connection to depict the gravitational field became difficult. For example, the Levi-Civita connection which has vanishing torsion and non-metricity was selected by Einstein such that the connection is fully determined by the ten components of the metric tensor. The gravitational field is described by its curvature.

Nevertheless, this is not the only possible choice. The other possibilities are to choose a flat connection that have non-vanishing torsion or non-metricity tensor. The gravitational theory that appears from these choice are the teleparallel gravity and symmetric teleparallel gravity. In both cases, curvature is supposed to vanish from the very beginning while the gravitational effect is depicted by a force, and particle trajectories are force equations with torsion or non-metricity act as force instead of geodesics. But it cannot be denied that Einstein's choice seems to be most natural in terms of universality. Gravitation can be simply visualized by assuming that it generates a curvature in spacetime causing all particles follow a geodesic on the curved spacetime regardless of their masses. In this approach, the universality of free fall is obviously merged into gravitation. The notion of force is substituted by geometry while the trajectories are solutions to a geodesic equation rather than a force equation. However, since such a geometrization is based on the equivalence principle, the general-relativistic depiction of gravitation would fail in the absence of universality.

On the flip side, general relativity and symmetric teleparallel gravity are found to produce equivalent depiction of the gravitational interaction, despite their conceptual differences. This equivalence has the direct consequence that curvature and non-metricity are essentially different ways of characterising the gravitational field. The fact that the symmetric matter energy-momentum tensor appears as the source of curvature in general relativity and non-metricity in symmetric teleparallel gravity supports this idea. Both general relativity and symmetric teleparallel gravity, according to this explanation, are complete theories, and Einstein did not make a mistake by ignoring non-metricity.

There is a common perception that gravity causes a curvature in spacetime, based on the geometric description of general relativity which uses the Levi-Civita connection. As a result, the entire Universe should be curved. Nevertheless, with the arrival of symmetric teleparallel gravity, this idea is no longer applicable. In fact, due to the equivalence of general relativity and symmetric teleparallel gravity, describing the gravitational interaction in terms of curvature or non-metricity has become a matter of convention. This means that attributing curvature to spacetime is a model-dependent statement rather than an absolute. Certainly, cosmology due to general relativity is not inaccurate in the least. Nevertheless, an assessment due to symmetric teleparallel gravity may suggest a new perspective on how to perceive and understand the cosmos. We may then argue that symmetric teleparallel gravity is a new way of looking at all gravitational phenomenon, including those that shape the Universe itself, instead of simply a theory that is equivalent to general relativity.

CHAPTER 4: GEODESIC DEVIATION EQUATION IN $f(Q)$

4.1 Introduction

In this chapter, we begin by deriving the Jacobi equation or also commonly known as geodesic deviation equation, which is an ordinary differential equation satisfied by the variation field of any one-parameter family of geodesics. A vector field satisfying this equation along a geodesic is called a Jacobi field. This section is mainly inspired by J. M. Lee (1997).

In section 4.3, we show the construction of the standard cosmology model, called the Friedmann-Lemaître-Robertson-Walker (FLRW) model based on Heller (1992); O'neil (1983). We first introduce the notion of perfect fluids which are idealized matter models that have no viscosity and shear stresses. Next, the concept of warped product is introduced, which is then followed by the Robertson-Walker spacetime. We end this section with a brief discussion of the observational aspects of the standard cosmological model.

Next, in section 4.4, we express the GDE in $f(Q)$-gravity with the background of FLRW cosmology. We also show the generalized Friedmann equations in section 4.5 as a comparison to the Fridemann equations in GR. Two special cases of the GDE which are the GDE for fundamental observers and for past-directed null vector fields will be studied in section 4.6 and 4.7 respectively.

4.2 Geodesic Deviation Equation

In this section, (M, g) denotes an arbitrary pseudo-Riemannian manifold.

Definition 4.1. Let $I, J \subseteq \mathbb{R}$ be intervals, a smooth map $\Gamma : I \times J \to M$ is called a *one-parameter family of curves* where the partial maps $t \mapsto \Gamma_s(t) = \Gamma(t, s)$ are called the main curves, and the curves $s \mapsto \Gamma(t, s)$ are called the transverse curves.

This implies that if $X \in \mathfrak{X}(\Gamma)$, then we can calculate the covariant derivative of X along either the main curves or the transverse curves resulting the vector field along Γ denoted by $D_t X$ and $D_s X$ respectively. We express the velocity vector fields of the main and transverse curves by $\partial_t \Gamma(t,s)$ and $\partial_s \Gamma(t,s)$ respectively, which are examples of the vector field along Γ.

Lemma 4.2. Suppose $\Gamma : I \times J \to M$ is a one-parameter family of curves in M. Then for every vector field X along Γ,

$$D_s D_t X - D_t D_s X = R(\partial_s \Gamma, \partial_t \Gamma) X\,. \tag{4.1}$$

Proof. For each $(t,s) \in I \times J$, consider a coordinate chart defined on $\Gamma(t,s)$ and write

$$\Gamma(t,s) = \big(\gamma^1(t,s), \ldots, \gamma^n(t,s)\big), \qquad X(t,s) = X^\mu(t,s)\partial_\mu|_{\Gamma(t,s)}. \tag{4.2}$$

By using formula (2.30), we get

$$D_t X = \frac{\partial X^\mu}{\partial t}\partial_\mu + X^\mu D_t \partial_\mu\,. \tag{4.3}$$

Thus, by applying (2.30), we obtain

$$D_s D_t X = \frac{\partial^2 X^\mu}{\partial s \partial t}\partial_\mu + \frac{\partial X^\mu}{\partial t} D_s \partial_\mu + \frac{\partial X^\mu}{\partial s} D_t \partial_\mu + X^\mu D_s D_t \partial_\mu\,. \tag{4.4}$$

Interchanging the indices s and t and subtracting, the only survive terms are

$$D_s D_t X - D_t D_s X = X^\mu (D_s D_t \partial_\mu - D_t D_s \partial_\mu). \tag{4.5}$$

