NOVEL CONTRIBUTIONS TO FRACTIONAL CALCULUS: COMPUTATIONAL METHODS AND ANALYSIS

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FACULTY OF SCIENCE
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NOVEL CONTRIBUTIONS TO FRACTIONAL CALCULUS: COMPUTATIONAL METHODS AND ANALYSIS

ABSTRACT

Various computational methods are proposed and employed for solving various classes of fractional differential equations. In addition to the computational part, an analysis part is provided for some interesting research problems. First, novel exact soliton solutions of the $(3+1)$-dimensional conformable Wazwaz-Benjamin-Bona-Mahony equation are investigated via the generalized Kudryashov and $\exp(-\phi(\aleph))$ Then, a modified nonlinear Schrödinger equation with spatio-temporal dispersion is constructed in the contexts of both Caputo fractional derivative and conformable derivative. This proposed equation is solved via a new generalized double Laplace transform coupled with Adomian decomposition method. Second, the multivariable calculus is investigated in the sense of conformable derivative. The conformable derivative of a real-valued function of several variables and all related properties are also investigated. An extension to vector valued functions of several real variables is studied. The conformable chain rule for functions of several variables is also introduced. The conformable implicit function theorem for several variables is established. In addition, a new definition of generalized fractional derivative, named Abu-Shady–Kaabar fractional derivative, is also proposed. Third, the solutions' existence and stability analysis of a newly proposed fractional boundary value problem are investigated for an implicit nonlinear variable order fractional differential equation with the help of both Krasnoselskii's fixed point theorem and the criterion of Ulam–Hyers–Rassias stability. Fourth, a generalized version of the Mittag–Leffler–Hyers–Ulam stability of quadratic fractional integral equation is investigated. Fifth, the existence of extremal solutions for a novel class of ψ-Caputo fractional differential equation with nonlinear boundary conditions is studied by employing the monotone iterative technique together with the method of upper and lower solutions. Sixth, the oscillation of even-order nonlinear differential equations with mixed nonlinear neutral terms is investigated, and new oscillation criteria are established.

Keywords: Fractional Calculus, Fractional Differential Equations, Mathematical Methods, Mathematical Analysis, Stability Analysis.

SUMBANGAN NOVEL KEPADA KALKULUS PECAHAN: KAEDAH DAN ANALISIS PENGIRAAN

ABSTRAK

Pelbagai kaedah perhitungan dicadangkan dan digunakan dalam menyelesaikan pelbagai kelas persamaan pembezaan pecahan. Sebagai tambahan kepada bahagian perhitungan, bahagian analisis disediakan untuk beberapa masalah penyelidikan yang menarik. Pada permulaan soliton tepat baharu bagi persamaan Wazwaz-Benjamin-Bona-Mahony dimensi-$(3+1)$ boleh laras diselidiki melalui persamaan Kudryashov dan $\exp(-\phi(\aleph))$ yang digeneralisasikan. Kemudian, persamaan Schrödinger tak linear yang diubah suai dengan serakan ruang-masa dibina dalam konteks kedua-dua terbitan pecahan Caputo dan terbitan boleh laras. Persamaan yang dicadangkan ini diselesaikan melalui penjelmaan Laplace berganda umum yang baharu ditambah dengan kaedah penguraian Adomian. Kedua, kalkulus berbilang pemboleh ubah disiasat dalam erti kata terbitan boleh laras. Terbitan boleh selaras bagi fungsi nilai sebenar bagi beberapa pemboleh ubah dan semua sifat berkaitan juga disiasat. Sambungan kepada fungsi bernilai vektor bagi beberapa pemboleh ubah nyata dikaji. Petua rantai boleh selaras untuk fungsi beberapa pemboleh ubah juga diperkenalkan. Teorem fungsi tersirat boleh laras untuk beberapa pemboleh ubah ditubuhkan. Di samping itu, takrifan baharu terbitan pecahan umum, dinamakan terbitan pecahan Abu-Shady–Kaabar, juga dicadangkan. Ketiga, kewujudan dan analisis kestabilan penyelesaian masalah nilai sempadan pecahan baharu dicadangkan kaji untuk persamaan pembezaan pecahan tertib pemboleh ubah tak linear tersirat dengan bantuan kedua-dua teorem titik tetap Krasnoselskii dan kriteria kestabilan Ulam–Hyers–Rassias. Keempat, versi umum bagi kestabilan Mittag–Leffler–Hyers–Ulam bagi persamaan kamiran pecahan kuadratik disiasat.

Kelima, kewujudan penyelesaian ekstrem untuk kelas novel persamaan pembezaan pecahan ψ-Caputo dengan keadaan sempadan tak linear dikaji dengan menggunakan teknik lelaran monoton bersama-sama kaedah penyelesaian atas dan bawah. Keenam, ayunan bagi persamaan pembezaan tak linear tertib genap dengan sebutan neutral tak linear bercampur disiasat, dan kriteria ayunan baru diwujudkan.

Kata kunci: Kalkulus Pecahan, Persamaan Pembezaan Pecahan, Kaedah Matematik, Analisis Matematik, Analisis Kestabilan.

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$\zeta-DF$	: $\zeta-$Differentiable Function
$\ni$	: Such That
w.r.t.	: With Respect To
ASK	: Abu-Shady–Kaabar
ADcM	: Adomian Decomposition Method
BFPThm	: Banach's Fixed Point Theorem
BS	: Banach Space
BeBoMaEq	: Benjamin-Bona-Mahony Equation
BPM	: Bernstein Polynomials Method
BVPs	: Boundary Value Problems
Cp	: Caputo
CpDLTr	: Caputo Double Laplace Transform
CpFD	: Caputo Fractional Derivative
CpFDfEq	: Caputo Fractional Differential Equation
ChR	: Chain Rule
ComD	: Conformable Derivative
CmDLTr	: Conformable Double Laplace Transform
ComV	: Conformable Version
CF	: Continuous Function
CoMp	: Contraction Mapping
DFs	: Differentiable Functions
DEqsOB	: Differential Equations' Oscillatory Behavior

EHPM	: Enhanced Homotopy Perturbation Method
ExM	: Exp($-\phi(\aleph)$) Method
FBVP	: Fractional Boundary Value Problem
FrCL	: Fractional Calculus
FDfEq	: Fractional Differential Equation
FIE	: Fractional Integral Equation
GF	: Gamma Function
GCMSp	: Generalized Complete Metric Space
GeKM	: Generalized Kudryashov Method
GMr	: Generalized Metric
HBPrp	: Homogeneous Balance Principle
H-U	: Hyers–Ulam
H-U-R	: Hyers–Ulam–Rassias
ImFThm	: Implicit Function Theorem
IABMM	: Improved Adams-Bashforth-Moulton Method
IEs	: Integral Equations
IFDLT	: Inverse Fractional Double Laplace Transform
KFPThm	: Krasnoselskii's Fixed Point Theorem
KMNC	: Kuratowski Measure of Noncompactness
LSo	: Lower Solution
MF	: Matrix Form
MeVaThm	: Mean Value Theorem
ML	: Mittag–Leffler

MLFs	: Mittag–Leffler Functions
MNLNTs	: Mixed Nonlinear Neutral Terms
ML-H-U	: ML–Hyers–Ulam
ML-H-U-R	: ML–Hyers–Ulam–Rassias
MHPM	: Modified Homotopy Perturbation Method
MoMeVaThm	: Modified Mean Value Theorem
MoNLSEq	: Modified Nonlinear Schrödinger Equation
MuCL	: Multivariable calculus
NComD	: Non-Conformable Derivative
NLBCs	: Nonlinear Boundary Conditions
NLDfEq	: Nonlinear Differential Equations
NLEEq	: Nonlinear Evolution Equation
NLPrDfEqs	: Nonlinear Partial Differential Equations
OB	: Open Ball
OS	: Open Set
ODEq	: Ordinary Differential Equation
PaDr	: Partial Derivative
PrDfEqs	: Partial Differential Equations
QIEs	: Quadratic Integral Equations
ReVaF	: Real-Valued Function
RLb	: Riemann–Lebesgue
RL	: Riemann-Liouville

RLFr	:	Riemann-Liouville fractional
RoThm	:	Rolle's Theorem
SeReVs	:	Several Real Variables
SeVs	:	Several Variables
U-H-R	:	Ulam–Hyers–Rassias
UqCF	:	Unique Continuous Function
UF	:	Unique Function
USo	:	Upper Solution
VOFBVP	:	Variable Order Fractional Boundary Value Problem
VOFDfEq	:	Variable Order Fractional Differential Equation
VORLFr	:	Variable Order Riemann-Liouville Fractional
VeVaFs	:	Vector Valued Functions
WaBeBoMaEq	:	Wazwaz-Benjamin-Bona-Mahony Equation
WLOG	:	Without Loss of Generality

LIST OF APPENDICES

CHAPTER 1: INTRODUCTION

1.1 Introduction

Fractional calculus was theoretically initiated in the seventeenth century during a mathematical conversation between two well-known mathematicians: L'Hôpital and Leibniz concerning the possibility of extending the derivative of integer order to the derivative of order 0.5. After this proposed discussion, other mathematicians investigated the fractional order derivative via the nth derivative of the power function expression (Almeida et al., 2019). The definitions of fractional derivative are categorized into two parts: global (classical) nature and local nature. The global fractional derivative is nonlocal with a memory and expressed as integral transformations. The most common examples of this category are Riemann-Liouville and Caputo. However, the local fractional derivative is locally defined through certain incremental ratios such as conformable derivative. Fractional calculus has shown a high capability in the applications of various research topics related to physics, electromagnetics, mechanics, signal processing, biology, and economics. The standard derivatives' properties cannot be satisfied by many classical fractional derivatives. The property that has been satisfied in almost all derivatives are the linear property.

In many definitions of fractional derivatives, the non-locality property is essential for studying many scientific phenomena. However, all fractional definitions have both advantages and disadvantages. Therefore, there is no single suitable definition that can work perfectly for all models. This makes fractional calculus to be considered as an open research problem. One of the most employed definitions in studying real-world processes is Riemann-Liouville (RL) which is expressed as: (Atangana & Secer, 2013):

Definition 1. *For $\zeta \in [n-1, n)$ where $n \in \mathbb{N}$, the ζ-derivative of $\Psi(t)$ is:*

$$\mathfrak{D}^{RL}\Psi(t) = \frac{1}{\Gamma(n-\zeta)} \frac{d^n}{dx^n} \int_a^t \frac{\Psi(x)}{(t-x)^{\zeta-n+1}} dx. \tag{1.1}$$

The second commonly used definition in engineering is Caputo (Cp) definition which is expressed as:

Definition 2. *For $\zeta \in [n-1, n)$ where $n \in \mathbb{N}$, the ζ-derivative of $\Psi(t)$ is:*

$$\mathfrak{D}^{C}\Psi(t) = \frac{1}{\Gamma(n-\zeta)} \int_a^t \frac{\frac{d^n\Psi(x)}{dx^n}}{(t-x)^{\zeta-n+1}} dx. \tag{1.2}$$

While RL and Cp derivatives share the linear property of derivatives and have been employed in various applications, they are not suitable for all applications. The RL derivative of constant is not zero, but the Cp derivative of constant is zero. In RL type, if there is a constant arbitrary function at the origin, its RL derivative has singularity at the origin like Mittag-Leffler and exponential functions. However, in Cp type, the differentiability needs higher regularity conditions. To find the Cp derivative of a function, its derivative is needed to be found at first where Cp derivative is defined only for differentiable functions.

In (Khalil et al., 2014), a recent initiated locally-defined fractional derivative, simply called conformable derivative (ComD), relied on limit-based derivative's definition. ComD is written as:

Definition 3. *For a function $\Psi : [0, \infty) \to \Re$ such that (∋) $\forall\, t > 0$, the ComD of order $\zeta \in (0, 1]$ of Ψ can be expressed as:*

$$\mathfrak{D}_t^{\zeta}(\Psi(t)) = \lim_{\Omega \to 0} \frac{\Psi(t + \Omega t^{1-\zeta}) - \Psi(t)}{\Omega}. \tag{1.3}$$

Suppose that Ψ *is* ζ*-differentiable function* ($\zeta-$*DF) in some* $(0, \varrho)$*, where* $\varrho > 0$*, and the limit of* $D_t^{\zeta}(\Psi(t))$ *exists as* $t \longrightarrow 0^{+}$*, then from Eq.* (1.3)*, we get:*

$$\mathfrak{D}_t^{\zeta}(\Psi(0)) = \lim_{t \to 0^{+}} D_t^{\zeta}(\Psi(t)). \tag{1.4}$$

Note that by substituting $\zeta = 1$, the usual limit-based derivative is obtained. One of the advantages of ComD is that many properties of derivatives are satisfied by employing this definition in comparison to other definitions. Therefore, ComD has been actively employed in many applications in physics, natural sciences, and engineering. ComD has also been applied in studying partial differential equations to provide exact soliton solutions to many nonlinear models in mathematical physics.

Guzman et al. (2018) another local definition proposed, named non-conformable derivative (NComD), which is similar to ComD, and the only difference is that the functional kernel in NComD is $e^{t^{-\zeta}}$ instead of the root function $t^{1-\zeta}$ in ComD. Many NComD results have been provided in (Guzmán et al., 2020; Valdés et al., 2018).

Recently, some new studies have discussed a possible extension of the theory of the fractional calculus of constant order, known as the fractional calculus of variable order. In this recent topic, the studied system's order, which is represented as a function of either independent or dependent variables, is varied continuously to describe the memory's changes with respect to either time or space (Baleanu et al., 2011). Initially, the variable-order fractional operators have been proposed by (Lorenzo & Hartley, 2000) to interpret the diffusion process's behaviors. More applications about this interesting topic have been discussed in (Sheng et al., 2011; Sun et al., 2009). Bouazza et al. (2021) proposed a multi-term variable order fractional boundary value problem and showed that under some conditions, there exists exactly one solution to the investigated system. Recently, limited research works on investigating the fractional constant order boundary value problems (BVPs). However, the solutions' existence to fractional BVPs of variable order have been rarely studied (see (Sousa & de Oliveira, 2018; Tavares et al., 2016; Yang et al., 2018)).

Integral equations can be encountered in modeling scenarios arising in natural sciences, physics, and engineering. One of the most useful types of such equations is the quadratic integral equation due to the variety of its applications in traffic, neutron transport, and queuing theories (Argyros, 1985; Busbridge, 1960). Therefore, this equation can be well-investigated in the context of fractional calculus to provide a suitable modeling tool for many real-world applications.

1.2 Research Questions (Statements of Research Problems)

Our research work has been conducted based on the following important research questions:

* What is the physical meaning and importance of the obtained exact and approximate-analytical solutions?
* Which properties should be provided to investigate the proposed fractional differential equations' solutions?
* What are the applications of the investigated fractional differential equations' solutions in physics, engineering, or in any related area?
* What are the theoretical frameworks for investigating the obtained solutions' uniqueness, existence, and stability?
* Can we propose a new technique or generalized definition of fractional derivative that can provide more efficient solutions than the other studied methods?

* Will the proposed methods or definitions be effective in solving fractional differential equations in comparison to other existing methods?
* Is it possible to propose a new technique to investigate the oscillation of nonlinear differential equations?

Various classes of differential equations have been studied in this research work to provide a new path to investigate many interesting phenomena arising from physics and engineering.

1.3 Research Objectives

The following are the research objectives to be accomplished in this research work:

* To formulate several fractional differential equations' classes in the context of fractional derivatives/integrals or a combination of different fractional definitions.
* To construct novel exact and approximate-analytical solutions for the proposed fractional differential equations.
* To provide theoretical and numerical investigations of several fractional formulations for various phenomena from physics and engineering.
* To develop and propose new methods and generalized definitions of fractional derivatives for studying and solving fractional differential equations.
* To compare various analytical and approximate analytical techniques for various types of fractional derivatives for certain interesting models.
* To construct a comprehensive investigation of the obtained solutions to various classes of fractional differential equations via fractional calculus properties, fixed point theorems, and other related essential approaches.

* To study the oscillation of nonlinear differential equations via a newly proposed technique.

1.4 Scopes of Research

The following is a list of our research scopes in this research study:

* Two nonlinear partial differential equations: $(3 + 1)$-dimensional Wazwaz–Benjamin–Bona–Mahony equation formulated in the context of ComD, and modified nonlinear Schrödinger equation with spatio-temporal dispersion formulated in the contexts of both Cp fractional derivative and ComD.
* A detailed investigation of multivariable calculus formulated in the context of ComD.
* The new generalized fractional derivative definition, named as Abu-Shady-Kaabar fractional derivative, to solve fractional differential equations.
* The existence and Ulam–Hyers–Rassias stability of fractional BVP's solutions for an implicit nonlinear variable-order fractional differential equation.
* An investigation of the quadratic fractional integral equation using a generalized Mittag-Leffler function.
* An investigation of the ψ-Caputo fractional differential equation with nonlinear boundary conditions by studying the extremal solutions' existence.
* A unique study of the oscillation of even-order nonlinear differential equations with mixed nonlinear neutral terms.

1.5 Novelty of Our Proposed Research Work

The novelty of our proposed research work is listed as follows:

* Investigating nonlinear partial differential equations can physically interpret various models and the dynamics of their solutions arising in mathematical

physics. Therefore, investigating the Wazwaz–Benjamin–Bona–Mahony and modified nonlinear Schrödinger equation with spatio-temporal dispersion is very helpful in understanding many scientific phenomena arising in oceanography, optics, electromagnetism, and optical communication.

* A detailed investigation of multivariable conformable calculus provides a novel tool for modeling phenomena in physics and engineering due to the need for mathematical analysis in many modeling scenarios.
* ComD satisfies some important properties that cannot be satisfied in RL and Cp definitions. Therefore, there is always a great need to propose a new generalized fractional derivative.
* Abu-Shady-Kaabar fractional derivative is proposed in a purpose to overcome all issues of other derivatives to obtain efficiently fractional differential equations' solutions.
* Diverse applications of variable-order spaces of fractional type require a series of systematic approaches to investigate fractional differential equation's solutions such as existence, uniqueness, and stability.
* Studying quadratic fractional integral equations provides a significant tool in modeling scientific scenarios due to the essential properties of fractional calculus in investigating systems' dynamics and behavior.
* A rare technique, known as monotone iterative technique, has been employed along with upper and lower solutions' technique to investigate the ψ-Caputo fractional differential equation with nonlinear boundary conditions.
* The investigation of differential equations' oscillation with nonlinear neutral terms which has been rarely mentioned in other research works.

Motivated by all recent research studies, this research work provides novel contributions to fractional calculus by proposing new computational methods or developing techniques for solving various types of fractional differential equations. In addition, the investigation of solutions to various classes of fractional differential equations are fully studied in this research. While there are some research studies that have attempted to investigate the computational method and mathematical analysis of fractional calculus, there are still many open research problems that have not been investigated in any research works. Therefore, there is a great need to investigate such problems due to their important role in various applications in natural sciences and engineering.

