A TWO-WAREHOUSE INVENTORY MODEL WITH REWORK PROCESS AND TIME-VARYING DEMAND

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A TWO-WAREHOUSE INVENTORY MODEL WITH REWORK PROCESS AND TIME-VARYING DEMAND

ABSTRACT

It is commonly assumed that in a classical inventory model, all items produced during a production cycle are of perfect items. However, contradictory to reality, production processes are not perfect at all times, hence resulting in the production of defectives. In order to minimise the cost incurred and reduce wastage in a production process, we might consider a rework process to be implemented on the defective items. Besides that, most classical inventory models also consider the assumption that the available storage facility has an unlimited capacity. Conversely, in reality an additional storage space commonly known as a rented warehouse (RW) is needed to store excessive inventory. Another common assumption is that the demand rate is constant. However, contrary to reality the demand rate is known to fluctuate due to several factors such as trends and seasons. Therefore, in order to take into consideration all of the aforementioned common assumptions, we proposed a two-warehouse inventory model with deteriorating items and rework process with time varying demand rate. We also considered the Last-In-First-Out (LIFO) and the First-In-First Out (FIFO) policies with the assumption that the holding cost is higher in the rented warehouse as compared to the owned warehouse. The objective of the proposed model is to determine the optimum values of the time period in a cycle of stage i that will minimise the total relevant cost, TRC of a production cycle. This dissertation is divided into several sections, where we will discuss the mathematical formulation of the model followed by numerical example and a sensitivity analysis to illustrate the derived results.

Keywords: Two-warehouse, Rework Process, LIFO Policy, Time-varying Demand, Deterioration.

MODEL INVENTORI DUA GUDANG BERSAMA PROSES KERJA SEMULA DENGAN KADAR PERMINTAAN BERUBAH

ABSTRAK

Adalah suatu andaian am di mana dalam sebuah model inventori klasik, semua item yang dihasilkan dalam suatu kitaran pengeluaran adalah sempurna. Walau bagaimanapun, bertentangan dengan realiti, proses pengeluaran adalah tidak sempurna pada setiap masa, justeru menyebabkan penghasilan barangan rosak. Bagi meminimumkan kos tanggungan dan mengurangkan pembaziran dalam suatu proses pengeluaran, kita boleh mengambil kira satu proses kerja semula untuk dilaksanakan ke atas barangan yang rosak. Selain itu, kebanyakan model inventori klasik juga mengambil kira andaian di mana gudang yang sedia ada mempunyai kapasiti tanpa had. Sebaliknya, secara realiti suatu ruang penyimpanan tambahan secara umum dikenali sebagai gudang yang disewa (RW), adalah diperlukan bagi menyimpan inventori berlebihan. Suatu andaian umum lagi ialah kadar permintaan adalah malar. Walau bagaimanapun, secara realitinya, kadar permintaan diketahui tidak sekata atas beberapa faktor seperti trend dan musim. Oleh itu, bagi mengambil kira kesemua andaian am yang disebut, kami mencadangkan suatu inventori model dua gudang bersama barangan yang merosot dan proses kerja semula dengan kadar permintaan berubah dengan masa. Kami juga mengambil kira polisi *Last-In-First-Out* (LIFO) dan polisi *First-In-First-Out* (FIFO) atas andaian di mana kos pemegangan di gudang yang disewa adalah lebih tinggi berbanding dengan di gudang yang dimiliki. Objektif model cadangan ini adalah untuk mengenal pasti nilai optimum masa dalam suatu kitaran pada peringkat i yang dapat meminimumkan jumlah kos relevan, TRC dalam satu kitaran proses pengeluaran. Disertasi ini dibahagikan kepada beberapa bahagian, di mana kami akan membincangkan formulasi matematik model ini diikuti oleh contoh berangka dan analisis sensitiviti untuk mengilustrasikan keputusan-keputusan yang diperoleh.

Kata kunci: Dua-gudang; Proses Kerja Semula, Polisi LIFO, Permintaan Berubah Dengan Masa: Kemerosotan.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF APPENDICES

CHAPTER 1: INTRODUCTION

1.1 Background of Study

Producers or manufacturers involved in a production line pay close attention in production planning as it is an important key to minimising their costs and maximising their profits. There are several factors to be considered in a production process.

For instance, the number of optimal items to be produced, the storage space to store the products, managing demand and supply, handling defectives and unrepairable items and many other factors which lead to the profit gain or loss in a business.

Let?s begin by looking at one of the essential parts in a production line, which is storage space. Prior to being distributed to the market, produced items need to be stored in a storage facility known as a warehouse. In most cases, it is more economical for producers or manufacturers to possess their own warehouses, hence the name owned warehouse.

However due to certain circumstances, the storage space in an existing warehouse may be insufficient due to limited capacity. In this case, a separate warehouse can be rented, which can be referred as a rented warehouse. The difference in the quality or the facilities offered by the two types of warehouses results in the difference of the costs in storing items in those spaces. This cost is known as the holding cost.

The main objective of a business is to always minimise cost and maximise profit. Hence, it is vital to determine the right policy in which items are being stored and distributed. In terms of ensuring the cost is minimised while keeping products in a storage, a manufacturer would choose to store items longer in the warehouse which holds a lower holding cost.

This phenomenon can be determined by utilising policies known as the Last-In-First-Out (LIFO) policy and First-In-First-Out (FIFO) policy. As its name would suggest, in a LIFO policy, items which are stored last would be exhausted or utilised first. Conversely, items which are stored first, would be exhausted first in a FIFO policy.

The decision-making process in choosing the best policy that suits a manufacturer's business strategy would be contributed by several factors such as storage cost, the type of produced items and the perishability rate of items. We can see that this approach has been considered by several researches such as Yadav & Swami (2013) whom exercised the FIFO policy in their model and Lee (2006) whom considered both LIFO and FIFO policies in their model.

Another important aspect of this study would be the presence of defective items. Realistically, produced items are not always 100% perfect. Producing items which are imperfect or defective is inevitable due to several reasons such as machine and human errors. Hence in order to reduce costs, some manufacturers may consider to repair or put the defectives under a rework process instead of disposing or discarding the items as it is more cost-effective.

Lastly, it is also vital to understand that most items deteriorate with time. Products such as blood and consumable items like food and medicine, have an estimated shelf life, which means that the quality will reduce over time and will eventually expire. This is commonly labelled on the products in the market as expiry date.

1.2 Problem Statement

In most existing inventory models, a two-warehouse model and the incorporation of LIFO and FIFO policies were usually considered separately. In addition, a rework process is also commonly considered in a single warehouse model.

Considering the gaps within the area of studying the aforementioned factors simultaneously and in reference to the model developed by Lee (2006), the objective of this study is to develop a similar two-warehouse model by incorporating the LIFO and FIFO policies while considering a rework process in our study.

The first approach to our study is to consider an increasing demand rate instead of the commonly used constant demand rate. This approach would allow inventory operators to plan their production accordingly when launching new items into the market. Following the current trend, it is common that a newly launched product such as cosmetics, fashion items and mobile phones will experience a linearly increasing demand rate at the beginning of the launching period to a certain extent.

The second dilemma we have encountered is that some of the researches have only considered a perfect production process. In other words, the presence of defective items is neglected. Hence, we have attempted to include a more realistic condition in which the production process is imperfect, hence producing defective items. In order to reduce the total relevant cost of the inventory model, a rework process is introduced in this study.

In addition, we have also proposed to separate perfect items from the defectives and assumed that the items undergo rework process only in the rented warehouse, once the production period has ended. This would be beneficial and convenient to manufacturers who have limited number of machines as they are able to focus on the production process first and the rework process later.

The final problem that motivated this study would be the assumption that a storage facility or warehouse has an unlimited capacity. This is unrealistic as a storage space would be quickly filled up during an ongoing production process. Hence, we have included a more realistic approach in which the first warehouse labelled as the owned warehouse would have a limited capacity. This will allow excess items to be stored in a second warehouse known as the rented warehouse, once the owned warehouse has reached its maximum capacity.

1.3 Objective of Study

In view of the aforementioned factors above, this study aims to achieve the following objectives:

1. To develop a two-warehouse inventory model with deteriorating items and rework process with time varying demand rate.
2. To incorporate the Last-In-First-Out (LIFO) and the First-In-First Out (FIFO) policies with the assumption that the holding cost is higher in the rented warehouse as compared to the owned warehouse.
3. To determine the optimum values of the time period in a cycle of stage i that will minimise the total relevant cost, TRC of a production cycle and to compare the values of TRC between LIFO and FIFO policies.

1.4 Limitation of Study

There are limitations in our study. The effectiveness of our proposed model has yet to be tested and confirmed in real production processes.

The demand function is limited to a linearly increasing demand rate on the basis that it would be appropriate for items which are newly launched into the market. Changing the demand function to a quadratic, exponential or other functions would mean that changes need to be made to the derivation of the total relevant cost.

In addition, limiting the rented warehouse to only storing the reworked items may dampen the optimisation of the storage space. More items can be produced if the rented warehouse were assumed to store non-defective items until it reaches its maximum capacity.

The limitations aforementioned may be the motivating factor for further research in developing models incorporating these factors.

1.5 Significance of Study

The following are the significance of the study to the producers or manufacturers in the production industry:

1. The research provides comparison between the Last-In-First-Out and First-In-First-Out policies which can be implemented in a production line depending on the specific type of items produced.
2. The research incorporated the rework process which is one of the approach that can be undertaken to minimise the total cost in a production process.

1.6 Thesis Organisation

This thesis consists of five chapters where Chapter 1 provides the introduction and overview of the study. Problem statements, research objectives and the significance of the study are also discussed in this chapter.

Following the introduction, Chapter 2 provides a comprehensive overview of the literature review, listing the past research papers related to the study. The papers are relevant to the study in which they incorporate the LIFO or FIFO policy, a rework process and two-warehouse inventory models.

Chapter 3 presents the model which incorporates the LIFO policy. An introduction of the model is discussed followed by the mathematical formulations section. Notations and assumptions for this model is listed under this section accordingly. Next, a detailed mathematical formulation derived from the equations governing each curve in the graph are presented. Finally, the equation of the total relevant cost, TRC is obtained and presented.

The next section discusses the numerical examples, solution procedure and the sensitivity analysis of the model. The derived results and the effect of changes in parameters are studied and illustrated in table and graph forms. The derived total relevant cost of the production cycle, TRC is further supported using the Microsoft Excel Solver and Mathematica by achieving an optimal unique solution for the TRC equation.

In this research, we have chosen the generalized reduced gradient search as the solving method. Conclusion of the research of the model is presented at the end of this chapter.

In chapter 4, the FIFO policy is introduced where the sequence in this chapter follows the layout of chapter 3. The comparison between both LIFO and FIFO policies is presented in this chapter as well.

Finally, Chapter 5 presents the conclusion of the study and provides recommendations of future research relevant to this study.

CHAPTER 2: LITERATURE REVIEW

Many researches have been carried out, where two warehouses inventory models and rework process were considered separately. Only a few researchers had incorporated both factors simultaneously in the same model.

We will now discuss some of the proposed models by other researches relative to our research. Lee (2006) modified Pakkala & Achary's (1992) two-warehouse LIFO model and proposed a FIFO dispatching two-warehouse model with deterioration and a constant demand rate. Inventory items that were stored first in the OW will be consumed before those in the RW. They proved that when deterioration rates in both OW and RW are the same, FIFO would be less expensive than LIFO provided that the holding cost in the RW is lower than in the OW. The modified model is also proven to have a lower cost than the aforementioned model when the deteriorating rate in OW is significantly less than the rate in the RW provided that the holding cost in the OW is less than that in the RW. They concluded that the deterioration rates and holding costs are the main keys in choosing between the LIFO or FIFO policy.

Wee & Widyadana (2012) developed an Economic Production Quantity (EPQ) model for deteriorating items with rework and stochastic preventive maintenance time. They considered the Last-In-First-Out (LIFO) policy, hence serviceable items during the rework up time are consumed first followed by the serviceable items during the production up time. The rework process is carried out in RW and they also considered lost sales in their paper. They also considered two distributions of the probability of machine preventive maintenance time namely, uniform and exponential distribution.

Wee & Widyadana (2013) developed a production model using the First-In-First-Out (FIFO) rule for deteriorating items with stochastic preventive maintenance time and re-work. They assumed that the deterioration rates for both serviceable and recoverable items to be the same. They also showed two different cases of maintenance time, namely uniform distribution and exponential distribution. Due to the fact that machine unavailability time is stochastic, analytical solution is complicated to be used. Hence they approached the problem using a simple search procedure to solve their model. The aforementioned papers discussed LIFO and/or FIFO policy in their papers.

The LIFO and FIFO policy could be applied by manufacturers depending on the type of items they produce. For instance, manufacturers who deal with perishable items would rather opt for the FIFO policy than LIFO as it is a common practice to ensure that items are dispatched while they are at its optimum condition or freshness.

Alamri & Syntetos (2018) proposed a new policy entitled Allocation-In-Fraction-Out (AIFO). Unlike LIFO and FIFO, AIFO implies simultaneous consumption fractions associated with RW and OW. The goods at both warehouses experience simultaneous consumption fractions, which indicate that the goods are depleted by the end of the same cycle. They developed and compared three general two-warehouse inventory models for items that are subject to inspection for imperfect quality. They demonstrated that the LIFO and FIFO policy might not be suitable for perishable products. They also illustrated the impact of considering different transportation costs associated with the two warehouses and the incorporation of varying demand, deterioration, defective and screening rates on the optimal order quantity.

Panda et el. (2012) proposed a two-warehouse inventory model for deteriorating items where the demand rate is assumed to be an exponential function of time. Items are transferred from RW to OW in a continuous release pattern. They assumed that both OW and RW have different deterioration rates, in which RW is assumed to have a better preserving facility.

Yadav & Swami (2013) developed a two-warehouse inventory model for decaying items. The demand is assumed to be exponentially increasing with time. They also considered different time varying holding costs for both warehouses. Shortages are allowed in this model, where the backlogging rate of unfulfilled demand is assumed to be a decreasing function of the waiting time. Through numerical example, they concluded that the "with-shortage" case has a higher optimum average profit compared to the "without-shortage" case. Therefore, the "with-shortage" model is considered to be better economically. The time horizon of this inventory model is assumed to be infinite.

Pakkala & Achary (1991) developed a two-warehouse inventory model for deteriorating items with the assumption that the demand follows a probabilistic function. The production rate is assumed to be infinite. The authors also considered a single-warehouse system with deteriorating items and a two-warehouse system without deteriorating items. Shortages are allowed and backlogged in this model. They concluded that the single-warehouse model with finite storage capacity is appropriate for lower shortage costs. However, for the two-warehouse model, higher shortage costs results in a lower expected total cost.

Benkherouf (1997) worked on relaxing the assumptions made by Sarma (1987) where Sarma assumed that their model is a two warehouse inventory model with fixed cycle length and the quantity to be stocked in OW is known. It is assumed that RW offers better preserving facilities than the OW, hence resulting in a lower rate of deterioration and higher holding cost in RW. Benkherouf suggested an arbitrary demand rate function where the cycles are assumed to form a regenerative process. The author then provided the numerical example of three different types of demand rates namely constant demand, linear demand and exponential demand rates. The four aforementioned papers, permits shortages where the shortages are backlogged.

Agrawal & Banerjee (2011) were the first to propose a two-warehouse inventory model where the demand is assumed to follow a general ramp-type function of time. Aside from the two-warehouse system, the authors also considered the single warehouse system and an alternative system where they assumed that OW is filled to its maximum capacity. They concluded that the decision on whether to use a single warehouse, a single warehouse filled to its maximum capacity or the usage of two warehouses is dependent on the relative values of shortage costs.

Agrawal et el. (2013) developed a two-warehouse inventory model by extending the work of Agrawal & Banerjee (2011). They considered the existence of deteriorating items and provided options in choosing a single or two-warehouse system. The demand rate is assumed to be a general ramp-type function of time. They concluded that the optimal solution for single warehouse system is independent of the form of demand function, while the two-warehouses is dependent. The authors also concluded that the cost acquired at OW due to high deterioration rate could be balanced out by purchasing more items to be stored in RW, hence reducing the shortage cost.

Bhunia et el. (2014) developed a two-warehouse inventory model for a single deteriorating item. They assumed that the demand is constant and shortages are allowed and partially backlogged with a rate dependent on the duration of waiting time up to the arrival of next lot. They also considered delay payment and lost of sales in their paper. They concluded that an increment in the average profit would result from an increment in the OW capacity as well as in the order quantity. It is assumed that the planning horizon is infinite in this model. The three models aforementioned allow shortages and are partially backlogged.

Rong et el. (2008) were the first to introduce fuzzy lead time in a two-warehouse inventory model for a single deteriorating item with infinite time horizon. Shortages are allowed in this paper where two models were considered, namely, Model 1 (Model for partially backlogged shortages) and Model 2 (Model for fully backlogged shortages). The demand rate is assumed to be price dependent. Inventories are transferred from RW to OW in a bulk release pattern and the holding cost at RW is assumed to be dependent on the distance between both the warehouses.

The approach of the aforementioned papers which consider shortage in their models is more realistic. Shortages occur when the demand for a certain item is greater than the supply in the market. This may occur due to an increase in demand or a decrease in supply or production.

Lee & Ma (2000) developed an optimal inventory policy for a two-warehouse inventory model. They considered a constant planning horizon in a continuous release pattern along with free form time-dependent demand function and deteriorating items to improve models before theirs. They reconsidered the assumption of regenerated cycles of identical cycle length being treated as a decision variable. They assumed that both warehouses hold different deterioration rates. It is also assumed that all items are to be distributed directly from each warehouse where they are stored. From their numerical analysis, they found that the ordering cost is a critical factor in deciding whether, when and how much RW is needed during the planning horizon.

Lee & Hsu (2009) extended the model of Lee & Ma (2000) to the finite replenishment rate condition. They suggested a two-warehouse inventory model for deteriorating items with a free form time-dependent demand over a finite planning horizon. They also developed an algorithm with an approach that allows variation in production cycle times to determine the number of production cycles and the times for replenishment. The results indicated that the performance of their model is greater than the heuristic approach proposed by Lee & Ma (2000). The proposed model results in a lower total system cost than the heuristic approach of equal cycle times, especially when the production rate is high.

Sett et el. (2012) developed a two-warehouse inventory model where they considered a quadratically increasing demand rate. The model is formulated by incorporating time-dependent deterioration rate for different warehouses and unequal length of the cycle time. The authors derived the minimised total cost of the whole system by proposing solution algorithm since the cost function is highly nonlinear and cannot be solved analytically.

Xu et el. (2017) developed a two-warehouse inventory problem for deteriorating items with a constant demand rate over a finite time horizon. They compared their model with the LIFO, MLIFO and FIFO inventory models and derived the critical conditions of the production cycle number, inventory holding and deterioration costs in the two warehouses. They have concluded that when the deterioration rates are the same in the two warehouses, the difference in the holding costs plays an important role in the comparison of the four inventory models. Whereas, when the deterioration rates are different, the difference in the total inventory, holding and deterioration costs plays the key role in comparing the four models. We observe that shortages are not permitted in the four aforementioned papers.

Wee et el. (2005) developed a two-warehouse inventory model where the demand rate is assumed to be constant while the deterioration rate follows a two-parameter Weibull distribution. The authors derived the optimal replenishment policy using the Discounted Cash Flow (DCF) and classical optimisation technique. They also concluded that the total present values of the total relevant cost per unit time are lower for the two-warehouse model as compared to the model for a single rented or owned warehouse.

Singh et el. (2013) proposed a two-warehouse inventory model with imperfect production process. Two cases were considered namely, (i) model that begins with shortages and (ii) model that ends with shortages. They assumed that the demand rate is time dependent, while the production rate is dependent on the demand rate. The deterioration rate is assumed to follow a Weibull distribution. The authors also considered the effect of learning on production cost.