To compute the commutator in parentheses, we first write

$$\partial_t\Gamma = \frac{\partial\gamma^\lambda}{\partial t}\partial_\lambda\,, \qquad \partial_s\Gamma = \frac{\partial\gamma^\nu}{\partial s}\partial_\nu\,. \tag{4.6}$$

Since ∂_μ is extendible,

$$D_t\partial_\mu = \nabla_{\partial_t\Gamma}\partial_\mu = \frac{\partial\gamma^\nu}{\partial t}\nabla_{\partial_\nu}\partial_\mu\,, \tag{4.7}$$

and also because $\nabla_{\partial_\nu}\partial_\mu$ is extendible,

$$\begin{aligned} D_sD_t\partial_\mu &= D_s\left(\frac{\partial\gamma^\nu}{\partial t}\nabla_{\partial_\nu}\partial_\mu\right) \\ &= \frac{\partial^2\gamma^\nu}{\partial s\partial t}\nabla_{\partial_\nu}\partial_\mu + \frac{\partial\gamma^\nu}{\partial t}\nabla_{\partial_s\Gamma}(\nabla_{\partial_\nu}\partial_\mu) \\ &= \frac{\partial^2\gamma^\nu}{\partial s\partial t}\nabla_{\partial_\nu}\partial_\mu + \frac{\partial\gamma^\nu}{\partial t}\frac{\partial\gamma^\lambda}{\partial s}\nabla_{\partial_\lambda}\nabla_{\partial_\nu}\partial_\mu\,. \end{aligned} \tag{4.8}$$

Interchanging $s \leftrightarrow t$ and $\nu \leftrightarrow \lambda$ and subtracting, we have

$$\begin{aligned} D_sD_t\partial_\mu - D_tD_s\partial_\mu &= \frac{\partial\gamma^\nu}{\partial t}\frac{\partial\gamma^\lambda}{\partial s}(\nabla_{\partial_\lambda}\nabla_{\partial_\nu}\partial_\mu - \nabla_{\partial_\nu}\nabla_{\partial_\lambda}\partial_\mu) \\ &= \frac{\partial\gamma^\nu}{\partial t}\frac{\partial\gamma^\lambda}{\partial s}R(\partial_\lambda,\partial_\nu)\partial_\mu \\ &= R(\partial_s\Gamma,\partial_t\Gamma)\partial_\mu\,. \end{aligned} \tag{4.9}$$

Substituting this into (4.5) yields the result. □

Lemma 4.3. Suppose $\Gamma : I \times J \to M$ is a one-parameter family of curves in M. Then

$$D_s\partial_t\Gamma = D_t\partial_s\Gamma\,. \tag{4.10}$$

Proof. In local coordinates (x^μ), we write the components of Γ as

$$\Gamma(t,s) = (x^1(t,s), \ldots, x^n(t,s)) \tag{4.11}$$

and then

$$\partial_t\Gamma = \frac{\partial x^\lambda}{\partial t}\partial_\lambda \,, \qquad \partial_s\Gamma = \frac{\partial x^\lambda}{\partial s}\partial_\lambda \,. \tag{4.12}$$

By applying formula (2.30), we get

$$D_s\partial_t\Gamma = \left(\frac{\partial^2 x^\lambda}{\partial s\partial t} + \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}\Gamma^\lambda_{\nu\mu}\right)\partial_\lambda \tag{4.13}$$

$$D_t\partial_s\Gamma = \left(\frac{\partial^2 x^\lambda}{\partial t\partial s} + \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\Gamma^\lambda_{\nu\mu}\right)\partial_\lambda \,. \tag{4.14}$$

By interchanging μ and ν in the second equation above and using the symmetry condition $\Gamma^\lambda_{\nu\mu} = \Gamma^\lambda_{\mu\nu}$, we conclude the proof. □

Definition 4.4. A *variation* of a smooth curve $\gamma : I \to M$ is a one-parameter family of curves $\Gamma : I \times (-\epsilon, \epsilon) \to M$ such that $\gamma(t) = \Gamma_0(t)$ for all $t \in I$. The curve γ is called the *centre curve*. The *variation field* of Γ is the vector field $\eta(t) = \partial_s\Gamma(t,0)$ along γ.

Definition 4.5. A variation Γ such that each of the main curves $t \mapsto \Gamma_s(t)$ is a geodesic is called a *one-parameter of geodesics* or *geodesic congruence*.

Definition 4.6. Let γ be a geodesic. A vector field $X \in \mathfrak{X}(\gamma)$ along γ that satisfies the *Jacobi equation*:

$$D_t^2 X = R(\gamma', X)\gamma' \tag{4.15}$$

is called a *Jacobi field*.

Theorem 4.7. *Suppose* Γ *is a one-parameter of geodesics. The variation field of* Γ *is a Jacobi field.*

Proof. From the geodesic equation, we have $D_t\partial_t\Gamma = 0$ and so $D_sD_t\partial_t\Gamma = 0$. By using Lemma 4.2 and Lemma 4.3, we compute

$$\begin{aligned} 0 &= D_sD_t\partial_t\Gamma \\ &= D_tD_s\partial_t\Gamma + R(\partial_s\Gamma, \partial_t\Gamma)\partial_t\Gamma \\ &= D_tD_t\partial_s\Gamma + R(\partial_s\Gamma, \partial_t\Gamma)\partial_t\Gamma\,. \end{aligned} \tag{4.16}$$

Evaluating at $s = 0$, where $\partial_s\Gamma(t, 0) = \eta(t)$ and $\partial_t\Gamma(t, 0) = \gamma'(t)$, we obtain

$$D_t^2\eta = R(\gamma', \eta)\gamma'\,. \tag{4.17}$$

□

Due to this result, the Jacobi equation is also called the *geodesic deviation equation*. Intuitively, if we imagine a geodesic congruence Γ of γ as a one-parameter family of freely falling particles, then the variation field η gives the position, relative to γ, of arbitrarily nearby particles. Hence, the derivative $D_t\eta$ gives relative velocity and $D_t^2\eta$ relative acceleration. That means the Jacobi equation can be interpreted as Newton's second law with the curvature vector $R(\gamma', \eta)\gamma'$ play the role of force, called the tidal force.