1.6 Importance and relevance of the study

All our expected results will provide a major contribution to the field of fractional calculus and its applications due to the importance of the fractional differential equations in various natural sciences and engineering phenomena. The degree of novelty of our expected research works will be very high, and our work will be highly appreciated and accepted in top-quality scientific journals due to the expected original and novel results that will be supported by many simulation results and application problems from all aspects of science and engineering. In addition, our results which will be obtained in this work will be useful for physicists and engineers who use nonlinear partial differential equations formulated in the sense of fractional derivatives to propose new mathematical models that explain various natural phenomena. According to the best of our knowledge, none of our expected results have been obtained in any other previous works concerning the computational methods and analysis of fractional calculus because we will propose new problems formulated in the sense of fractional calculus. The existence, uniqueness, and stability of obtained solutions will be investigated in detail with many illustrative numerical examples that will be also provided in our work to validate our results' applicability. Our focus in our research on the importance of proposed problems and their wide applications in natural sciences and engineering. We always make sure that our proposed problems have never been introduced in any other previous works. If they are found to be proposed previously in some related works, our goal is to extend these problems to something more interesting by formulating it into new or generalized derivatives and applying new techniques to solve them. Comparisons will be provided to show the validity of all proposed techniques in solving fractional differential equations. Tables and graphical representations will be provided to support our results. All in all, we are very sure that our excellent expected results will attract a global research interest in reading our works and citing all of them in other possible future works based on our results.

1.7 Description of Conceptual Framework and Research Methodology

In this research work, various fractional calculus formulations such as ComD, RL, Cp, Abu-Shady-Kaabar, and many other related fractional definitions will be investigated to study many interesting problems and scientific phenomena. Various properties of these fractional definitions will be studied and extended in our work. New propositions related to these definitions will be presented to obtain our results. Symbolic computational computer programs such as MAPLE, Wolfram Mathematica, and MATLAB will be utilized in our work to obtain various types of solutions to our proposed fractional differential equations. In addition, the 3D and 2D graphical representations will be drawn with the aid of these software programs for showing the dynamics and physical behavior of our obtained solutions. Our main goal in this work is to provide novel contributions to the computational methods and mathematical analysis of fractional calculus, and to shed light on the importance of this field of research in modeling many important scientific phenomena by investigating the proposed problems theoretically and numerically. The dynamics and physical behavior of the proposed systems will be also presented through numerical experiments and illustrative examples. The following are the steps of our conceptual framework:

* Formulate various fractional differential equations' classes.
* Choose appropriate analytical or approximate-analytical techniques to the proposed problems in order to formulate novel exact and approximate-analytical solutions for the proposed fractional differential equations.
* Develop and propose new techniques and generalized definitions of fractional derivatives for studying and solving fractional differential equations.
* Compare various analytical and approximate analytical techniques for various types of fractional derivatives for certain interesting models.
* Investigate the obtained solutions to various fractional differential equations' classes via fractional calculus properties, fixed point theorems, and other related essential approaches.
* Perform numerical experiments of the obtained results.

To perform the proposed study, the following steps of methodology are needed:

1.7.1 Problem Formulation

We formulate our research problems in the context of fractional derivatives or integrals.

1.7.2 Theoretical Investigation

We perform an investigation on the proposed research problems by studying the obtained solutions to various classes of fractional differential equations via fractional calculus properties, fixed point theorems, and other related essential approaches.

For new proposed fractional definitions, we prove all proposed theorems related to our definitions.

1.7.3 Computation

We select appropriate analytical or approximate-analytical techniques to the proposed problems. In addition, we develop and propose new techniques and generalized definitions of fractional derivatives for studying and solving fractional differential equations. Graphical representations of the obtained solutions are provided in our study. MAPLE, Wolfram Mathematica, and MATLAB software programs are used to obtain symbolic computational results and represent solutions graphically.

1.7.4 Comparative Study

Various methods are compared with each other to show the accuracy of the proposed methods in obtaining solutions to fractional differential equations.

1.7.5 Numerical Validation

We validate our results using numerical experiments by providing many illustrative examples in our research study.

1.7.6 Discussion of Results

All results are discussed in detail, and each proposed problem has been accepted by strong journals for publications.

1.8 Research Outline

This thesis is consists of 10 chapters. Chapter 1 introduces and gives an overview of fractional calculus with some interesting topics of research in this field. The novelty of this research work is mentioned with a detailed description of motivation, scope, conceptual framework, objectives, and research methodology. Literature review on all previous research works related to fractional calculus is discussed in Chapter 2. Two nonlinear partial differential equations are formulated and solved vis three methods in the context of ComD and Cp fractional derivatives in Chapter 3. The multivariable calculus is investigated in the context of conformable derivative in Chapter 4. A new generalized fractional derivative definition, named as Abu-Shady-Kaabar fractional derivative, is proposed and employed for some functions in Chapter 5. With the help of Krasnoselskii's fixed point theorem, the implicit nonlinear variable order fractional differential equation is studied in Chapter 6. The generalized Mittag-Leffler function is employed to study the stability of quadratic integral equation in the context of fractional calculus is discussed in Chapter 7. In Chapter 8, the ψ-Caputo fractional differential equation with nonlinear boundary conditions is investigated via a novel and rare technique. The even-order nonlinear differential equations' oscillation with mixed nonlinear neutral terms is studied via a newly proposed method in Chapter 9. In Chapter 10, some conclusions are given, and suggestions for future research works are also given based on the results in this thesis

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

Fractional differential equations (FDfEqs) are considered as an extended version of fractional calculus (FrCL) that was first defined in the 17th century during a mathematical discussion between two mathematicians to define the $\frac{1}{2}$-order derivative by extending the integerâĂŹs definition. The *n*th derivative of the power function expression was one of the first expressions of fractional derivatives that was studied and proposed by Euler and Lacroix (Almeida et al., 2019). The introduced definitions of fractional derivative are categorized into two parts: global (classical) nature and local nature. The fractional derivative in the global category, which has a non-local property with a memory, is expressed as transformations in terms of integral, Mellin, or Fourier. However, the fractional derivative in the local category is basically based on a local-type definition involving incremental ratios.

The FDfEqs' qualitative analysis including the solution's existence and uniqueness of FDfEqs is considered as the most interesting research problems in fractional calculus analysis. The theories of fixed points are considered as important tools in investigating these problems (see (Abbas & Ragusa, 2020; Akdemir et al., 2021)).

Due to this global category, FrCL has been existed in the history of mathematics since Laplace's, Euler's, and Fourier's time, until RL and Cp fractional definitions have been proposed in the modern era. FrCL has shown a high ability to be applied in studying various research topics' applications related to electromagnetics, mechanics, fluid dynamics, signal processing, electric circuits, heat transfer, epidemiology, nonlinear optics, theoretical physics, biology, and control theory (Miller & Ross, 1993).

While the traditional FrCL definitions attempt to satisfy the usual derivatives' properties, none of them have been successful in satisfying most of them. However, there is only the property of linearity that has been shared commonly between all of them (Hammad & Khalil, 2014). Many physical and engineering systems have been studied in (Almeida et al., 2019) via some techniques to obtain the fractional equations' solutions analytically or approximate analytically in the context of fractional derivatives and integrals (Amoupour et al., 2018).

Many systems' behavior can be discussed in better way using FrCL than the integer-order ones because fractional definitions' nonlocality and memory effects can be seen in some systems that give this topic a special importance in modeling. Due to the variety of modeling scenarios in science and engineering that can benefit greatly from FrCL where many physical systems have naturally some FrCL characteristics. Therefore, a particular interest has been paid to study the FDfEqs because of their applicabilities in the fractional-order modeling (Afshari et al., 2015; Boutiara et al., 2020). . FDfEqs are regarded as a distinguished tool in many modeling scenarios, but this tool is associated with many challenges such as obtaining analytical solutions of FDfEqs and finding efficient techniques to solve such equations analytically. Some of the attempts to solve these challenges have started with the ComD definition to provide such analytical solutions (Khalil et al., 2014).

The physical interpretation for ComD is a directional derivative, based on usual derivative with some modifications in both magnitude and direction (Silva et al., 2018). Khalil et al. (2019) presented the fractional cords' example to show the interpretation of ComD geometrically.

From the ComD's definition, with the help of mathematical tools in analysis, some notable studies concerning the functions of a real variable have been investigated, and some other related works concerning the complex-valued functions of a real variable, Green's function, Rolle's Theorem, Mean Value Theorem, integration by parts, power series expansion, Sturm's theorems, and the definitions of single and double Laplace transforms have been studied in (Kaabar, 2020; Khalil et al., 2014; Martínez et al., 2020).

In FrCL, the arbitrary order integrals and derivatives are considered. Fractional derivatives have a unique behavior which inspires new research works concerning the developments of their theoretical frameworks. However, this research field has recently been considered as an essential interdisciplinary subject in natural and engineering sciences.

To provide a better understanding the scientific phenomena' mechanisms, it is essential to propose new models constructed in the FrCL context involving their obtained solutions and properties. In addition, the mathematical analysis of some interesting FDfEqs are needed to be studied in order to answer many of the open research questions such as the solutions' existence, uniqueness, and stability of our proposed systems.

This chapter provides a survey of all previous research studies that have been done on FrCL including computational methods and analysis. This chapter consists of six sections. Some previous interesting research works on studying the nonlinear FDfEqs are discussed in Section 2.2. Multivariable conformable calculus is presented in Section 2.3. The FDfEqs of variable order are discussed in Section 2.4. In Section 2.5, the ψ-Caputo FDfEq is mentioned. Some previous works on differential equations' oscillation are discussed in Section 2.6. A brief conclusion about all investigated studies is provided in Section 2.7.

2.2 Nonlinear Fractional Partial Differential Equations

Partial differential equations (PrDfEqs) have motivated many researchers of natural sciences and engineering to investigate them due to their essential role in modeling various scientific phenomena in chemistry, physics, and signal processing. Due to the nonlinearity of many systems, nonlinear PrDfEqs have motivated scientists to prove the existence of exact solutions. Finding novel exact solutions for nonlinear PrDfEqs can help significantly understand systems physically and dynamically. Therefore, new techniques have been recently developed for obtaining the nonlinear PrDfEqs' solutions exactly such as the technique of generalized Kudryashov in solving some interesting nonlinear PrDfEqs (Kaplan et al., 2016), and the techniques of extended simplest equation and modified simple equation to solve the nonlinear Fokas equation (Al-Amr & El-Ganaini, 2017). To solve nonlinear integrable equations, the Hirota bilinear technique, was first introduced in (Ma, 2020; Ma et al., 2021) to find new lump solutions, which are considered as an alternative kind of exact solutions that are rationally and spatially localized, for the investigated equations (Ma, 2022). Soliton solutions are basically analytic functions that are exponentially localized, which represent multifarious wave phenomena. The motivation is that nonlinear PrDfEqs can be constructed in the context of fractional derivatives which will provide new and novel results to the soliton theory and mathematical physics by investigating the exact solutions and their dynamics via new techniques.

One of the most notable examples of these equations is the Schrödinger equation which is commonly seen in studying nonlinear optics due to its ability to interpret the optical soliton propagation's dynamics in optical fibers. Some studies concerning the complex Ginzburg-Landau equation's optical solutions with Kerr Law nonlinearity constructed via the truncated M-fractional and beta derivatives were obtained in

(Yusuf et al., 2019). Moreover, the Kundu-Eckhaus equation's optical solutions were investigated with the help of the techniques of extended Jacobi elliptic function expansion and modified tanh coth (Baleanu et al., 2017), and the technique of first integral (Wazwaz, 2017), respectively. Different other related models' optical solutions were studied in (Al-Amr & El-Ganaini, 2017; Inc et al., 2018). As a result, finding nonlinear fractional PrDfEqs' novel solutions, especially for Schrödinger equation's modified forms, has attracted the interests of all concerned researchers due to the applicability of these equations in electromagnetism, nonlinear optics, and fluid dynamics (Ghanbari & Gómez-Aguilar, 2019). There are also many other interesting nonlinear partial differential equations that are needed to be investigated to understand the dynamics of their obtained solutions.

Another interesting nonlinear PrDfEqs is the Benjamin-Bona-Mahony equation (BeBoMaEq), which is basically an extended formulation of the Korteweg-de-Vries equation. In a shallow water channel, the unidirectional propagation of long waves with small amplitude is represented by this equation (Bekir et al., 2021, 2020).

Wazwaz (2017) initiated the idea of three-dimensional BeBoMaEq, named as the Wazwaz-Benjamin-Bona-Mahony equation (WaBeBoMaEq), by the procedures of coupling or generalized formulations. WaBeBoMaEq can describe many research problems with a variety of applications that involve higher dimensional systems (Bekir et al., 2021, 2020). The WaBeBoMaEq's exact solutions are highly needed to be further investigated to understand the behavior and dynamics of many systems. Some recent research works on WaBeBoMaEq include the study of the conformable version of WaBeBoMaEq such as (Seadawy et al., 2019) using the procedure of simple ansatz and (Bilal et al., 2021) using the procedure of generalized exponential rational function. In addition, the conformable WaBeBoMaEq has been investigated using other different techniques such as the techniques of modified simple equation

(Bekir et al., 2021), Riccati-Bernolli Sub-ODE (Bekir et al., 2021), and (G'/G)-expansion (Bekir et al., 2020).

2.3 Multivariable Conformable Calculus

Several studies have been recently conducted on studying conformable derivatives and integrals. A two-dimensional wave equation, constructed in the context of ComD, representing a circular membrane undergoing axisymmetric vibrations was investigated in (Kaabar, 2020). This equation was solved via three novel methods: Separation of Variables Method, Differential Transform Method, and Double Laplace Transform Method. Novel results on Laplacian, constructed in the conformable context, was investigated in (Martínez et al., 2021b) where the results were discussed with Dirichlet and Neumann conditions. The generalized mean value theorems were studied in the context of ComD (Martínez et al., 2021a). Some new results on complex fractional integration are investigated in (Martínez et al., 2020) via the establishment of sufficient conditions for a continuous function in order to get a conformable antiderivative.

ComD has been employed in many scientific applications, particularly theoretical physics. The conformable version of analytic functions has been discussed in (Khalil et al., 2018). The contour conformable version of integral has been studied in (Martínez et al., 2020; Uçar et al., 2019). In 2018, the idea of the conformable version of multivariable calculus has been initiated in (N. Y. Gözütok & Gözütok, 2017). In 2019, the differential geometry of curves has been studied in the contexts of ComDs and integrals (U. Gözütok et al., 2019). In 2020, the behavior of the linear differential systems' stability has been investigated in the context of ComD. In 2021, the behavior of the ComDs of functions has been investigated in arbitrary Banach spaces (Kiskinov et al., 2021).

2.4 Variable Order Fractional Differential Equation

Fractional models of variable and constant orders have been investigated by (Sun et al., 2011) to study the proposed systems' memory. A nonlinear alcoholism model, constructed in the context of variable-order fractional operator, has been proposed and solved by (Gómez-Aguilar, 2018). A multi-term variable-order fractional BVP has been studied in detail by (Bouazza et al., 2021). Then, the variable-order fractional BVPs in the context of Atangana-Baleanu fractional operator has been proposed by defining a kernal function with the help of polynomial form (X. Li et al., 2020). A linear variable-order FDfEq, constructed in the context of Cp, has been investigated in (Derakhshan, 2021). In a very recent study, a variable-order fractional BVP, constructed in the context of Hadamard, has been studied by (Refice et al., 2021) via the technique of the Kuratowski measure of noncompactness.

In (Benchohra & Lazreg, 2014), the implicit nonlinear FDfEq has been investigated concerning its existence-uniqueness in the framework of constant order as follows:

$$\begin{cases} {}^{c}\mathfrak{D}^{u}_{0^+}x(t) = m(t, x(t), {}^{c}\mathfrak{D}^{u}_{0^+}x(t)), \ \ t \in \mathfrak{J} := [0, \Omega], \ \ 0 < \Omega < +\infty, \ \ 1 < u \leq 2 \\ x(0) = x_0, \ \ x(\Omega) = x_1 \end{cases}$$

where $m : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function that belongs to $C(\mathfrak{J} \times \mathbb{R}^2, \mathbb{R})$, x_0, $x_1 \in \mathbb{R}$, and the Cp fractional derivative is represented by ${}^{c}\mathfrak{D}^{u}_{0^+}$.

2.5 ψ-Caputo Fractional Differential Equation with Nonlinear Boundary Conditions

In FrCL, there are many interesting fractional operators such as Cp, RL, and Hadamard that can be applicable for many modeling scenarios. However, none of these definitions can be universally applied for all models. To overcome this issue in FrCL, new definitions have been initiated to provide a one universal definition that can contain some of the above definitions (Caputo & Fabrizio, 2015). One of the successful attempts in formulating such universal definition is the ψ-fractional operator which has been proposed in (Almeida, 2017; Almeida et al., 2018) as a universal platform by including other definitions in its integral kernel's function by substituting it with certain functions. Another advantage is that this universal definition preserves its nonlocality and the property of semigroup. Some applications have been conducted using this definition in (Abdo et al., 2019; Jarad et al., 2020).

2.6 The Oscillatory Behavior of Differential Equations

Differential equations' oscillatory behavior (DEqsOB) with a linear neutral term has been recently investigated in various research works. Some of the most interesting works that have been conducted on studying DEqsOB are the even-order quasilinear neutral functional DEqsOB (Baculikova et al., 2011), 2nd-order superlinear Emden-Fowler neutral DEqsOB (T. Li & Rogovchenko, 2017), and 2nd-order nonlinear neutral delay differential equations solutions' asymptotic behavior (Graef et al., 1991). For the higher-order systems, the neutral delay DEqsOB has been investigated in (W. N. Li, 2000). However, both of the even-order and nonlinear neutral DEqsOBs with variable coefficients have been discussed in (Zafer, 1998) and (Q. Zhang et al., 2010), respectively.

2.7 Conclusion

Motivated by all previous research works, all our results in this thesis provide a major contribution to the field of fractional calculus and its applications due to the importance of the FDfEqs in various natural sciences and engineering phenomena. According to our comprehension of all previous other research studies, none of our results have been obtained in any other previous research works concerning the computational methods and mathematical analysis of FrCL because new problems are proposed and formulated in the context of FrCL. The existence, uniqueness, and stability of obtained solutions are investigated in detail with many illustrative numerical examples that will be also provided in our work to validate the applicability of our theoretical results. Our focus in our work on the importance of proposed problems and their wide applications. We always make sure that our proposed problems have never been introduced in any other previous works. If they are found to be proposed previously in some related works, these problems are extended to something more interesting by formulating it into new derivatives and applying new techniques to solve them. Comparisons are provided to validate all proposed techniques in obtaining FDfEqs' solutions. Tables and graphical representations are provided to support our results. All in all, we are very sure that our results will attract a global research interest in reading our works and citing them in other possible future works based on our results. In addition, our work will open new insights to develop new mathematical models and will motivate all other researchers to investigate this interesting field of research.