Kumar et el. (2013) developed a two-warehouse inventory model with deteriorating items where the demand rate is assumed to follow a combination of linearly time varying function and on-hand inventory level dependent demand namely a multivariate demand. They assumed that the transferring of items from RW to OW follows a bulk release (K-release) rule. Deterioration rate in OW is assumed to be a time dependent function, while the rate in RW follows a Weibull distribution.

Dey et el. (2008) developed a two-warehouse inventory model for deteriorating items under the influence of inflation and time value of money. The model is assumed to have an interval valued lead-time over finite time horizon. The demand is time dependent and assumed to be increasing with time at a decreasing rate. They considered three cases namely; (i) a model where shortages are allowed at the end of each cycle, (ii) a model where shortages are allowed in each cycle with the exception of the last cycle and (iii) a model where shortages are allowed at the end of each cycle but not backlogged at the end of the last cycle.

Jaggi et el. (2011) presented a two-warehouse inventory model for deteriorating items and also included the single-warehouse model in their paper. They assumed that the demand rate is a linearly increasing function of time. Both OW and RW are assumed to have different deterioration rates. Items are transferred from RW to OW in a continuous release pattern.

Singh et el. (2009) developed a two-warehouse inventory model for deteriorating items under the influence of inflation and time-value of money with infinite replenishment rate. The demand rate is assumed to be a linearly increasing function of time. The deterioration rate for both OW and RW are different, where in OW the rate is time dependent while in RW the rate follows a two-parameter Weibull distribution. Items are transferred from RW to OW in a continuous release pattern with a k-release rule.

Yang (2004) proposed a two-warehouse inventory model under the influence of inflation by introducing a non-traditional shortage model. Unlike the traditional model (Model 1) where each replenishment cycle starts with an instant order and ends with shortages, the alternative model, known as Model 2 begins with shortages and ends without shortages. The author assumed that the deterioration rate is constant and permits shortages. It is observed that the proposed model, Model 2 is more economical to operate compared to Model 1, provided that the inflation rate is greater than zero.

Yang (2006) extended their own paper, Yang (2004) by incorporating partial backlogging to the existing model. The backlogging rate is assumed to be a decreasing function of waiting time. Yang also compared the two models introduced in the latter work, namely Model 1 and 2 based on the minimum cost approach. They revealed that Model 2 is still less expensive to operate compared to Model 1, where Model 2 begins with shortage and ends without shortages.

Yang (2012) extended their model, Yang (2006) by incorporating the assumption that the deterioration rate follows a three-parameter Weibull distribution. Shortages are allowed in this model and partially backlogged, where the rate of backlogging is time varying. It is observed that the operation cost is still cheaper for Model 2 compared to Model 1.

Researchers of the nine aforementioned models considered the influence of inflation in their respective models. Shortages are also allowed and partially backlogged in papers by Wee et el. (2005), Singh et el. (2013), Kumar et el. (2013), Dey et el. (2008) and Jaggi et el. (2011). While in the model by Singh et el. (2009), it is completely backlogged.

Inflation relates to the increase in the prices of goods or services such as food, clothing and housing, to name a few. It occurs when the prices rise however the purchasing power decreases for a certain period of time. This is an important field of study as the demand, supply and expectations of goods in a market has an effect towards inflation rates.

The following papers considered transportation cost in their models. This is a more realistic approach when dealing with models with two warehouses. In reality, there would be a difference in distance between two storage facilities or the two warehouses. This shall mean that a transportation cost is payable to move inventory items from one warehouse to the other.

Bhunia & Maiti (1998) developed a deterministic two-warehouse inventory model where the demand rate is assumed to be a linearly increasing function of time. Both warehouses are also assumed to have different deterioration rates. Inventory items are transferred from RW to OW in a continuous release pattern and the associated transportation cost is taken into consideration. Shortages are allowed in OW and excess demand is backlogged. They developed the two models namely single warehouse and two-warehouse models and found the optimal solution for both models.

Goswami & Chaudhuri (1992) made significant changes to the model developed by Sarma (1983) by introducing more realistic assumptions. The authors developed a deterministic two-warehouse inventory model with a linearly increasing demand. They considered two cases namely a model with shortage and a model without shortage. Unlike Sarma, they assumed that the transportation cost of items from RW to OW is dependent on the quantity being transported.

Zhou & Yang (2005) developed a deterministic model with inventory-dependent demand rate and two separate warehouses; OW and RW. The demand rate is assumed to be a polynomial form of current inventory level. A bulk release pattern is utilized to transfer stock from the RW to the OW and the transportation cost is assumed to be dependent on the amount transported. Shortages are not permitted in this model.

Kar et el. (2001) developed a deterministic model with two levels of storage facilities and fixed time horizon where they assumed that the demand rate follows a linearly increasing function. Shortages are allowed in this model and completely backlogged. Items are transferred from RW to OW in a continuous release pattern and the transportation cost is considered. The authors assumed that the model has an infinite replenishment rate. A case without shortages was also discussed in this paper.

The following papers incorporated trade credit policy in their models where they permit delay in payment. Chung & Huang (2007) modified Huang's (2003) model by developing a two-warehouse inventory model for deteriorating items under permissible delay in payments with infinite time horizon. They assumed that the supplier would provide the retailer a delay period while the retailer would also adopt the trade credit policy to increase their demand. The demand rate is assumed to be constant and shortages are not allowed in this model. Items are transported from RW to OW in a continuous release pattern however the transportation cost is neglected.

Liang & Zhou (2011) developed a two-warehouse model where they incorporated the existence of deteriorating items with the condition delay in payment permitted and the demand rate is assumed to be constant. They assumed that the RW has a higher holding cost compared to the OW as the RW is assumed to possess better conserving facility. Hence, in order to minimise the total cost, the best decision is to use all items from RW first followed by items in OW.

Liao et el. (2012) developed a deterministic two-warehouse inventory model for deteriorating items. They considered and incorporated two mathematical models developed by Chung & Huang (2006) and Chung & Liao (2004). The authors took delay in payments into consideration, where the delay in payment is dependent on the order quantity. It is assumed that if the order quantity is less than that at which delayed payment is allowed, payment has to be made instantly. On the contrary, if the order quantity is more than that at which delayed payment is allowed, then the fixed trade credit period is allowed.

Yang & Chang (2013) extended a paper by Yang (2006) where Yang considered a two-warehouse inventory model for deteriorating items with partial backlogging under inflation. They extended Yang's paper by considering Model 2 in which that the model is less expensive to be operated compared to Model 1. Hence, they incorporated a permissible delay in payment into the model of Yang and considered only Model 2, where the inventory model begins with shortages and ends without shortages. They derived the retailer's optimal replenishment policy that maximises the net present value of the profit per unit time.

Liao et el. (2013) developed a two-warehouse inventory model for deteriorating items by extending the work of Chung & Huang (2007). The authors made the assumption that the deterioration rate in RW is higher than the rate in OW. Contradictory to the assumption made by Chung & Huang (2007) where they assumed that both warehouses have identical deterioration rates. They made an assumption that the supplier offers the retailer a permissible delay period while in return the retailer provides a trade credit period to their customers. They assumed that items from RW are transported from OW in a continuous release pattern.

The assumption that the deterioration rate in RW is higher than the rate in OW is not commonly used in previous studies. This is due to the assumption that the RW offers a better preserving facility and services, hence resulted in the higher holding cost in RW as compared to the OW.

Singh & Pattnayak (2014) discussed a two-warehouse inventory model for a single deteriorating item under conditionally permissible delay in payment. They considered a non-traditional approach by assuming a linearly increasing demand. Since it is assumed that RW offers better preserving storage facilities, higher holding costs will be charged for items in the RW. However, the deterioration rate will be lower in the RW compared to the OW. Hence, the authors concluded that it would be more economical to consume items from RW first rather than in OW.

The following are some papers that incorporated the assumption of imperfect production process in their models. This is a realistic approach in real life situations as a production process is imperfect due to factors such as human errors, machine failure and many other unpredictable situations.

Chung et el. (2009) suggested an inventory model that incorporates the idea of the two-warehouse system and the existence of defectives due to an imperfect quality production process. Their paper generalises the model of Salameh & Jaber (2000). The authors attempted the relaxation of two basic assumptions in a traditional EOQ model; (i) all units produced are of perfect quality, and (ii) the inventories are stored by a single warehouse with unlimited capacity. They assumed that the defectives are sold as a single batch at a discounted price.

Lin & Chin (2011) modified the model of Chung et. al. (2009) by correcting the expression for the optimal order quantity and expected profit per unit time. The authors further extended the model by introducing a 100% inspection process with screening errors and penalty costs. The demand is assumed to be constant and shortages are not allowed in this model. It is shown in the results that the order lot size and expected total profit is less than that in the model of Chung et. al. (2009).

Pal & Mahaptra (2017) developed and EOQ model with imperfect quality and shortage backordering under inspection errors and deterioration. They have considered price discounts and return cost in their model. They developed a three layer model involving the supplier, manufacturer and retailer where their objective is to minimise the total joint annual costs incurred at each layer as well as the integrated three layer. They made an assumption that the production process is imperfect, hence producing defective items. In addition, they also assumed that there is an inspection error at the retailerś layer due to imperfect screening process.

Al-Salamah (2019) developed EPQ models for imperfect manufacturing process and a flexible rework process. The shortages are backordered and defective items are reworked at the same cycle. The rework process is assumed to be perfect, hence producing only perfect reworked items. The author also proposed two kinds of configurations for the rework process, namely the asynchronous and synchronous rework configurations. The asynchronous rework configuration works by letting the defective items to be reworked only after the lot has been completed. While the latter, permits the rework of the defective items as soon as they are produced.

Sarkar (2019) developed a model where a multi-stage production model with work-in-process items and finished products were considered to reduce defective items through two different rework approaches. The rework process was considered to take place within each cycle avoiding any shortages, or at the end of each stage cycle. The aim of the model is to diminish wastes by considering the rework strategies to minimise the total cost. Two-stage inspection is also considered to detect faulty products and ensure zero faulty items.

Rework process is an essential initiative to manufacturers as it is one of the approaches in reducing production cost and maintaining the standards and quality of a product. The items that are defective or does not meet the quality standards can be reprocessed and be as good as new. In return, this will reduce wastage and indirectly reduces a manufacturer's cost.

In this study, we propose a two warehouse inventory model with deteriorating items and a rework process. The production process is conducted in OW only during the first interval, where a portion of the produced items is assumed to be defective. All defective items are separated and sent to RW to be reworked. Both OW and RW share the same deterioration rate, α.

A Last-In-First-Out (LIFO) policy is considered where the reworked items in RW will be consumed and fully depleted first, followed by items in OW. Next, a First-In-First-Out (FIFO) policy is also considered where items stored in OW will be consumed and fully depleted first, followed by the reworked items in RW. Shortages are not allowed in the proposed model.

CHAPTER 3: LAST-IN-FIRST-OUT (LIFO)

3.1 Introduction

In this chapter, we will look at a two-warehouse inventory model while considering the Last-In-First-Out (LIFO) policy. The first warehouse is known as the owned warehouse. When the owned warehouse has reached its maximum capacity, an alternative warehouse also known as the rented warehouse is required to store the excess items.

Assuming that a rented warehouse provides a better service and facility in storing inventory, the cost of the rented warehouse may be higher than that of the owned warehouse. Hence, it is more economical that items are stored longer in the owned warehouse instead of the rented warehouse.

We have incorporated the LIFO policy in the model, where inventories that are produced last are the ones to be exhausted first. In other words, all items that are produced the latest will be used or distributed first, to fulfil demand. We have also assumed that the production process is imperfect, hence resulting in the production of defective items.

In order to ensure the smoothness of the production process, the defective items are kept aside and sent to the rented warehouse to undergo the rework process once the production process has ended. Since we have made the assumption that the rented warehouse holds a higher holding cost than the owned warehouse, hence the reworked items in the rented warehouse will be exhausted first followed by items in the owned warehouse. This phenomenon is known as the Last-In-First-Out (LIFO) policy.

3.2 Mathematical Formulations

3.2.1 Notations

Listed below are the notations used in the LIFO policy model.

a : the initial rate of demand, where $f(t) = a + bt$

b : the rate at which the demand rate increases, where $f(t) = a + bt$

$f(t)$: linearly increasing demand rate $f(t) = a + bt$

D : total demand, where $D = \int_{t_0}^{T} f(t)\,dt$, units

P : constant production rate, units per unit time where $P > f(t)$ for all t

R : rework process rate, units per unit time where $R > f(t)$ for all t

x : product defect rate, units per unit time

α : deteriorating rate in OW and RW, units per unit time where $0 \leq \alpha \leq 1$

W : maximum inventory of OW, units

Q : maximum inventory of RW, units

t_i : time period in a cycle of stage i, where $0 \leq i \leq 3$

T : batch cycle time period

$I_i(t)$: inventory level at time t_i, where $0 \leq i \leq 5$

A_i : area under the curve $I_i(t)$, where $0 \leq i \leq 5$

S : production setup cost, \$ per setup

c_P : processing cost per unit item, \$

c_R : rework processing cost per unit item, \$

c_D : deterioration cost per unit item, \$

h_1, h_2 : holding costs per unit item in OW and RW respectively

TRC : total relevant costs per unit time, \$

3.2.2 Abbreviations

Listed below are the abbreviations used in the LIFO policy model.

OW : owned warehouse

RW : rented warehouse

LIFO : last-in-first-out

GRG : generalized reduced gradient

3.2.3 Assumptions

The assumptions adopted in this study includes:

1. Lead time is zero and the replenishment rate is finite.
2. The production rate, P is larger than the demand rate, $f(t)$.
3. The rented warehouse, RW has an unlimited capacity.
4. Only perfect items are stored in OW, while all defective items are sent to RW to be reworked.
5. The rework process is assumed to be perfect since special care is given to the process, hence all reworked items are perfect. The total reworked items is equal to the total defective items.
6. The production and rework rates, P and R respectively, are assumed to be different.
7. The RW is located near the OW, hence the transportation cost between the warehouses is negligible.

3.3 Modelling LIFO

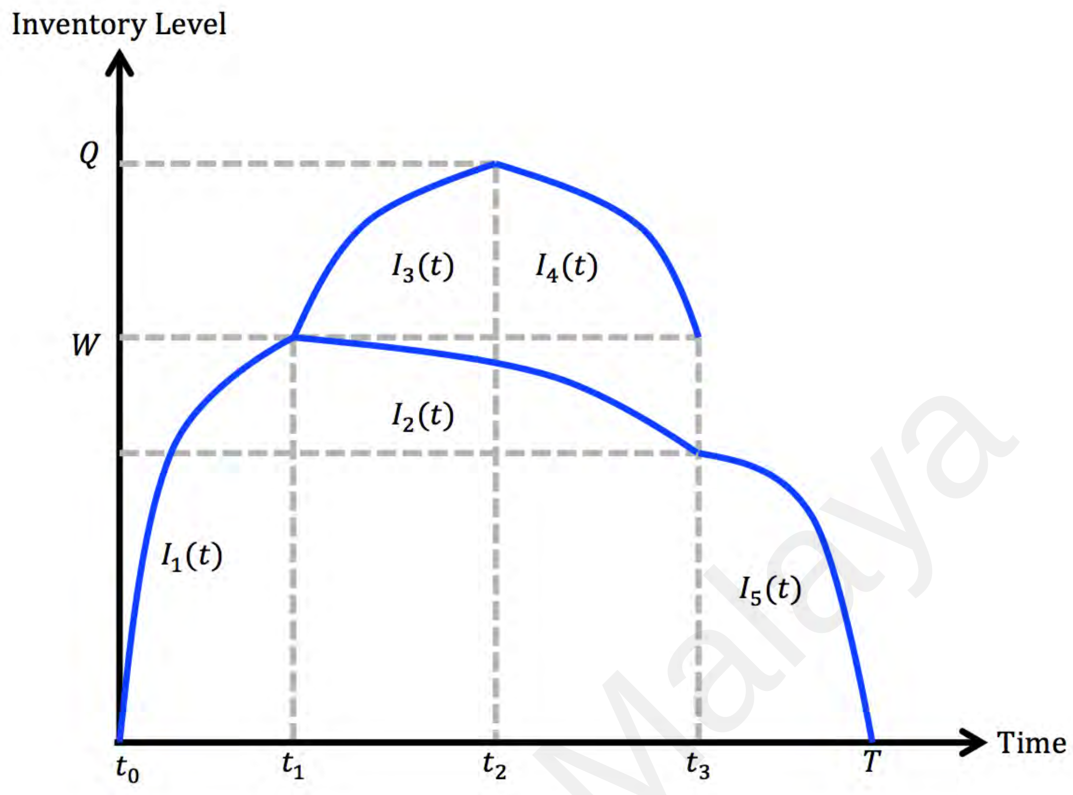

Figure 3.1: Inventory level of items with LIFO policy.

The level of inventory items in both warehouses, OW and RW is as illustrated in Figure 3.1. The interval $t_0 \leq t \leq t_1$ represents the production uptime period in OW, where items produced in this interval are a mixture of perfect and defective items. All defectives are separated to be sent to RW to undergo rework process. While all perfect items produced are subjected to deterioration and used in satisfying demands. The inventory level in this interval is represented by $I_1(t)$.

The following interval, $t_1 \leq t \leq t_2$ represents the rework uptime period in RW. At t_1, all defectives from OW are sent to RW to be reworked. Since the rework process is assumed to be free of flaws, all reworked items are assumed to be perfect and as good as new. Hence, the total defective items is assumed to be equal to the total items reworked. The inventory level of items in this interval is represented by $I_3(t)$. Note that, all items in this interval are subjected to deterioration and fulfilment of demand.

Rework uptime is then followed by rework downtime in the interval $t_2 \le t \le t_3$. The inventory level in this interval decreases due to deterioration and satisfying of demand which is represented by the equation $I_4(t)$. At t_3, the inventory level in RW is equal to zero since we assume that all items have been consumed completely.

On the other hand, in the interval $t_1 \le t \le t_3$, the inventory level of items stored in OW decreases due to deterioration only. The inventory level of items in this interval is represented by $I_2(t)$. The following interval, $t_3 \le t \le T$ represents the production down time period in OW. In this interval, items are depleted completely due to deterioration and satisfying of demand. The inventory level of items in this interval is represented by $I_5(t)$.

Note that, items are said to enter the owned warehouse first since they are produced in the OW up to t_1. After which, all collected defective items are then sent to RW to be reworked. At the end of the production cycle, items are fully depleted in the rented warehouse first, before all items are cleared in the owned warehouse. This phenomenon is known as the Last-In-First-Out, LIFO policy.

Subsections are divided based on the intervals $t_0 \le t \le t_1$, $t_1 \le t \le t_3$, $t_1 \le t \le t_2$, $t_2 \le t \le t_3$ and $t_3 \le t \le T$.

3.3.1 Production up time in the interval $t_0 \le t \le t_1$

The production process begins in this interval, where some items produced are defective. All perfect items are stored in OW, while all defectives are separated and sent to RW to be reworked at t_1. Non-defective items are utilised to satisfy demands and are also subjected to deterioration. The maximum inventory level of OW is reached at t_1, where it is represented by W. Hence, the net replenishment rate can be formulated as

$$\frac{dI_1(t)}{dt} = P - x - f(t) - \alpha I_1(t), \text{ where } I_1(0) = 0 \text{ and } I_1(t_1) = W$$

$$
\begin{aligned}
\frac{dI_1(t)}{dt} &= P - x - a - bt - \alpha I_1(t) \\
e^{\alpha t} I_1(t) &= \int e^{\alpha t}(P - x - a - bt)dt \\
&= \frac{1}{\alpha}(P - x - a)e^{\alpha t} - \frac{bte^{\alpha t}}{\alpha} + \frac{be^{\alpha t}}{\alpha^2} + C
\end{aligned}
$$

Given that the initial condition, $I_1(t_0) = 0$, we have

$$
C = -\frac{(P - x - a)}{\alpha} - \frac{b}{\alpha^2}
$$

Then,

$$
\begin{aligned}
e^{\alpha t} I_1(t) &= \frac{(P - x - a)e^{\alpha t}}{\alpha} - \frac{bte^{\alpha t}}{\alpha} + \frac{be^{\alpha t}}{\alpha^2} - \frac{(P - x - a)}{\alpha} - \frac{b}{\alpha^2} \\
I_1(t) &= \left[\frac{(P - x - a)}{\alpha} + \frac{b}{\alpha^2}\right]\left(1 - e^{-\alpha t}\right) - \frac{bt}{\alpha}
\end{aligned}
$$

$I_1(t)$ represents the inventory level at time t, within the interval $t_0 \leq t \leq t_1$, in which the inventory level is equivalent to the number of produced items excluding the defective and deteriorated items as well as the demand which has been fulfilled, at time t.