4.3 Friedmann-Lemaître-Robertson-Walker Cosmology

Definition 4.8. A *perfect fluid* in spacetime is the triple (u, ρ, p), where u is a unit timelike vector field, called the flow vector field, ρ is the energy density, and p is the pressure. In local coordinates (x^μ), the energy-momentum tensor of the perfect fluid is defined as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,. \tag{4.18}$$

In this form, the perfect fluid is isotropic which is free from shear and viscosity. It follows that the components of the energy-momentum tensor are

$$T_{00} = \rho\,, \qquad T_{ij} = p g_{ij}\,, \quad (i, j = 1, 2, 3). \tag{4.19}$$

Hence, the trace of T is

$$T = 3p - \rho\,. \tag{4.20}$$

If $p = 0$, the perfect fluid is called *dust*, and if $p = \frac{1}{3}\rho$, the perfect fluid is called *radiation*.

Definition 4.9. Let (B, g_B) and (F, g_F) be pseudo-Riemannian manifolds and $f > 0$ be a smooth function on B. The *warped product* $M = B \times_f F$ is the product manifold $B \times F$ equipped with metric tensor

$$g = \mathrm{pr}_1^*(g_B) + (f \circ \mathrm{pr}_1)^2\, \mathrm{pr}_2^*(g_F) \tag{4.21}$$

where $\mathrm{pr}_1 : B \times F \to B$ and $\mathrm{pr}_2 : B \times F \to F$ are projection maps.

Commonly, B is called the base of M, F is the fibre and f is the warping function. The function $\tilde{f} = f \circ \mathrm{pr}_1$ defined on $M = B \times_f F$ is called the lift of the function f to the warped product M. Likewise, if $X \in \mathfrak{X}(B)$, the lift of X to M is the unique vector field $\tilde{X}$ of $\mathfrak{X}(M)$ which is pr_1-related to X. Thus, without confusion, we can always write X to represent both the vector field on the base and its lift.

Definition 4.10. Let I be an open interval of the $\mathbb{R}^1_1$, S be a connected three-dimensional Riemannian manifold with constant curvature $k = -1, 0$, or $+1$, and $a(t) > 0, t \in I$ be a smooth function on I. The warped product $\mathcal{M} = I \times_a S$ is called a *Robertson-Walker (RW) spacetime*.

Thus, a RW spacetime is a manifold $I \times S$ with the line element

$$ds^2 = -dt^2 + a^2(t)d\sigma^2 \tag{4.22}$$

where $d\sigma^2$ is the line element of S lifted to $I \times S$. The value of k gives the spatial curvature of the metric, where $k = 1$ implies that the Universe has positive spatial curvature, $k = -1$ implies that the Universe has negative spatial curvature and $k = 0$ implies that the Universe is spatially flat. In this case, the warping function $a(t)$ is called the *scaling function* and its value $a(t_0)$, for an instant $t_0 \in I$, called the *scale factor*. In addition, let d/dt be the standard vector field on $I \subseteq \mathbb{R}^1_1$ and $u = \partial_t$ be its lifting to $I \times S$. For each $p \in S$, the curve $I \times p$ parametrized by $\gamma_p(t) = (t, p)$ is an observer, called a *fundamental observer*. Hence, the parameter t represent the proper time of fundamental observer, which is usually called the *cosmic time*. It can be easily shown that $\langle u, u \rangle = -1$ and $u \perp S(t)$, for all $t \in I$. That means for each $t \in I$, $S(t)$ is a spacelike hypersurface, called the *spacelike slice*.

A RW spacetime is spatially isotropic and homogeneous. Intuitively, the concept of spatial isotropic said that there are no privileged directions with respect to a point, while the concept of spatial homogeneous means there are no privileged points in the given space. This two results play an important rule in observational motivation of the choice of a RW spacetime for the standard cosmological model. We can express the line element

$d\sigma^2$ of RW spacetime in standard spherical coordinates as

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{4.23}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. By putting $dt = a(t)d\tau$, the RW metric (4.23) can be rewritten to the form

$$ds^2 = a^2(t)\left(-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{4.24}$$

which is clearly conformally flat in the case of $k = 0$. It immediately implies that the Weyl tensor of the RW metric vanishes in the spatially flat case. In fact, the Weyl tensor also vanishes for $k = -1$ and $k = +1$ (Lihoshi et al., 2007). By using the RW metric, the Ricci tensor and Ricci scalar can be easily calculated, and assuming the cosmological background of the Universe as a perfect fluid where the energy momentum tensor is given in (4.18), followed by substituting the values into EFE with the cosmological constant yields the *Friedmann equations*

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{4.25}$$

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6} + \frac{\Lambda}{3}\,. \tag{4.26}$$

The *Hubble parameter H* which measures the rate of expansion of the Universe is a time dependent scalar that is constant in space defined by

$$H = \frac{\dot{a}(t)}{a(t)}\,. \tag{4.27}$$

Given any RW spacetime, $H > 0$ implies an expanding Universe. Let $t = 0$ as the beginning of time for an expanding RW Universe, set $a(0) = a_0$ and t_0 to denote present time, representing the age of the Universe. It is clear that $a(t_0) > 0$ and $\dot{a}(t_0) > 0$ which represents an expansive Universe, but in fact expansion is accelerating which means $\ddot{a}(t_0) > 0$. The *redshift parameter* z which is characterized by the relative difference between the observed and emitted wavelengths of light beam is defined by

$$z = \frac{a(t_0)}{a(t)} - 1 \,. \tag{4.28}$$

The fact that the Universe is expanding implies that $a(t_0) > a(t)$ and so $z > 0$. The *density parameter* expressed as

$$\Omega = \frac{\rho + \Lambda}{3H^2} = 1 + \frac{k}{a^2H^2} \tag{4.29}$$

which is to determines the overall geometry of the Universe. Note that if $k = 0$, then $\Omega = 1$ and the Universe is said to be flat. Based on the observational evidence, the Universe is believed to be nearly flat. That means the Universe should be well approximated by a model where the spatial curvature k is zero. Therefore, with $k = 0$, we obtain the density parameter

$$\Omega = \frac{\rho}{3H_0^2} \tag{4.30}$$

where H_0 represents $H(t_0)$, and the Friedmann equations can be rewritten as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} \tag{4.31}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6} \tag{4.32}$$

where Λ is ignored for comparison in the following sections.