CHAPTER 3: NOVEL TECHNIQUES FOR SOLVING CONFORMABLE AND FRACTIONAL NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

3.1 Introduction

In this chapter, three novel techniques are discussed for obtaining solutions of nonlinear partial differential equations (NLPrDfEqs) in the context of ComD and Cp derivative. In the first part of this chapter, the $(3 + 1)$-dimensional WaBeBoMaEq is formulated via ComD. The introduced equation is solved via the generalized Kudryashov method (GeKM) and $\exp(-\phi(\aleph))$ method (ExM). MAPLE software has been employed to perform all algebraic computations. 3D and 2D graphical profiles are presented to show all obtained solutions' behavior and dynamics at various parameters' values and orders via Wolfram Mathematica. In the second part of this chapter, a modified version of the nonlinear Schrödinger equation with spatio-temporal dispersion is formulated in the contexts of both Cp derivative and ComD. The proposed equation is solved via a new generalized double Laplace transform coupled with Adomian decomposition method. The 3D plots of the real and imaginary parts with their corresponding contour plots of the obtained approximate analytical solutions are also provided.

3.2 Fundamental Preliminaries and Methodology

Consider the following $(3 + 1)$-dimensional modified form of BeBoMaEq:

$$\Psi_t + \Psi_x + \Psi^2\Psi_y - \Psi_{xzt} = 0. \tag{3.1}$$

Eq. (3.1) was first proposed by (Wazwaz, 2017) via the formulation of 3-dimensional modified version of BeBoMaEqs, named as WaBeBoMaEq, via coupling or various generalized contexts or as their combination. Higher dimensional problems with a variety of applications can be mentioned via the WaBeBoMaEq (Bekir et al., 2021, 2020). To interpret WaBeBoMaEq physically and understand its dynamics, finding exact solutions for WaBeBoMaEq is very essential. Various soliton solutions for conformable WaBeBoMaEq have been obtained by (Seadawy et al., 2019) and (Bilal et al., 2021) via the procedures of simple ansatz and generalized exponential rational function, respectively.

Therefore, we are motivated to obtain new exact solitary solutions for WaBeBoMaEq in the context of ComD via two novel procedures: GeKM and ExM. The general fractional formulation of WaBeBoMaEq can be expressed as:

$$\mathfrak{D}_t^{\zeta}\Psi + \mathfrak{D}_x^{\zeta}\Psi + \mathfrak{D}_y^{\zeta}\Psi - \mathfrak{D}_{xzt}^{3\zeta}\Psi = 0, \tag{3.2}$$

where $\mathfrak{D}^{\zeta}$ is the conformable operator of order: $\zeta \in (0, 1]$. The exact solutions of Eq. (3.2) have been previously studied via the (G'/G)-expansion method (Bekir et al., 2020), modified simple equation method (Bekir et al., 2021), and Riccati-Bernolli Sub-ODE method (Bekir et al., 2021). However, no previous research works has studied Eq. (3.2) in the context of ComD via GeKM and ExM. Therefore, all our results are original and novel.

Let us now introduce some essential notions about ComD.

The theorem (Khalil et al., 2014) below shows that $\mathfrak{D}_t^{\zeta}$ is a ComD in Eq. 1.3 satisfies the known limit-based derivative's properties as follows:

Theorem 1. *For $\zeta \in (0, 1]$, let functions: Ψ and Φ be $\zeta-DF$ at a point t, then we get:*

(a) $\mathfrak{D}_t^{\zeta}(\Psi(t)\Phi(t)) = \Psi(t)\mathfrak{D}_t^{\zeta}(\Phi(t)) + \Phi(t)\mathfrak{D}_t^{\zeta}(\Psi(t))$.

(b) $\mathfrak{D}_t^{\zeta}(m\Psi(t) + w\Phi(t)) = m\mathfrak{D}_t^{\zeta}\Psi(t) + w\mathfrak{D}_t^{\zeta}\Phi(t)$, $\forall m, w \in \mathfrak{R}$.

(c) $\mathfrak{D}_t^{\zeta}\left(\frac{\Psi(t)}{\Phi(t)}\right) = \frac{\Phi(t)\mathfrak{D}_t^{\zeta}(\Psi(t)) - \Psi(t)\mathfrak{D}_t^{\zeta}(\Phi(t))}{\Phi^2(t)}$.

(d) $\mathfrak{D}_t^{\zeta}(t^k) = kt^{k-\zeta}, \forall\, k \in \mathfrak{R}$.

(e) If $\Psi(t)$ *is supposed to be a differentiable function, then* $\mathfrak{D}_t^{\zeta}(\Psi(t)) = t^{1-\zeta}\frac{d\Psi}{dt}$.

(f) $\mathfrak{D}_t^{\zeta}(\nu) = 0$, $\forall$ *constant functions* $\Psi(t) = \nu$.

Then, the methodology of GeKM and ExM are mentioned as follows:

Consider the nonlinear evolution equation (NLEEq), with 4 independent variables: x, y, z, and t, formulated generally in the context of ComD as follows:

$$T(\Psi, \mathfrak{D}_t^{\zeta}\Psi, \mathfrak{D}_x^{\zeta}\Psi, \mathfrak{D}_y^{\zeta}\Psi, \mathfrak{D}_z^{\zeta}\Psi, \mathfrak{D}_t^{2\zeta}\Psi, \mathfrak{D}_x^{2\zeta}\Psi, \mathfrak{D}_y^{2\zeta}\Psi, \mathfrak{D}_z^{2\zeta}\Psi, , ...) = 0; 0 < \zeta \leq 1, \tag{3.3}$$

where $\Psi = \Psi(x, y, z, t)$ is an unknown function, and T is a polynomial of Ψ and its partial derivatives in which all of the nonlinear terms and highest-order derivatives are included in Eq. (3.3). First, to solve Eq. (3.3), we use traveling wave transformations for ComD as follows:

For ComD:

$$\Psi(x, y, z, t) = \Psi(\aleph);\;\; \aleph = p\frac{x^{\zeta}}{\zeta} + q\frac{y^{\zeta}}{\zeta} + \gamma\frac{z^{\zeta}}{\zeta} - \delta\frac{t^{\zeta}}{\zeta}, \tag{3.4}$$

where p, q, γ, and δ are all constants with the condition: $p, q, \gamma, \delta \neq 0$, and δ is the wave speed.

From the above, Eq. (3.3) is reduced to the ordinary differential equation (ODEq) as:

$$L(\Psi, \Psi', \Psi'', \Psi''', ...) = 0. \tag{3.5}$$

The derivative with respect to (w.r.t.) $\aleph$ is represented by a prime. Eq. (3.5) should be integrated term by term one or more times.

Schrödinger equation is another important NLPrDfEq that has been applied in various applications in physics and engineering due to its essential role nonlinear optics which can successfully explain the dynamics of optical soliton propagation in optical fibers.

Various fractional formulations have been introduced in (Ghanbari & Gómez-Aguilar, 2019; Yaşar & Yaşar, 2018) to obtain exact optical soliton solutions for modified nonlinear Schrödinger equation (MoNLSEq) with spatio-temporal dispersion. Finding analytical and approximate analytical solutions for the modified forms of nonlinear fractional Schrödinger equation have become a common research interest for physicists and applied mathematicians in the field of optical soliton propagation because of the applications of this equation in plasma, optics, electromagnetism, fluid dynamics, and optical communication (Ghanbari & Gómez-Aguilar, 2019; Yaşar & Yaşar, 2018). The dynamics of optical soliton propagation in optical fiber can be interpreted from the MoNLSEq with second-order spatio-temporal dispersion and group velocity dispersion coefficients (Ghanbari & Gómez-Aguilar, 2019). Given $\Psi(x,t)$ as a complex-valued wave function that represents the macroscopic property of wave profile of the spatial and temporal variables which are expressed as x and t, respectively. Then, MoNLSEq can be written as (Ghanbari & Gómez-Aguilar, 2019):

$$i\left(\frac{\partial \Psi}{\partial x} + \omega_1 \frac{\partial \Psi}{\partial t}\right) + \omega_2 \frac{\partial^2 \Psi}{\partial t^2} + \omega_3 \frac{\partial^2 \Psi}{\partial x^2} + |\Psi|^2 \Psi = 0, \tag{3.6}$$

whereω_1 is proportional to the ratio of group speed;

ω_2 is a group velocity dispersion coefficient;

ω_3 is a spatial dispersion coefficient;

To formulate Eq. (3.6) in the sense of fractional derivatives, let us first define

Caputo fractional derivative as follows:

Definition 4. *For $\xi, \gamma > 0$, given two functions: $h(x)$ and $h(t)$ such that for $x, t > 0$, the Caputo fractional derivative (CpFD) of h of order ξ and γ, denoted by $\mathfrak{D}_x^{\xi}(h)(x)$ and $\mathfrak{D}_t^{\gamma}(h)(t)$, respectively where $\mathfrak{D}_x^{\xi}$ and $\mathfrak{D}_t^{\gamma}$ are Cp derivative operators which can be simply expressed as (Hamed et al., 2014):*

$$\mathfrak{D}_x^{\xi} h(x) = \frac{1}{\Gamma(\Omega - \xi)} \int_0^x (x - \eta)^{\Omega - \xi - 1} h^{(\Omega)}(\eta) \mathrm{d}\eta; \;\; \Omega - 1 < \xi \leq \Omega \; \textit{for} \; \Omega \in \mathbb{N}, \tag{3.7}$$

$$\mathfrak{D}_t^{\gamma} h(t) = \frac{1}{\Gamma(w - \gamma)} \int_0^t (t - \mu)^{w - \gamma - 1} h^{(w)}(\mu) \mathrm{d}\mu; \;\; w - 1 < \gamma \leq w \; \textit{for} \; w \in \mathbb{N}. \tag{3.8}$$

If $\xi = \Omega$ and $\gamma = w$ where $\Omega, w \in \mathbb{N}$, then $\mathfrak{D}_x^{\xi} h(x) = \frac{d^{\Omega}}{dx^{\Omega}} h(x)$ and $\mathfrak{D}_t^{\gamma} h(t) = \frac{d^w}{dt^w} h(t)$. CpFD is very useful in science and engineering due to their important properties such as the inclusion of initial and boundary conditions in its formulation (Almeida et al., 2019; Odibat et al., 2008). Let us now define the Mittag-Leffler function:

Definition 5. *The Mittag-Leffler function, denoted by $E_{\xi,\zeta}(t)$, can be expressed as follows (Dhunde & Waghmare, 2016):*

$$E_{\xi,\zeta}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\xi m + \zeta)}, \textit{where } t, \zeta \in \mathbb{C} \textit{ and } \Re(\xi) > 0. \tag{3.9}$$

From Definition (5), the Mittag-Leffler function, denoted by $E(t, h, c)$, can be written (Hamed et al., 2014) as: $E(t, h, c) = t^h E_{1,h+1}(ct)$, and the fractional derivative of Mittag-Leffler function can also be expressed (Hamed et al., 2014) as: $\frac{\partial^{\delta}}{\partial t^{\delta}}(t^{\zeta-1} E_{\xi,\zeta}(ct^{\xi})) = t^{\zeta-\delta-1} E_{\xi,\zeta-\delta}(ct^{\xi})$ where $\delta \geq 0$.

The approximate-analytical solutions for Eq. (3.6) are obtained using double Laplace transform method in the sense of CpFD and ComD.

From Definition (4), Eq. (3.6) can be formulated in the sense of CpFD as follows:

$$
\begin{aligned}
&i\left(\frac{\partial^{\gamma}\Psi(x,t)}{\partial x^{\gamma}}+\omega_1\frac{\partial^{\delta}\Psi(x,t)}{\partial t^{\delta}}\right)+\omega_2\frac{\partial^{2\delta}\Psi(x,t)}{\partial t^{2\delta}}+\omega_3\frac{\partial^{2\gamma}\Psi(x,t)}{\partial x^{2\gamma}}+|\Psi|^2\Psi=0;\\
&i\frac{\partial^{\gamma}\Psi(x,t)}{\partial x^{\gamma}}+\omega_1 i\frac{\partial^{\delta}\Psi(x,t)}{\partial t^{\delta}}+\omega_2\frac{\partial^{2\delta}\Psi(x,t)}{\partial t^{2\delta}}+\omega_3\frac{\partial^{2\gamma}\Psi(x,t)}{\partial x^{2\gamma}}+|\Psi|^2\Psi=0;\\
&\text{where } i=\sqrt{-1},\ 0<\gamma,\delta\leq 1,\ t,x>0.
\end{aligned}
\tag{3.10}
$$

From Eq. 1.3, Let us formulate Eq. (3.6) in the sense of ComD as follows:

$$
\begin{aligned}
&i\left(\mathfrak{D}_x^{\gamma}\Psi\left(\frac{x^{\gamma}}{\gamma},\frac{t^{\delta}}{\delta}\right)+\omega_1\mathfrak{D}_t^{\delta}\Psi\left(\frac{x^{\gamma}}{\gamma},\frac{t^{\delta}}{\delta}\right)\right)+\omega_2\mathfrak{D}_t^{2\delta}\Psi\left(\frac{x^{\gamma}}{\gamma},\frac{t^{\delta}}{\delta}\right)+\omega_3\mathfrak{D}_x^{2\gamma}\Psi\left(\frac{x^{\gamma}}{\gamma},\frac{t^{\delta}}{\delta}\right)\\
&+|\Psi|^2\Psi=0;\\
&i\mathfrak{D}_x^{\gamma}\Psi\left(\frac{x^{\gamma}}{\gamma},\frac{t^{\delta}}{\delta}\right)+\omega_1 i\mathfrak{D}_t^{\delta}\Psi\left(\frac{x^{\gamma}}{\gamma},\frac{t^{\delta}}{\delta}\right)+\omega_2\mathfrak{D}_t^{2\delta}\Psi\left(\frac{x^{\gamma}}{\gamma},\frac{t^{\delta}}{\delta}\right)+\omega_3\mathfrak{D}_x^{2\gamma}\Psi\left(\frac{x^{\gamma}}{\gamma},\frac{t^{\delta}}{\delta}\right)\\
&+|\Psi|^2\Psi=0;\\
&\text{where } i=\sqrt{-1},\ 0<\gamma,\delta\leq 1,\ t,x>0.
\end{aligned}
\tag{3.11}
$$

3.2.1 The GeKM

The obtained solution for the reduced equation using GeKM is formulated via a polynomial in $\hbar(\aleph)$ as (Kaplan et al., 2016; Kudryashov, 2012):

$$
\Psi(\aleph)=\frac{\sum\limits_{k=0}^{J}p_k\hbar^k(\aleph)}{\sum\limits_{l=0}^{W}q_l\hbar^l(\aleph)},
\tag{3.12}
$$

where $p_k(k=0,1,...,J), q_l(l=0,1,...,W)$ are constants which are needed to be determined $\ni p_J\neq 0$, $q_W\neq 0$, and $L=L(\aleph)$ is the solution of the following equation:

$$\frac{d\hbar}{d\aleph} = \hbar^2(\aleph) - \hbar(\aleph). \tag{3.13}$$

The solution of Eq. (3.13) can be expressed as:

$$\hbar(\aleph) = \frac{1}{1 + I_1 e^{\aleph}}, \; I_1 \text{ is integration constant.} \tag{3.14}$$

From the homogeneous balance principle (HBPrp), the positive integers: J and W in Eq. (3.12) can be obtained using Eq. (3.5). In addition, a polynomial, $\hbar$, can be determined by the substitution of Eq. (3.12) into Eq. (3.5) along with Eq. (3.13). Now, by equating all the coefficients of polynomial $\hbar$ to 0 in order to build a system of algebraic equations. This system is solved using MAPLE to find the values of $p_k(k = 0, 1, ..., J), q_l(l = 0, 1, ..., W)$. All soliton-type solutions of the reduced Eq. (3.5) can be obtained by the substitution of these obtained values and Eq. (3.13) into Eq. (3.12).

3.2.2 The ExM

From ExM (Roshid et al., 2014), the obtained solution for the reduced equation is formulated via a polynomial in $\exp(-\Phi(\aleph))$ as follows:

$$\Psi(\aleph) = \sum_{j=0}^{w} p_j (\exp(-\phi(\aleph)))^j, \tag{3.15}$$

where p_J ($p_W \neq 0$) are constants which are needed to be found, and $\phi(\aleph)$ satisfies the following auxiliary ODEq:

$$\phi'(\aleph) = \exp(-\phi(\aleph)) + \vartheta \exp(\phi(\aleph)) + \chi. \tag{3.16}$$

Note that Eq. (3.16) has distinct solutions which are expressed as:

CASE I: When $\chi^2 - 4\vartheta > 0$ and $\vartheta \neq 0$, the hyperbolic function solutions are expressed as:

$$\phi_1(\aleph) = \ln\left(\frac{-\sqrt{\chi^2 - 4\vartheta}\tanh(\frac{\sqrt{\chi^2-4\vartheta}}{2}(\aleph + I)) - \chi}{2\vartheta}\right). \tag{3.17}$$

CASE II: When $\chi^2 - 4\vartheta < 0$ and $\vartheta \neq 0$, the trigonometric function solutions are expressed as:

$$\phi_2(\aleph) = \ln\left(\frac{\sqrt{4\vartheta - \chi^2}\tan(\frac{\sqrt{4\vartheta-\chi^2}}{2}(\aleph + C)) - \chi}{2\vartheta}\right). \tag{3.18}$$

CASE III: When $\chi^2 - 4\vartheta > 0$, $\vartheta = 0$ and $\chi \neq 0$, the hyperbolic function solutions are expressed as:

$$\phi_3(\aleph) = -\ln\left(\frac{\chi}{\cosh(\chi(\aleph + I)) + \sinh(\chi(\aleph + I)) - 1}\right). \tag{3.19}$$

CASE IV: When $\chi^2 - 4\vartheta = 0, \vartheta \neq 0$ and $\chi \neq 0$, the rational function solutions are expressed as:

$$\phi_4(\aleph) = \ln(-\frac{2(\chi(\aleph + I) + 2)}{\chi^2(\aleph + I)}). \tag{3.20}$$

CASE V: When $\chi^2 - 4\vartheta = 0, \vartheta = 0$ and $\chi = 0$, we have:

$$\phi_5(\aleph) = \ln(\aleph + I). \tag{3.21}$$

From the above cases, the integration constant is represented by I. By the substitution of Eq. (3.15) into the reduced Eq. (3.5) and collecting all terms together that are in the same order of $\exp(-\phi(\aleph))^j (j = 0, 1, 2, ...)$, the polynomial in terms of

$\exp(-\phi(\aleph))$ is verified. Then, by equating all coefficients to 0, a set of algebraic equations is constructed for $p_j(j = 0, 1, ..m), \chi, \delta$, and ϑ . With the help of MAPLE, the system can be solved to obtain diverse exact solutions for Eq. (3.4).