A_1 represents the area under the curve $I_1(t)$ in the interval $t_0 \leq t \leq t_1$, during the production uptime period in OW, which is given as

$$
\begin{aligned}
A_1 &= \int_0^{t_1} I_1(t)\, dt \\
&= \left(\frac{P - x - a}{\alpha} + \frac{b}{\alpha^2}\right) t_1 + \frac{e^{-\alpha t_1}}{\alpha}\left(\frac{P - x - a}{\alpha} + \frac{b}{\alpha^2}\right) - \frac{bt_1{}^2}{2\alpha} \\
&- \frac{1}{\alpha}\left(\frac{P - x - a}{\alpha} + \frac{b}{\alpha^2}\right) \\
&= \left(\frac{P - x - a}{\alpha} + \frac{b}{\alpha^2}\right) t_1 - \frac{bt_1{}^2}{2\alpha} + \left(\frac{P - x - a}{\alpha^2} + \frac{b}{\alpha^3}\right)(e^{-\alpha t_1} - 1) \\
&= \frac{(P - x)t_1}{\alpha} - \frac{1}{\alpha}\left(at_1 + \frac{b}{2}t_1{}^2\right) + \frac{bt_1}{\alpha^2} + \left(\frac{P - x - a}{\alpha^2} + \frac{b}{\alpha^3}\right)(e^{-\alpha t_1} - 1)
\end{aligned}
$$

We also note that the maximum inventory level of OW which is represented by W is governed by the equation $W = I_1(t_1)$. Hence, we have

$$W = \left[\frac{(P-x-a)}{\alpha} + \frac{b}{\alpha^2}\right]\left(1 - e^{-\alpha t_1}\right) - \frac{bt_1}{\alpha}$$

3.3.2 Interval $t_1 \leq t \leq t_3$

The interval $t_1 \leq t \leq t_2$ represents the rework uptime period in RW, while the interval $t_2 \leq t \leq t_3$ represents the rework downtime period in RW. Produced items in OW from the previous interval, $t_0 \leq t \leq t_1$ are kept in storage during rework up time and down time periods in RW. Hence, the inventory level of the items in OW decreases solely due to deterioration within the interval $t_1 \leq t \leq t_3$, while demands are fulfilled by items in RW. The net replenishment rate in OW can be formulated as

$$\frac{dI_2(t)}{dt} = -\alpha I_2(t), \text{ where } I_2(t_1) = W$$

$$\begin{aligned} \frac{dI_2(t)}{dt} &= -\alpha I_2(t) \\ \frac{dI_2(t)}{dt} + \alpha I_2(t) &= 0 \\ e^{\alpha t} I_2(t) &= C \end{aligned}$$

$I_2(t)$ represents the inventory level at time t within the interval $t_1 \leq t \leq t_3$, in which the inventory level is equivalent to the number of items stored in OW after the interval $t_0 \leq t \leq t_1$ minus the deteriorated items at time t.

Given the boundary condition, $I_2(t_1) = W$, we have

$$\begin{aligned} e^{\alpha t_1} W &= C \\ I_2(t) &= We^{\alpha(t_1 - t)} \end{aligned}$$

A_2 represents the area under the curve $I_2(t)$ in the interval $t_1 \leq t \leq t_3$ which is given as

$$\begin{aligned} A_2 &= \int_{t_1}^{t_3} I_2(t)\, dt \\ &= -\frac{1}{\alpha} W e^{\alpha(t_1 - t_3)} + \frac{1}{\alpha} W \\ &= \frac{W}{\alpha}\left[1 - e^{\alpha(t_1 - t_3)}\right] \end{aligned}$$

3.3.3 Rework up time in the interval $t_1 \leq t \leq t_2$

All defective items sent to RW will undergo rework process during this interval with the rework rate, R. Since the rework process is assumed to be perfect, all items reworked are also assumed to be non-defective and as good as new. The inventory net replenishment rate in this interval can be formulated as

$$\tfrac{dI_3(t)}{dt} = R - f(t) - \alpha I_3(t), \text{ where } I_3(t_1) = 0$$

$$\begin{aligned} \frac{dI_3(t)}{dt} &= R - a - bt - \alpha I_3(t) \\ \frac{dI_3(t)}{dt} + \alpha I_3(t) &= R - a - bt \\ e^{\alpha t} I_3(t) &= \int (R - a - bt) e^{\alpha t} dt \\ &= \frac{1}{\alpha}(R - a) e^{\alpha t} - \frac{bt}{\alpha} e^{\alpha t} + \frac{b}{\alpha^2} e^{\alpha t} + C \end{aligned}$$

Given the boundary condition, $I_3(t_1) = 0$, we have

$$\begin{aligned} C &= -\left[\frac{1}{\alpha}(R - a - bt_1) + \frac{b}{\alpha^2}\right] e^{\alpha t_1} \\ I_3(t) &= \frac{1}{\alpha}(R - a - bt) + \frac{b}{\alpha^2} - \left[\frac{1}{\alpha}(R - a - bt_1) + \frac{b}{\alpha^2}\right] e^{\alpha(t_1 - t)} \end{aligned}$$

$I_3(t)$ represents the inventory level at time t within the interval $t_1 \le t \le t_2$, in which the inventory level is equivalent to the number of items in RW which have been reworked excluding deteriorated items and demand which has been fulfilled, at time t.

A_3 represents the area under the curve $I_3(t)$ in the interval $t_1 \le t \le t_2$, during the rework up time period in RW which is given as

$$
\begin{aligned}
A_3 &= \int_{t_1}^{t_2} I_3(t)\, dt \\
&= \frac{1}{\alpha}\left[(R-a)t - \frac{b}{2}t^2 + \frac{1}{\alpha}(R-a-bt_1)e^{\alpha(t_1-t)}\right] + \frac{b}{\alpha^2}\left[t + \frac{1}{\alpha}e^{\alpha(t_1-t)}\right] \\
&= \frac{1}{\alpha}\left[(R-a)(t_2-t_1) - \frac{b}{2}\left({t_2}^2 - {t_1}^2\right) + \frac{1}{\alpha}(R-a-bt_1)\left[e^{\alpha(t_1-t_2)} - 1\right]\right] \\
&+ \frac{b}{\alpha^2}\left[(t_2-t_1) + \frac{1}{\alpha}\left[e^{\alpha(t_1-t_2)} - 1\right]\right] \\
&= \frac{1}{\alpha}(R-a)(t_2-t_1) - \frac{b}{2\alpha}\left({t_2}^2 - {t_1}^2\right) + \frac{1}{\alpha^2}(R-a-bt_1)\left[e^{\alpha(t_1-t_2)} - 1\right] \\
&+ \frac{b}{\alpha^2}(t_2-t_1) + \frac{b}{\alpha^3}\left[e^{\alpha(t_1-t_2)} - 1\right] \\
&= \frac{1}{\alpha}R(t_2-t_1) - \frac{1}{\alpha}\left(at_2 + \frac{b}{2}{t_2}^2\right) + \frac{1}{\alpha}\left(at_1 + \frac{b}{2}{t_1}^2\right) + \frac{b}{\alpha^2}(t_2-t_1) \\
&+ \left[\frac{1}{\alpha^2}(R-a-bt_1) + \frac{b}{\alpha^3}\right]\left[e^{\alpha(t_1-t_2)} - 1\right]
\end{aligned}
$$

3.3.4 Rework down time in the interval $t_2 \le t \le t_3$

All perfectly reworked items in RW are depleted in this interval due to the occurrence of deterioration and fulfilment of demand where the items will be completely depleted at t_3. The inventory net replenishment rate can be formulated as

$$\frac{dI_4(t)}{dt} = -f(t) - \alpha I_4(t), \text{ where } I_4(t_3) = 0$$

$$\begin{aligned}
\frac{dI_4(t)}{dt} &= -f(t) - \alpha I_4(t) \\
\frac{dI_4(t)}{dt} + \alpha I_4(t) &= -a - bt \\
e^{\alpha t} I_4(t) &= \int (-a - bt) e^{\alpha t}\, dt \\
&= \left[-\frac{a}{\alpha} - \frac{bt}{\alpha} + \frac{b}{\alpha^2} \right] e^{\alpha t} + C
\end{aligned}$$

Given the boundary condition, $I_4(t_3) = 0$, we have

$$\begin{aligned}
C &= -\left[-\frac{a}{\alpha} - \frac{bt_3}{\alpha} + \frac{b}{\alpha^2} \right] e^{\alpha t_3} \\
I_4(t) &= \left[\frac{1}{\alpha}(a + bt_3) - \frac{b}{\alpha^2} \right] e^{\alpha(t_3 - t)} - \frac{(a + bt)}{\alpha} + \frac{b}{\alpha^2}
\end{aligned}$$

$I_4(t)$ represents the inventory level at time t within the interval $t_2 \le t \le t_3$, in which the inventory level is equivalent to the remaining number of reworked items after t_2 excluding deteriorated items and demand which has been fulfilled, at time t.

A_4 represents the area under the curve $I_4(t)$ in the interval $t_2 \le t \le t_3$, during the rework down time period in RW which is given as

$$\begin{aligned}
A_4 &= \int_{t_2}^{t_3} I_4(t)\, dt \\
&= -\frac{1}{\alpha} \left\{ \frac{1}{\alpha}(a + bt_3) \left[1 - e^{\alpha(t_3 - t_2)} \right] + a(t_3 - t_2) + \frac{b}{2}({t_3}^2 - {t_2}^2) \right\} \\
&+ \frac{b}{\alpha^2} \left\{ (t_3 - t_2) + \frac{1}{\alpha} \left[1 - e^{\alpha(t_3 - t_2)} \right] \right\} \\
&= \frac{1}{\alpha} \left(at_2 + \frac{b}{2}{t_2}^2 \right) - \frac{1}{\alpha} \left(at_3 + \frac{b}{2}{t_3}^2 \right) + \frac{b}{\alpha^2}(t_3 - t_2) \\
&+ \left[\frac{1}{\alpha^2}(a + bt_3) - \frac{b}{\alpha^3} \right] \left[e^{\alpha(t_3 - t_2)} - 1 \right]
\end{aligned}$$

3.3.5 Production down time in the interval $t_3 \le t \le T$

Once all items in RW have been depleted and consumed completely at t_3, inventory items stored in OW will decrease due to deterioration and fulfilment of demands in the interval $t_3 \leq t \leq T$. The production cycle ends at T where all items in OW are also consumed completely. The inventory net replenishment rate can be formulated as

$$\frac{dI_5(t)}{dt} = -f(t) - \alpha I_5(t), \text{ where } I_5(T) = 0$$

$$\begin{aligned}
\frac{dI_5(t)}{dt} + \alpha I_5(t) &= -a - bt \\
e^{\alpha t} I_5(t) &= \int (-a - bt)e^{\alpha t}\, dt \\
&= \left[-\frac{(a+bt)}{\alpha} + \frac{b}{\alpha^2}\right] e^{\alpha t} + C
\end{aligned}$$

Given the boundary condition, $I_5(T) = 0$, we have

$$\begin{aligned}
C &= \left[\frac{(a+bT)}{\alpha} - \frac{b}{\alpha^2}\right] e^{\alpha T} \\
I_5(t) &= \left[\frac{(a+bT)}{\alpha} - \frac{b}{\alpha^2}\right] e^{\alpha(T-t)} - \frac{(a+bt)}{\alpha} + \frac{b}{\alpha^2}
\end{aligned}$$

$I_5(t)$ represents the inventory level at time t within the interval $t_3 \leq t \leq T$, in which the inventory level is equivalent to the remaining number of items in OW after t_3 excluding deteriorated items and demand which has been fulfilled, at time t.

A_5 represents the area under the curve $I_5(t)$ in the interval $t_3 \leq t \leq T$, during a production down time period in OW which is given as

$$\begin{aligned}
A_5 &= \int_{t_3}^{T} I_5(t)\, dt \\
&= \left[1 - e^{\alpha(T-t_3)}\right]\left[\frac{b}{\alpha^3} - \frac{1}{\alpha^2}(a+bT)\right] - \frac{1}{\alpha}\left[a(T-t_3) + \frac{b}{2}\left(T^2 - {t_3}^2\right)\right] \\
&+ \frac{b(T-t_3)}{\alpha^2} \\
&= \frac{1}{\alpha}\left(at_3 + \frac{b}{2}{t_3}^2\right) - \frac{1}{\alpha}\left(aT + \frac{b}{2}T^2\right) + \frac{b}{\alpha^2}(T-t_3) \\
&+ \left[\frac{1}{\alpha^2}(a+bT) - \frac{b}{\alpha^3}\right]\left[e^{\alpha(T-t_3)} - 1\right]
\end{aligned}$$

Hence, the governing differential equations stating the inventory levels within the production cycle are given as follows:

$$\frac{dI_1(t)}{dt} = P - x - f(t) - \alpha I_1(t) \qquad ; \qquad t_0 \le t \le t_1 \tag{1}$$
$$\frac{dI_2(t)}{dt} = -\alpha I_2(t) \qquad ; \qquad t_1 \le t \le t_3 \tag{2}$$
$$\frac{dI_3(t)}{dt} = R - f(t) - \alpha I_3(t) \qquad ; \qquad t_1 \le t \le t_2 \tag{3}$$
$$\frac{dI_4(t)}{dt} = -f(t) - \alpha I_4(t) \qquad ; \qquad t_2 \le t \le t_3 \tag{4}$$
$$\frac{dI_5(t)}{dt} = -f(t) - \alpha I_5(t) \qquad ; \qquad t_3 \le t \le T \tag{5}$$

where $f(t) = a + bt$.

Solving the differential equations (1) to (5) above, we have

$$I_1(t) = \left[\frac{(P - x - a)}{\alpha} + \frac{b}{\alpha^2}\right]\left(1 - e^{-\alpha t}\right) - \frac{bt}{\alpha} \tag{6}$$
$$I_2(t) = We^{\alpha(t_1 - t)} \tag{7}$$
$$I_3(t) = \frac{1}{\alpha}(R - a - bt) + \frac{b}{\alpha^2} - \left[\frac{1}{\alpha}(R - a - bt_1) + \frac{b}{\alpha^2}\right]e^{\alpha(t_1 - t)} \tag{8}$$
$$I_4(t) = \left[\frac{1}{\alpha}(a + bt_3) - \frac{b}{\alpha^2}\right]e^{\alpha(t_3 - t)} - \frac{(a + bt)}{\alpha} + \frac{b}{\alpha^2} \tag{9}$$
$$I_5(t) = \left[\frac{(a + bT)}{\alpha} - \frac{b}{\alpha^2}\right]e^{\alpha(T - t)} - \frac{(a + bt)}{\alpha} + \frac{b}{\alpha^2} \tag{10}$$

where the boundary conditions are given as follows;

$I_1(t_0) = 0$, $I_1(t_1) = W = I_2(t_1)$, $I_3(t_1) = 0$, $I_3(t_2) = I_4(t_2)$, $I_4(t_3) = 0$, $I_2(t_3) = I_5(t_3)$ and $I_5(T) = 0$.

Area under the curve $I_i(t)$, A_i in each interval where $i = 1, 2, 3, 4$ and 5 are given as follows:

In the interval $t_0 \leq t \leq t_1$,

$$A_1 = \frac{(P-x)t_1}{\alpha} - \frac{1}{\alpha}\left(at_1 + \frac{b}{2}{t_1}^2\right) + \frac{bt_1}{\alpha^2} + \left(\frac{P-x-a}{\alpha^2} + \frac{b}{\alpha^3}\right)(e^{-\alpha t_1} - 1) \tag{11}$$

In the interval $t_1 \leq t \leq t_3$,

$$A_2 = \frac{W}{\alpha}\left[1 - e^{\alpha(t_1 - t_3)}\right] \tag{12}$$

In the interval $t_1 \leq t \leq t_2$,

$$\begin{aligned} A_3 =& \frac{R}{\alpha}(t_2 - t_1) - \frac{1}{\alpha}\left(at_2 + \frac{b}{2}{t_2}^2\right) + \frac{1}{\alpha}\left(at_1 + \frac{b}{2}{t_1}^2\right) + \frac{b}{\alpha^2}(t_2 - t_1) \\ &+ \left[\frac{1}{\alpha^2}(R - a - bt_1) + \frac{b}{\alpha^3}\right]\left[e^{\alpha(t_1 - t_2)} - 1\right] \end{aligned} \tag{13}$$

In the interval $t_2 \leq t \leq t_3$,

$$\begin{aligned} A_4 =& \frac{1}{\alpha}\left(at_2 + \frac{b}{2}{t_2}^2\right) - \frac{1}{\alpha}\left(at_3 + \frac{b}{2}{t_3}^2\right) + \frac{b}{\alpha^2}(t_3 - t_2) \\ &+ \left[\frac{1}{\alpha^2}(a + bt_3) - \frac{b}{\alpha^3}\right]\left[e^{\alpha(t_3 - t_2)} - 1\right] \end{aligned} \tag{14}$$

In the interval $t_3 \leq t \leq T$,

$$\begin{aligned} A_5 =& \frac{1}{\alpha}\left(at_3 + \frac{b}{2}{t_3}^2\right) - \frac{1}{\alpha}\left(aT + \frac{b}{2}T^2\right) + \frac{b}{\alpha^2}(T - t_3) \\ &+ \left[\frac{1}{\alpha^2}(a + bT) - \frac{b}{\alpha^3}\right]\left[e^{\alpha(T - t_3)} - 1\right] \end{aligned} \tag{15}$$

3.3.6 Finding t_2

Next, assuming that all defective items are sent to RW to be reworked and the rework process is perfect, we may use the following equality to solve for t_2;

Total Defective Items = Total Reworked Items

Hence, we have the following equation

$$\begin{aligned} xt_1 &= R(t_2 - t_1) \\ xt_1 + Rt_1 &= Rt_2 \\ t_2 &= \frac{(R+x)t_1}{R} \end{aligned}$$

3.3.7 Costs Involved

3.3.7.1 Setup Production, Processing and Rework Processing Costs

The setup production, processing and rework processing costs per unit time are computed as below respectively.

$$\begin{aligned} \text{Setup Production Cost} &= \frac{S}{T} \\ \text{Processing Cost} &= \frac{c_P P(t_1 - t_0)}{T} = \frac{c_P P t_1}{T} \\ \text{Rework Processing Cost} &= \frac{c_R x(t_1 - t_0)}{T} = \frac{c_R x t_1}{T} \end{aligned}$$

3.3.7.2 Inventory Carrying Costs

Next, the total inventory carrying cost is given as the sum of the holding costs in OW and RW, where we have

1. The total holding cost in OW per unit time = $\frac{h_1}{T}(A_1 + A_2 + A_5)$

The total holding cost in OW per unit time

$$
\begin{aligned}
&= \frac{h_1}{T}\Bigg\{\frac{1}{\alpha}\left[(P-x)t_1 - \left(at_1 + \frac{b}{2}{t_1}^2\right) + \left(at_3 + \frac{b}{2}{t_3}^2\right) - \left(aT + \frac{b}{2}T^2\right)\right] \\
&- \frac{1}{\alpha^2}(a+bt_3) + \frac{b}{\alpha^3} - \frac{W}{\alpha}e^{\alpha(t_1-t_3)} + \left[\frac{1}{\alpha^2}(a+bT) - \frac{b}{\alpha^3}\right]e^{\alpha(T-t_3)}\Bigg\}
\end{aligned}
$$