4.4 GDE For FLRW Universe In $f(Q)$-Gravity

Let $(\mathcal{M}, g)$ be a time-oriented Robertson-Walker spacetime and Γ be a one-parameter family of geodesics with the centre geodesic $\Gamma_0 = \gamma$ parametrized by the arc length τ. To simplify the notation, we denote $V(\tau) = \gamma'(\tau)$ the velocity vector field, so $\langle V, V\rangle = V_\alpha V^\alpha = \epsilon$, where $\epsilon = +1, 0, -1$, if the geodesic γ is spacelike, null, or timelike respectively. Furthermore, we restrict the variation field $\eta \in \mathfrak{X}(\gamma)$ such that $\eta(\tau) \perp V(\tau)$ for all τ. Hence, we have $\langle \eta, V\rangle = \eta_\alpha V^\alpha = 0$. We may decompose the vector field V into

$$V = Eu + Pe \tag{4.33}$$

where e is a unit spacelike vector field orthogonal to u, that is, $\langle e, e\rangle = 1$, $\langle u, e\rangle = 0$, and $E = -\langle V, u\rangle$, $P = (\epsilon + E)^{1/2}$. Since V and u might not parallel to each other, so additionally to $\langle \eta, V\rangle = 0$, we also set $\langle \eta, u\rangle = 0$ which means the vector field η lies in the two-dimensional spacelike slice orthogonal to both V and u.

With these setting, we now proceed to derive the GDE in $f(Q)$ gravity. For convenience, the rest of the calculation will be carried out in component form. Recall from Chapter 2, the field equations in $f(Q)$ gravity can be written as

$$f_Q(\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathring{R}) + \frac{1}{2} g_{\mu\nu}(Q f_Q - f) + 2 f_{QQ} \nabla_\lambda Q P^\lambda{}_{\mu\nu} = \kappa T_{\mu\nu}\,. \tag{4.34}$$

Contracting with $g_{\mu\nu}$ and rearrange, we obtain the Ricci scalar

$$\mathring{R} = \frac{1}{f_Q}(2Q f_Q - 2f + 2 f_{QQ} P^{\lambda\rho}{}_\rho \nabla_\lambda Q - \kappa T)\,. \tag{4.35}$$

Inserting this formula back into the field equations, we have the Ricci tensor

$$\mathring{R}_{\mu\nu} = \frac{1}{f_Q}\left[\frac{1}{2}g_{\mu\nu}(Qf_Q - f + 2f_{QQ}P^{\lambda\rho}{}_{\rho}\nabla_\lambda Q - \kappa T) - 2f_{QQ}P^{\lambda}{}_{\mu\nu}\nabla_\lambda Q + \kappa T_{\mu\nu}\right]. \quad (4.36)$$

As mentioned in previous section, RW spacetime is conformally flat and so the Weyl tensor vanishes, that is, $C_{\alpha\beta\gamma\delta} = 0$. Hence, from (2.42) and the above equations, we can express the curvature tensor as

$$\begin{aligned}\mathring{R}_{\alpha\beta\gamma\delta} =&\frac{1}{2f_Q}\Bigg[\kappa(g_{\alpha\gamma}T_{\delta\beta} - g_{\alpha\delta}T_{\gamma\beta} + g_{\beta\delta}T_{\gamma\alpha} - g_{\beta\gamma}T_{\delta\alpha})\\ &+\left(\frac{Qf_Q}{3} - \frac{f}{3} - \frac{2\kappa T}{3} + \frac{4}{3}f_{QQ}P^{\lambda\rho}{}_{\rho}\nabla_\lambda Q\right)(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta})\\ &+(g_{\alpha\gamma}\mathcal{D}_{\delta\beta} - g_{\alpha\delta}\mathcal{D}_{\gamma\beta} + g_{\beta\delta}\mathcal{D}_{\gamma\alpha} - g_{\beta\gamma}\mathcal{D}_{\delta\alpha})f_Q\Bigg] \quad (4.37)\end{aligned}$$

where

$$\mathcal{D}_{\mu\nu} := -2P^{\lambda}{}_{\mu\nu}\nabla_\lambda Q\partial_Q\,. \quad (4.38)$$

By taking the perfect fluid form of the energy-momentum tensor stated in (4.18) and (4.20), the above equation reduces to

$$\begin{aligned}\mathring{R}_{\alpha\beta\gamma\delta} =&\frac{1}{2f_Q}\Bigg[\kappa(\rho + p)(g_{\alpha\gamma}u_\delta u_\beta - g_{\alpha\delta}u_\gamma u_\beta + g_{\beta\delta}u_\gamma u_\alpha - g_{\beta\gamma}u_\delta u_\alpha)\\ &+\left(\frac{Qf_Q}{3} - \frac{f}{3} + \frac{2\kappa\rho}{3} + \frac{4}{3}f_{QQ}P^{\lambda\rho}{}_{\rho}\nabla_\lambda Q\right)(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta})\\ &+(g_{\alpha\gamma}\mathcal{D}_{\delta\beta} - g_{\alpha\delta}\mathcal{D}_{\gamma\beta} + g_{\beta\delta}\mathcal{D}_{\gamma\alpha} - g_{\beta\gamma}\mathcal{D}_{\delta\alpha})f_Q\Bigg]. \quad (4.39)\end{aligned}$$