3.2.3 The Double Laplace Transform method

Kaabar (2020) defined the conformable double Laplace transform (CmDLTr) as follows:

Definition 6. *Given a function,* $\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right) : [0, \infty) \to \Re$ *such that for all* $x, t > 0$, *the CmDLTr of order* $\gamma, \delta \in (0, 1]$ *of* $\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right)$, *denoted by* $\ell^{xt}_{\gamma\delta}\left[\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right)\right]$, *starting from* 0 *can be expressed as follows:*

$$\begin{aligned} \ell^{xt}_{\gamma\delta}\left[\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right)\right] &= \ell^{x}_{\gamma}\ell^{t}_{\delta}\left[\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right)\right] = \tilde{\Psi}^{xt}_{\gamma\delta}(s_1, s_2) \\ &= \int_0^\infty \int_0^\infty e^{-(s_1 \frac{x^\gamma}{\gamma} + s_2 \frac{t^\delta}{\delta})} \Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right) x^{\gamma-1} t^{\delta-1}\, dx\, dt. \end{aligned} \tag{3.22}$$

where $s_1, s_1 \in \mathbb{C}$. If the integral in the above definition exists, then this definition holds true.

To define the double Laplace transform in the sense of Caputo partial fractional derivatives, let us assume that $\tilde{\Psi}^{xt}(s_1, s_2) = \int_0^\infty \int_0^\infty e^{-(s_1 x + s_2 t)} \Psi(x, t)\, dx\, dt$. From (Dhunde & Waghmare, 2016), Theorems 3.1 and 3.3 in (Anwar et al., 2013), and Theorem 2 in (Khan et al., 2019), the Caputo double Laplace transform (CpDLTr) can be defined as follows:

Definition 7. *Given a function,* $\Psi(x, t) : [0, \infty) \to \Re$ *such that for all* $x, t > 0$, *the double Laplace transform of the Caputo partial fractional derivatives of* $\Psi(x, t)$ *of orders* ξ *and* γ *where* $\xi \in (j - 1, j]$ *and* $\gamma \in (b - 1, b]$ *such that* $\xi, \gamma > 0$ *and*

$j, b \in \mathbb{N}$, *denoted by* $\frac{\partial^\xi}{\partial x^\xi}\Psi(x,t)$ *and* $\frac{\partial^\gamma}{\partial t^\gamma}\Psi(x,t)$, *respectively can be expressed as:*

$$\ell_\xi^{xt}\left[\frac{\partial^\xi}{\partial x^\xi}\Psi(x,t)\right] = s_1^\xi \tilde{\Psi}_\xi^{xt}(s_1, s_2) - \sum_{i=0}^{j-1} s_1^{\xi-1-i} \ell_t \left[\frac{\partial^i \Psi(0,t)}{\partial x^i}\right]. \tag{3.23}$$

$$\ell_\gamma^{xt}\left[\frac{\partial^\gamma}{\partial t^\gamma}\Psi(x,t)\right] = s_2^\gamma \tilde{\Psi}_\gamma^{xt}(s_1, s_2) - \sum_{a=0}^{b-1} s_1^{\gamma-1-a} \ell_x \left[\frac{\partial^a \Psi(x,0)}{\partial t^a}\right]. \tag{3.24}$$

The double Laplace transform in the sense of conformable partial fractional derivatives can be similarly defined (Kaabar, 2020) as follows:

Definition 8. *Given a function,* $\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right) : [0, \infty) \to \Re$ *such that for all* $x, t > 0$, *the double Laplace transform of the conformable partial fractional derivatives of* $\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right)$ *of orders* γ *and* δ *where* $\gamma, \delta \in (0, 1]$, *denoted by* $\frac{\partial^\gamma}{\partial x^\gamma}\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right)$ *and* $\frac{\partial^\delta}{\partial t^\delta}\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right)$, *respectively can be written as:*

$$\ell_\gamma^{xt}\left[\frac{\partial^\gamma}{\partial x^\gamma}\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right)\right] = s_1^\gamma \tilde{\Psi}_\gamma^{xt}(s_1, s_2) - \sum_{i=0}^{j-1} s_1^{\gamma-1-i} \ell_t \left[\frac{\partial^i \Psi(0,t)}{\partial x^i}\right]. \tag{3.25}$$

$$\ell_\delta^{xt}\left[\frac{\partial^\delta}{\partial t^\delta}\Psi\left(\frac{x^\gamma}{\gamma}, \frac{t^\delta}{\delta}\right)\right] = s_2^\delta \tilde{\Psi}_\delta^{xt}(s_1, s_2) - \sum_{a=0}^{b-1} s_1^{\delta-1-a} \ell_x \left[\frac{\partial^a \Psi(x,0)}{\partial t^a}\right]. \tag{3.26}$$

The existence and uniqueness of CmDLTr have been proven in (Kaabar, 2020), while the existence and uniqueness of CpDLTr have been discussed in (Anwar et al., 2013). It is obvious that the formulas of the double Laplace transform in Definition (7) and Definition (8) are the same when $j, b = 1$. However, for CpDLTr definition, j and b can be any natural number without any restrictions, while in CmDLTr definition, the γ and δ are restricted to the maximum order's value of 1. Therefore, the general definition of CpDLTr coincides with the general definition of CmDLTr when $j, b = 1$. The properties of CmDLTr and CpDLTr have been discussed in (Özkan & Kurt, 2018) and (Omran & Kiliçman, 2017), respectively. For $\gamma, \delta \in (0,1]$, let us now define the formula of the inverse fractional double Laplace transform (Dhunde & Waghmare, 2016) for both ComD and CpFD, denoted by $(\ell_{\gamma\delta}^{xt})^{-1}[\tilde{\Psi}_{\gamma\delta}^{xt}(s_1, s_1)]$, as follows:

Definition 9. *Given an analytic function:* $\tilde{\Psi}_{\gamma\delta}^{xt}(s_1, s_2)$*, for all* $s_1, s_2 \in \mathbb{C}$ *and for* $\gamma, \delta \in (0,1]$ *such that* $Re\{s_1 \geq \eta\}$ *and* $Re\{s_2 \geq \sigma\}$*, where* $\eta, \sigma \in \Re$*, then, the inverse fractional double Laplace transform (IFDLT) can be expressed (Kaabar, 2020) as follows:*

$$
\begin{aligned}
&(\ell_{\gamma\delta}^{xt})^{-1}[\tilde{\Psi}_{\gamma\delta}^{xt}(s_1, s_2)] = (\ell_{\gamma}^{x})^{-1}(\ell_{\delta}^{t})^{-1}[\tilde{\Psi}_{\gamma}^{xt}(s_1, s_2)] \\
&= \frac{-1}{4\pi^2}\int_{\varrho-i\infty}^{\varrho+i\infty}\int_{\varsigma-i\infty}^{\varsigma+i\infty} e^{s_1 x} e^{s_2 t}\tilde{\Psi}_{\gamma\delta}^{xt}(s_1, s_2)\, ds_1\, ds_2
\end{aligned} \tag{3.27}
$$

To solve the Eq. (3.6) in the senses of CpFD and ComD via CpDLTr and CmDLTr. respectively, let us first re-write both Eq. (3.10) and Eq. (3.11) as follows:

$$
\frac{\partial^{2\gamma}\Psi(x,t)}{\partial x^{2\gamma}} = -\frac{\omega_2}{\omega_3}\frac{\partial^{2\delta}\Psi(x,t)}{\partial t^{2\delta}} - \frac{i}{\omega_3}\frac{\partial^{\gamma}\Psi(x,t)}{\partial x^{\gamma}} - \frac{\omega_1}{\omega_3} i\frac{\partial^{\delta}\Psi(x,t)}{\partial t^{\delta}} - \frac{1}{\omega_3}|\Psi|^2\Psi.
$$

subject to the following initial and boundary conditions:

$\Psi(x,0) = a_0(x)$ and $\dfrac{\partial\Psi(x,0)}{\partial t} = a_1(x)$.

$\Psi(0,t) = b_0(t)$ and $\dfrac{\partial\Psi(0,t)}{\partial x} = b_1(t)$.

where $i = \sqrt{-1}$, $0 < \gamma, \delta \leq 1$, $t, x > 0$; $x, t \in \Re^+$, and $a_0, a_1, b_0, b_1 \in \mathbb{C}(\Re^+, \Re^+)$.

(3.28)

$$\mathfrak{D}_x^{2\gamma}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)=-\frac{\omega_2}{\omega_3}\mathfrak{D}_t^{2\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)-\frac{i}{\omega_3}\mathfrak{D}_x^{\gamma}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)-\frac{\omega_1}{\omega_3}i\mathfrak{D}_t^{\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)$$
$$-\frac{1}{\omega_3}|\Psi|^2\Psi.$$

subject to the following initial and boundary conditions:

$$\Psi\left(\frac{x^\gamma}{\gamma},0\right)=n_0\left(\frac{x^\gamma}{\gamma}\right) \text{ and } \mathfrak{D}_t\Psi\left(\frac{x^\gamma}{\gamma},0\right)=n_1\left(\frac{x^\gamma}{\gamma}\right).$$
$$\Psi\left(0,\frac{t^\delta}{\delta}\right)=m_0\left(\frac{t^\delta}{\delta}\right) \text{ and } \mathfrak{D}_x\Psi\left(0,\frac{t^\delta}{\delta}\right)=m_1\left(\frac{t^\delta}{\delta}\right).$$

where $i=\sqrt{-1}$, $0<\gamma,\delta\leq 1$, $t,x>0$; $x,t\in\mathfrak{R}^+$, and $n_0,n_1,m_0,m_1\in\mathbb{C}(\mathfrak{R}^+,\mathfrak{R}^+)$.

(3.29)

By applying the single Laplace transform to initial and boundary conditions in Eq. (3.28) and Eq. (3.29), respectively, we obtain the following:

$$\begin{aligned}\ell[\Psi(x,0)]&=\ell[a_0(x)]=\tilde{a_0}(s_1);\ell\left[\frac{\partial\Psi(x,0)}{\partial t}\right]=\tilde{a_1}(s_1).\\ \ell[\Psi(0,t)]&=\ell[b_0(t)]=\tilde{b_0}(s_2);\ell\left[\frac{\partial\Psi(0,t)}{\partial x}\right]=\tilde{b_1}(s_2).\end{aligned} \tag{3.30}$$

$$\ell[\Psi(x,0)]=\ell[n_0(x)]=\tilde{n_0}(s_1);\ell\left[\mathfrak{D}_t\Psi\left(\frac{x^\gamma}{\gamma},0\right)\right]=\ell\left[n_1\left(\frac{x^\gamma}{\gamma}\right)\right]=\tilde{n_1}(s_1).$$
$$\ell[\Psi(0,t)]=\ell[m_0(t)]=\tilde{m_0}(s_2);\ell\left[\mathfrak{D}_x\Psi\left(0,\frac{t^\delta}{\delta}\right)\right]=\ell\left[m_1\left(\frac{t^\delta}{\delta}\right)\right]=\tilde{m_1}(s_2).$$

(3.31)

Let us now apply the CpDLTr to both left-hand and right-hand sides of Eq. (3.28), we obtain:

$$\ell^x\ell^t\left[\frac{\partial^{2\gamma}\Psi(x,t)}{\partial x^{2\gamma}}\right]=\tilde{\Psi}(s_1,s_2)=\frac{s_1^{2\gamma-1}}{s_1^{2\gamma}}\ell^x\ell^t\left[b_0(t)\right]+\frac{s_1^{s\gamma-2}}{s_1^{2\gamma}}\ell^x\ell^t\left[b_1(t)\right]$$
$$-\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\omega_2}{\omega_3}\frac{\partial^{2\delta}\Psi(x,t)}{\partial t^{2\delta}}+\frac{i}{\omega_3}\frac{\partial^{\gamma}\Psi(x,t)}{\partial x^{\gamma}}+i\frac{\omega_1}{\omega_3}\frac{\partial^{\delta}\Psi(x,t)}{\partial t^{\delta}}\right]+\ell^x\ell^t\left[\frac{1}{\omega_3}\left|\Psi\right|^2\Psi\right]\right].$$

By simplifying the above, we obtain:

$$\ell^x\ell^t\left[\frac{\partial^{2\gamma}\Psi(x,t)}{\partial x^{2\gamma}}\right]=\tilde{\Psi}(s_1,s_2)=\frac{1}{s_1}\ell^x\ell^t\left[b_0(t)\right]+\frac{1}{s_1^{2}}\ell^x\ell^t\left[b_1(t)\right]$$
$$-\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\omega_2}{\omega_3}\frac{\partial^{2\delta}\Psi(x,t)}{\partial t^{2\delta}}+\frac{i}{\omega_3}\frac{\partial^{\gamma}\Psi(x,t)}{\partial x^{\gamma}}+i\frac{\omega_1}{\omega_3}\frac{\partial^{\delta}\Psi(x,t)}{\partial t^{\delta}}\right]+\ell^x\ell^t\left[\frac{1}{\omega_3}\left[\Psi^2\Psi^*\right]\right]\right].$$

where $\left|\Psi\right|^2\Psi=\Psi^2\Psi^*$ such that Ψ^* is the conjugate of Ψ.

(3.32)

According to the Adomian decomposition method (ADcM), Eq. (3.32) is written according to the following standard operator form for NLPrDfEqs: $L\Psi(x,t)+R\Psi(x,t)+N\Psi(x,t)=S(x,t)$ where N represents the nonlinear differential operator, L represents the 2nd-order partial differential operator, R represents the remaining linear operator, and $S(x,t)$ represents a source term. Therefore, by applying the method of CpDLTr coupled with ADcM, the decomposition infinite series can be expressed for both linear and nonlinear terms as follows:

$$\Psi(x,t)=\sum_{i=0}^{\infty}\Psi_i(x,t)$$
$$N(\Psi(x,t))=\sum_{i=0}^{\infty}\phi_i(\Psi(x,t)). \tag{3.33}$$

where the above nonlinear term, denoted by $N(\Psi(x,t))$, is represented by infinite series of the Adomian polynomials, donated by ϕ_i, which can be expressed (Nuruddeen et al., 2018) as follows:

$$\phi_i = \frac{1}{i!}\frac{d^i}{d\Omega^i}\left[N\left(\sum_{j=0}^{\infty}\Omega^j\phi_j\right)\right]_{\Omega=0}, i = 0, 1, 2, 3, ... \tag{3.34}$$

so, we can write some of those terms as follows:

$$\phi_0 = N(\Psi_0);\ \phi_1 = \Psi_1 N'(\Psi_0);\ \phi_2 = \Psi_2 N'(\Psi_0) + \frac{1}{2!}\Psi_1^2 N''(\Psi_0).$$

By applying the standard NLPrDfEqs operator form, and Eq. (3.33) to Eq. (3.32), we obtain:

$$\ell^x\ell^t\left[\sum_{i=0}^{\infty}\Psi_i(x,t)\right] = \frac{1}{s_1}\ell^x\ell^t[b_0(t)] + \frac{1}{s_1^2}\ell^x\ell^t[b_1(t)]$$
$$-\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[R\left[\sum_{i=0}^{\infty}\Psi_i(x,t)\right]\right] + \ell^x\ell^t\left[\sum_{i=0}^{\infty}\phi_i\right]\right]. \tag{3.35}$$

where $R[\Psi(x,t)] = \frac{\omega_2}{\omega_3}\frac{\partial^{2\delta}\Psi(x,t)}{\partial t^{2\delta}} + \frac{i}{\omega_3}\frac{\partial^{\gamma}\Psi(x,t)}{\partial x^{\gamma}} + i\frac{\omega_1}{\omega_3}\frac{\partial^{\delta}\Psi(x,t)}{\partial t^{\delta}}$

and $\phi_i[\Psi(x,t)] = \frac{1}{\omega_3}\Psi^2\Psi^*$.

Let us now write some of the Adomian polynomials, $\phi_i's$ using the formula in Eq. (3.34) as follows:

$$\phi_0 = \frac{1}{\omega_3}\Psi_0^2\Psi_0^*,$$
$$\phi_1 = \frac{2}{\omega_3}\Psi_0\Psi_1\Psi_0^* + \frac{1}{\omega_3}\Psi_0^2\Psi_1^*,$$
$$\phi_2 = \frac{2}{\omega_3}\Psi_0\Psi_2\Psi_0^* + \frac{1}{\omega_3}\Psi_1^2\Psi_0^* + \frac{2}{\omega_3}\Psi_0\Psi_1\Psi_1^* + \frac{1}{\omega_3}\Psi_0^2\Psi_2^*.$$

By applying the inverse double Laplace transform to the left-hand and right-hand sides of Eq. (3.35), we obtain the following general solution to Eq. (3.28), recursively:

$$\Psi_0(x,t) = b_0(t) + x b_1(t),$$

$$\Psi_1(x,t) = -(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\omega_2}{\omega_3}\frac{\partial^{2\delta}\Psi_0(x,t)}{\partial t^{2\delta}}+\frac{i}{\omega_3}\frac{\partial^{\gamma}\Psi_0(x,t)}{\partial x^{\gamma}}+i\frac{\omega_1}{\omega_3}\frac{\partial^{\delta}\Psi_0(x,t)}{\partial t^{\delta}}\right]+\ell^x\ell^t\left[\phi_0(\Psi(x,t))\right]\right]\right],$$

$$\Psi_2(x,t) = -(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\omega_2}{\omega_3}\frac{\partial^{2\delta}\Psi_1(x,t)}{\partial t^{2\delta}}+\frac{i}{\omega_3}\frac{\partial^{\gamma}\Psi_1(x,t)}{\partial x^{\gamma}}+i\frac{\omega_1}{\omega_3}\frac{\partial^{\delta}\Psi_1(x,t)}{\partial t^{\delta}}\right]+\ell^x\ell^t\left[\phi_1(\Psi(x,t))\right]\right]\right],$$

.

.

.

$$\Psi_{i+1}(x,t) = -(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\ell^x\ell^t\left[R[\Psi_i(x,t)]\right]+\ell^x\ell^t\left[\phi_i(\Psi(x,t))\right]\right], \text{for } i \geq 0. \tag{3.36}$$

Similarly, we can apply the CmDLTr to both sides of Eq. (3.29), we have:

$$\ell^x\ell^t\left[\mathfrak{D}_x^{2\gamma}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]=\tilde{\Psi}(s_1,s_2)=\frac{s_1^{2\gamma-1}}{s_1^{2\gamma}}\ell^x\ell^t\left[m_0(t)\right]+\frac{s_1^{s\gamma-2}}{s_1^{2\gamma}}\ell^x\ell^t\left[m_1\left(\frac{t^\delta}{\delta}\right)\right]$$
$$-\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\omega_2}{\omega_3}D_t^{2\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+\frac{i}{\omega_3}D_x^{\gamma}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\frac{\omega_1}{\omega_3}D_t^{\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]+\ell^x\ell^t\left[\frac{1}{\omega_3}|\Psi|^2\Psi\right]\right].$$

After simplifications, we have:

$$\ell^x\ell^t\left[\mathfrak{D}_x^{2\gamma}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]=\tilde{\Psi}(s_1,s_2)=\frac{1}{s_1}\ell^x\ell^t\left[m_0(t)\right]+\frac{1}{s_1^2}\ell^x\ell^t\left[m_1\left(\frac{t^\delta}{\delta}\right)\right]$$
$$-\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\omega_2}{\omega_3}D_t^{2\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+\frac{i}{\omega_3}D_x^{\gamma}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\frac{\omega_1}{\omega_3}D_t^{\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]+\ell^x\ell^t\left[\frac{1}{\omega_3}\left[\Psi^2\Psi^*\right]\right]\right]. \tag{3.37}$$

where $|\Psi|^2\Psi=\Psi^2\Psi^*$ such that Ψ^* is the conjugate of Ψ.