2. The total holding cost in RW per unit time $= \frac{h_2}{T}(A_3 + A_4)$

The total holding cost in RW per unit time

$$
\begin{aligned}
&= \frac{h_2}{T}\Bigg\{\frac{1}{\alpha}\left[R(t_2-t_1) + \left(at_1 + \frac{b}{2}{t_1}^2\right) - \left(at_3 + \frac{b}{2}{t_3}^2\right)\right] - \frac{R}{\alpha^2} \\
&+ \left[\frac{1}{\alpha^2}(R-a-bt_1) + \frac{b}{\alpha^3}\right]e^{\alpha(t_1-t_2)} + \left[\frac{1}{\alpha^2}(a+bt_3) - \frac{b}{\alpha^3}\right]e^{\alpha(t_3-t_2)}\Bigg\}
\end{aligned}
$$

Therefore, the total holding costs for both OW and RW per unit time, HC is given as

$$
\begin{aligned}
HC &= \frac{h_1}{T}\Bigg\{\frac{1}{\alpha}\left[(P-x)t_1 - \left(at_1 + \frac{b}{2}{t_1}^2\right) + \left(at_3 + \frac{b}{2}{t_3}^2\right) - \left(aT + \frac{b}{2}T^2\right)\right] \\
&- \frac{1}{\alpha^2}(a+bt_3) + \frac{b}{\alpha^3} - \frac{W}{\alpha}e^{\alpha(t_1-t_3)} + \left[\frac{1}{\alpha^2}(a+bT) - \frac{b}{\alpha^3}\right]e^{\alpha(T-t_3)}\Bigg\} \\
&+ \frac{h_2}{T}\Bigg\{\frac{1}{\alpha}\left[R(t_2-t_1) + \left(at_1 + \frac{b}{2}{t_1}^2\right) - \left(at_3 + \frac{b}{2}{t_3}^2\right)\right] - \frac{R}{\alpha^2} \\
&+ \left[\frac{1}{\alpha^2}(R-a-bt_1) + \frac{b}{\alpha^3}\right]e^{\alpha(t_1-t_2)} + \left[\frac{1}{\alpha^2}(a+bt_3) - \frac{b}{\alpha^3}\right]e^{\alpha(t_3-t_2)}\Bigg\}
\end{aligned}
$$

3.3.7.3 Deteriorating Cost

The total number of deteriorated items is equal to the product of deterioration rate and the total area under the curves $I_1(t), I_2(t), I_3(t), I_4(t)$, and $I_5(t)$. Hence, we have the equality

The total number of deteriorated items, $G = \alpha$(The total area under the curves)

$$
\begin{aligned}
G &= \alpha(A_1 + A_2 + A_3 + A_4 + A_5) \\
&= \left[(P-x)t_1 + R(t_2 - t_1) + W[1 - e^{\alpha(t_1 - t_3)}] - \left(aT + \frac{b}{2}T^2\right)\right] \\
&- \frac{1}{\alpha}[P + R - x - b(t_1 - t_3)] + \left(\frac{P - x - a}{\alpha} + \frac{b}{\alpha^2}\right) e^{-\alpha t_1} \\
&+ \left[\frac{1}{\alpha}(R - a - bt_1) + \frac{b}{\alpha^2}\right] e^{\alpha(t_1 - t_2)} + \left[\frac{1}{\alpha}(a + bt_3) - \frac{b}{\alpha^2}\right] e^{\alpha(t_3 - t_2)} \\
&+ \left[\frac{1}{\alpha}(a + bT) - \frac{b}{\alpha^2}\right] e^{\alpha(T - t_3)}
\end{aligned}
$$

Hence, the deteriorating cost per unit time, DC is given as

$$
\begin{aligned}
DC &= \frac{c_D G}{T} \\
&= \frac{c_D}{T}\Bigg\{ \left[(P-x)t_1 + R(t_2 - t_1) + W[1 - e^{\alpha(t_1 - t_3)}] - \left(aT + \frac{b}{2}T^2\right)\right] \\
&- \frac{1}{\alpha}[P + R - x - b(t_1 - t_3)] + \left(\frac{P - x - a}{\alpha} + \frac{b}{\alpha^2}\right) e^{-\alpha t_1} \\
&+ \left[\frac{1}{\alpha}(R - a - bt_1) + \frac{b}{\alpha^2}\right] e^{\alpha(t_1 - t_2)} + \left[\frac{1}{\alpha}(a + bt_3) - \frac{b}{\alpha^2}\right] e^{\alpha(t_3 - t_2)} \\
&+ \left[\frac{1}{\alpha}(a + bT) - \frac{b}{\alpha^2}\right] e^{\alpha(T - t_3)} \Bigg\}
\end{aligned}
$$

3.3.8 The Total Relevant Cost, TRC

The total relevant cost per unit time for the LIFO policy is given as

$$
\begin{aligned}
TRC \;=\;& \text{Setup Production Cost} + \text{Processing Cost} + \text{Rework Processing Cost} \\
+\;& \text{Holding Cost in OW} + \text{Holding Cost in RW} + \text{Deteriorating Cost} \\
=\;& \frac{S}{T} + \frac{c_P P t_1}{T} + \frac{c_R x t_1}{T} \\
+\;& \frac{h_1}{T}\left\{\frac{1}{\alpha}\left[(P-x)t_1 - \left(at_1 + \frac{b}{2}{t_1}^2\right) + \left(at_3 + \frac{b}{2}{t_3}^2\right) - \left(aT + \frac{b}{2}T^2\right)\right]\right. \\
-\;& \left.\frac{1}{\alpha^2}(a+bt_3) + \frac{b}{\alpha^3} - \frac{W}{\alpha}e^{\alpha(t_1-t_3)} + \left[\frac{1}{\alpha^2}(a+bT) - \frac{b}{\alpha^3}\right]e^{\alpha(T-t_3)}\right\} \\
+\;& \frac{h_2}{T}\left\{\frac{1}{\alpha}\left[R(t_2-t_1) + \left(at_1 + \frac{b}{2}{t_1}^2\right) - \left(at_3 + \frac{b}{2}{t_3}^2\right)\right] - \frac{R}{\alpha^2}\right. \\
+\;& \left.\left[\frac{1}{\alpha^2}(R-a-bt_1) + \frac{b}{\alpha^3}\right]e^{\alpha(t_1-t_2)} + \left[\frac{1}{\alpha^2}(a+bt_3) - \frac{b}{\alpha^3}\right]e^{\alpha(t_3-t_2)}\right\} \\
+\;& \frac{c_D}{T}\left\{\left[(P-x)t_1 + R(t_2-t_1) + W[1-e^{\alpha(t_1-t_3)}] - \left(aT + \frac{b}{2}T^2\right)\right]\right. \\
-\;& \frac{1}{\alpha}[P+R-x-b(t_1-t_3)] + \left(\frac{P-x-a}{\alpha} + \frac{b}{\alpha^2}\right)e^{-\alpha t_1} \\
+\;& \left[\frac{1}{\alpha}(R-a-bt_1) + \frac{b}{\alpha^2}\right]e^{\alpha(t_1-t_2)} + \left[\frac{1}{\alpha}(a+bt_3) - \frac{b}{\alpha^2}\right]e^{\alpha(t_3-t_2)} \\
+\;& \left.\left[\frac{1}{\alpha}(a+bT) - \frac{b}{\alpha^2}\right]e^{\alpha(T-t_3)}\right\} \qquad (16)
\end{aligned}
$$

3.4 Numerical Examples, Solution Procedure and Sensitivity Analysis

3.4.1 Numerical Examples

The following parameters are considered to show a clear illustration of the LIFO policy model. The production rate per unit time, P is 3000, rework rate per unit time, R is 1000, defective rate per unit time, x is 500, deterioration rate per unit time, α is 0.04, the initial rate of demand, a is 550 and the rate with which the demand rate increases, b is 200, where $f(t) = a + bt$ is the demand rate per unit time.

The costs involved in this model are given as follows, where the setup production cost per setup, S is \$1000, the processing cost per unit item, c_P is \$2, rework processing cost per unit item, c_R is \$3, deterioration cost per unit item, d is \$2.50, holding cost per unit item in OW, h_1 is \$1.50 and holding cost per unit item in RW, h_2 is \$2.50.

We will now discuss on the equations involved in OW in the following subsections.

3.4.1.1 Total Items Produced

The total items produced is equal to the sum of the total demand and deteriorated items. Hence, we have the equality

$$
\begin{aligned}
P(t_1 - t_0) &= \int_{t_0}^{T} f(t)\,dt + \alpha(A_1 + A_2 + A_3 + A_4 + A_5), \text{ where} \\
\alpha A_1 &= (P - x)t_1 - \int_{t_0}^{t_1} f(t)\,dt - W \\
\alpha A_2 &= W - I_2(t_3) \\
\alpha A_3 &= R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - I_3(t_2) \\
\alpha A_4 &= I_4(t_2) - \int_{t_2}^{t_3} f(t)\,dt \\
\alpha A_5 &= I_5(t_3) - \int_{t_3}^{T} f(t)\,dt
\end{aligned}
$$

Focusing on the RHS of the equation, we have the following

$$
\begin{aligned}
RHS &= \int_{t_0}^{T} f(t)\,dt + \alpha(A_1 + A_2 + A_3 + A_4 + A_5) \\
&= \int_{t_0}^{T} f(t)\,dt + \left[(P-x)t_1 - \int_{t_0}^{t_1} f(t)\,dt - W\right] + [W - I_2(t_3)] \\
&+ \left[R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - I_3(t_2)\right] + \left[I_4(t_2) - \int_{t_2}^{t_3} f(t)\,dt\right] \\
&+ \left[I_5(t_3) - \int_{t_3}^{T} f(t)\,dt\right] \\
&= \int_{t_0}^{T} f(t)\,dt + (P-x)t_1 - W + W - I_2(t_3) + R(t_2 - t_1) - I_3(t_2) + I_4(t_2) + I_5(t_3) \\
&- \int_{t_0}^{t_1} f(t)\,dt - \int_{t_1}^{t_2} f(t)\,dt - \int_{t_2}^{t_3} f(t)\,dt - \int_{t_3}^{T} f(t)\,dt \\
&= \int_{t_0}^{T} f(t)\,dt + (P-x)t_1 + R(t_2 - t_1) - \int_{t_0}^{T} f(t)\,dt \\
&= (P-x)t_1 + xt_1 \\
&= Pt_1 \\
&= LHS
\end{aligned}
$$

Hence, the equality is true given that, $R(t_2 - t_1) = xt_1$, $I_3(t_2) = I_4(t_2)$ and $I_2(t_3) = I_5(t_3)$. Numerically, the equality has a value of 766.82 (correct to 2 decimal places).

We also have the equality where the total items produced is equal to the sum of the total defective items, the total demand in OW and the total deteriorated items in OW.

$$
P(t_1 - t_0) = x(t_1 - t_0) + \left[\int_{t_0}^{t_1} f(t)\,dt + \int_{t_3}^{T} f(t)\,dt\right] + \alpha(A_1 + A_2 + A_5)
$$

Focusing on the RHS of the equation, we have

$$\begin{aligned}
RHS &= x(t_1 - t_0) + \left[\int_{t_0}^{t_1} f(t)\,dt + \int_{t_3}^{T} f(t)\,dt\right] + \alpha(A_1 + A_2 + A_5) \\
&= xt_1 + \left[\int_{t_0}^{t_1} f(t)\,dt + \int_{t_3}^{T} f(t)\,dt\right] + \left[(P-x)t_1 - \int_{t_0}^{t_1} f(t)\,dt - W\right] \\
&+ [W - I_2(t_3)] + \left[I_5(t_3) - \int_{t_3}^{T} f(t)\,dt\right] \\
&= xt_1 + (P-x)t_1 \\
&= Pt_1 \\
&= LHS
\end{aligned}$$

The equality holds given that $I_2(t_3) = I_5(t_3)$, where the value is 766.82 (correct to 2 decimal places).

3.4.1.2 Maximum Inventory of OW, W

The maximum inventory of OW, W is equal to sum of the total deteriorated items in the interval t_1 to T and the total demand in the interval t_3 to T.

$$W = \alpha A_2 + \alpha A_5 + \int_{t_3}^{T} f(t)dt$$

Focusing on the RHS of the equation, we have

$$\begin{aligned}
RHS &= [W - I_2(t_3)] + \left[I_5(t_3) - \int_{t_3}^{T} f(t)\,dt\right] + \int_{t_3}^{T} f(t)\,dt \\
&= W \\
&= LHS
\end{aligned}$$

Hence, the equality holds given that $I_2(t_3) = I_5(t_3)$, where $W = 489.38$ (correct to 2 decimal places).

Besides that, we also have the equality where the maximum inventory of OW, W is equal to the total items produced subtract the sum of the total defective items, total demand from t_0 to t_1 and total deteriorated items from t_0 to t_1.

$$W = P(t_1 - t_0) - x(t_1 - t_0) - \int_{t_0}^{t_1} f(t)dt - \alpha A_1$$

The RHS of the equality is given as

$$\begin{aligned} RHS &= P(t_1 - t_0) - x(t_1 - t_0) - \int_{t_0}^{t_1} f(t)dt - \left[(P - x)t_1 - \int_{t_0}^{t_1} f(t)\,dt - W\right] \\ &= (P - x)t_1 - \int_{t_0}^{t_1} f(t)dt - (P - x)t_1 + \int_{t_0}^{t_1} f(t)\,dt + W \\ &= W \\ &= LHS \end{aligned}$$

Hence, the equality holds where $W = 489.38$ (correct to 2 decimal places). Numerically, we obtained the value of W where $W = I_1(t_1) = I_2(t_1) = 489.38$ (correct to 2 decimal places).

3.4.1.3 Total Deteriorated Items from t_1 to t_3 in OW

The total deteriorated items from t_1 to t_3 in OW is governed by the equality

$$\begin{aligned} \alpha A_2 &= I_2(t_1) - I_2(t_3) \\ &= I_1(t_1) - I_5(t_3) \end{aligned}$$

Substituting t_1 and t_3 into $\alpha A_2 = I_2(t_1) - I_2(t_3)$, the RHS of the equality is

$$\begin{aligned} RHS &= We^{\alpha(t_1-t_1)} - We^{\alpha(t_1-t_3)} \\ &= W - We^{\alpha(t_1-t_3)} \\ &= W\left[1 - e^{\alpha(t_1-t_3)}\right] \\ &= \alpha A_2 \\ &= LHS \end{aligned}$$

Similarly, substituting $I_1(t_1) = W$ and t_3 into $\alpha A_2 = I_1(t_1) - I_5(t_3)$, we have

$$\begin{aligned} RHS &= W - We^{\alpha(t_1-t_3)} \\ &= W\left[1 - e^{\alpha(t_1-t_3)}\right] \\ &= \alpha A_2 \\ &= LHS \end{aligned}$$

Numerically, the total deteriorated items from t_1 to t_3 in OW is 4.00 (correct to 2 decimal places).

We will now discuss on the equations involved in RW in the following subsections.

3.4.1.4 Total Reworked Items

In this model, we assumed that the rework process is perfect, hence the total reworked items is equal to the total deteriorated items. Numerically, the total reworked items and the total defective items are given as follows

$$R(t_2 - t_1) = x(t_1 - t_0) = 127.80 \text{ (correct to 2 decimal places)}$$

Therefore, the following equality holds,

$$R(t_2 - t_1) = x(t_1 - t_0)$$

We also have the equality where the total reworked items is equal to the sum of the total demand in RW and the total deteriorated items in RW.

$$R(t_2 - t_1) = \int_{t_1}^{t_3} f(t)\, dt + \alpha(A_3 + A_4)$$

Focusing on the RHS of the equation, we have

$$\begin{aligned}
RHS &= \int_{t_1}^{t_3} f(t)\, dt + \alpha A_3 + \alpha A_4 \\
&= \int_{t_1}^{t_3} f(t)\, dt + \left[R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\, dt - I_3(t_2)\right] + \left[I_4(t_2) - \int_{t_2}^{t_3} f(t)\, dt\right] \\
&= R(t_2 - t_1) + \int_{t_1}^{t_3} f(t)\, dt - \left[\int_{t_1}^{t_2} f(t)\, dt - \int_{t_2}^{t_3} f(t)\, dt\right] - I_3(t_2) + I_4(t_2) \\
&= R(t_2 - t_1) + \int_{t_1}^{t_3} f(t)\, dt - \int_{t_1}^{t_3} f(t)\, dt - I_3(t_2) + I_4(t_2) \\
&= R(t_2 - t_1) \\
&= LHS
\end{aligned}$$

Hence, the following equality holds where $\int_{t_1}^{t_3} f(t)\, dt = \int_{t_1}^{t_2} f(t)\, dt + \int_{t_2}^{t_3} f(t)\, dt$ and $I_3(t_2) = I_4(t_2)$. Numerically, the equality has a value of 127.80 (correct to 2 decimal places).

Next, we can also see that the total reworked items is equal to the total items produced subtract the sum of the total demand in OW and the total deteriorated items in OW. The equality is given as

$$R(t_2 - t_1) = P(t_1 - t_0) - \left[\left(\int_{t_0}^{t_1} f(t)\, dt + \int_{t_3}^{T} f(t)\right) dt + \alpha(A_1 + A_2 + A_5)\right]$$

Focusing on the RHS of the equation, we have

$$\begin{aligned}
RHS &= P(t_1 - t_0) - \left[\left(\int_{t_0}^{t_1} f(t)\,dt + \int_{t_3}^{T} f(t)\,dt\right) + \alpha A_1 + \alpha A_2 + \alpha A_5\right] \\
&= P(t_1 - t_0) - \left(\int_{t_0}^{t_1} f(t)\,dt + \int_{t_3}^{T} f(t)\,dt\right) - \left[(P - x)t_1 - \int_{t_0}^{t_1} f(t)\,dt - W\right] \\
&- [W - I_2(t_3)] - \left[I_5(t_3) - \int_{t_3}^{T} f(t)\,dt\right] \\
&= P(t_1 - t_0) - \int_{t_0}^{t_1} f(t)\,dt - \int_{t_3}^{T} f(t)\,dt - (P - x)t_1 + \int_{t_0}^{t_1} f(t)\,dt + W \\
&- W + I_2(t_3) - I_5(t_3) + \int_{t_3}^{T} f(t)\,dt \\
&= xt_1 + I_2(t_3) - I_5(t_3) \\
&= xt_1 \\
&= R(t_2 - t_1) \\
&= LHS
\end{aligned}$$

Hence, the equality holds given that $I_2(t_3) = I_5(t_3)$ and $xt_1 = R(t_2 - t_1)$. Numerically, the equality has a value of 127.80 (correct to 2 decimal places).

3.4.1.5 Inventory Level at t_i

By formulation, we observed that the inventory level of RW at t_2, $I_3(t_2)$ is equal to the difference between the total reworked items and the sum of the demand from t_1 to t_2 and the deteriorated items from t_1 to t_2 in RW. Hence, the equality is

$$I_3(t_2) = R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - \alpha A_3$$

Substituting (13) into the equality, we have

$$\begin{aligned} RHS &= R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - \left[R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - I_3(t_2)\right] \\ &= R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - R(t_2 - t_1) + \int_{t_1}^{t_2} f(t)\,dt + I_3(t_2) \\ &= I_3(t_2) \\ &= LHS \end{aligned}$$

Similarly, at t_2 another equality can be obtained. The inventory level of RW at t_2, $I_4(t_2)$ is equal to the sum of the demand from t_2 to t_3 and the total deteriorated items from t_2 to t_3 in RW. Hence, we have

$$I_4(t_2) = \int_{t_2}^{t_3} f(t)\,dt + \alpha A_4$$

Focusing on the RHS of the equality, we have

$$\begin{aligned} RHS &= \int_{t_2}^{t_3} f(t)\,dt + \left[I_4(t_2) - \int_{t_2}^{t_3} f(t)\,dt\right] \\ &= I_4(t_2) \\ &= LHS \end{aligned}$$

Therefore, both equality holds where $I_3(t_2) = I_4(t_2)$ and the numerical value is given as 49.22 (correct to 2 decimal places).