Contracting with $V^\beta V^\delta$ and since $V_\alpha V^\alpha = \epsilon$, we have

$$\begin{aligned}\mathring{R}_{\alpha\beta\gamma\delta}V^\beta V^\delta =& \frac{1}{2f_Q}\bigg[\kappa(\rho+p)[g_{\alpha\gamma}(u_\beta V^\beta)^2 - 2(u_\beta V^\beta)V_{(\alpha}u_{\gamma)} + \epsilon u_\alpha u_\gamma] \\ &+ \left(\frac{Qf_Q}{3} - \frac{f}{3} + \frac{2\kappa\rho}{3} + \frac{4}{3}f_{QQ}P^{\lambda\rho}{}_\rho \nabla_\lambda Q\right)(\epsilon g_{\alpha\gamma} - V_\alpha V_\gamma) \\ &+ [(g_{\alpha\gamma}\mathcal{D}_{\delta\beta} - g_{\alpha\delta}\mathcal{D}_{\gamma\beta} + g_{\beta\delta}\mathcal{D}_{\gamma\alpha} - g_{\beta\gamma}\mathcal{D}_{\delta\alpha})f_Q]V^\beta V^\delta\bigg]. \end{aligned} \tag{4.40}$$

By raising the α index in the curvature tensor and contracting with η^γ, we obtain

$$\begin{aligned}\mathring{R}^\alpha{}_{\beta\gamma\delta}V^\beta \eta^\gamma V^\delta =& \frac{1}{2f_Q}\bigg[\kappa(\rho+p)[(u_\beta V^\beta)^2\eta^\alpha - (u_\beta V^\beta)V^\alpha(u_\gamma\eta^\gamma) \\ &- (u_\beta V^\beta)u^\alpha(V_\gamma\eta^\gamma) + \epsilon u^\alpha(u_\gamma\eta^\gamma)] \\ &+ \left(\frac{Qf_Q}{3} - \frac{f}{3} + \frac{2\kappa\rho}{3} + \frac{4}{3}f_{QQ}P^{\lambda\rho}_\rho \nabla_\lambda Q\right)(\epsilon\eta^\alpha - V^\alpha(V_\gamma\eta^\gamma)) \\ &+ [(\delta^\alpha_\gamma\mathcal{D}_{\delta\beta} - \delta^\alpha_\delta\mathcal{D}_{\gamma\beta} + g_{\beta\delta}\mathcal{D}^\alpha_\gamma - g_{\beta\gamma}\mathcal{D}^\alpha_\delta)f_Q]V^\beta\eta^\gamma V^\delta\bigg]. \end{aligned} \tag{4.41}$$

Since $V_\alpha u^\alpha = -E$ and $\eta_\alpha u^\alpha = \eta_\alpha V^\alpha = 0$, so

$$\begin{aligned}&\mathring{R}^\alpha{}_{\beta\gamma\delta}V^\beta \eta^\gamma V^\delta \\ &= \frac{1}{2f_Q}\left[\kappa(\rho+p)E^2 + \epsilon\left(\frac{2\kappa\rho}{3} + \frac{Qf_Q}{3} - \frac{f}{3} + \frac{4}{3}f_{QQ}P^{\lambda\rho}{}_\rho\nabla_\lambda Q\right)\right]\eta^\alpha \\ &+ \frac{1}{2f_Q}\left[(\delta^\alpha_\gamma\mathcal{D}_{\delta\beta} - \delta^\alpha_\delta\mathcal{D}_{\gamma\beta} + g_{\beta\delta}\mathcal{D}^\alpha_\gamma - g_{\beta\gamma}\mathcal{D}^\alpha_\delta)f_Q V^\beta V^\delta\right]\eta^\gamma . \end{aligned} \tag{4.42}$$

To further simplify, we have to deal with the $\nabla_\lambda Q$. As shown in (3.46), the non-metricity scalar Q is not in a very simplified form and hence $\nabla_\lambda Q$. Thus, we have to impose a particular coordinate system so that Q can be simplified. But as mentioned in Chapter 3, the field equations are only valid in some specific coordinate systems such that $\Gamma^\lambda_{\mu\nu} = 0$. A simplest example that agree on both is the Cartesian coordinate system. Therefore, we consider the spatially flat RW metric in Cartesian coordinate where the line element can be written as

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\,. \tag{4.43}$$

This implies that the only non-vanishing metric components are

$$g_{tt} = -1, \qquad g_{xx} = g_{yy} = g_{zz} = a^2(t)\,. \tag{4.44}$$

Hence, we obtain

$$Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} = -\nabla_\lambda g_{\mu\nu}\nabla^\lambda g^{\mu\nu} = -12H^2 \tag{4.45}$$

$$Q_{\lambda\mu\nu}Q^{\mu\lambda\nu} = -\nabla_\lambda g_{\mu\nu}\nabla^\mu g^{\lambda\nu} = 0 \tag{4.46}$$

$$Q_\lambda Q^\lambda = (g_{\mu\rho}\nabla_\lambda g^{\mu\rho})(g_{\nu\gamma}\nabla^\lambda g^{\nu\gamma}) = -36H^2 \tag{4.47}$$

$$Q_\lambda \tilde{Q}^\lambda = (g_{\mu\rho}\nabla_\lambda g^{\mu\rho})(\nabla_\nu g^{\lambda\nu}) = 0\,. \tag{4.48}$$

It follows from (3.46), we have

$$Q = -\frac{1}{4}(-12H^2) + \frac{1}{4}(-36H^2) = -6H^2\,. \tag{4.49}$$

Therefore, the scalar Q is only time-dependent and consequently

$$\nabla_\lambda Q = 12H\dot{H}u_\lambda\,. \tag{4.50}$$

After a cumbersome calculation as shown in Appendices, we found that

$$\frac{1}{2f_Q}\left[(\delta^\alpha_\gamma\mathcal{D}_{\delta\beta}-\delta^\alpha_\delta\mathcal{D}_{\gamma\beta}+g_{\beta\delta}\mathcal{D}^\alpha_\gamma-g_{\beta\gamma}\mathcal{D}^\alpha_\delta)f_QV^\beta V^\delta\right]\eta^\gamma = \frac{1}{2f_Q}[-24H^2\dot{H}f_{QQ}(2\epsilon+E^2)]\eta^\alpha$$