Let us now apply the standard NLPrDfEqs operator form and Eq. (3.33) to Eq.

(3.37), we have:

$$
\begin{aligned}
&\ell^x\ell^t\left[\sum_{i=0}^{\infty}\Psi_i(x,t)\right]=\frac{1}{s_1}\ell^x\ell^t\left[m_0(t)\right]+\frac{1}{s_1^2}\ell^x\ell^t\left[m_1\left(\frac{t^\delta}{\delta}\right)\right]\\
&-\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[R\left[\sum_{i=0}^{\infty}\Psi_i(x,t)\right]\right]+\ell^x\ell^t\left[\sum_{i=0}^{\infty}\phi_i\right]\right].\\
&\text{where } R[\Psi(x,t)]=\left[\frac{\omega_2}{\omega_3}\mathfrak{D}_t^{2\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+\frac{i}{\omega_3}\mathfrak{D}_x^{\gamma}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\frac{\omega_1}{\omega_3}\mathfrak{D}_t^{\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]\\
&\text{and } \phi_i[\Psi(x,t)]=\frac{1}{\omega_3}\Psi^2\Psi^*.
\end{aligned}
\tag{3.38}
$$

We apply the inverse double Laplace transform to both sides of Eq. (3.38) to obtain the general solution to Eq. (3.29), recursively as follows:

$$
\begin{aligned}
&\Psi_0(x,t)=m_0\left(\frac{t^\delta}{\delta}\right)+\frac{x^\gamma}{\gamma}m_1\left(\frac{t^\delta}{\delta}\right),\\
&\Psi_1(x,t)=\\
&-(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\omega_2}{\omega_3}\mathfrak{D}_t^{2\delta}\Psi_0\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+\frac{i}{\omega_3}\mathfrak{D}_x^{\gamma}\Psi_0\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\frac{\omega_1}{\omega_3}D_t^{\delta}\Psi_0\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]+\ell^x\ell^t\left[\phi_0(\Psi(x,t))\right]\right]\right],\\
&\Psi_2(x,t)=\\
&-(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\omega_2}{\omega_3}\mathfrak{D}_t^{2\delta}\Psi_1\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+\frac{i}{\omega_3}\mathfrak{D}_x^{\gamma}\Psi_1\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\frac{\omega_1}{\omega_3}D_t^{\delta}\Psi_1\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]+\ell^x\ell^t\left[\phi_1(\Psi(x,t))\right]\right]\right],\\
&\vdots\\
&\Psi_{i+1}(x,t)=-(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\ell^x\ell^t\left[R[\Psi_i(x,t)]\right]+\ell^x\ell^t\left[\phi_i(\Psi(x,t))\right]\right],\text{ for } i\geq 0.
\end{aligned}
\tag{3.39}
$$

3.3 Solutions of the $(3+1)$-Dimensional Conformable WaBeBoMaEq

Exact soliton solutions of the proposed Eq. (3.2) are obtained in this section via GeKM and ExM.

$$p\gamma\delta\Psi''' + q(\Psi^3)' + (-\gamma + p)\Psi' = 0. \tag{3.40}$$

By integrating Eq. (3.40) once with respect to $\aleph$, we obtain:

$$p\gamma\delta\Psi'' + q\Psi^3 + (-\delta + p)\Psi = 0. \tag{3.41}$$

3.3.1 Exact Solutions via the GeKM

According to the HBPrp, the highest-order derivative and nonlinear terms in Eq. (3.41) are balanced. Thus, we get: $J = W + 1$. Let us set $W = 1$, we obtain: $J = 2$. Thus, the solution can be written as:

$$\Psi(\aleph) = \frac{p_0 + p_1\hbar + p_2\hbar^2}{q_0 + q_1\hbar}, \tag{3.42}$$

where $\hbar = \hbar(\aleph)$ is the solution of the Eq. (3.13). As a result, by substituting Eq. (3.42) into Eq. (3.41) and using Eq. (3.13), we obtain system of algebraic equations by equating all coefficients of the functions $\hbar^0, \hbar^1, \hbar^2, \hbar^3, \hbar^4, \hbar^5, \hbar^6$ to 0. Now, p_0, p_1, p_2, q_0, and q_1 are all parameters.

$$\hbar^6 \quad : \quad qp_2^3{+}2p\gamma\delta p_2q_1^2{=}0,$$

$$\hbar^5 \quad : \quad 3qp_1p_2^2{-}3p\gamma\delta p_2q_1^2{+}6p\gamma\delta p_2q_0q_1{=}0,$$

$$\hbar^4 \quad : \quad -9p\gamma\delta p_2q_0q_1{-}\delta p_2q_1^2{+}pp_2q_1^2{+}6p\gamma\delta p_2q_0^2{+}p\gamma\delta p_2q_1^2{+}3qp_0p_2^2{+}3qp_1^2p_2{=}0,$$

$$\hbar^3 \quad : \quad -\delta p_1q_1^2{+}2p\gamma\delta p_1q_0^2{+}qp_1^3{-}10p\gamma\delta p_2q_0^2{+}p\gamma\delta\hbar_1q_0q_1{-}p\gamma\delta q_1^2p_0{+}pp_1q_1^2$$
$$-2\delta p_2q_0q_1{+}3p\gamma\delta p_2q_0q_1{+}6qp_0p_1p_2{+}2pp_2q_0q_1{-}2p\gamma\delta q_1p_0q_0{=}0,$$

$$\hbar^2 \quad : \quad -2\delta p_1q_0q_1{-}p\gamma\delta p_1q_0q_1{+}pp_0q_1^2{+}3qp_0^2p_2{-}\delta p_0q_1^2{+}3qp_0p_1^2{-}3p\gamma\delta p_1q_0^2$$
$${+}2pp_1q_0q_1{+}pp_2q_0^2{-}\delta p_2q_0^2{+}4p\gamma\delta p_2q_0^2{+}p\gamma\delta q_1^2p_0{+}3p\gamma\delta q_1p_0q_0{=}0,$$

$$\hbar^1 \quad : \quad -p\gamma\delta q_1p_0q_0{+}2pp_0q_0q_1{+}p\gamma\delta p_1q_0^2{-}\delta p_1q_0^2{+}3qp_0^2p_1{-}2\delta p_0q_0q_1{+}pp_1q_0^2{=}0,$$

$$\hbar^0 \quad : \quad pp_0q_0^2{+}qp_0^3{=}0$$

From the above set of algebraic equations, various cases are presented as follows:

CASE I:

$$p_0{=}0, p_1{=}\pm pq_1\sqrt{-\frac{\gamma}{2q+pq\gamma}}, p_2{=}\pm\frac{2p\gamma q_1}{(p\gamma{+}2)q\sqrt{-\frac{\gamma}{2q+pq\gamma}}}, q_0{=}0, q_1{=}q_1, \delta{=}\frac{2p}{p\gamma+2}$$

Then, by the substitution of the obtained values into Eq. (3.42) with Eq. (3.14), the soliton-type solutions of the following WaBeBoMaEq in the context of ComD:

$$\Psi_1(x,y,z,t) = \pm\frac{(1 - I_1\,(\cosh(\aleph) + \sinh(\aleph)))\,p\gamma}{\sqrt{-\frac{\gamma}{q(p\gamma+2)}}\,q\,(p\gamma+2)\,(1 + I_1\cosh(\aleph) + I_1\sinh(\aleph))}, \quad (3.43)$$

where $\aleph = p\frac{x^\zeta}{\zeta} + q\frac{y^\zeta}{\zeta} + \gamma\frac{z^\zeta}{\zeta} - \frac{2p}{p\gamma+2}\frac{t^\zeta}{\zeta}$ for ComD. I_1 is an arbitrary constant.

CASE II:

$$p_0=0, p_1=\pm 2q_0 p\sqrt{\frac{2\gamma}{-q+pq\gamma}}, p_2=\mp 2q_0 p\sqrt{\frac{2\gamma}{-q+pq\gamma}}, q_0=q_0, q_1=-2q_0, \delta=\frac{p}{1-p\gamma}$$

Then, by the substitution of the obtained values into Eq. (3.42) with Eq. (3.14), the soliton-type solutions of the following WaBeBoMaEq in the context of ComD:

$$\Psi_2(x,y,z,t) = \pm\frac{2pI_1\sqrt{\frac{2\gamma}{q(-1+p\gamma)}}\left(\cosh(\aleph)+\sinh(\aleph)\right)}{-1+I_1^2\left(2\cosh(\aleph)\sinh(\aleph)+2\cosh^2(\aleph)-1\right)}, \tag{3.44}$$

where $\aleph = p\frac{x^\zeta}{\zeta} + q\frac{y^\zeta}{\zeta} + \gamma\frac{z^\zeta}{\zeta} - \frac{p}{1-p\gamma}\frac{t^\zeta}{\zeta}$ for ComD. I_1 is a constant.

3.3.2 Exact Solutions via the ExM

The ExM is a very helpful technique to construct the solution of Eq. (3.5) in the following form:

$$\Psi(\xi) = p_0 + p_1\exp(-\phi(\aleph)). \tag{3.45}$$

Let us substitute Eq. (3.41) into Eq. (3.5) and collect the coefficient of each power of $\exp(-\Phi(\aleph))^j$. Now, a set of algebraic equations of $p, q, \gamma, \delta, p_0, p_1, \chi$ and ϑ is obtained by equating all coefficients to 0.

$$\begin{aligned}
\exp(3\aleph) &: \; qp_0^3 - \delta p_0 + p\gamma\delta p_1\vartheta\chi + pp_0 = 0, \\
\exp(2\aleph) &: \; 3qp_0^2 p_1 - \delta p_1 + 2p\gamma\delta p_1\vartheta + p\gamma\delta p_1\chi^2 + pp_1 = 0, \\
\exp(\aleph) &: \; 3p\gamma\delta p_1\chi + 3qp_0p_1^2 = 0, \\
\exp(0\aleph) &: \; 2p\gamma\delta p_1 + qp_1^3 = 0.
\end{aligned} \tag{3.46}$$

By using MAPLE software, we obtain the following solution:

$$p_0 = \pm\chi p\sqrt{\frac{\gamma}{-pq\gamma\chi^2-2q+4pq\gamma\vartheta}}, p_1 = \pm 2p\sqrt{\frac{\gamma}{-pq\gamma\chi^2-2q+4pq\gamma\vartheta}}, \delta = \frac{2p}{p\gamma\chi^2+2-4p\gamma\vartheta} \tag{3.47}$$

From all the above obtained values, the technique's algorithm, and its auxiliary equations, different cases for the conformable WaBeBoMaEq are given as follows:

CASE I:

When $\chi^2 - 4\vartheta > 0$ and $\vartheta \neq 0$, the hyperbolic function solutions are expressed as:

$$\Psi_1(x, y, z, t) = \pm \frac{p\sqrt{\frac{\gamma}{q\left(-p\gamma\chi^2-2+4p\gamma\vartheta\right)}}\left(\chi^2+\chi\tanh\left(\frac{\sqrt{\chi^2-4\vartheta}}{2}(\aleph+I)\right)\sqrt{\chi^2-4\vartheta}-4\vartheta\right)}{\chi+\tanh\left(\frac{\sqrt{\chi^2-4\vartheta}}{2}(\aleph+I)\right)\sqrt{\chi^2-4\vartheta}}, \tag{3.48}$$

where $\aleph = p\frac{x^\zeta}{\zeta} + q\frac{y^\zeta}{\zeta} + \gamma\frac{z^\zeta}{\zeta} - \frac{2p}{p\gamma\chi^2+2-4p\gamma\vartheta}\frac{t^\zeta}{\zeta}$ for ComD. I is a constant.

CASE II:

When $\chi^2 - 4\vartheta < 0$ and $\vartheta \neq 0$, the trigonometric function solutions are expressed as:

$$\Psi_2(x,y,z,t) = \pm \frac{p\sqrt{\frac{\gamma}{q\left(-p\gamma\chi^2-2+4p\gamma\vartheta\right)}}\left(-\chi^2+\chi\tan\left(\frac{\sqrt{4\vartheta-\chi^2}}{2}(\aleph+I)\right)\sqrt{4\vartheta-\chi^2}+4\vartheta\right)}{-\chi+\tan\left(\frac{\sqrt{4\vartheta-\chi^2}}{2}(\aleph+I)\right)\sqrt{4\vartheta-\chi^2}}, \tag{3.49}$$

where $\aleph = p\frac{x^\zeta}{\zeta} + q\frac{y^\zeta}{\zeta} + \gamma\frac{z^\zeta}{\zeta} - \frac{2p}{p\gamma\chi^2+2-4p\gamma\vartheta}\frac{t^\zeta}{\zeta}$ for ComD. I is a constant.

CASE III:

When $\chi^2 - 4\vartheta > 0$, $\vartheta = 0$ and $\chi \neq 0$, the hyperbolic function solutions are expressed as:

$$\Psi_3(x,y,z,t) = \pm \frac{\chi p\sqrt{\frac{\gamma}{q(-p\gamma\chi^2-2+4p\gamma\vartheta)}}\left(\cosh\left(\chi\left(\aleph+I\right)\right)+\sinh\left(\chi\left(\aleph+I\right)\right)+1\right)}{\cosh\left(\chi\left(\aleph+I\right)\right)+\sinh\left(\chi\left(\aleph+I\right)\right)-1}, \tag{3.50}$$

where $\aleph = p\frac{x^\zeta}{\zeta} + q\frac{y^\zeta}{\zeta} + \gamma\frac{z^\zeta}{\zeta} - \frac{2p}{p\gamma\chi^2+2-4p\gamma\vartheta}\frac{t^\zeta}{\zeta}$ for ComD. I is a constant.

CASE IV:

When $\chi^2 - 4\vartheta = 0, \vartheta \neq 0$ and $\chi \neq 0$, the rational function solutions are expressed as:

$$\Psi_4(x,y,z,t) = \pm \frac{2\chi p\sqrt{\frac{\gamma}{q\left(-p\gamma\chi^2-2+4p\gamma\vartheta\right)}}}{\chi\left(\aleph+I\right)+2}, \tag{3.51}$$

where $\aleph = p\frac{x^\zeta}{\zeta} + q\frac{y^\zeta}{\zeta} + \gamma\frac{z^\zeta}{\zeta} - \frac{2p}{p\gamma\chi^2+2-4p\gamma\vartheta}\frac{t^\zeta}{\zeta}$ for ComD. I is a constant.

CASE V:

Finally, when $\chi^2 - 4\vartheta = 0, \vartheta = 0$ and $\chi = 0$, we have:

$$\Psi_5(x,y,z,t) = \pm \frac{\sqrt{\frac{\gamma}{q\left(-p\gamma\chi^2-2+4p\gamma\vartheta\right)}}\,p\left(\chi\left(\aleph+I\right)+2\right)}{\aleph+I}, \tag{3.52}$$

where $\aleph = p\frac{x^\zeta}{\zeta} + q\frac{y^\zeta}{\zeta} + \gamma\frac{z^\zeta}{\zeta} - \frac{2p}{p\gamma\chi^2+2-4p\gamma\vartheta}\frac{t^\zeta}{\zeta}$ for ComD. I is a constant.

3.4 Solutions of the Nonlinear Fractional Schrödinger Equation

Let us start this section by introducing two numerical experiments to find the solutions of proposed NLPrDfEq.

Numerical Experiment 1:

By applying definitions and properties of CpFD and double Laplace transform, the following numerical experiment will solve Eq. (3.28) analytically: Let $\omega_1 = \omega_2 = \omega_3 = 1$, and $b_0(t) = e^{it}; b_1(t) = 0; a_0(x) = a_1(x) = 0$ in (16), we have:

$$\frac{\partial^{2\gamma}\Psi(x,t)}{\partial x^{2\gamma}} = -\frac{\partial^{2\delta}\Psi(x,t)}{\partial t^{2\delta}} - i\frac{\partial^{\gamma}\Psi(x,t)}{\partial x^{\gamma}} - i\frac{\partial^{\delta}\Psi(x,t)}{\partial t^{\delta}} - |\Psi|^2\,\Psi.$$

subject to the following initial and boundary conditions:

$$\Psi(x,0) = 0 \text{ and } \frac{\partial\Psi(x,0)}{\partial t} = 0. \tag{3.53}$$

$$\Psi(0,t) = e^{it} \text{ and } \frac{\partial\Psi(0,t)}{\partial x} = 0.$$

where $i = \sqrt{-1}$, $0 < \gamma, \delta \leq 1$, $t, x > 0$; $x, t \in \Re^+$.