Next, we also have the equality where the inventory level a t_3 is equal to difference between the maximum inventory of OW and the deteriorated items from t_1 to t_3. The equality is given as

$$I_2(t_3) = W - \alpha A_2$$

Focusing on the RHS of the equation, we have

$$\begin{aligned} RHS &= W - [W - I_2(t_3)] \\ &= W - W + I_2(t_3) \\ &= I_2(t_3) \\ &= LHS \end{aligned}$$

Similarly, the inventory level at t_3 is equal to the sum of the demand from t_3 to T and the deteriorated items from t_3 to T in OW. The equality is given as

$$I_5(t_3) = \int_{t_3}^{T} f(t)\,dt + \alpha A_5$$

Substituting (15) into the equality, we have

$$\begin{aligned} RHS &= \int_{t_3}^{T} f(t)\,dt + I_5(t_3) - \int_{t_3}^{T} f(t)\,dt \\ &= I_5(t_3) \\ &= LHS \end{aligned}$$

Hence, both equality holds where $I_2(t_3) = I_5(t_3)$ and the numerical value is given as 485.38 (correct to 2 decimal places).

3.4.2 Solution Procedure

Numerical algorithms for constrained nonlinear optimization can be broadly categorized into gradient-based methods and direct search methods. Gradient search methods use first derivatives (gradients) or second derivatives (Hessians) information, while direct search methods do not use derivative information.

In this research, generalized reduced gradient (GRG) has been chosen as the solving method. Hence, the Microsoft Excel Solver is used as a solution tool. GRG converts the constrained problem into and unconstrained problem. The GRG method is an extension of the reduced gradient method to accommodate nonlinear inequality constraints. In this method, a search direction is found such that for any small move, the current active constraints remain precisely active.

The following algorithm is used

1. Set $t_0 = 0$.
2. Determine the values of t_1, t_3 and T which satisfy the following constraints: $I_1(t_0) = 0$, $I_1(t_1) = I_2(t_1)$, $I_3(t_1) = 0$, $I_3(t_2) = I_4(t_2)$, $I_4(t_3) = 0$, $I_2(t_3) = I_5(t_3)$ and $I_5(T) = 0$
3. Compute $t_2 = \frac{(R+x)t_1}{R}$.
4. Compute TRC using the equation (16).

Aside from the GRG method, we have utilized the Wolfram Language function which solves for numeric local constrained optimization which is known as the **FindMinimum** function. Hence, we have used this built-in function in Mathematica software to verify our results and note that both the Microsoft Excel Solver and Mathematica, provide the same results. The coding has been included in the appendix section of this research.

3.4.3 Sensitivity Analysis

We will now look at the sensitivity analysis of the parameters incorporated in this model. We observed that the total relevant cost, TRC^* changes significantly with the changes in the values of the selective parameters by ±25% of the optimal values in these models.

As the production rate, P increases, items are produced at a faster rate in a shorter period of time. Hence, fewer items are produced resulting in the fewer number of defective items. The decrement of the processing cost, rework processing cost and holding cost in RW, results in the decrement of TRC^*.

As the rework rate, R increases, the rework process requires a shorter duration to complete. The significant increment in the setup production cost and holding cost in RW results in the increment of TRC^*.

As the defective rate, x increases, the amount of total defective items increases. Hence, resulting in a significant increment of the rework processing cost and holding cost in RW and a slight increment in the setup production cost. Hence, resulting in the increment of TRC^*.

As the deterioration rate α increases, the total number of deteriorated items increases. In return, resulting in the increment of the deterioration cost. All other costs involved also increases except the holding costs in OW and RW. Hence, resulting in the increment of TRC^*.

As the initial rate of demand, a increases, we observed that the total items produced, total demand and total deteriorated items increase as well. As a result, all costs involved in the system showed increment except for the holding costs in OW and RW. Hence, as a increases, TRC^* increases as well.

As the rate at which the demand rate b increases, we observed that all costs involved in this model increases, except for the holding costs in OW and RW as well as the deteriorating cost. Hence, resulting in the increment of TRC^*.

The graphical representation for the LIFO system of TRC^* against the discussed parameters above is shown in Figure 3.2.

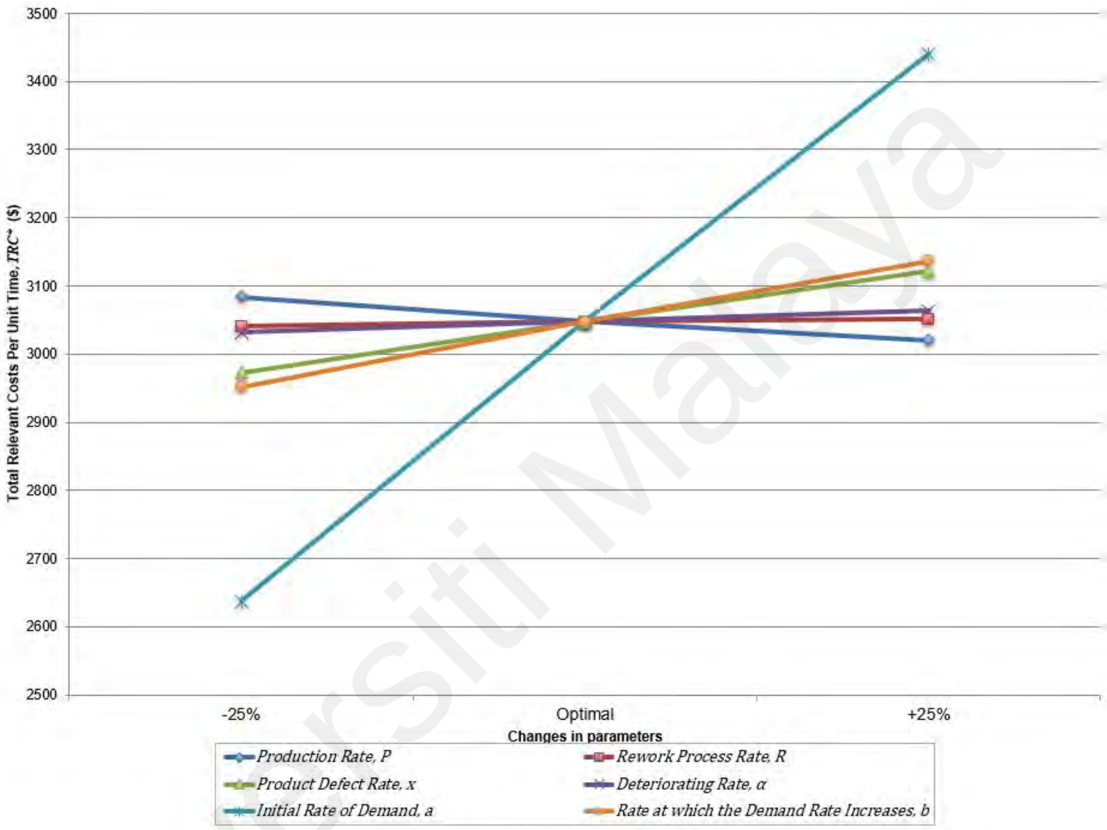

Figure 3.2: The optimal value of TRC^* with varying parameters.

Table 3.1 shows the changes in TRC^* as parameters are reduced and increased by 25% of the optimal values in the LIFO system.

Table 3.1: Analysis of change in various parameters on the total relevant cost, TRC^*.

Parameters	-25%, Optimal, 25%	TRC^* (2 d.p)	t_1	t_3	T	Total Items Produced	Total Defective Items	Total Demand	Total Deteriorated Items
P	2250	3083.67	0.3590	0.6349	1.1889	807.8162	179.5147	795.2416	12.5746
	3000	3047.39	0.2556	0.4609	1.1354	766.8200	127.8033	753.3743	13.4457
	3750	3020.29	0.1994	0.3636	1.1103	747.8007	99.7068	733.9174	13.8830
R	750	3040.90	0.2572	0.4638	1.1415	771.6592	128.6099	758.1646	13.4946
	1000	3047.39	0.2556	0.4609	1.1354	766.8200	127.8033	753.3743	13.4457
	1250	3051.26	0.2548	0.4594	1.1323	764.3777	127.3963	750.9474	13.4303
x	375	2972.45	0.2561	0.4116	1.1367	768.1875	96.0234	754.4014	13.7861
	500	3047.39	0.2556	0.4609	1.1354	766.8200	127.8033	753.3743	13.4457
	625	3121.25	0.2554	0.5097	1.1351	766.2228	159.6298	753.1308	13.0920
α	0.03	3031.63	0.2575	0.4642	1.1468	772.5589	128.7598	762.2700	10.2889
	0.04	3047.39	0.2556	0.4609	1.1354	766.8200	127.8033	753.3743	13.4457
	0.05	3063.01	0.2536	0.4573	1.1237	760.7275	126.7879	744.2683	16.4592
a	412.5	2637.03	0.2099	0.4297	1.1666	629.7372	104.9562	617.3395	12.3977
	550.0	3047.39	0.2556	0.4609	1.1354	766.8200	127.8033	753.3743	13.4457
	687.5	3440.62	0.3016	0.4979	1.1146	904.7851	150.7975	890.5478	14.2373
b	150	2951.18	0.2630	0.4797	1.2086	788.9703	131.4950	774.2570	14.7133
	200	3047.39	0.2556	0.4609	1.1354	766.8200	127.8033	753.3743	13.4457
	250	3136.46	0.2497	0.4454	1.0760	748.9665	124.8277	736.5201	12.4456

3.5 Conclusion

The total relevant cost, TRC is a nonlinear equation where its second derivative with respect to t_1, t_3 and T is complicated. In this research, a generalized reduced gradient (GRG) method has been chosen as the solving method in which it converts the constrained problem into and unconstrained problem. The GRG method is an extension of the reduced gradient method to accommodate nonlinear inequality constraints.

In this method, a search direction is found such that for any small move, the current active constraints remain precisely active. By using Microsoft Excel Solver as a solution tool, we are able to obtain and justify the optimal solution numerically and observed that the equation of TRC is convex and has optimal unique solution at ${t_1}^* = 0.2556$, ${t_3}^* = 0.4609$ and $T^* = 1.1354$ (correct to 4 decimal places).

Alternatively, we have also utilized the Wolfram Language function which solves for numeric local constrained optimization, also known as the **FindMinimum** function. This built in function in Mathematica software is used to verify the results obtained above, in which both the Microsoft Excel Solver and Mathematica software, provides the same results.

We further note that several past researches by Sett et el. (2012) and Lee & Hsu (2009) to name a few, exhibit similar results as obtained in this study. They have achieved a unique optimal solution for their proposed model. Similarly, for this particular set of parameters, we have obtained the optimal unique solution for TRC^* at the optimal times, where TRC^* is equal to \$3047.39.

CHAPTER 4: FIRST-IN-FIRST-OUT (FIFO)

4.1 Introduction

In this chapter, we will look at a two-warehouse inventory model while considering the First-In-First-Out (FIFO) policy. The mechanism of this policy is the opposite of the LIFO policy that have been discussed in Chapter 3.

Similar to the model introduced in the previous chapter, the first warehouse is known as the owned warehouse. When the owned warehouse has reached its maximum capacity, an alternative warehouse also known as the rented warehouse is required to store the excess items.

We have assumed that the production process is imperfect, hence resulting in the production of defective items. In order to ensure the smoothness of the production process, the defective items are kept aside and sent to the rented warehouse to undergo the rework process once the production process has ended.

We have incorporated the FIFO policy in the model, where inventories that are produced first are the ones to be exhausted first. The FIFO policy may be preferred by manufacturers who deal with inventories with high deterioration rate.

Items that deteriorate at a faster rate should be consumed or sold the soonest. Hence, in this case, assuming that the rented warehouse offers better facilities, items stored in the rented warehouse may last longer. The number of deteriorated items may also be lower. Hence, it is more cost efficient if the items that are stored in the owned warehouse first, are the items to be exhausted first. This phenomenon is known as the First-In-First-Out (FIFO) policy.

4.2 Mathematical Formulations

4.2.1 Notations

Listed below are the notations used in the FIFO policy model.

a	:	the initial rate of demand, where $f(t) = a + bt$
b	:	the rate at which the demand rate increases, where $f(t) = a + bt$
$f(t)$	:	linearly increasing demand rate $f(t) = a + bt$
D	:	total demand, where $D = \int_{t_0}^{T} f(t)\, dt$, units
P	:	constant production rate, units per unit time where $P > f(t)$ for all t
R	:	rework process rate, units per unit time where $R > f(t)$ for all t
x	:	product defect rate, units per unit time
α	:	deteriorating rate in OW and RW, units per unit time where $0 \leq \alpha \leq 1$
W	:	maximum inventory of OW, units
Q	:	maximum inventory of RW, units
t_i	:	time period in a cycle of stage i, where $0 \leq i \leq 3$
T	:	batch cycle time period
$I_i(t)$	:	inventory level at time t_i, where $0 \leq i \leq 6$
A_i	:	area under the curve $I_i(t)$, where $0 \leq i \leq 6$
S	:	production setup cost, \$ per setup
c_P	:	processing cost per unit item, \$
c_R	:	rework processing cost per unit item, \$
c_D	:	deterioration cost per unit item, \$
h_1, h_2	:	holding costs per unit item in OW and RW respectively
TRC	:	total relevant costs per unit time, \$

4.2.2 Abbreviations

Listed below are the abbreviations used in the FIFO policy model.

OW : owned warehouse

RW : rented warehouse

FIFO : first-in-first-out

GRG : generalized reduced gradient

4.2.3 Assumptions

The assumptions adopted in this study includes:

1. Lead time is zero and the replenishment rate is finite.
2. The production rate, P is larger than the demand rate, $f(t)$.
3. The rented warehouse, RW has an unlimited capacity.
4. Only perfect items are stored in OW, while all defective items are sent to RW to be reworked.
5. The rework process is assumed to be perfect since special care is given to the process, hence all reworked items are perfect. The total reworked items is equal to the total defective items.
6. The production and rework rates, P and R respectively, are assumed to be different.
7. The RW is located near the OW, hence the transportation cost between the warehouses is negligible.

4.3 Modelling FIFO

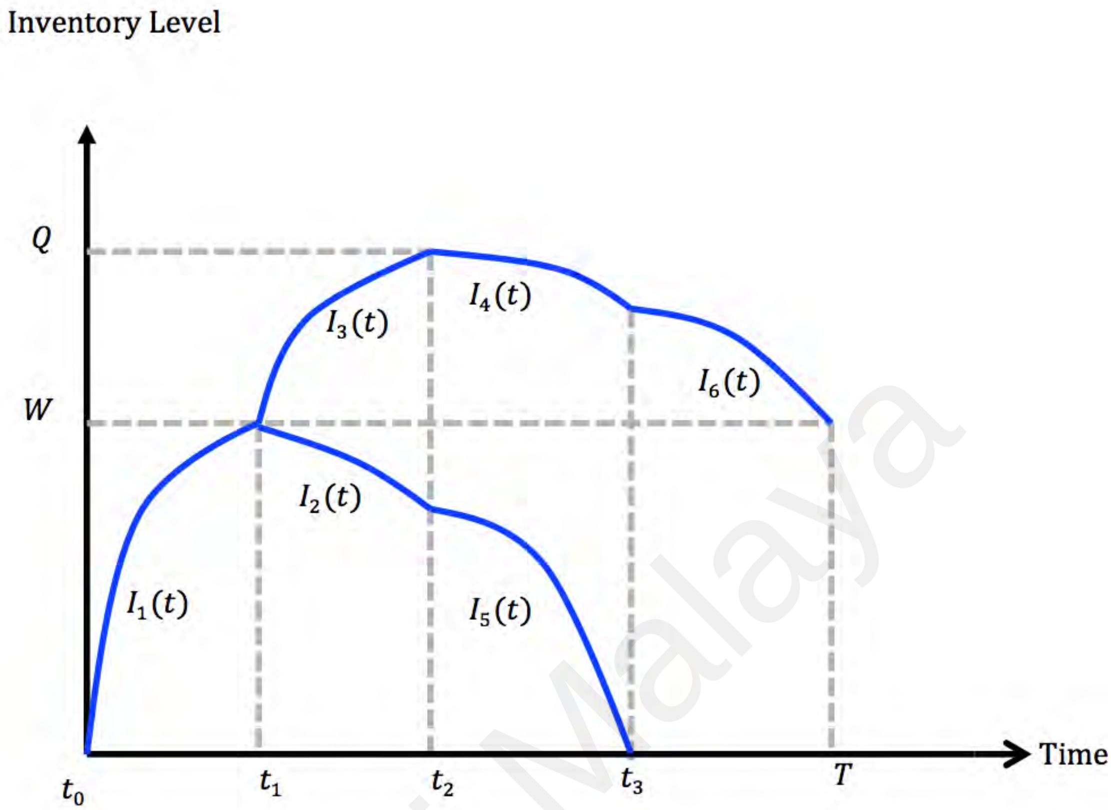

Figure 4.1: Inventory level of items with FIFO policy.

The level of inventory items in both warehouses, OW and RW is as illustrated in Figure 4.1. The interval $t_0 \leq t \leq t_1$ represents the production uptime period in OW, where some items produced in this interval are assumed to be non-defective or perfect and some are defective.

All defectives are separated to be sent to RW to undergo rework process. On the other hand, all perfect items produced are subjected to deterioration and used in satisfying demands. The inventory level in this interval is represented by $I_1(t)$.

The following interval, $t_1 \leq t \leq t_2$ represents the rework uptime period in RW. At t_1, all defectives from OW are sent to RW to be reworked. Since the rework process is assumed to be free of flaws, all reworked items are assumed to be perfect and as good as new. Hence, the total defective items is assumed to be equal to the total items reworked. The inventory level of items in this interval is represented by $I_3(t)$.

Note that, all item in this interval is subjected to deterioration and fulfilment of demand. On the other hand, items held in OW during the interval $t_1 \leq t \leq t_2$ will be subjected to deterioration only. The inventory level in this interval is represented by $I_2(t)$.

Since the FIFO policy is considered in this model, items in OW will be utilised completely before demands are satisfied by items in RW. Hence, in the interval $t_2 \leq t \leq t_3$ items in OW are used to satisfy demand and also subjected to deterioration until they are completely utilised.

The inventory level during the production down time period in OW is represented by $I_5(t)$. All items are assumed to be depleted completely at t_3. While in RW, items are subjected to deterioration only where the inventory level is represented by $I_4(t)$.

The final interval, $t_3 \leq t \leq T$ represents the rework down time period in RW. Items in this interval are depleted due to the satisfying of demand and deterioration. The inventory level in this interval is given as $I_6(t)$. All items will be fully depleted at T.

Note that, items are said to enter the owned warehouse first since they are produced in the OW up to t_1. After which, all collected defective items are then sent to RW to be reworked. At the end of the production cycle, items are fully depleted in the owned warehouse first, before all items are cleared in the rented warehouse. This phenomenon is known as the First-In-First-Out, FIFO policy.

The following subsections are divided based on the intervals $t_0 \leq t \leq t_1$, $t_1 \leq t \leq t_2$, $t_2 \leq t \leq t_3$, and $t_3 \leq t \leq T$.