(4.51)

and

$$\frac{4}{3}\eta^{\alpha} f_{QQ} P^{\lambda\nu}{}_{\nu} \nabla_{\lambda} Q = 48 H^2 \dot{H} f_{QQ} \epsilon \eta^{\alpha} \,. \tag{4.52}$$

Thus, (4.42) reduces to

$$\mathring{R}^{\alpha}{}_{\beta\gamma\delta} V^{\beta} \eta^{\gamma} V^{\delta} = \frac{1}{2 f_Q}\left[(\kappa\rho + \kappa p - 24 H^2 \dot{H} f_{QQ}) E^2 + \left(\frac{2\kappa\rho}{3} + \frac{Q f_Q}{3} - \frac{f}{3} \right) \epsilon \right] \eta^{\alpha} \tag{4.53}$$

which is considered to be the generalized Pirani equation. Finally, the GDE in $f(Q)$ gravity can be written as

$$\frac{D^2 \eta^{\alpha}}{D\tau^2} = -\frac{1}{2 f_Q}\left[(\kappa\rho + \kappa p - 24 H^2 \dot{H} f_{QQ}) E^2 + \left(\frac{2\kappa\rho}{3} + \frac{Q f_Q}{3} - \frac{f}{3} \right) \epsilon \right] \eta^{\alpha} \,. \tag{4.54}$$

Notice that in this GDE only the magnitude of the deviation vector η^{α} is changed along the geodesics, which reflects the homogeneity and isotropy of the FLRW universe.

4.5 Generalized Friedmann equations

Given the previous construction of FLRW cosmology using EFE, the next logical step is to consider deriving a similar equation using the $f(Q)$ field equations presented in (3.56). Consider the previously defined spatially flat RW metric in (4.43) and the non-vanishing metric components in (4.44), the only non-vanishing Christoffel symbols are

$$\mathring{\Gamma}^{0}_{ii} = a\dot{a} \,, \qquad \mathring{\Gamma}^{i}_{0i} = \frac{\dot{a}}{a} = \mathring{\Gamma}^{i}_{i0} \,. \tag{4.55}$$

Consequently, the (00) and (ii) components of the Ricci tensor are

$$\mathring{R}_{00} = -3\frac{\ddot{a}}{a} \tag{4.56}$$

$$\mathring{R}_{ii} = a\ddot{a} + 2\dot{a}^2 \tag{4.57}$$

and so the Ricci scalar

$$\mathring{R} = 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right). \tag{4.58}$$

Substitute the above results of the Ricci tensor and Ricci scalar into the field equation (3.56) while considering the perfect fluid (4.18), followed by some lengthy rearrangement, we obtain the *generalized Friemann equations* as

$$3H^2 = \frac{1}{f_Q}\left[\kappa\rho + \frac{1}{2}(Qf_Q - f)\right] \tag{4.59}$$

$$2\dot{H} + 3H^2 = -\frac{1}{f_Q}\left[\kappa p - \frac{1}{2}(Qf_Q - f) - 24 f_{QQ} H^2 \dot{H}\right]. \tag{4.60}$$

In both the case of GR and $f(Q)$, the conservation of energy-momentum applies which means that $f(Q)$ gravity is only a modification of the gravitational theory rather than an alteration to the matter content. Nevertheless, note that the generalized Friedmann equations are far more complex than the GR case, which implies that there are much fewer exact cosmological solutions.

4.6 GDE for fundamental observers

In this situation, we have $V^\alpha = u^\alpha$ for the centre geodesic. That means the arclength parameter coincides with the proper time of the centre fundamental observer, that is, $\tau = t$.

Therefore,

$$\epsilon = V_\alpha V^\alpha = u_\alpha u^\alpha = -1 \tag{4.61}$$

$$E = -V_\alpha u^\alpha = -u_\alpha u^\alpha = 1\,. \tag{4.62}$$

Inserting these two values into (4.53), we obtain

$$\mathring{R}^\alpha_{\beta\gamma\delta} u^\beta \eta^\gamma u^\delta = \frac{1}{f_Q}\left(\frac{\kappa\rho}{6} + \frac{\kappa p}{2} - \frac{Q f_Q}{6} + \frac{f}{6} - 12H^2 \dot{H} f_{QQ}\right)\eta^\alpha\,. \tag{4.63}$$

If we set the variation field $\eta^\alpha = l e^\alpha$, where e^α is parallel transported along t, then

$$\frac{De^\alpha}{Dt} = 0 \tag{4.64}$$

and so

$$\frac{D^2\eta^\alpha}{Dt^2} = \frac{d^2 l}{dt^2} e^\alpha\,. \tag{4.65}$$

Thus, by using (4.63) and (4.65), we get

$$\frac{d^2 l}{dt^2} = -\frac{1}{f_Q}\left(\frac{\kappa\rho}{6} + \frac{\kappa p}{2} - \frac{Q f_Q}{6} + \frac{f}{6} - 12H^2 \dot{H} f_{QQ}\right) l\,. \tag{4.66}$$

By letting $l = a(t)$, we have

$$\frac{\ddot{a}}{a} = -\frac{1}{f_Q}\left(\frac{\kappa\rho}{6} + \frac{\kappa p}{2} - \frac{Q f_Q}{6} + \frac{f}{6} - 12H^2 \dot{H} f_{QQ}\right). \tag{4.67}$$

This equation can be considered as a special case of the generalized Raychaudhuri equation.