To solve Eq. (3.53), we use our result in Eq. (3.36) as follows:

$$\Psi_0(x,t) = e^{it},$$

$$\Psi_1(x,t) = -(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\partial^{2\delta}\Psi_0(x,t)}{\partial t^{2\delta}} + i\frac{\partial^{\gamma}\Psi_0(x,t)}{\partial x^{\gamma}} + i\frac{\partial^{\delta}\Psi_0(x,t)}{\partial t^{\delta}}\right] + \ell^x\ell^t\left[\phi_0(\Psi(x,t))\right]\right]\right]$$

$$= -(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\partial^{2\delta}\Psi_0(x,t)}{\partial t^{2\delta}} + i\frac{\partial^{\gamma}\Psi_0(x,t)}{\partial x^{\gamma}} + i\frac{\partial^{\delta}\Psi_0(x,t)}{\partial t^{\delta}}\right] + \ell^x\ell^t\left[\Psi_0^2\Psi_0^*\right]\right]\right]$$

$$= -\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\left[\left[it^{1-2\delta}E_{1,2-2\delta}(it) - t^{1-\delta}E_{1,2-\delta}(it)\right] + e^{it}\right]$$

$$= -\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\left[\left[iE(t, 1-2\delta, i) - E(t, 1-\delta, i)\right] + e^{it}\right],$$

$$
\begin{aligned}
\Psi_2(x,t) &= -(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\partial^{2\delta}\Psi_1(x,t)}{\partial t^{2\delta}}+i\frac{\partial^{\gamma}\Psi_1(x,t)}{\partial x^{\gamma}}+i\frac{\partial^{\delta}\Psi_1(x,t)}{\partial t^{\delta}}\right]+\ell^x\ell^t\left[\phi_1(\Psi(x,t))\right]\right]\right] \\
&= -(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\frac{\partial^{2\delta}\Psi_1(x,t)}{\partial t^{2\delta}}+i\frac{\partial^{\gamma}\Psi_1(x,t)}{\partial x^{\gamma}}+i\frac{\partial^{\delta}\Psi_1(x,t)}{\partial t^{\delta}}\right]+\ell^x\ell^t\left[2\Psi_0\Psi_1\Psi_0^*+\Psi_0^2\Psi_1^*\right]\right]\right] \\
&= -\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\left[-\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\frac{\partial^{2\delta}}{\partial t^{2\delta}}\left([iE(t,1-2\delta,i)-E(t,1-\delta,i)]+e^{it}\right)\right. \\
&- i\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\left[\left[\left([iE(t,1-2\delta,i)-E(t,1-\delta,i)]+e^{it}\right)\right]\frac{\partial^{\gamma}}{\partial x^{\gamma}}\left(-\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\right)\right] \\
&- i\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\left[-\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\frac{\partial^{\delta}}{\partial t^{\delta}}\left([iE(t,1-2\delta,i)-E(t,1-\delta,i)]+e^{it}\right)\right] \\
&- \frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\left[(2e^{it})\left(-\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\right)\left([iE(t,1-2\delta,i)-E(t,1-\delta,i)]+e^{it}\right)(e^{-it})\right] \\
&- \frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\left[(e^{2it})\left(-\frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\right)\left([-iE(t,1-2\delta,i)-E(t,1-\delta,i)]+e^{-it}\right)(e^{-it})\right] \\
&= \frac{x^{4\gamma}}{\Gamma(4\gamma+1)}\left[iE(t,1-4\delta,i)-E(t,1-3\delta,i)+iE(t,1-2\delta,i)\right] \\
&+ \frac{x^{3\gamma}}{\Gamma(3\gamma+1)}\left[E(t,1-2\delta,i)-iE(t,1-\delta,i)+ie^{it}\right] \\
&+ \frac{x^{4\gamma}}{\Gamma(4\gamma+1)}\left[-E(t,1-3\delta,i)-iE(t,1-2\delta,i)-E(t,1-\delta,i)\right] \\
&+ \frac{x^{4\gamma}}{\Gamma(4\gamma+1)}\left[-2iE(t,1-2\delta,i)-E(t,1-\delta,i)+e^{it}\right] \\
&+ \frac{x^{4\gamma}}{\Gamma(4\gamma+1)}\left[-ie^{2it}E(t,1-2\delta,i)-e^{2it}E(t,1-\delta,i)+e^{it}\right] \\
&= \frac{x^{3\gamma}}{\Gamma(3\gamma+1)}\left[E(t,1-2\delta,i)-iE(t,1-\delta,i)+ie^{it}\right] \\
&+ \frac{x^{4\gamma}}{\Gamma(4\gamma+1)}\{iE(t,1-4\delta,i)-E(t,1-3\delta,i)+iE(t,1-2\delta,i)-E(t,1-3\delta,i) \\
&- iE(t,1-2\delta,i)-E(t,1-\delta,i)-2iE(t,1-2\delta,i)-E(t,1-\delta,i)+e^{it} \\
&- ie^{2it}E(t,1-2\delta,i)-e^{2it}E(t,1-\delta,i)+e^{it}\}.
\end{aligned}
$$

$$\vdots$$

and so on.

By using all above obtained results, the general approximate-analytical solution to Eq. (3.53) can be written as follows:

$$\Psi(x,t) = e^{it} - \frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\left[[iE(t,1-2\delta,i) - E(t,1-\delta,i)] + e^{it}\right]$$
$$+ \frac{x^{3\gamma}}{\Gamma(3\gamma+1)}\left[E(t,1-2\delta,i) - iE(t,1-\delta,i) + ie^{it}\right] + \frac{x^{4\gamma}}{\Gamma(4\gamma+1)}\{iE(t,1-4\delta,i)$$
$$- E(t,1-3\delta,i) + iE(t,1-2\delta,i) - E(t,1-3\delta,i) - iE(t,1-2\delta,i) - E(t,1-\delta,i)$$
$$- 2iE(t,1-2\delta,i) - E(t,1-\delta,i) + e^{it} - ie^{2it}E(t,1-2\delta,i) - e^{2it}E(t,1-\delta,i) + e^{it}\}.$$
$$+ \dots \tag{3.54}$$

Hence, the approximate-analytical solution for the Eq. (3.6) in the sense of CpFD has been easily obtained via the double Laplace transform coupled with the ADcM.

Numerical Experiment 2:

By applying the definitions and properties of ComD and double Laplace transform, the following numerical experiment solves Eq. (3.29) analytically: Let $\omega_1 = \omega_2 = \omega_3 = 1$, and $m_0(t) = e^{i\frac{t^\delta}{\delta}}; m_1(t) = 0; n_0(x) = n_1(x) = 0$ in Eq. 3.29, we have:

$$\mathfrak{D}_x^{2\gamma}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right) = -\mathfrak{D}_t^{2\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right) - i\mathfrak{D}_x^{\gamma}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right) - i\mathfrak{D}_t^{\delta}\Psi\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)$$
$$- |\Psi|^2\,\Psi.$$

subject to the following initial and boundary conditions:

$$\Psi\left(\frac{x^\gamma}{\gamma},0\right) = 0 \text{ and } \mathfrak{D}_t\Psi\left(\frac{x^\gamma}{\gamma},0\right) = 0. \tag{3.55}$$
$$\Psi\left(0,\frac{t^\delta}{\delta}\right) = e^{i\frac{t^\delta}{\delta}} \text{ and } \mathfrak{D}_x\Psi\left(0,\frac{t^\delta}{\delta}\right) = 0.$$

where $i = \sqrt{-1}$, $0 < \gamma, \delta \le 1$, $t, x > 0$; $x, t \in \mathfrak{R}^+$.

To solve Eq. (3.55), we use our result in Eq. 3.39 as follows:

$$\Psi_0(x,t) = e^{i\frac{t^\delta}{\delta}},$$

$$\Psi_1(x,t) = -(\ell^x)^{-1}(\ell^t)^{-1}\times$$

$$\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\mathfrak{D}_t^{2\delta}\Psi_0\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\mathfrak{D}_x^{\gamma}\Psi_0\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\mathfrak{D}_t^{\delta}\Psi_0\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]+\ell^x\ell^t\left[\phi_0(\Psi(x,t))\right]\right]\right]$$

$$=-(\ell^x)^{-1}(\ell^t)^{-1}\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\mathfrak{D}_t^{2\delta}\Psi_0\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\mathfrak{D}_x^{\gamma}\Psi_0\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\mathfrak{D}_t^{\delta}\Psi_0\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]+\ell^x\ell^t\left[\Psi_0^2\Psi_0^*\right]\right]\right]$$

$$= -\frac{x^{2\gamma-1}}{\gamma^{2\gamma-1}\Gamma(2\gamma)}\left[\left[-e^{i\frac{t^\delta}{\delta}} - e^{i\frac{t^\delta}{\delta}}\right] + e^{i\frac{t^\delta}{\delta}}\right]$$

$$= -\frac{x^{2\gamma-1}}{\gamma^{2\gamma-1}\Gamma(2\gamma)}\left[-e^{i\frac{t^\delta}{\delta}}\right]$$

$$\Psi_2(x,t) = -(\ell^x)^{-1}(\ell^t)^{-1}\times$$

$$\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\mathfrak{D}_t^{2\delta}\Psi_1\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\mathfrak{D}_x^{\gamma}\Psi_1\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\mathfrak{D}_t^{\delta}\Psi_1\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]+\ell^x\ell^t\left[\phi_1(\Psi(x,t))\right]\right]\right]$$

$$= -(\ell^x)^{-1}(\ell^t)^{-1}\times$$

$$\left[\frac{1}{s_1^{2\gamma}}\left[\ell^x\ell^t\left[\mathfrak{D}_t^{2\delta}\Psi_1\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\mathfrak{D}_x^{\gamma}\Psi_1\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)+i\mathfrak{D}_t^{\delta}\Psi_1\left(\frac{x^\gamma}{\gamma},\frac{t^\delta}{\delta}\right)\right]+\ell^x\ell^t\left[2\Psi_0\Psi_1\Psi_0^*+\Psi_0^2\Psi_1^*\right]\right]\right]$$

$$=-\frac{x^{2\gamma-1}}{\gamma^{2\gamma-1}\Gamma(2\gamma)}\{\left(-\frac{x^{2\gamma-1}}{\gamma^{2\gamma-1}\Gamma(2\gamma)}\right)e^{i\frac{t^\delta}{\delta}}+(2\gamma-1)x^{\gamma-1}\left(i\frac{e^{i\frac{t^\delta}{\delta}}}{\gamma^{2\gamma-1}\Gamma(2\gamma)}\right)-\left(\frac{x^{2\gamma-1}}{\gamma^{2\gamma-1}\Gamma(2\gamma)}\right)e^{i\frac{t^\delta}{\delta}}$$

$$-\left(\frac{2x^{2\gamma-1}}{\gamma^{2\gamma-1}\Gamma(2\gamma)}\right)e^{-i\frac{t^\delta}{\delta}}+\left(\frac{x^{2\gamma-1}}{\gamma^{2\gamma-1}\Gamma(2\gamma)}\right)e^{i\frac{t^\delta}{\delta}}\}$$

$$=\frac{x^{4\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}e^{i\frac{t^\delta}{\delta}}-i\frac{(2\gamma-1)x^{3\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}e^{i\frac{t^\delta}{\delta}}+\frac{x^{4\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}e^{i\frac{t^\delta}{\delta}}+\frac{2x^{4\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}e^{-i\frac{t^\delta}{\delta}}-\frac{x^{4\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}e^{i\frac{t^\delta}{\delta}}$$

$$= -i\frac{(2\gamma-1)x^{3\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}e^{i\frac{t^\delta}{\delta}} + \frac{x^{4\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}\left[e^{i\frac{t^\delta}{\delta}} + e^{i\frac{t^\delta}{\delta}} + 2e^{-i\frac{t^\delta}{\delta}} - e^{i\frac{t^\delta}{\delta}}\right]$$

$$= -i\frac{(2\gamma-1)x^{3\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}e^{i\frac{t^\delta}{\delta}} + \frac{x^{4\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}\left[e^{i\frac{t^\delta}{\delta}} + 2e^{-i\frac{t^\delta}{\delta}}\right].$$

$$\vdots$$

and so on.

By using the above obtained results, the general approximate-analytical solution to Eq. (3.55) can be written as follows:

$$\Psi(x,t) = e^{i\frac{t^{\delta}}{\delta}} - \frac{x^{2\gamma-1}}{\gamma^{2\gamma-1}\Gamma(2\gamma)}\left[-e^{i\frac{t^{\delta}}{\delta}}\right] - i\frac{(2\gamma-1)x^{3\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}e^{i\frac{t^{\delta}}{\delta}} + \frac{x^{4\gamma-2}}{\gamma^{4\gamma-2}\Gamma(4\gamma)}\left[e^{i\frac{t^{\delta}}{\delta}} + 2e^{-i\frac{t^{\delta}}{\delta}}\right] + \ldots \tag{3.56}$$

Hence, the approximate-analytical solution for the Eq. (3.6) in the sense of ComD has also been easily obtained via the double Laplace transform coupled with the ADcM.

3.5 The Graphical Comparisons of Solutions

A variety of soliton solutions in Eqs. (3.43),(3.44),(3.48),(3.49),(3.50),(3.51), (3.52) are graphically represented using Wolfram Mathematica and compared in both 3D and 2D plots in Figs. (3.1),(3.2),(3.3),(3.4),(3.5),(3.6),(3.7) for various parameters' values and ζ to show the WaBeBoMaEq solutions' dynamics and behavior. The obtained approximate solutions in both Eq. (3.54) and Eq. (3.56) have been graphically compared for various values of γ and δ (see Figs. (3.8)−(3.17),) where each graph shows both real and imaginary parts of solution. From our results, solving NLPrDfEqs in the sense of CpFD and ComD are highly recommended. Exploring the definition of other fractional derivatives are also essential, and any new mathematical definition has to be investigated further.

3.6 Conclusion

In the first part of this chapter, GeKM and ExM have been employed to investigate exact soliton solutions of the (3 + 1)-dimensional conformable WaBeBoMaEq in the context of ComD. The obtained solutions are new which indicate that GeKM and ExM provide efficient and reliable results. 2D and 3D graphical profiles have been represented for all obtained solutions at various parameters' values and orders.

The GeKM and ExM can be further employed to solve various NLPrDfEqs arising in several nonlinear scientific phenomena. A possible extension of our obtained results into higher dimensional NLPrDfEqs is a new direction of study in the future, which will provide a novel contributions to mathematical physics. In the second part, a powerful mathematical tool have been successfully proposed to solve the nonlinear Schrödinger equation in the context of CpFD and ComD. The generalized double Laplace transform method can be efficiently applied in solving MoNLSEq and all other fractional-order NLPrDfEqs.

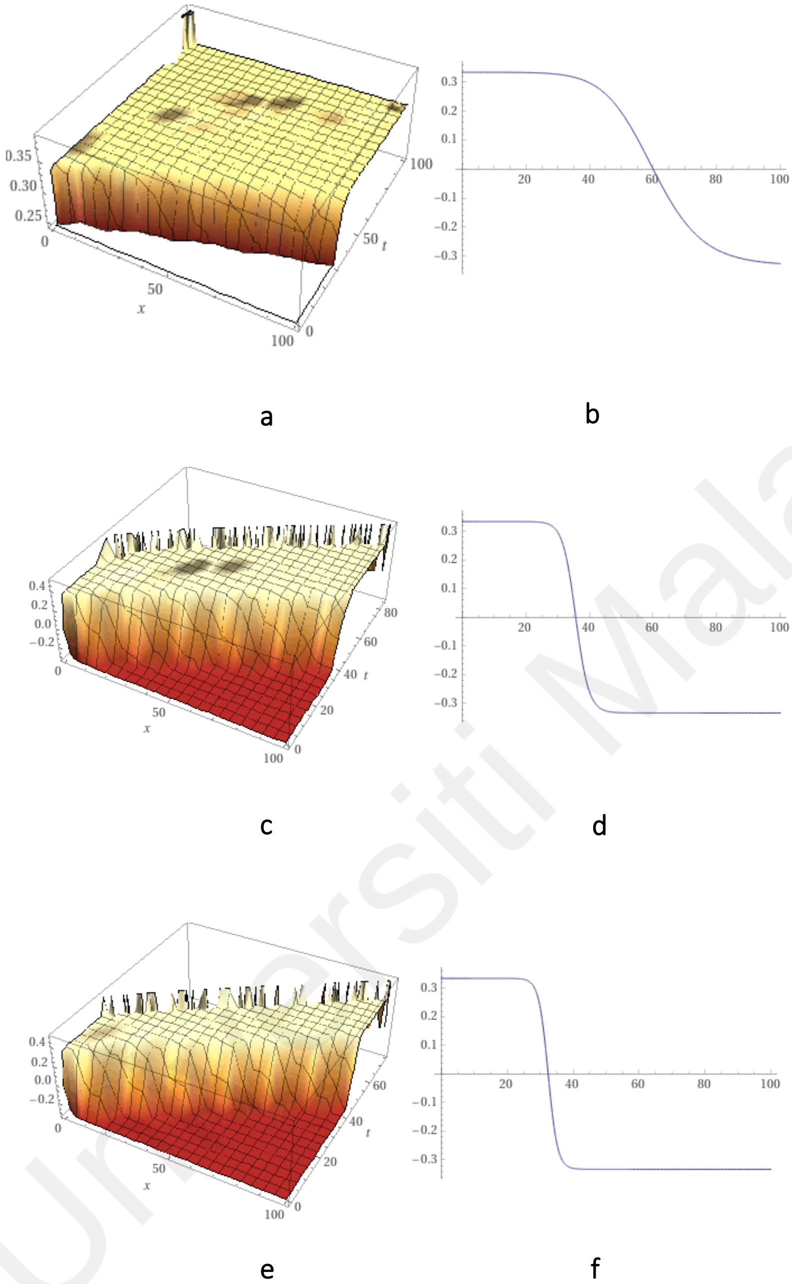

Figure 3.1: The plots of Eq. (3.43) are represented in 3D in (a), (c), (e) and in 2D in (b), (d), (f) for $\gamma = -1; q = p = I_1 = 1; t = 15; y = z = 0; \zeta = 0.50; \zeta = 0.8; \zeta = 0.90$ **for ComD, respectively.**

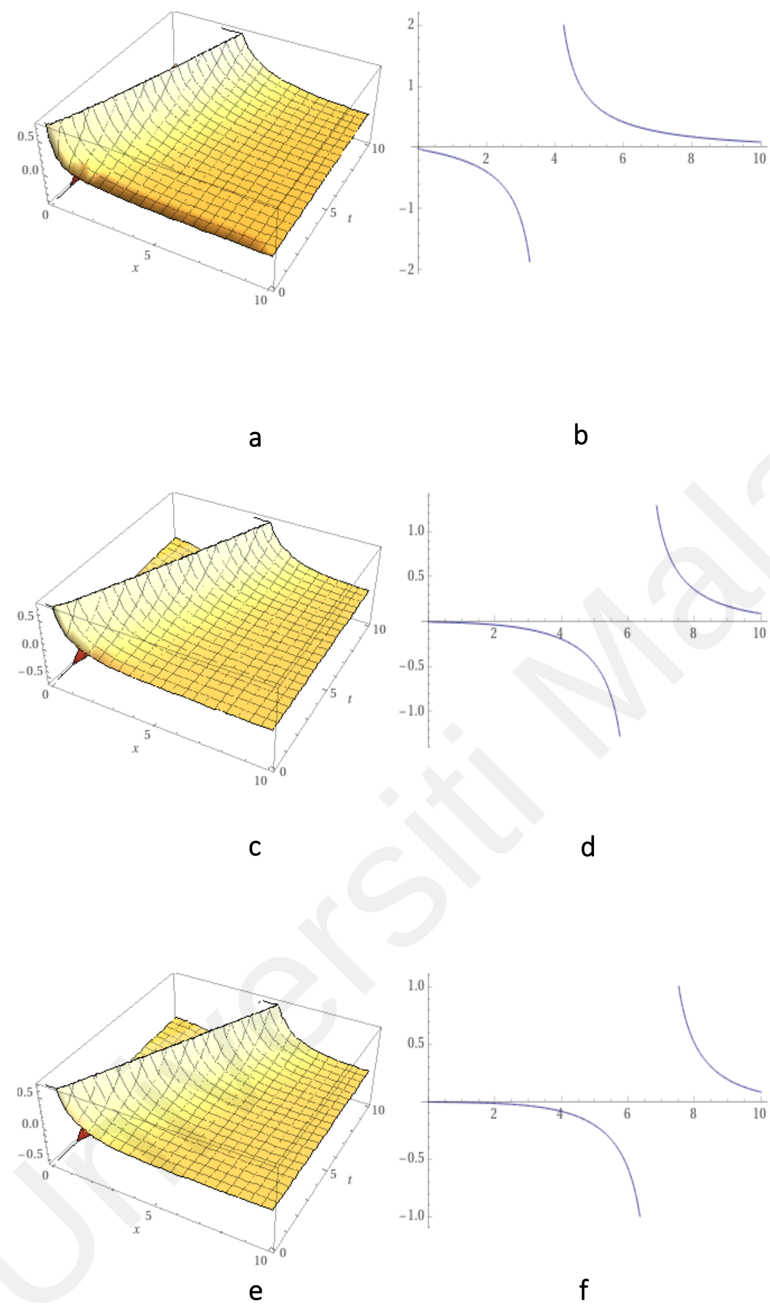

Figure 3.2: The plots of Eq. (3.44) are represented in 3D in (a), (c), (e) and in 2D in (b), (d), (f) for $\gamma = -1; q = p = I_1 = 1; t = 15; y = z = 0; \zeta = 0.50; \zeta = 0.80; \zeta = 0.90$ **for ComD, respectively.**