4.3.1 Production up time in the interval $t_0 \leq t \leq t_1$

The production process begins in this interval, where some items produced are defective. All perfect items are stored in OW, while all defectives are separated and sent to RW to be reworked at t_1. Non-defective items are utilised to satisfy demands and are also subjected to deterioration. The maximum inventory level of OW is reached at t_1, where it is represented by W. Hence, the net replenishment rate can be formulated as

$$\frac{dI_1(t)}{dt} = P - x - f(t) - \alpha I_1(t), \text{ where } I_1(0) = 0 \text{ and } I_1(t_1) = W$$

$$\begin{aligned}
\frac{dI_1(t)}{dt} &= P - x - a - bt - \alpha I_1(t) \\
e^{\alpha t} I_1(t) &= \int e^{\alpha t}(P - x - a - bt)dt \\
&= \frac{1}{\alpha}(P - x - a)e^{\alpha t} - \frac{bte^{\alpha t}}{\alpha} + \int \frac{b}{\alpha}e^{\alpha t}dt + c_1 \\
&= \frac{1}{\alpha}(P - x - a)e^{\alpha t} - \frac{bte^{\alpha t}}{\alpha} + \frac{be^{\alpha t}}{\alpha^2} + C
\end{aligned}$$

Given that the initial condition, $I_1(t_0) = 0$, we have

$$\begin{aligned}
0 &= \frac{(P - x - a)}{\alpha} + \frac{b}{\alpha^2} + C \\
C &= -\frac{(P - x - a)}{\alpha} - \frac{b}{\alpha^2} \\
e^{\alpha t} I_1(t) &= \frac{(P - x - a)e^{\alpha t}}{\alpha} - \frac{bte^{\alpha t}}{\alpha} + \frac{be^{\alpha t}}{\alpha^2} - \frac{(P - x - a)}{\alpha} - \frac{b}{\alpha^2} \\
I_1(t) &= \frac{(P - x - a)}{\alpha} - \frac{bt}{\alpha} + \frac{b}{\alpha^2} - \frac{(P - x - a)}{\alpha}e^{-\alpha t} - \frac{b}{\alpha^2}e^{-\alpha t} \\
I_1(t) &= \left[\frac{(P - x - a)}{\alpha} + \frac{b}{\alpha^2}\right]\left(1 - e^{-\alpha t}\right) - \frac{bt}{\alpha}
\end{aligned}$$

$I_1(t)$ represents the inventory level at time t, within the interval $t_0 \le t \le t_1$, in which the inventory level is equivalent to the number of produced items excluding the defective and deteriorated items as well as the demand which has been fulfilled, at time t.

A_1 represents the area under the curve $I_1(t)$ in the interval $t_0 \le t \le t_1$, during the production uptime period in OW which is given as

$$\begin{aligned}
\int_0^{t_1} I_1(t)\,dt &= \left(\frac{P-x-a}{\alpha}+\frac{b}{\alpha^2}\right)t_1 + \frac{e^{-\alpha t_1}}{\alpha}\left(\frac{P-x-a}{\alpha}+\frac{b}{\alpha^2}\right) - \frac{bt_1{}^2}{2\alpha} \\
&- \frac{1}{\alpha}\left(\frac{P-x-a}{\alpha}+\frac{b}{\alpha^2}\right) \\
&= \left(\frac{P-x-a}{\alpha}+\frac{b}{\alpha^2}\right)t_1 - \frac{bt_1{}^2}{2\alpha} + \left(\frac{P-x-a}{\alpha^2}+\frac{b}{\alpha^3}\right)(e^{-\alpha t_1}-1) \\
A_1 &= \frac{(P-x)t_1}{\alpha} - \frac{1}{\alpha}\left(at_1 + \frac{b}{2}t_1{}^2\right) + \frac{bt_1}{\alpha^2} + \left(\frac{P-x-a}{\alpha^2}+\frac{b}{\alpha^3}\right)(e^{-\alpha t_1}-1)
\end{aligned}$$

We also note that the maximum inventory level of OW which is represented by W is governed by the equation $W = I_1(t_1)$. Hence, we have

$$W = \left[\frac{(P-x-a)}{\alpha}+\frac{b}{\alpha^2}\right]\left(1-e^{-\alpha t_1}\right) - \frac{bt_1}{\alpha}$$

We can see that the formulations in the production uptime period for this model is the same as the formulations in the LIFO policy model.

4.3.2 Interval $t_1 \le t \le t_2$ in OW

Produced items in OW from the previous interval, $t_0 \le t \le t_1$ are kept in storage during rework up time period in RW. The inventory level of the items in OW within the interval $t_1 \le t \le t_2$ decreases solely due to deterioration, while demands are fulfilled by items in RW. The net replenishment rate can be formulated as

$$\frac{dI_2(t)}{dt} = -\alpha I_2(t), \text{ where } I_2(t_1) = W$$

$$
\begin{aligned}
\frac{dI_2(t)}{dt} &= -\alpha I_2(t) \\
\frac{dI_2(t)}{dt} + \alpha I_2(t) &= 0 \\
e^{\alpha t} I_2(t) &= C
\end{aligned}
$$

Given the boundary condition, $I_2(t_1) = W$, we have

$$
\begin{aligned}
e^{\alpha t_1} W &= C \\
e^{\alpha t} I_2(t) &= e^{\alpha t_1} W \\
I_2(t) &= W e^{\alpha(t_1 - t)}
\end{aligned}
$$

$I_2(t)$ represents the inventory level at time t within the interval $t_1 \leq t \leq t_2$, in which the inventory level is equivalent to the number of items stored in OW after the interval $t_0 \leq t \leq t_1$ minus the deteriorated items at time t.

A_2 represents the area under the curve $I_2(t)$ in the interval $t_1 \leq t \leq t_2$, which is given as

$$
\begin{aligned}
\int_{t_1}^{t_2} I_2(t)\, dt &= -\frac{1}{\alpha} W e^{\alpha(t_1 - t_2)} + \frac{1}{\alpha} W \\
A_2 &= \frac{W}{\alpha} \left[1 - e^{\alpha(t_1 - t_2)} \right]
\end{aligned}
$$

4.3.3 Rework up time in the interval $t_1 \leq t \leq t_2$

All defective items sent to RW will undergo rework process during this interval with the rework rate, R. Since the rework process is assumed to be perfect, all items reworked are also assumed to be non-defective and as good as new. The inventory net replenishment rate in this interval can be formulated as

$$
\frac{dI_3(t)}{dt} = R - f(t) - \alpha I_3(t), \text{ where } I_3(t_1) = 0
$$

$$
\begin{aligned}
\frac{dI_3(t)}{dt} &= R - a - bt - \alpha I_3(t) \\
\frac{dI_3(t)}{dt} + \alpha I_3(t) &= R - a - bt \\
e^{\alpha t} I_3(t) &= \int (R - a - bt) e^{\alpha t} dt \\
&= \frac{1}{\alpha}(R - a) e^{\alpha t} - \int bt e^{\alpha t} dt + c_1 \\
&= \frac{1}{\alpha}(R - a) e^{\alpha t} - \frac{bt}{\alpha} e^{\alpha t} + \int \frac{b}{\alpha} e^{\alpha t} dt + c_2 \\
&= \frac{1}{\alpha}(R - a) e^{\alpha t} - \frac{bt}{\alpha} e^{\alpha t} + \frac{b}{\alpha^2} e^{\alpha t} + C
\end{aligned}
$$

Given the boundary condition, $I_3(t_1) = 0$, we have

$$
\begin{aligned}
C &= -\left[\frac{1}{\alpha}(R - a - bt_1) + \frac{b}{\alpha^2} \right] e^{\alpha t_1} \\
I_3(t) &= \frac{1}{\alpha}(R - a - bt) + \frac{b}{\alpha^2} - \left[\frac{1}{\alpha}(R - a - bt_1) + \frac{b}{\alpha^2} \right] e^{\alpha(t_1 - t)}
\end{aligned}
$$

$I_3(t)$ represents the inventory level at time t, within the interval $t_1 \leq t \leq t_2$, in which the inventory level is equivalent to the number of reworked items excluding the defective and deteriorated items as well as the demand which has been fulfilled, at time t.

A_3 represents the area under the curve $I_3(t)$ in the interval $t_1 \leq t \leq t_2$, during the rework up time period in RW which is given as

$$
\begin{aligned}
\int_{t_1}^{t_2} I_3(t)\,dt &= \frac{1}{\alpha}\left[(R-a)t - \frac{b}{2}t^2 + \frac{1}{\alpha}(R-a-bt_1)e^{\alpha(t_1-t)}\right] + \frac{b}{\alpha^2}\left[t + \frac{1}{\alpha}e^{\alpha(t_1-t)}\right] \\
A_3 &= \frac{1}{\alpha}\left[(R-a)(t_2-t_1) - \frac{b}{2}\left(t_2{}^2 - t_1{}^2\right) + \frac{1}{\alpha}(R-a-bt_1)\left[e^{\alpha(t_1-t_2)} - 1\right]\right] \\
&+ \frac{b}{\alpha^2}\left[(t_2-t_1) + \frac{1}{\alpha}\left[e^{\alpha(t_1-t_2)} - 1\right]\right] \\
&= \frac{1}{\alpha}(R-a)(t_2-t_1) - \frac{b}{2\alpha}\left(t_2{}^2 - t_1{}^2\right) + \frac{1}{\alpha^2}(R-a-bt_1)\left[e^{\alpha(t_1-t_2)} - 1\right] \\
&+ \frac{b}{\alpha^2}(t_2-t_1) + \frac{b}{\alpha^3}\left[e^{\alpha(t_1-t_2)} - 1\right] \\
&= \frac{1}{\alpha}R(t_2-t_1) - \frac{1}{\alpha}\left(at_2 + \frac{b}{2}t_2{}^2\right) + \frac{1}{\alpha}\left(at_1 + \frac{b}{2}t_1{}^2\right) + \frac{b}{\alpha^2}(t_2-t_1) \\
&+ \left[\frac{1}{\alpha^2}(R-a-bt_1) + \frac{b}{\alpha^3}\right]\left[e^{\alpha(t_1-t_2)} - 1\right]
\end{aligned}
$$

We can see that the formulation in the production uptime period for the FIFO policy model is the same as the formulation in the LIFO policy model.

4.3.4 Interval $t_2 \leq t \leq t_3$ in RW

Reworked items in RW from the previous interval, $t_1 \leq t \leq t_2$ are kept in storage during the production down time period in OW. Hence, the inventory level of the items in RW decreases solely due to deterioration only, while demands are fulfilled by items in OW. The net replenishment rate can be formulated as

$$\frac{dI_4(t)}{dt} = -\alpha I_4(t), \text{ where } I_4(t_2) = Q$$

$$
\begin{aligned}
\frac{dI_4(t)}{dt} &= -\alpha I_4(t) \\
\frac{dI_4(t)}{dt} + \alpha I_4(t) &= 0 \\
e^{\alpha t} I_4(t) &= C
\end{aligned}
$$

Given the boundary condition, $I_4(t_2) = Q$, we have

$$
\begin{aligned}
e^{\alpha t_2} Q &= C \\
e^{\alpha t} I_4(t) &= e^{\alpha t_2} Q \\
I_4(t) &= Q e^{\alpha(t_2 - t)}
\end{aligned}
$$

$I_4(t)$ represents the inventory level at time t, within the interval $t_2 \leq t \leq t_3$, in which the inventory level is equivalent to the number of reworked items stored in RW after the interval $t_1 \leq t \leq t_2$ excluding the deteriorated items at time t.

A_4 represents the area under the curve $I_4(t)$ in the interval $t_2 \leq t \leq t_3$, which is given as

$$
\begin{aligned}
\int_{t_2}^{t_3} I_4(t)\, dt &= -\frac{1}{\alpha} Q e^{\alpha(t_2 - t_3)} + \frac{1}{\alpha} Q \\
A_4 &= \frac{Q}{\alpha}\left[1 - e^{\alpha(t_2 - t_3)}\right]
\end{aligned}
$$

We also note that the maximum inventory level of RW, which is represented by Q is governed by the equation $Q = I_3(t_2)$. Hence, we have

$$
Q = \frac{1}{\alpha}(R - a - bt_2) + \frac{b}{\alpha^2} - \left[\frac{1}{\alpha}(R - a - bt_1) + \frac{b}{\alpha^2}\right] e^{\alpha(t_1 - t_2)}
$$

4.3.5 Production down time in the interval $t_2 \leq t \leq t_3$

All items in OW are now depleted completely in this interval due to the occurrence of deterioration and fulfilment of demand. While the reworked items are kept in RW until all items in OW has been completely consumed at t_3. The inventory net replenishment rate can be formulated as

$\frac{dI_5(t)}{dt} = -f(t) - \alpha I_5(t)$, where $I_5(t_3) = 0$

$$\begin{aligned}
\frac{dI_5(t)}{dt} &= -f(t) - \alpha I_5(t) \\
\frac{dI_5(t)}{dt} + \alpha I_5(t) &= -a - bt \\
e^{\alpha t} I_5(t) &= \int (-a - bt) e^{\alpha t} dt \\
&= -\frac{a}{\alpha} e^{\alpha t} - \frac{bt}{\alpha} e^{\alpha t} + \int \frac{b}{\alpha} e^{\alpha t} dt + c_1 \\
&= -\frac{a}{\alpha} e^{\alpha t} - \frac{bt}{\alpha} e^{\alpha t} + \frac{b}{\alpha^2} e^{\alpha t} + C \\
&= \left[-\frac{a}{\alpha} - \frac{bt}{\alpha} + \frac{b}{\alpha^2} \right] e^{\alpha t} + C
\end{aligned}$$

Given the boundary condition, $I_5(t_3) = 0$, we have

$$\begin{aligned}
C &= -\left[-\frac{a}{\alpha} - \frac{bt_3}{\alpha} + \frac{b}{\alpha^2} \right] e^{\alpha t_3} \\
&= \left[\frac{(a + bt_3)}{\alpha} - \frac{b}{\alpha^2} \right] e^{\alpha t_3} \\
I_5(t) &= \left[-\frac{(a + bt)}{\alpha} + \frac{b}{\alpha^2} \right] + \left[\frac{(a + bt_3)}{\alpha} - \frac{b}{\alpha^2} \right] e^{\alpha (t_3 - t)} \\
I_5(t) &= \left[\frac{1}{\alpha}(a + bt_3) - \frac{b}{\alpha^2} \right] e^{\alpha (t_3 - t)} - \frac{(a + bt)}{\alpha} + \frac{b}{\alpha^2}
\end{aligned}$$

$I_5(t)$ represents the inventory level at time t, within the interval $t_2 \leq t \leq t_3$, in which the inventory level is equivalent to the number of remaining items stored in OW after the interval $t_1 \leq t \leq t_2$ excluding the deteriorated items and demand which has been fulfilled, at time t.

A_5 represents the area under the curve $I_5(t)$ in the interval $t_2 \leq t \leq t_3$, during the production down time period in OW which is given as

$$\begin{aligned}
\int_{t_2}^{t_3} I_5(t)\,dt &= -\frac{1}{\alpha}\left\{\frac{1}{\alpha}(a+bt_3)\left[1-e^{\alpha(t_3-t_2)}\right]+a(t_3-t_2)+\frac{b}{2}({t_3}^2-{t_2}^2)\right\} \\
&+ \frac{b}{\alpha^2}\left\{(t_3-t_2)+\frac{1}{\alpha}\left[1-e^{\alpha(t_3-t_2)}\right]\right\} \\
A_5 &= -\frac{1}{\alpha^2}(a+bt_3)\left[1-e^{\alpha(t_3-t_2)}\right]-\frac{a(t_3-t_2)}{\alpha}-\frac{b}{2\alpha}({t_3}^2-{t_2}^2) \\
&+ \frac{b}{\alpha^2}(t_3-t_2)+\frac{b}{\alpha^3}\left[1-e^{\alpha(t_3-t_2)}\right] \\
&= \frac{1}{\alpha}\left(at_2+\frac{b}{2}{t_2}^2\right)-\frac{1}{\alpha}\left(at_3+\frac{b}{2}{t_3}^2\right)+\frac{b}{\alpha^2}(t_3-t_2) \\
&+ \left[\frac{1}{\alpha^2}(a+bt_3)-\frac{b}{\alpha^3}\right]\left[e^{\alpha(t_3-t_2)}-1\right]
\end{aligned}$$

4.3.6 Rework down time in the interval $t_3 \leq t \leq T$

Once all items in OW have been depleted and consumed completely at t_3, inventory items stored in RW will decrease due to deterioration and fulfilment of demands. The production cycle ends at T where all items in RW are also consumed completely. The inventory net replenishment rate can be formulated as

$$\frac{dI_6(t)}{dt} = -f(t) - \alpha I_6(t), \text{ where } I_6(T) = 0$$

$$\begin{aligned}
\frac{dI_6(t)}{dt} + \alpha I_6(t) &= -a - bt \\
e^{\alpha t} I_6(t) &= \int (-a-bt)e^{\alpha t}dt \\
&= -\frac{a}{\alpha}e^{\alpha t} - \int bte^{\alpha t}dt + c_1 \\
&= -\frac{a}{\alpha}e^{\alpha t} - \frac{bt}{\alpha}e^{\alpha t} + \int \frac{b}{\alpha}e^{\alpha t}dt + c_2 \\
&= -\frac{a}{\alpha}e^{\alpha t} - \frac{bt}{\alpha}e^{\alpha t} + \frac{b}{\alpha^2}e^{\alpha t} + C \\
&= \left[-\frac{(a+bt)}{\alpha} + \frac{b}{\alpha^2}\right]e^{\alpha t} + C
\end{aligned}$$

Given the boundary condition, $I_6(T) = 0$, we have

$$
\begin{aligned}
C &= \left[\frac{(a+bT)}{\alpha} - \frac{b}{\alpha^2}\right] e^{\alpha T} \\
I_6(t) &= \left[-\frac{(a+bt)}{\alpha} + \frac{b}{\alpha^2}\right] + \left[\frac{(a+bT)}{\alpha} - \frac{b}{\alpha^2}\right] e^{\alpha(T-t)} \\
I_6(t) &= \left[\frac{(a+bT)}{\alpha} - \frac{b}{\alpha^2}\right] e^{\alpha(T-t)} - \frac{(a+bt)}{\alpha} + \frac{b}{\alpha^2}
\end{aligned}
$$

$I_6(t)$ represents the inventory level at time t, within the interval $t_3 \leq t \leq T$, in which the inventory level is equivalent to the number of remaining items stored in RW after the interval $t_2 \leq t \leq t_3$ excluding the deteriorated items and demand which has been fulfilled, at time t.

A_6 represents the area under the curve $I_6(t)$ in the interval $t_3 \leq t \leq T$, during the rework down time period in RW which is given as

$$
\begin{aligned}
\int_{t_3}^{T} I_6(t)\,dt &= \left[1 - e^{\alpha(T-t_3)}\right]\left[\frac{b}{\alpha^3} - \frac{1}{\alpha^2}(a+bT)\right] - \frac{1}{\alpha}\left[a(T-t_3) + \frac{b}{2}\left(T^2 - {t_3}^2\right)\right] \\
&+ \frac{b(T-t_3)}{\alpha^2} \\
A_6 &= \frac{1}{\alpha}\left(at_3 + \frac{b}{2}{t_3}^2\right) - \frac{1}{\alpha}\left(aT + \frac{b}{2}T^2\right) + \frac{b}{\alpha^2}(T-t_3) \\
&+ \left[\frac{1}{\alpha^2}(a+bT) - \frac{b}{\alpha^3}\right]\left[e^{\alpha(T-t_3)} - 1\right]
\end{aligned}
$$

Hence, the governing differential equations stating the inventory levels within the production cycle are given as follows:

$$\frac{dI_1(t)}{dt} = P - x - f(t) - \alpha I_1(t) \qquad ; \qquad t_0 \leq t \leq t_1 \qquad (1)$$

$$\frac{dI_2(t)}{dt} = -\alpha I_2(t) \qquad ; \qquad t_1 \leq t \leq t_2 \qquad (2)$$

$$\frac{dI_3(t)}{dt} = R - f(t) - \alpha I_3(t) \qquad ; \qquad t_1 \leq t \leq t_2 \qquad (3)$$

$$\frac{dI_4(t)}{dt} = -\alpha I_4(t) \qquad ; \qquad t_2 \leq t \leq t_3 \qquad (4)$$

$$\frac{dI_5(t)}{dt} = -f(t) - \alpha I_5(t) \qquad ; \qquad t_2 \leq t \leq t_3 \qquad (5)$$

$$\frac{dI_6(t)}{dt} = -f(t) - \alpha I_6(t) \qquad ; \qquad t_3 \leq t \leq T \qquad (6)$$

where $f(t) = a + bt$.