4.7 GDE for past-directed null vector fields

Under these circumstances, we have $V^{\alpha} = k^{\alpha}$, where k is a past-directed null vector fields, $k_{\alpha}k^{\alpha} = 0$ and $k^0 < 0$. This implies that

$$\epsilon = V_{\alpha}V^{\alpha} = k_{\alpha}k^{\alpha} = 0. \tag{4.68}$$

Hence, (4.53) becomes

$$\mathring{R}^{\alpha}{}_{\beta\gamma\delta}k^{\beta}\eta^{\gamma}k^{\delta} = \frac{1}{2f_Q}(\kappa\rho + \kappa p - 24H^2\dot{H}f_{QQ})E^2\eta^{\alpha}\,. \tag{4.69}$$

This equation can be explained as the Ricci focusing in $f(Q)$ gravity. If we consider $\eta^{\alpha} = \eta e^{\alpha}$, $e_{\alpha}e^{\alpha} = 1$, $e_{\alpha}u^{\alpha} = e_{\alpha}k^{\alpha} = 0$ and $D_{\tau}e^{\alpha} = k^{\beta}\nabla_{\beta}e^{\alpha} = 0$, in which e^{α} is parallel transported and orthogonal to u^{α} and k^{α}, then the GDE can be written in a new form

$$\frac{d^2\eta}{d\tau^2} = -\frac{1}{2f_Q}(\kappa\rho + \kappa p - 24H^2\dot{H}f_{QQ})E^2\eta\,. \tag{4.70}$$

As in the case of GR in Ellis and Elst (1997), all past-directed null geodesics experience focusing if $\kappa(\rho + p) > 0$ except the special case with the equation of state $p = -\rho$. Thus, it is clear that (4.70) indicates the focusing condition for the $f(Q)$-gravity, which is

$$\frac{\kappa(\rho + p)}{f_Q} > \frac{24H^2\dot{H}f_{QQ}}{f_Q}\,. \tag{4.71}$$

When involving dynamical problem, we have to express the quantities in (4.70) in term of the redshift parameter z, defined in (4.28). First, we write

$$\frac{d}{d\tau} = \frac{dz}{d\tau}\frac{d}{dz} \tag{4.72}$$

which implies that

$$\begin{aligned}\frac{d^2}{d\tau^2} &= \frac{dz}{d\tau}\frac{d}{dz}\left(\frac{d}{d\tau}\right) \\ &= \left(\frac{d\tau}{dz}\right)^{-2}\left[-\left(\frac{d\tau}{dz}\right)^{-1}\frac{d^2\tau}{dz^2}\frac{d}{dz}+\frac{d^2}{dz^2}\right]. \end{aligned} \tag{4.73}$$

For the null geodesics, we have

$$(1+z) = \frac{a_0}{a} = \frac{E}{E_0} \tag{4.74}$$

which implies that

$$\frac{dz}{1+z} = -\frac{da}{a} \tag{4.75}$$

where $a_0 = a(t_0)$ the present value of the scale factor. For the past-directed case, we set $E_0 = -1$, so

$$dz = -(1+z)\frac{1}{a}\frac{da}{d\tau}d\tau = -(1+z)\frac{\dot{a}}{a}E\,d\tau = H(1+z)^2 d\tau \tag{4.76}$$

and so

$$\frac{d\tau}{dz} = \frac{1}{H(1+z)^2}\,. \tag{4.77}$$

Consequently,

$$\frac{d^2\tau}{dz^2} = -\frac{1}{H(1+z)^3}\left[\frac{1}{H}(1+z)\frac{dH}{dz}+2\right] \tag{4.78}$$

where

$$\frac{dH}{dz} = \frac{d\tau}{dz}\frac{dt}{d\tau}\frac{dH}{dt} = -\frac{1}{H(1+z)}\frac{dH}{dt} \tag{4.79}$$

and we make use of $\frac{dt}{d\tau} = E = -(1+z)$. From (4.27), we get

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 \tag{4.80}$$

and so

$$\dot{H} = -\frac{1}{f'}\left(\frac{\kappa\rho}{6} + \frac{\kappa p}{2} - \frac{Qf_Q}{6} + \frac{f}{6} - 12H^2\dot{H}f_{QQ}\right) - H^2 . \tag{4.81}$$

Hence,

$$\frac{d^2\tau}{dz^2} = -\frac{3}{H(1+z)^3}\left[1 + \frac{1}{3H^2 f_Q}\left(\frac{\kappa\rho}{6} + \frac{\kappa p}{2} - \frac{Qf_Q}{6} + \frac{f}{6} - 12H^2\dot{H}f_{QQ}\right)\right]. \tag{4.82}$$

Putting this equation in (4.73), we have

$$\begin{aligned}\frac{d^2\eta}{d\tau^2} = (H(1+z)^2)^2\Bigg[\frac{d^2\eta}{dz^2} + \frac{3}{(1+z)}\Bigg[1 + \frac{1}{3H^2 f_Q}\Bigg(\frac{\kappa\rho}{6} + \frac{\kappa p}{2} \\ - \frac{Qf_Q}{6} + \frac{f}{6} - 12H^2\dot{H}f_{QQ}\Bigg)\Bigg]\frac{d\eta}{dz}\Bigg].\end{aligned} \tag{4.83}$$

Finally, by using (4.70), the null GDE can be written in the form

$$\begin{aligned}\frac{d^2\eta}{dz^2} + \frac{3}{(1+z)}\left[1 + \frac{1}{3H^2 f_Q}\left(\frac{\kappa\rho}{6} + \frac{\kappa p}{2} - \frac{Qf_Q}{6} + \frac{f}{6} - 12H^2\dot{H}f_{QQ}\right)\right]\frac{d\eta}{dz} \\ + \frac{\kappa(\rho+p) - 24H^2\dot{H}f_{QQ}}{2H^2(1+z)^2 f_Q}\eta = 0 .\end{aligned} \tag{4.84}$$