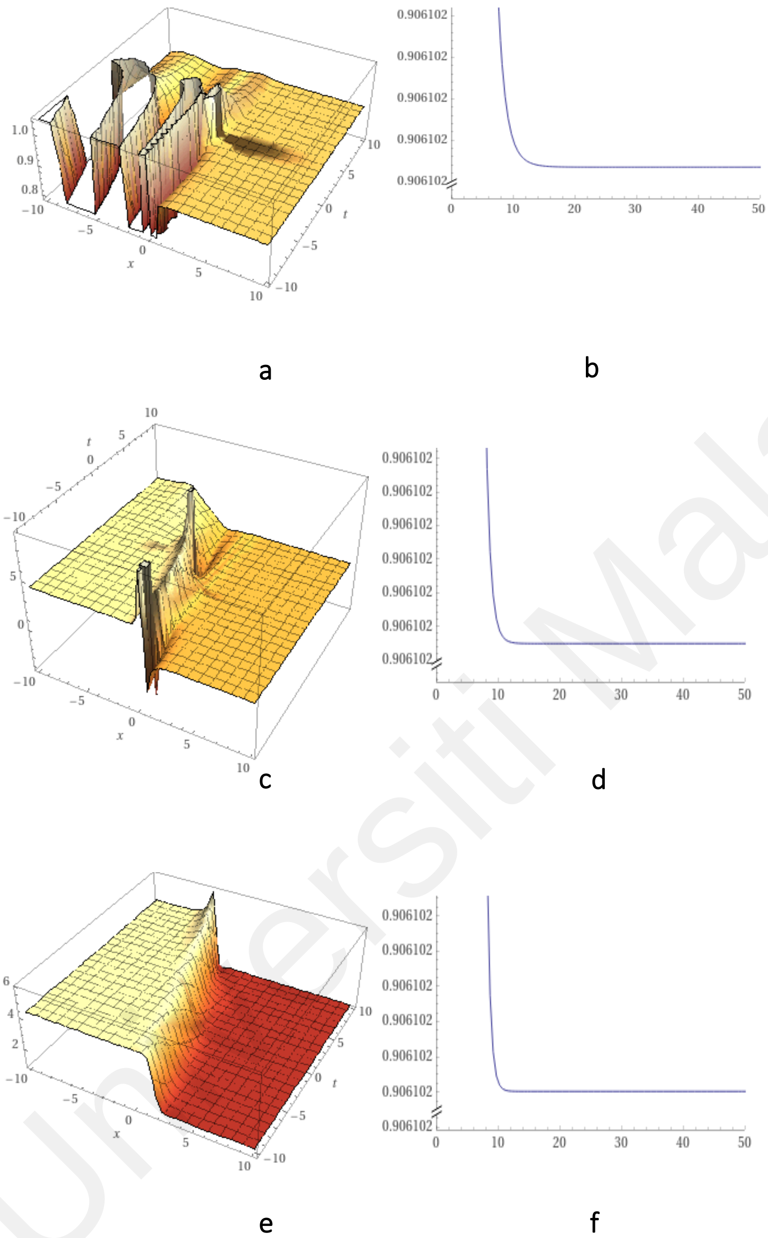

Figure 3.3: The plots of Eq. (3.48) are represented in 3D in (a), (c), (e) and in 2D in (b), (d), (f) for $\gamma = 1; \vartheta = -1; q = p = \chi = I = 1; t = 15; y = z = 0; \zeta = 0.50; \zeta = 0.75; \zeta = 0.90$ **for ComD, respectively.**

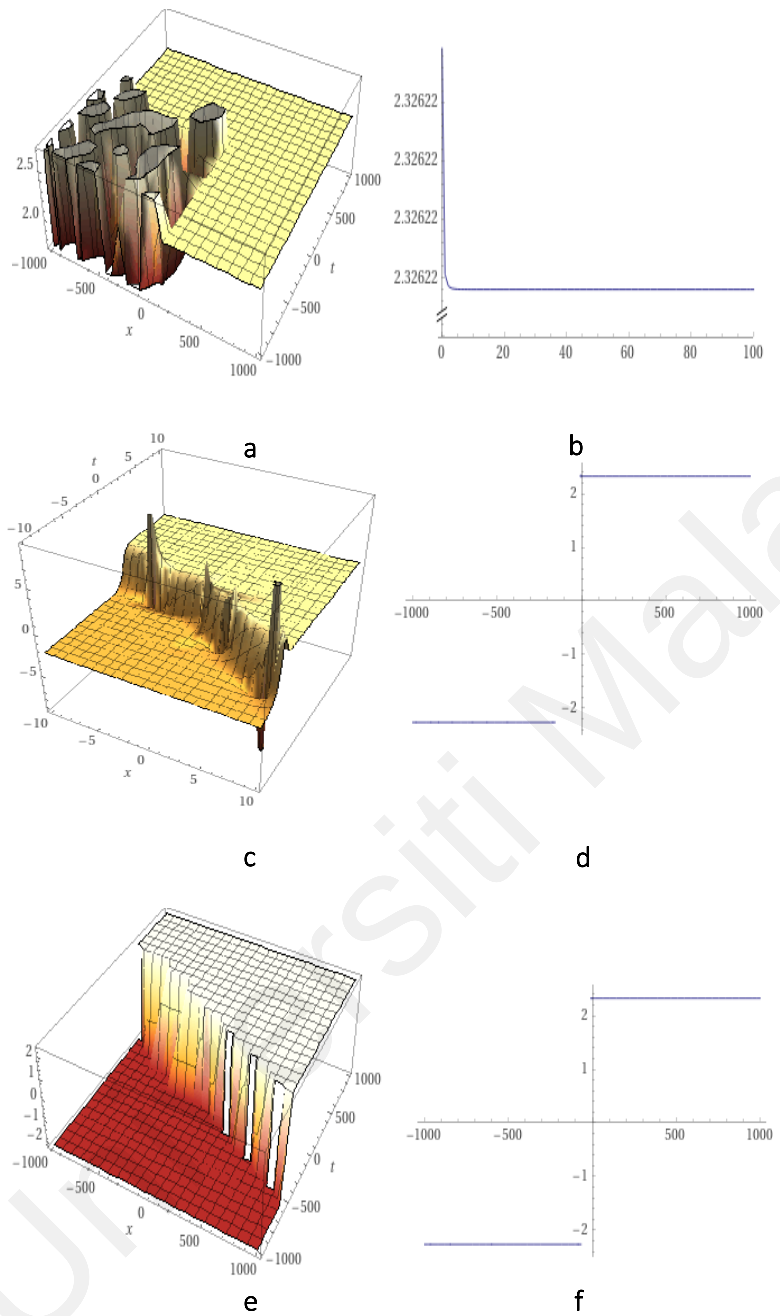

Figure 3.4: The plots of Eq. (3.49) are represented in 3D in (a), (c), (e) and in 2D in (b), (d), (f) for $\gamma = 1; \vartheta = 1; q = p = \chi = I = 1; t = 15; y = z = 0; \zeta = 0.50; \zeta = 0.75; \zeta = 0.90$ **for ComD, respectively.**

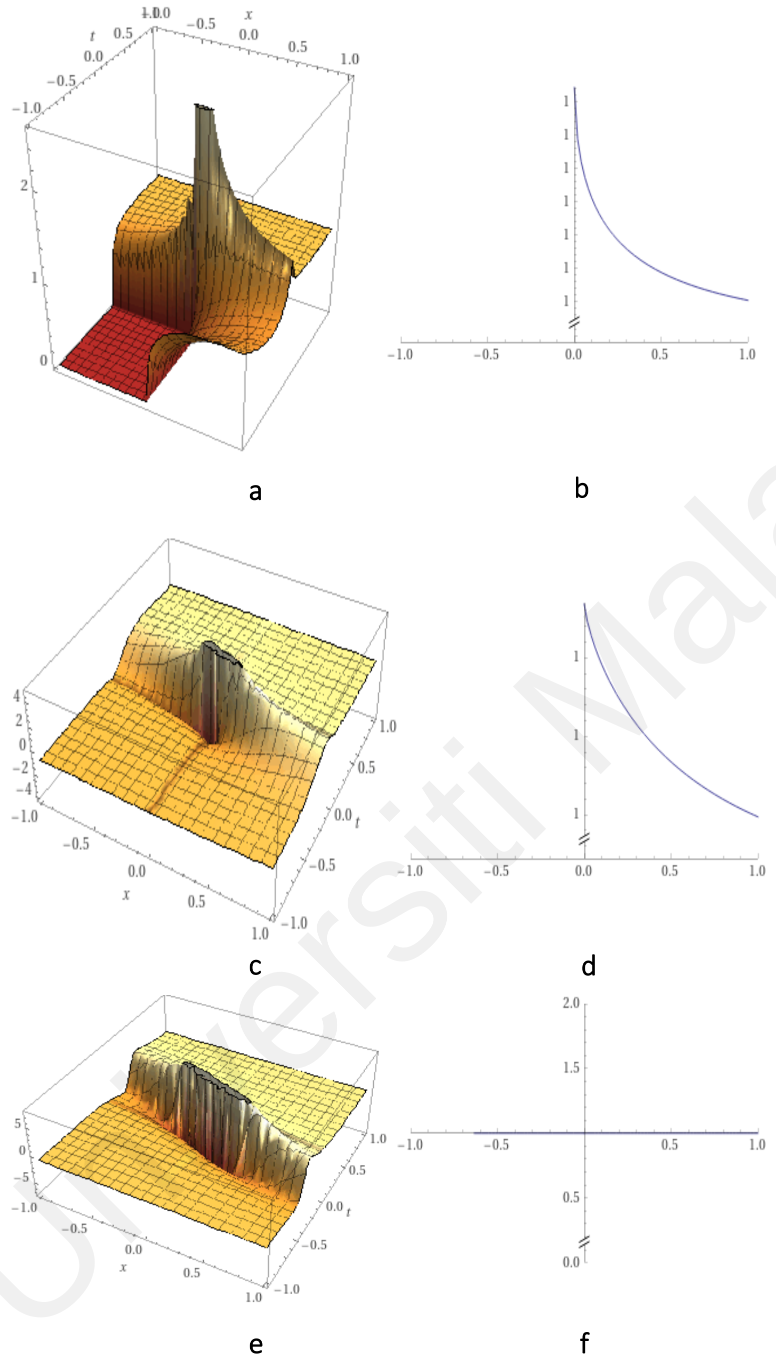

Figure 3.5: The plots of Eq. (3.50) are represented in 3D in (a), (c), (e) and in 2D in (b), (d), (f) for $\gamma = -1; \vartheta = 0; q = p = \chi = I = 1; t = 15; y = z = 0; \zeta = 0.50; \zeta = 0.75; \zeta = 0.90$ **for ComD, respectively.**

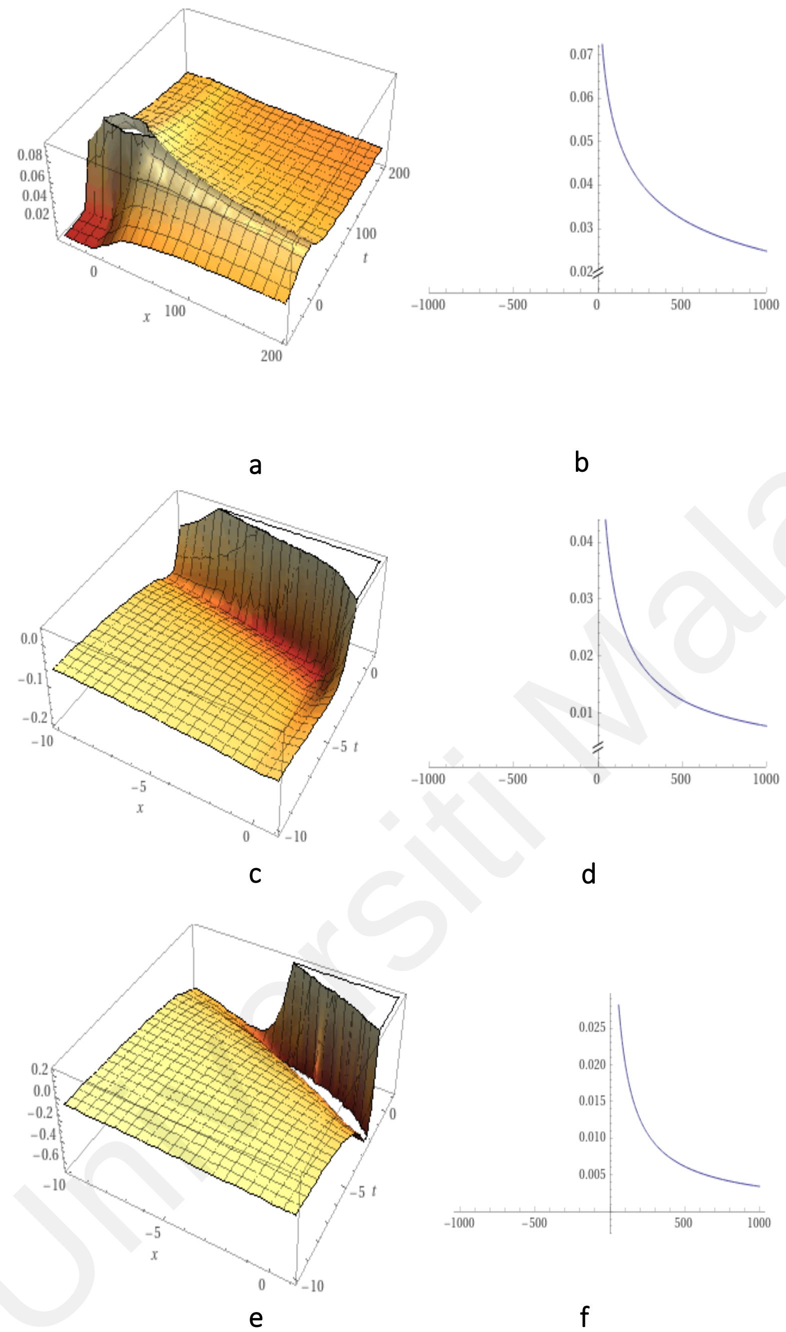

Figure 3.6: The plots of Eq. (3.51) are represented in 3D in (a), (c), (e) and in 2D in (b), (d), (f) for $\gamma = 1; \vartheta = 0; q = p = \chi = I = 1; t = 15; y = z = 0; \zeta = 0.50; \zeta = 0.75; \zeta = 0.90$ **for ComD, respectively.**

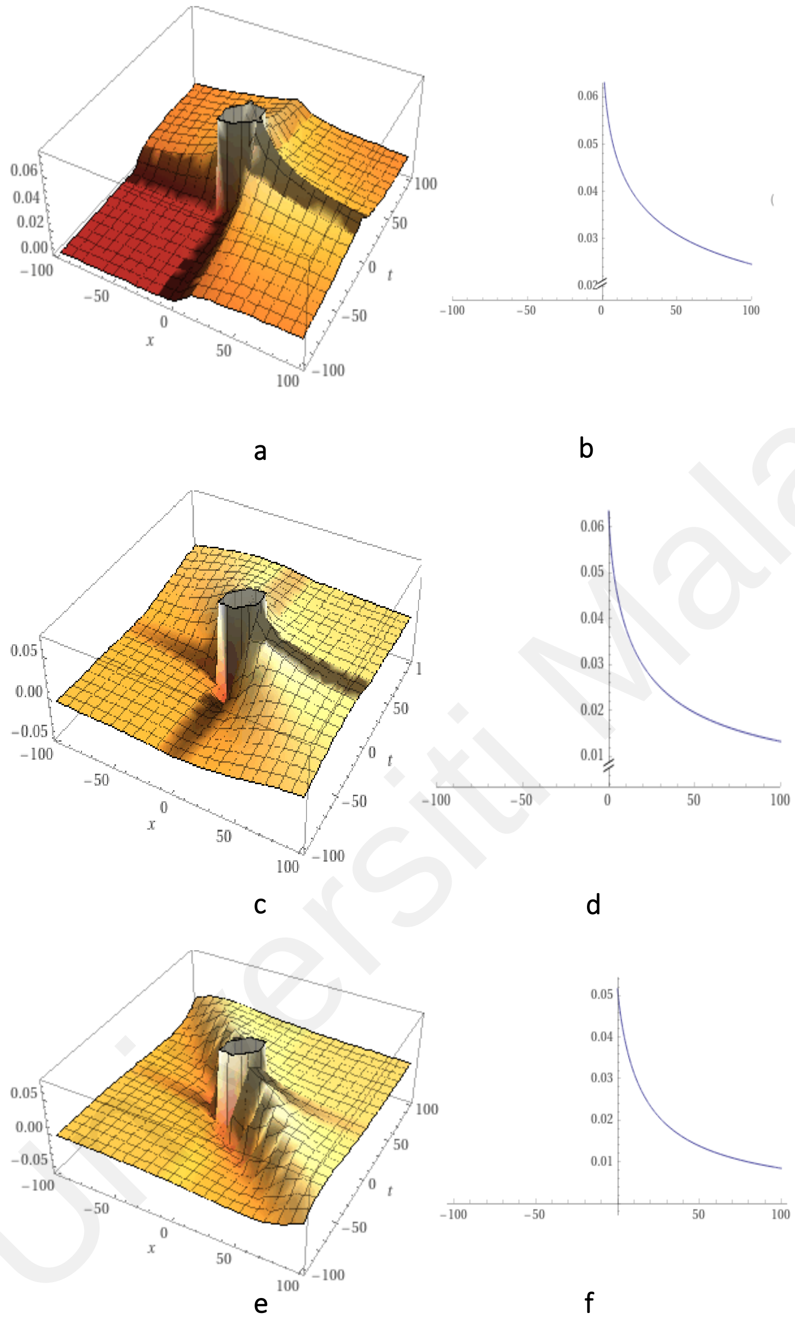

Figure 3.7: The plots of Eq. (3.52) are represented in 3D in (a), (c), (e) and in 2D in (b), (d), (f) for $\gamma = -1; \vartheta = 0; q = p = I = 1; \chi = 0; t = 15; y = z = 0; \zeta = 0.50; \zeta = 0.75; \zeta = 0.90$ **for ComD, respectively.**