Solving the differential equations (1) to (6) above, we have

$$I_1(t) = \left[\frac{(P - x - a)}{\alpha} + \frac{b}{\alpha^2}\right]\left(1 - e^{-\alpha t}\right) - \frac{bt}{\alpha} \qquad (7)$$

$$I_2(t) = We^{\alpha(t_1 - t)} \qquad (8)$$

$$I_3(t) = \frac{1}{\alpha}(R - a - bt) + \frac{b}{\alpha^2} - \left[\frac{1}{\alpha}(R - a - bt_1) + \frac{b}{\alpha^2}\right] e^{\alpha(t_1 - t)} \qquad (9)$$

$$I_4(t) = Qe^{\alpha(t_2 - t)} \qquad (10)$$

$$I_5(t) = \left[\frac{1}{\alpha}(a + bt_3) - \frac{b}{\alpha^2}\right] e^{\alpha(t_3 - t)} - \frac{(a + bt)}{\alpha} + \frac{b}{\alpha^2} \qquad (11)$$

$$I_6(t) = \left[\frac{(a + bT)}{\alpha} - \frac{b}{\alpha^2}\right] e^{\alpha(T - t)} - \frac{(a + bt)}{\alpha} + \frac{b}{\alpha^2} \qquad (12)$$

where the boundary conditions are given as follows;

$I_1(t_0) = 0$, $I_1(t_1) = W = I_2(t_1)$, $I_3(t_1) = 0$, $I_3(t_2) = Q = I_4(t_2)$,

$I_4(t_3) = I_6(t_3)$, $I_2(t_2) = I_5(t_2)$, $I_5(t_3) = 0$ and $I_6(T) = 0$.

The amount of inventory A_i under the curve $I_i(t)$ in each interval where $i = 1, 2, 3, 4, 5$ and 6 are given as follows:

In the interval $t_0 \le t \le t_1$,

$$A_1 = \left[\frac{(P-x)}{\alpha} + \frac{b}{\alpha^2}\right] t_1 - \frac{1}{\alpha}\left(at_1 + \frac{b}{2}{t_1}^2\right) + \left(\frac{P-x-a}{\alpha^2} + \frac{b}{\alpha^3}\right)\left(e^{-\alpha t_1} - 1\right) \tag{13}$$

In the interval $t_1 \le t \le t_2$,

$$A_2 = \frac{W}{\alpha}\left[1 - e^{\alpha(t_1 - t_2)}\right] \tag{14}$$

In the interval $t_1 \le t \le t_2$,

$$A_3 = \left[\frac{R}{\alpha} + \frac{b}{\alpha^2}\right](t_2 - t_1) + \frac{1}{\alpha}\left[\left(at_1 + \frac{b}{2}{t_1}^2\right) - \left(at_2 + \frac{b}{2}{t_2}^2\right)\right] + \left[\frac{1}{\alpha^2}(R - a - bt_1) + \frac{b}{\alpha^3}\right]\left[e^{\alpha(t_1 - t_2)} - 1\right] \tag{15}$$

In the interval $t_2 \le t \le t_3$,

$$A_4 = \frac{Q}{\alpha}\left[1 - e^{\alpha(t_2 - t_3)}\right] \tag{16}$$

In the interval $t_2 \le t \le t_3$,

$$A_5 = \frac{b}{\alpha^2}(t_3 - t_2) + \frac{1}{\alpha}\left[\left(at_2 + \frac{b}{2}{t_2}^2\right) - \left(at_3 + \frac{b}{2}{t_3}^2\right)\right] + \left[\frac{1}{\alpha^2}(a + bt_3) - \frac{b}{\alpha^3}\right]\left[e^{\alpha(t_3 - t_2)} - 1\right] \tag{17}$$

In the interval $t_3 \le t \le T$,

$$A_6 = \frac{b}{\alpha^2}(T - t_3) + \frac{1}{\alpha}\left[\left(at_3 + \frac{b}{2}{t_3}^2\right) - \left(aT + \frac{b}{2}T^2\right)\right] + \left[\frac{1}{\alpha^2}(a + bT) - \frac{b}{\alpha^3}\right]\left[e^{\alpha(T - t_3)} - 1\right] \tag{18}$$

4.3.7 Finding t_2

Next, assuming that all defective items are sent to RW to be reworked and the rework process is perfect, we may use the following equality to solve for t_2;

Total Defective Items = Total Reworked Items

Hence, we have the following equation

$$\begin{aligned} xt_1 &= R(t_2 - t_1) \\ xt_1 + Rt_1 &= Rt_2 \\ t_2 &= \frac{(R+x)t_1}{R} \end{aligned}$$

Note that, the formula for t_2 is also the same in the LIFO policy model.

4.3.8 Costs Involved in This Model

4.3.8.1 Setup Production, Processing and Rework Processing Costs

The setup production, processing and rework processing costs per unit time are computed as below respectively.

$$\begin{aligned} \text{Setup Production Cost} &= \frac{S}{T} \\ \text{Processing Cost} &= \frac{c_P P(t_1 - t_0)}{T} = \frac{c_P P t_1}{T} \\ \text{Rework Processing Cost} &= \frac{c_R x(t_1 - t_0)}{T} = \frac{c_R x t_1}{T} \end{aligned}$$

4.3.8.2 Inventory Carrying Cost

Next, the total inventory carrying cost is given as the sum of the holding costs in OW and RW, where we have

1. The total holding cost in OW per unit time = $\frac{h_1}{T}(A_1 + A_2 + A_5)$

   The total holding cost in OW per unit time

$$
\begin{aligned}
&= \frac{h_1}{T}\left\{\frac{1}{\alpha}\left[(P-x)t_1 - \left(at_1 + \frac{b}{2}{t_1}^2\right) + \left(at_2 + \frac{b}{2}{t_2}^2\right) - \left(at_3 + \frac{b}{2}{t_3}^2\right)\right]\right. \\
&\left. - \frac{1}{\alpha^2}(a + bt_2) + \frac{b}{\alpha^3} - \frac{W}{\alpha}e^{\alpha(t_1-t_2)} + \left[\frac{1}{\alpha^2}(a + bt_3) - \frac{b}{\alpha^3}\right]e^{\alpha(t_3-t_2)}\right\}
\end{aligned}
$$

2. The total holding cost in RW per unit time = $\frac{h_2}{T}(A_3 + A_4 + A_6)$

   The total holding cost in RW per unit time

$$
\begin{aligned}
&= \frac{h_2}{T}\left\{\frac{1}{\alpha}\left[R(t_2 - t_1) + \left(at_1 + \frac{b}{2}{t_1}^2\right) - \left(at_2 + \frac{b}{2}{t_2}^2\right) + \left(at_3 + \frac{b}{2}{t_3}^2\right) - \left(aT + \frac{b}{2}T^2\right)\right]\right. \\
&\left. - \frac{1}{\alpha^2}(a + bt_3) + \frac{b}{\alpha^3} - \frac{Q}{\alpha}e^{\alpha(t_2-t_3)} + \left[\frac{1}{\alpha^2}(a + bT) - \frac{b}{\alpha^3}\right]e^{\alpha(T-t_3)}\right\}
\end{aligned}
$$

Therefore, the total holding costs for both OW and RW per unit time, HC is given as

$$
\begin{aligned}
HC = &\frac{h_1}{T}\left\{\frac{1}{\alpha}\left[(P-x)t_1 - \left(at_1 + \frac{b}{2}{t_1}^2\right) - \left(at_3 + \frac{b}{2}{t_3}^2\right) + \left(at_2 + \frac{b}{2}{t_2}^2\right)\right] - \frac{1}{\alpha^2}(a + bt_2)\right. \\
&\left. + \frac{b}{\alpha^3} - \frac{W}{\alpha}e^{\alpha(t_1-t_2)} + \left[\frac{1}{\alpha^2}(a + bt_3) - \frac{b}{\alpha^3}\right]e^{\alpha(t_3-t_2)}\right\} \\
&+ \frac{h_2}{T}\left\{\frac{1}{\alpha}\left[R(t_2 - t_1) + \left(at_1 + \frac{b}{2}{t_1}^2\right) - \left(at_2 + \frac{b}{2}{t_2}^2\right) + \left(at_3 + \frac{b}{2}{t_3}^2\right) - \left(aT + \frac{b}{2}T^2\right)\right]\right. \\
&\left. - \frac{1}{\alpha^2}(a + bt_3) + \frac{b}{\alpha^3} - \frac{Q}{\alpha}e^{\alpha(t_2-t_3)} + \left[\frac{1}{\alpha^2}(a + bT) - \frac{b}{\alpha^3}\right]e^{\alpha(T-t_3)}\right\}
\end{aligned}
$$

4.3.8.3 Deteriorating Cost

The total number of deteriorated items is equal to the product of deterioration rate and the total area under the curves $I_1(t), I_2(t), I_3(t), I_4(t), I_5(t)$ and $I_6(t)$. Hence, we have the equality

The total number of deteriorated items, $G = \alpha$(The total area under the curves)

$$\begin{aligned} G &= \alpha(A_1 + A_2 + A_3 + A_4 + A_5 + A_6) \\ &= Pt_1 - \left(aT + \frac{b}{2}T^2\right) - \frac{1}{\alpha}(a + bt_2) - \frac{1}{\alpha}(a + bt_3) + \frac{2b}{\alpha^2} - We^{\alpha(t_1-t_2)} - Qe^{\alpha(t_2-t_3)} \\ &+ \left[\frac{1}{\alpha}(a + bt_3) - \frac{b}{\alpha^2}\right] e^{\alpha(t_3-t_2)} + \left[\frac{1}{\alpha}(a + bT) - \frac{b}{\alpha^2}\right] e^{\alpha(T-t_3)} \end{aligned}$$

Hence, the deteriorating cost per unit time, DC is given by

$$\begin{aligned} DC &= \frac{c_D G}{T} \\ &= \frac{c_D}{T}\Big\{Pt_1 - \left(aT + \frac{b}{2}T^2\right) - \frac{1}{\alpha}(a + bt_2) - \frac{1}{\alpha}(a + bt_3) + \frac{2b}{\alpha^2} - We^{\alpha(t_1-t_2)} - Qe^{\alpha(t_2-t_3)} \\ &+ \left[\frac{1}{\alpha}(a + bt_3) - \frac{b}{\alpha^2}\right] e^{\alpha(t_3-t_2)} + \left[\frac{1}{\alpha}(a + bT) - \frac{b}{\alpha^2}\right] e^{\alpha(T-t_3)}\Big\} \end{aligned}$$

4.3.9 The Total Relevant Cost, TRC

The total relevant cost per unit time for the FIFO policy is given as

$$
\begin{aligned}
TRC = {} & \text{Setup Production Cost} + \text{Processing Cost} + \text{Rework Processing Cost} \\
& + \text{Holding Cost in OW} + \text{Holding Cost in RW} + \text{Deteriorating Cost} \\
= {} & \frac{S}{T} + \frac{c_P P t_1}{T} + \frac{c_R x t_1}{T} \\
& + \frac{h_1}{T}\Bigg\{\frac{1}{\alpha}\left[(P-x)t_1 - \left(at_1 + \frac{b}{2}{t_1}^2\right) + \left(at_2 + \frac{b}{2}{t_2}^2\right) - \left(at_3 + \frac{b}{2}{t_3}^2\right)\right] - \frac{1}{\alpha^2}(a+bt_2) \\
& + \frac{b}{\alpha^3} - \frac{W}{\alpha}e^{\alpha(t_1-t_2)} + \left[\frac{1}{\alpha^2}(a+bt_3) - \frac{b}{\alpha^3}\right]e^{\alpha(t_3-t_2)}\Bigg\} \\
& + \frac{h_2}{T}\Bigg\{\frac{1}{\alpha}\left[R(t_2-t_1) + \left(at_1 + \frac{b}{2}{t_1}^2\right) - \left(at_2 + \frac{b}{2}{t_2}^2\right) + \left(at_3 + \frac{b}{2}{t_3}^2\right) - \left(aT + \frac{b}{2}T^2\right)\right] \\
& - \frac{1}{\alpha^2}(a+bt_3) + \frac{b}{\alpha^3} - \frac{Q}{\alpha}e^{\alpha(t_2-t_3)} + \left[\frac{1}{\alpha^2}(a+bT) - \frac{b}{\alpha^3}\right]e^{\alpha(T-t_3)}\Bigg\} \\
& + \frac{c_D}{T}\Bigg\{Pt_1 - \left(aT + \frac{b}{2}T^2\right) - \frac{1}{\alpha}(a+bt_2) - \frac{1}{\alpha}(a+bt_3) + \frac{2b}{\alpha^2} - We^{\alpha(t_1-t_2)} - Qe^{\alpha(t_2-t_3)} \\
& + \left[\frac{1}{\alpha}(a+bt_3) - \frac{b}{\alpha^2}\right]e^{\alpha(t_3-t_2)} + \left[\frac{1}{\alpha}(a+bT) - \frac{b}{\alpha^2}\right]e^{\alpha(T-t_3)}\Bigg\} \qquad (19)
\end{aligned}
$$

4.4 Numerical Examples, Solution Procedure and Sensitivity Analysis

4.4.1 Numerical Examples

The following parameters are considered to show a clear illustration of the FIFO policy model. The production rate per unit time, P is 3000, rework rate per unit time, R is 1000, defective rate per unit time, x is 500, deterioration rate per unit time, α is 0.04, the initial rate of demand, a is 550 and the rate with which the demand rate increases, b is 200, where $f(t) = a + bt$ is the demand rate per unit time.

The costs involved in this model are given as follows, where the setup production cost per setup, S is \$1000, the processing cost per unit item, c_P is \$2, rework processing cost per unit item, c_R is \$3, deterioration cost per unit item, d is \$2.50, holding cost per unit item in OW, h_1 is \$1.50 and holding cost per unit item in RW, h_2 is \$2.50.

We will now discuss on the equations involved in OW in the following subsections.

4.4.1.1 Total Items Produced

The total items produced is equal to the sum of the total demand and deteriorated items. Hence, we have the equality

$$
\begin{aligned}
P(t_1 - t_0) &= \int_{t_0}^{T} f(t)\,dt + \alpha(A_1 + A_2 + A_3 + A_4 + A_5 + A_6) \\
\text{where,} & \\
\alpha A_1 &= (P - x)t_1 - \int_{t_0}^{t_1} f(t)\,dt - W \\
\alpha A_2 &= W - I_2(t_2) \\
\alpha A_3 &= R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - I_3(t_2) \\
\alpha A_4 &= Q - I_4(t_3) \\
\alpha A_5 &= I_5(t_2) - \int_{t_2}^{t_3} f(t)\,dt \\
\alpha A_6 &= I_6(t_3) - \int_{t_3}^{T} f(t)\,dt
\end{aligned}
$$

Focusing on the RHS of the equation, we have the following

$$
\begin{aligned}
RHS &= \int_{t_0}^{T} f(t)\,dt + \alpha(A_1 + A_2 + A_3 + A_4 + A_5) \\
&= \int_{t_0}^{T} f(t)\,dt + \left[(P-x)t_1 - \int_{t_0}^{t_1} f(t)\,dt - W\right] + [W - I_2(t_2)] \\
&\quad + \left[R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - I_3(t_2)\right] + [Q - I_4(t_3)] \\
&\quad + \left[I_5(t_2) - \int_{t_2}^{t_3} f(t)\,dt\right] + \left[I_6(t_3) - \int_{t_3}^{T} f(t)\,dt\right] \\
&= Pt_1 - W + W + R(t_2 - t_1) - xt_1 + Q - I_3(t_2) \\
&\quad + I_5(t_2) - I_2(t_2) + I_6(t_3) - I_4(t_3) \\
&\quad + \int_{t_0}^{T} f(t)\,dt - \left[\int_{t_0}^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^{t_3} f(t)\,dt + \int_{t_3}^{T} f(t)\,dt\right] \\
&= Pt_1
\end{aligned}
$$

Hence, the equality is true given that, $R(t_2 - t_1) = xt_1$, $Q = I_3(t_2)$, $I_2(t_2) = I_5(t_2)$ and $I_4(t_3) = I_6(t_3)$. Numerically, the equality has a value of 754.71 (correct to 2 decimal places).

We also have the equality where the total items produced is equal to the sum of the total defective items, the total demand in OW and the total deteriorated items in OW.

$$
P(t_1 - t_0) = x(t_1 - t_0) + \left[\int_{t_0}^{t_1} f(t)\,dt + \int_{t_2}^{t_3} f(t)\,dt\right] + \alpha(A_1 + A_2 + A_5)
$$

Focusing on the RHS of the equation, we have

$$\begin{aligned}
RHS &= x(t_1 - t_0) + \left[\int_{t_0}^{t_1} f(t)\,dt + \int_{t_2}^{t_3} f(t)\,dt\right] + \alpha(A_1 + A_2 + A_5) \\
&= xt_1 + \left[\int_{t_0}^{t_1} f(t)\,dt + \int_{t_2}^{t_3} f(t)\,dt\right] + \left[(P - x)t_1 - \int_{t_0}^{t_1} f(t)\,dt - W\right] \\
&+ [W - I_2(t_2)] + \left[I_5(t_2) - \int_{t_2}^{t_3} f(t)\,dt\right] \\
&= Pt_1 \\
&= LHS
\end{aligned}$$

The equality holds given that $I_2(t_2) = I_5(t_2)$, where the value is 754.71 (correct to 2 decimal places).

4.4.1.2 Maximum Inventory of OW, W

The maximum inventory of OW, W is equal to sum of the total deteriorated items in the interval t_1 to t_3 and the total demand in the interval t_2 to t_3.

$$W = \alpha A_2 + \alpha A_5 + \int_{t_2}^{t_3} f(t)dt$$

Substituting equations (20) and (23) into the RHS of the equation, we have

$$\begin{aligned}
RHS &= [W - I_2(t_2)] + \left[I_5(t_2) - \int_{t_2}^{t_3} f(t)\,dt\right] + \int_{t_2}^{t_3} f(t)\,dt \\
&= W \\
&= LHS
\end{aligned}$$

Hence, the equality holds given that $I_2(t_2) = I_5(t_2)$, where $W = 481.79$ (correct to 2 decimal places).

Besides that, we also have the equality where the maximum inventory of OW, W is equal to the total items produced subtract the sum of the total defective items, total demand from t_0 to t_1 and total deteriorated items from t_0 to t_1.

$$W = P(t_1 - t_0) - x(t_1 - t_0) - \int_{t_0}^{t_1} f(t)dt - \alpha A_1$$

Focusing on the RHS of the equality, we have

$$\begin{aligned} RHS &= P(t_1 - t_0) - x(t_1 - t_0) - \int_{t_0}^{t_1} f(t)dt - \left[(P - x)t_1 - \int_{t_0}^{t_1} f(t)\, dt - W\right] \\ &= (P - x)t_1 - \int_{t_0}^{t_1} f(t)dt - (P - x)t_1 + \int_{t_0}^{t_1} f(t)\, dt + W \\ &= W \\ &= LHS \end{aligned}$$

Hence, the equality holds where $W = 481.79$ (correct to 2 decimal places).

Numerically, we obtained the value of W where $W = I_1(t_1) = I_2(t_1) = 481.79$ (correct to 2 decimal places).