The above equation is useful for cosmological applications. Assume the matter content of the universe is dust and radiation field, so the p and ρ can be be expressed as

$$\kappa p = H_0^2 \Omega_{r_0}(1+z)^4, \qquad \kappa\rho = 3H_0^2\Omega_{m_0}(1+z)^3 + 3H_0^2\Omega_{r_0}(1+z)^4 \tag{4.85}$$

where $p_m = 0$ and $p_r = \frac{1}{3}\rho_r$. From (4.85) , we could express H^2 as

$$H^2 = \frac{H_0^2}{f_Q}[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_{DE}] \tag{4.86}$$

where

$$\Omega_{DE} := \frac{1}{H_0^2}\left(\frac{Qf_Q}{6} - \frac{f}{6}\right) \tag{4.87}$$

is the Dark Energy parameter. Therefore, by applying (4.85) and (4.86), the null GDE in (4.84) can be written as

$$\frac{d^2\eta}{dz^2} + \mathcal{P}(H,\dot{H},z)\frac{d\eta}{dz} + \mathcal{Q}(H,\dot{H},z)\eta = 0 \tag{4.88}$$

where

$$\begin{aligned}\mathcal{P}(H,\dot{H},z) =& \frac{\frac{7}{2}\Omega_{m_0}(1+z)^3 + 4\Omega_{r_0}(1+z)^4 + 2\Omega_{DE}}{(1+z)[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_{DE}]} \\ &- \frac{\frac{12\dot{H}f''}{f'}[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_{DE}]}{(1+z)[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_{DE}]}\end{aligned} \tag{4.89}$$

$$\begin{aligned}\mathcal{Q}(H,\dot{H},z) =& \frac{3\Omega_{m_0}(1+z)^3 + 4\Omega_{r_0}(1+z)^4}{2(1+z)^2[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_{DE}]} \\ &- \frac{\frac{24\dot{H}f''}{f'}[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_{DE}]}{2(1+z)^2[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_{DE}]}\,.\end{aligned} \tag{4.90}$$

In a particular case, where $f(Q) = Q - 2\Lambda$, so $f_Q = 1$ and $f_{QQ} = 0$. Thus, Ω_{DE} in (4.87) reduces to

$$\Omega_{DE} = \frac{1}{H_0^2}\left(\frac{Q}{6} - \frac{Q-2\Lambda}{6}\right) = \frac{\Lambda}{3H_0^2} =: \Omega_\Lambda \,. \tag{4.91}$$

This implies that the H^2 in (4.86) becomes the same as the case in GR

$$H^2 = H_0^2[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_\Lambda] \tag{4.92}$$

which confirms the obtained results. Moreover, $\mathcal{P}$ (4.89) and $\mathcal{Q}$ (4.90) turns into

$$\mathcal{P}(z) = \frac{\frac{7}{2}\Omega_{m_0}(1+z)^3 + 4\Omega_{r_0}(1+z)^4 + 2\Omega_\Lambda}{(1+z)[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_\Lambda]} \tag{4.93}$$

$$\mathcal{Q}(z) = \frac{3\Omega_{m_0}(1+z) + 4\Omega_{r_0}(1+z)^2}{2[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_\Lambda]}. \tag{4.94}$$

Then, the GDE for null vector fields becomes

$$\begin{aligned}\frac{d^2\eta}{dz^2} &+ \frac{\frac{7}{2}\Omega_{m_0}(1+z)^3 + 4\Omega_{r_0}(1+z)^4 + 2\Omega_\Lambda}{(1+z)[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_\Lambda]}\frac{d\eta}{dz} \\ &+ \frac{3\Omega_{m_0}(1+z) + 4\Omega_{r_0}(1+z)^2}{2[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4 + \Omega_\Lambda]}\eta = 0\,.\end{aligned} \tag{4.95}$$

We set $\Omega_\Lambda = 0$ and $\Omega_{m_0} + \Omega_{r_0} = 1$ for the original Mattig relation, so we have

$$\begin{aligned}\frac{d^2\eta}{dz^2} &+ \frac{\frac{7}{2}\Omega_{m_0}(1+z)^3 + 4\Omega_{r_0}(1+z)^4}{(1+z)[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4]}\frac{d\eta}{dz} \\ &+ \frac{3\Omega_{m_0}(1+z) + 4\Omega_{r_0}(1+z)^2}{2[\Omega_{m_0}(1+z)^3 + \Omega_{r_0}(1+z)^4]}\eta = 0\,.\end{aligned} \tag{4.96}$$

This implies that (4.88) is the generalized Mattig relation in $f(Q)$ gravity. In FLRW universe, the angular diametral distance $D_A(z)$ is given by

$$D_A(z) = \sqrt{\left|\frac{dA(z)}{d\Omega}\right|} \tag{4.97}$$

where dA is the area of the object and $d\Omega$ is the solid angle. Thus, from (4.88), the GDE in terms of the angular diametral distance is

$$\frac{d^2 D_A}{dz^2} + \mathcal{P}(H, \dot{H}, z)\frac{dD_A}{dz} + \mathcal{Q}(H, \dot{H}, z) D_A = 0 \tag{4.98}$$

where $\mathcal{P}$ and $\mathcal{Q}$ is given in (4.89) and (4.90). This equation satisfies the initial conditions (for $z \geq z_0$)

$$D_A(z)|_{z=z_0} = 0 \tag{4.99}$$

$$\frac{dD_A}{dz}(z)|_{z=z_0} = \frac{H_0}{H(z_0)(1+z_0)} \tag{4.100}$$

where $H(z_0)$ is the modified Friedmann equation (4.86) at $z = z_0$.

CHAPTER 5: CONCLUSION

In this dissertation, we have provided the differential geometric constructions of Einstein's general relativity. We also established an equivalence between the general relativity and symmetry teleparallel gravity. We have shown the $f(Q)$ gravity field equations in a form analogous to the EFE. The $f(Q)$ field equations is then applied to calculate the Ricci tensor and Ricci scalar. By using the coincident gauge, the GDE in $f(Q)$ gravity with FLRW cosmology was shown. Moreover, the Friedmann equations were modified using the $f(Q)$ field equations yielding a generalized Friedmann equations describing a spatially flat Universe. Furthermore, we have focused on two particular cases, the GDE for fundamental observers and the past-directed null vector fields in FLRW universe. Within these cases, we have obtained the generalized Raychaudhuri equation, the generalized Mattig relation, and the diametric angular distance differential for $f(Q)$ gravity theory. In addition, the focusing condition for past-directed null geodesics for $f(Q)$ gravity is investigated. Though we have been quite optimistic, there are many more viabilities and obstacles that were not reviewed here. Thus, future work would include replacing $f(Q)$ with $f(Q,T)$ gravity theory, where the T represents the trace of the energy-momentum tensor, or considering the cosmology in anisotropic universe, often called the Bianchi models.

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