4.4.1.3 Total Deteriorated Items from t_1 to t_2 in OW

The total deteriorated items from t_1 to t_2 in OW is governed by the equality

$$\alpha A_2 = I_2(t_1) - I_2(t_2) \tag{20}$$

Substituting t_1 and t_2 into (8), the RHS of the equality (25) is

$$\begin{aligned} RHS &= We^{\alpha(t_1-t_1)} - We^{\alpha(t_1-t_2)} \\ &= W - We^{\alpha(t_1-t_2)} \\ &= W\left[1 - e^{\alpha(t_1-t_2)}\right] \\ &= \alpha A_2 \\ &= LHS \end{aligned}$$

Numerically, the total deteriorated items from t_1 to t_2 in OW is 2.42 (correct to 2 decimal places).

We will now discuss on the equations involved in RW in the following subsections.

4.4.1.4 Total Reworked Items

In this model, we assumed that the rework process is perfect, hence the total reworked items is equal to the total deteriorated items where we have the equality

$$R(t_2 - t_1) = x(t_1 - t_0) \text{ (correct to 2 decimal places)}$$

Therefore, the following equality holds,

$$R(t_2 - t_1) = x(t_1 - t_0)$$

We also have the equality where the total reworked items is equal to the sum of the total demand in RW and the total deteriorated items in RW.

$$R(t_2 - t_1) = \int_{t_1}^{t_2} f(t)\, dt + \int_{t_3}^{T} f(t)\, dt + \alpha(A_3 + A_4 + A_6)$$

Focusing on the RHS of the equation, we have

$$RHS = \int_{t_1}^{t_2} f(t)\,dt + \int_{t_3}^{T} f(t)\,dt + \alpha(A_3 + A_4 + A_6)$$

$$= \int_{t_1}^{t_2} f(t)\,dt + \int_{t_3}^{T} f(t)\,dt + \left[R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - I_3(t_2)\right]$$

$$+ [Q - I_4(t_3)] + \left[I_6(t_3) - \int_{t_3}^{T} f(t)\,dt\right]$$

$$= \int_{t_1}^{t_2} f(t)\,dt - \int_{t_1}^{t_2} f(t)\,dt + \int_{t_3}^{T} f(t)\,dt - \int_{t_3}^{T} f(t)\,dt + R(t_2 - t_1)$$

$$- I_3(t_2) + Q - I_4(t_3) + I_6(t_3)$$

$$= R(t_2 - t_1)$$

$$= LHS$$

Hence, the following equality holds where $I_3(t_2) = Q$ and $I_4(t_3) = I_6(t_3)$.

Next, we can also see that the total reworked items is equal to the total items produced subtract the sum of the total demand in and deteriorated items in OW. The equality is given as

$$R(t_2 - t_1) = P(t_1 - t_0) - \left[\left(\int_{t_0}^{t_1} f(t)\,dt + \int_{t_2}^{t_3} f(t)\right) dt + \alpha(A_1 + A_2 + A_5)\right]$$

Focusing on the RHS of the equality, we have

$$
\begin{aligned}
RHS &= P(t_1 - t_0) - \left[\left(\int_{t_0}^{t_1} f(t)\,dt + \int_{t_2}^{t_3} f(t)\right) dt + \alpha(A_1 + A_2 + A_5)\right] \\
&= P(t_1 - t_0) - \left(\int_{t_0}^{t_1} f(t)\,dt + \int_{t_2}^{t_3} f(t)\,dt\right) - \left[(P - x)t_1 - \int_{t_0}^{t_1} f(t)\,dt - W\right] \\
&- [W - I_2(t_2)] - \left[I_5(t_2) - \int_{t_2}^{t_3} f(t)\,dt\right] \\
&= Pt_1 - (P - x)t_1 - \int_{t_0}^{t_1} f(t)\,dt + \int_{t_0}^{t_1} f(t)\,dt - \int_{t_2}^{t_3} f(t)\,dt + \int_{t_2}^{t_3} f(t)\,dt \\
&+ W - W + I_2(t_2) - I_5(t_2) \\
&= xt_1 \\
&= R(t_2 - t_1) \\
&= LHS
\end{aligned}
$$

Hence, the equality holds given that $I_2(t_2) = I_5(t_2)$ and $xt_1 = R(t_2 - t_1)$. Numerically, the equalities above have a value of 125.78 (correct to 2 decimal places).

4.4.1.5 Inventory Level at t_i

By formulation, we observed that the inventory level of RW at t_2, $I_3(t_2)$ is equal to the difference between the total reworked items and the sum of the demand from t_1 to t_2 and the deteriorated items from t_1 to t_2 in RW. Hence, the equality is

$$I_3(t_2) = R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - \alpha A_3$$

Focusing on the RHS of the equality, we have

$$
\begin{aligned}
RHS &= R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - \left[R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - I_3(t_2)\right] \\
&= R(t_2 - t_1) - \int_{t_1}^{t_2} f(t)\,dt - R(t_2 - t_1) + \int_{t_1}^{t_2} f(t)\,dt + I_3(t_2) \\
&= I_3(t_2) \\
&= LHS
\end{aligned}
$$

Similarly, at t_2 another equality can be obtained. The inventory level of RW at t_2, $I_4(t_2)$ is equal to the total deteriorated items from t_2 to T and the sum of the demand from t_3 to T in RW. Hence, we have

$$I_4(t_2) = \alpha A_4 + \alpha A_6 + \int_{t_3}^{T} f(t)\, dt$$

Focusing on the RHS of the equality, we have

$$\begin{aligned} RHS &= Q - I_4(t_3) + I_6(t_3) - \int_{t_3}^{T} f(t)\, dt + \int_{t_3}^{T} f(t)\, dt \\ &= Q \\ &= I_4(t_2) \\ &= LHS \end{aligned}$$

Therefore, both equality holds where $I_4(t_3) = I_6(t_3)$ and the numerical value is given as 48.57 (correct to 2 decimal places).

Next, we also have the equality where the inventory level a t_3 is equal to difference between the maximum inventory of OW and the deteriorated items from t_1 to t_2. The equality is given as

$$I_2(t_2) = W - \alpha A_2$$

Focusing on the RHS of the equality, we have

$$\begin{aligned} RHS &= W - [W - I_2(t_2)] \\ &= W - W + I_2(t_2) \\ &= I_2(t_2) \\ &= LHS \end{aligned}$$

Similarly, the inventory level at t_2, $I_5(t_2)$ is equal to the sum of the demand from t_2 to t_3 and the deteriorated items from t_2 to t_3 in OW. The equality is given as

$$I_5(t_2) = \int_{t_2}^{t_3} f(t)\,dt + \alpha A_5$$

Substituting (23) into the equality, we have

$$\begin{aligned} RHS &= \int_{t_2}^{t_3} f(t)\,dt + I_5(t_2) - \int_{t_2}^{t_3} f(t)\,dt \\ &= I_5(t_2) \\ &= LHS \end{aligned}$$

Hence, both equality holds where $I_2(t_2) = I_5(t_2)$ and the numerical value is given as 479.37 (correct to 2 decimal places).

4.4.2 Solution Procedure

Numerical algorithms for constrained nonlinear optimization can be broadly categorized into gradient-based methods and direct search methods. Gradient search methods use first derivatives (gradients) or second derivatives (Hessians) information, while direct search methods do not use derivative information.

In this research, generalized reduced gradient (GRG) has been chosen as the solving method. Hence, the Microsoft Excel Solver is used as a solution tool. GRG converts the constrained problem into and unconstrained problem. The GRG method is an extension of the reduced gradient method to accommodate nonlinear inequality constraints. In this method, a search direction is found such that for any small move, the current active constraints remain precisely active.

The following algorithm is used

1. Set $t_0 = 0$.
2. Determine the values of t_1, t_3 and T which satisfy the following constraints: $I_1(t_0) = 0$, $I_1(t_1) = I_2(t_1)$, $I_3(t_1) = 0$, $I_3(t_2) = I_4(t_2)$, $I_4(t_3) = I_6(t_3)$, $I_2(t_2) = I_5(t_2)$, $I_5(t_3) = 0$ and $I_6(T) = 0$
3. Compute $t_2 = \frac{(R+x)t_1}{R}$.
4. Compute TRC using the equation (19).

Besides than the GRG method, we have utilized the Wolfram Language function which solves for numeric local constrained optimization which is known as the **FindMinimum** function. Hence, we have used this built-in function in Mathematica software to verify our results and note that both the Microsoft Excel Solver and Mathematica software, provide the same results. The coding has been included in the appendix section of this research.

4.4.3 Sensitivity Analysis

We will now look at the sensitivity analysis of the parameters incorporated in this model. We observed that the total relevant cost, TRC^* changes significantly with the changes in the values of the selective parameters by ±25% of the optimal values in these models.

As the production rate, P increases, items are produced at a faster rate in a shorter period of time. Hence, fewer items are produced resulting in the fewer number of defective items. The decrement of the processing cost, rework processing cost and holding cost in RW, results in the decrement of TRC^*.

As the rework rate, R increases, the rework process requires a shorter duration to complete. The significant increment in the setup production cost and holding cost in RW results in the increment of TRC^*.

As the defective rate, x increases, the amount of total defective items increases. Hence, resulting in a significant increment of the rework processing cost and holding cost in RW and a slight increment in the setup production cost. Hence, resulting in the increment of TRC^*.

As the deterioration rate α increases, the total number of deteriorated items increases. In return, resulting in the increment of the deterioration cost. All other costs involved also increases except the holding costs in OW and RW. Hence, resulting in the increment of TRC^*.

As the initial rate of demand, a increases, we observed that the total items produced, total demand and total deteriorated items increase as well. As a result all costs involved in the system showed increment except for the holding cost in RW. Hence, as a increases, TRC^* increases as well.

As the rate at which the demand rate b increases, we observed that all costs involved in this model increases, except for the holding costs in OW and RW as well as the deteriorating cost. Hence, resulting in the increment of TRC^*.

4.5 Comparison between LIFO and FIFO

In general, the TRC^* of the LIFO system is lower than the TRC^* of the FIFO system where $TRC^*{}_{LIFO} = \$3047.39 < TRC^*{}_{FIFO} = \$3076.34$. Based on the sensitivity analysis of both models, the following are the features that we have identified.

We observed that the value of t_1 in the FIFO system is slightly lower than the value of t_1 in the LIFO system. As a result, the total produced and defective items are also lower in the FIFO system.

Since T is also slightly lower in the FIFO system, we can see that the total demand is slightly lower in the FIFO system as well. Besides, the lower value of T in the FIFO system resulted in the higher setup production cost in the system as compared to the LIFO system.

In addition, the total produced and the defective items in the FIFO system is lower than the LIFO system. Hence, we can see that the processing cost and rework processing cost are also lower in the FIFO system as compared to the LIFO system.

The items are stored in RW longer in the FIFO system compared to the LIFO system. Hence, resulting in the higher holding cost in RW in the FIFO system compared to the LIFO system.

Taking every cost involved into account for both systems, the significant higher holding cost in RW for the FIFO system contributed to the higher TRC^* in the FIFO system.

Hence, we can conclude that given the same value of parameters, the LIFO system has a lower total relevant cost, TRC^* when making the assumption that RW holds a higher holding cost compared to the OW.

We also note that Lee (2006) has proven that when the deterioration rate in both the owned and rented warehouses is the same, FIFO is less expensive than LIFO, provided that the holding cost in RW is lower than OW. This is similar to the result that we have obtained, in which the total relevant cost, TRC in the FIFO policy model is more expensive than LIFO, assuming that the holding cost in RW is higher compared to OW.

The graphical representation for the FIFO system of TRC^* against the discussed parameters above is shown in Figure 4.2.

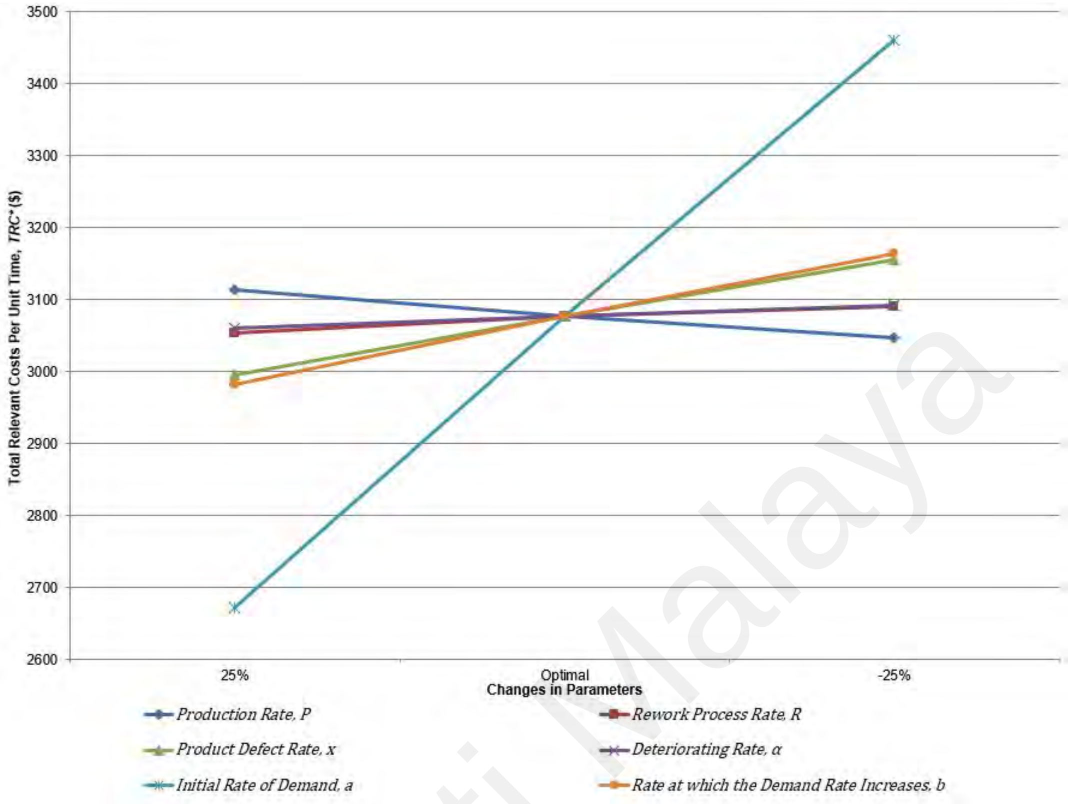

Figure 4.2: The optimal value of TRC^* with varying parameters.

Table 4.1 shows the changes in TRC^{*} as parameters is reduced and increased by 25% of the optimal values in the FIFO system.

Table 4.1: Analysis of change in various parameters on the total relevant cost, TRC^{*}.

Parameters	-25%, Optimal, 25%	TRC^{*} (2 d.p.)	t_1	t_3	T	Total Items Produced	Total Defective Items	Total Demand	Total Deteriorated Items
	2250	3113.57	0.3537	1.0939	1.1740	795.7450	176.8322	783.5066	12.2384
P	3000	3076.34	0.2516	1.0588	1.1203	754.7062	125.7844	741.6470	13.0592
	3750	3046.72	0.1963	1.0459	1.0958	736.2182	98.1624	722.7293	13.4889
	750	3054.10	0.2560	1.1089	1.1370	768.0176	128.0029	754.6393	13.3783
R	1000	3076.34	0.2516	1.0588	1.1203	754.7062	125.7844	741.6470	13.0592
	1250	3089.49	0.2490	1.0297	1.1107	747.1043	124.5174	734.2252	12.8791
	375	2995.95	0.2527	1.0777	1.1242	758.1528	94.7691	744.6952	13.4576
x	500	3076.34	0.2516	1.0588	1.1203	754.7062	125.7844	741.6470	13.0592
	625	3154.57	0.2508	1.0417	1.1180	752.5220	156.7754	739.8549	12.6671
	0.03	3060.93	0.2534	1.0700	1.1312	760.0954	126.6826	750.1093	9.9862
α	0.04	3076.34	0.2516	1.0588	1.1203	754.7062	125.7844	741.6470	13.0592
	0.05	3091.63	0.2498	1.0488	1.1097	749.4750	124.9125	733.4581	16.0169
	412.5	2672.17	0.2051	1.0611	1.1451	615.3688	102.5615	603.4772	11.8916
a	550.0	3076.34	0.2516	1.0588	1.1203	754.7062	125.7844	741.6470	13.0592
	687.5	3460.15	0.2992	1.0688	1.1070	897.5913	149.5986	883.5625	14.0288
	150	2982.44	0.2581	1.1199	1.1892	774.3726	129.0621	760.1604	14.2123
b	200	3076.34	0.2516	1.0588	1.1203	754.7062	125.7844	741.6470	13.0592
	250	3163.42	0.2463	1.0090	1.0642	739.0036	123.1673	726.8563	12.1473

4.6 Conclusion

Similar to the LIFO policy, the total relevant cost, TRC is a nonlinear equation where its second derivative with respect to t_1, t_3 and T is complicated. In this research, a generalized reduced gradient (GRG) method has been chosen as the solving method in which it converts the constrained problem into and unconstrained problem. The GRG method is an extension of the reduced gradient method to accommodate nonlinear inequality constraints.

In this method, a search direction is found such that for any small move, the current active constraints remain precisely active. By using Microsoft Excel Solver as a solution tool, we are able to obtain and justify the optimal solution numerically and observed that the equation of TRC is convex and has optimal unique solution at ${t_1}^* = 0.2516$, ${t_3}^* = 1.0588$ and $T^* = 1.1203$ (correct to 4 decimal places).

Alternatively, we have also utilized the Wolfram Language function which solves for numeric local contsrained optimization, also known as the **FindMinimum** function. This built-in function in Mathematica software is used to verify the results obtained above, in which both the Microsoft Excel Solver and Mathematica software, provides the same results.

We further note that several past researches by Sett et el. (2012) and Lee & Hsu (2009) to name a few, exhibit similar results as obtained in this study. They have achieved a unique optimal solution for their proposed model. Similarly, for this particular set of parameters, we have obtained the optimal unique solution for TRC^* at the optimal times, where TRC^* is equal to \$3076.34.

CHAPTER 5: CONCLUSION AND FURTHER RESEARCH

Most literatures on two-warehouse inventory model have assumed that all items produced during the production process are of perfect items. Hence, rework process was not considered in most models. Moreover, most researchers have also assumed that the replenishment rate is infinite in their models.

In this research, a two-warehouse inventory model with time varying demand rate and rework process is considered. Both LIFO and FIFO policies are also incorporated, where we assume that the holding cost in RW is higher than the holding cost in OW. This assumption is made on the basis that the RW provides better storage facilities while special care is given to the processes done in RW.

We have utilised Microsoft Excel Solver as a solution tool, where the generalized reduced gradient (GRG Nonlinear) has been chosen as the solving method. The optimum values of t_1, t_3 and T which resulted in the minimum value of TRC were obtained numerically using this method for both LIFO and FIFO systems. A sensitivity analysis was also conducted respectively for both systems to provide illustration on the derived results.

It has been proven that by varying the selected parameters, we observed that given the same changes are made to the parameters in both the LIFO and FIFO systems, a lower total relevant cost, TRC^* is obtained in the LIFO system. This shall mean that the LIFO system is less expensive than the FIFO system, provided that the holding cost in RW is higher than the holding cost in OW.

The LIFO flow of inventory suggests that items which have been stored the latest in the owned warehouse will be dispatched first. Realistically this is an important factor for producers or manufacturers as they are able to ensure that items are dispatched or distributed while they are at the optimal state of freshness. Besides that, this will also allow sufficient time for consumers to consume the items prior to the expiry date of the items.

5.1 Further Research

There are many other aspects that can be further explored and studied. Different types of demand rate such as multivariate demand rate, exponential demand rate, and linearly decreasing demand rate can be considered for future study to suit the type of items produced in an inventory system.

In addition, the presence of shortages and backlogged can also be incorporated to produce a more realistic model. Shortages and backlogged are commonly present in the market when the demand is higher than the supply. Incorporating this factor in the model would be beneficial in planning the right amount of items to be produced to meet the demand.

The deterioration rates in OW and RW can be further explored in which each warehouse holds a different deterioration rate. This is more realistic as different storage space may have different facilities, which may result in the difference in the rate items deteriorate.

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