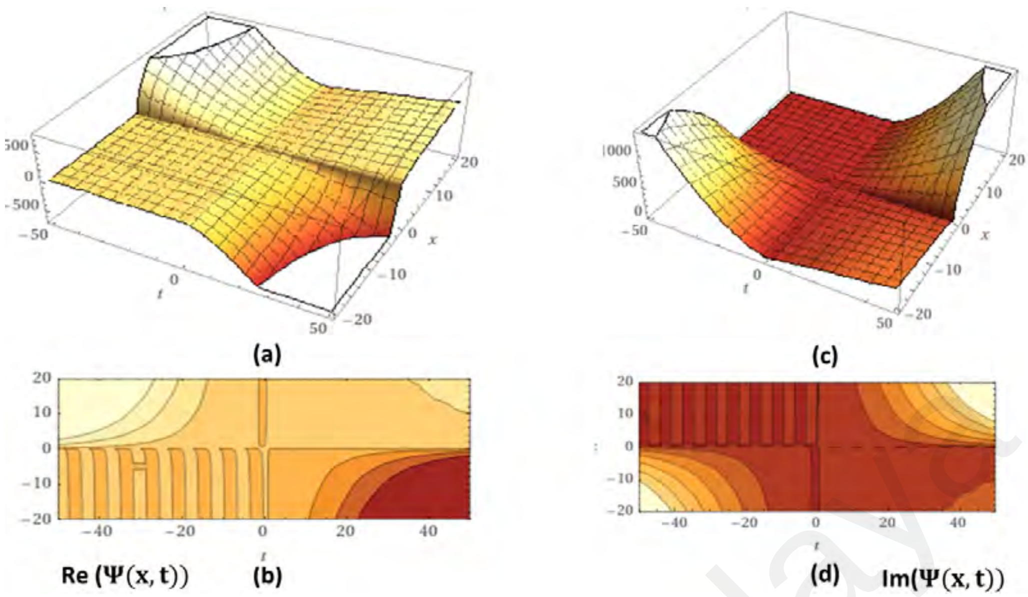

Figure 3.8: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.54) **for** $\gamma = \delta = 0.25$

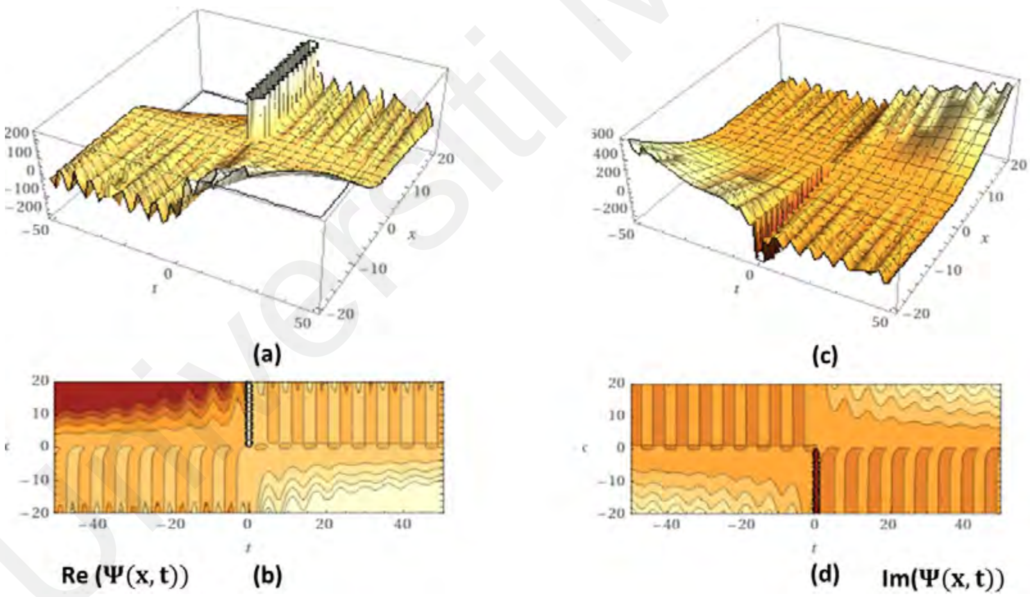

Figure 3.9: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.54) **for** $\gamma = \delta = 0.75$

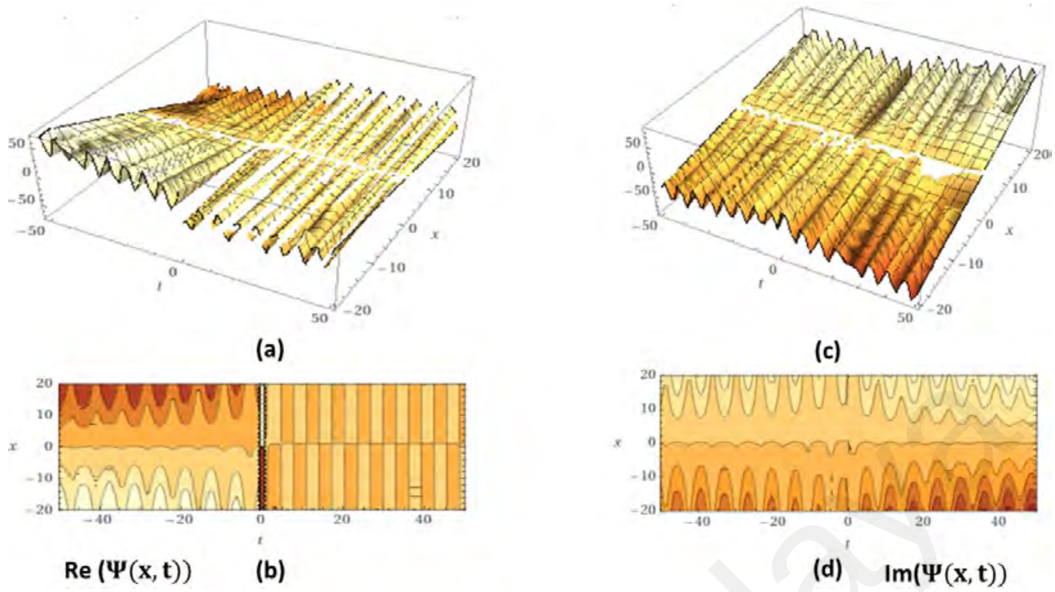

Figure 3.10: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.54) **for** $\gamma = 0.50; \delta = 0.85$

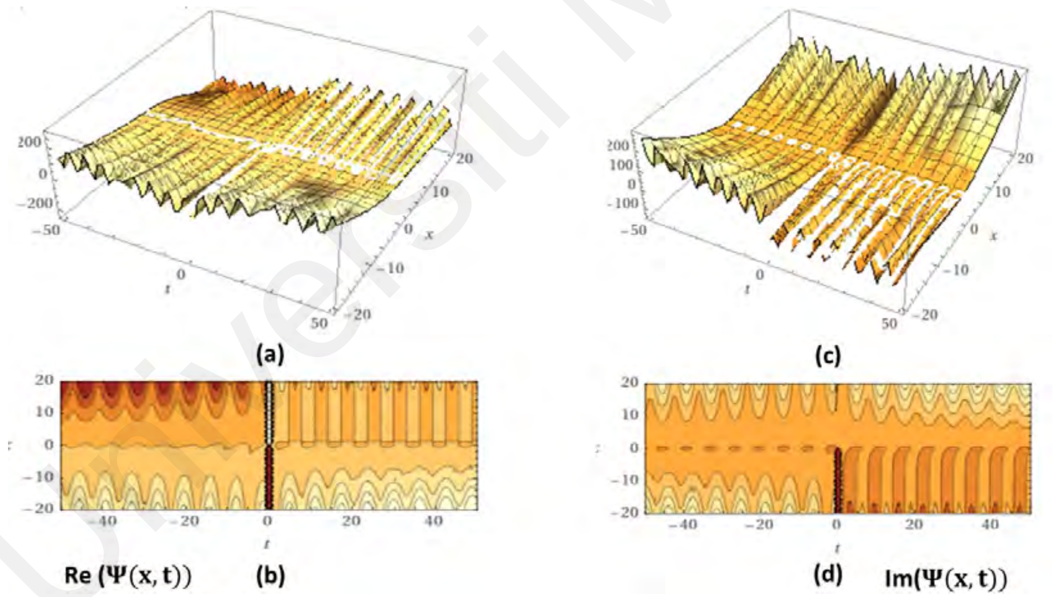

Figure 3.11: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.54) **for** $\gamma = 0.75; \delta = 0.85$

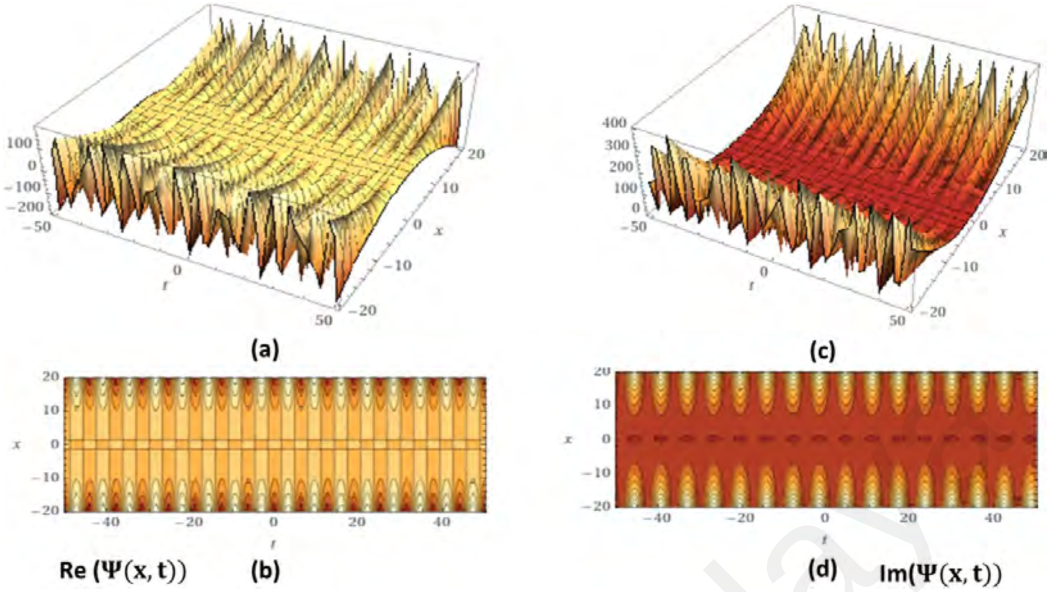

Figure 3.12: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.54) **for** $\gamma = \delta = 1$

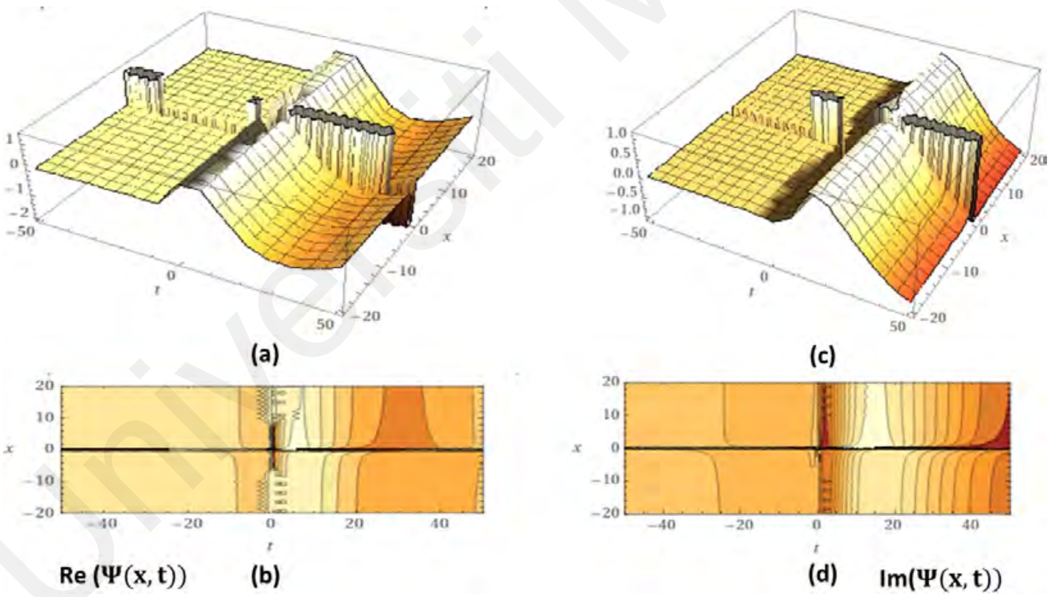

Figure 3.13: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.56) **for** $\gamma = \delta = 0.25$

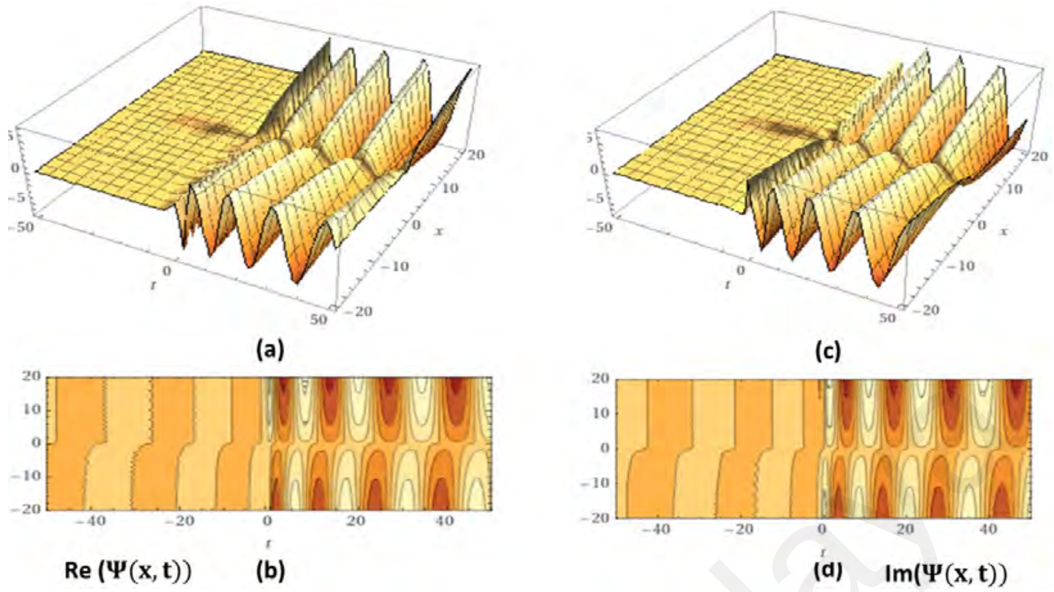

Figure 3.14: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.56) **for** $\gamma = \delta = 0.75$

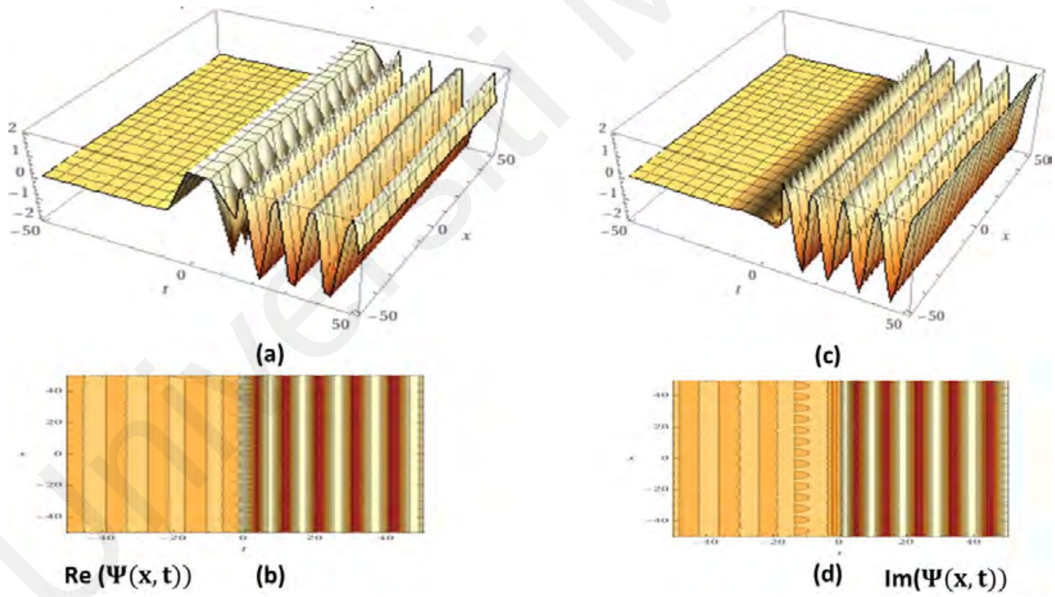

Figure 3.15: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.56) **for** $\gamma = 0.50; \delta = 0.85$

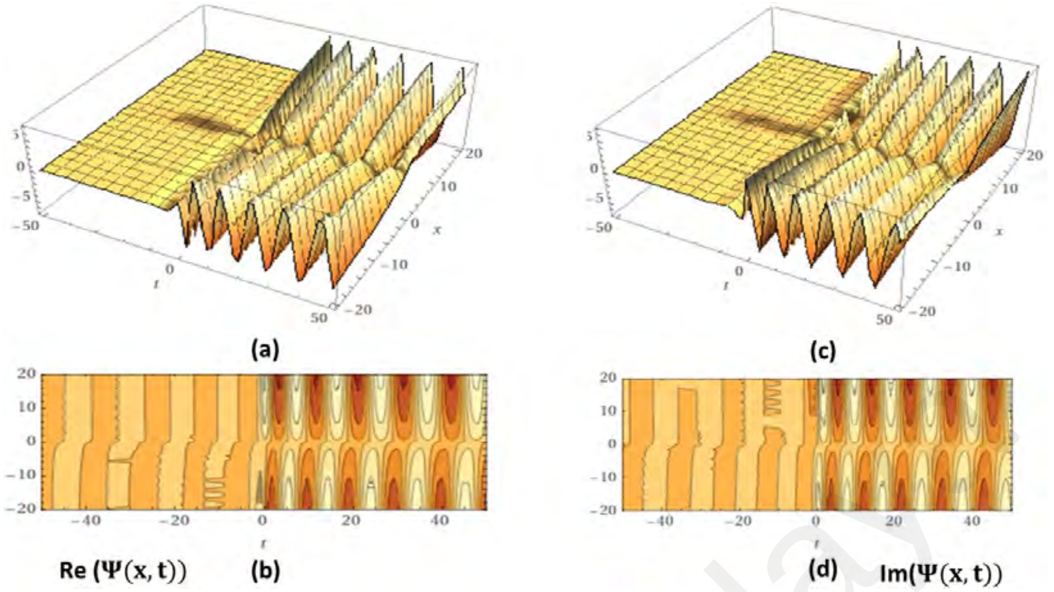

Figure 3.16: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.56) **for** $\gamma = 0.75; \delta = 0.85$

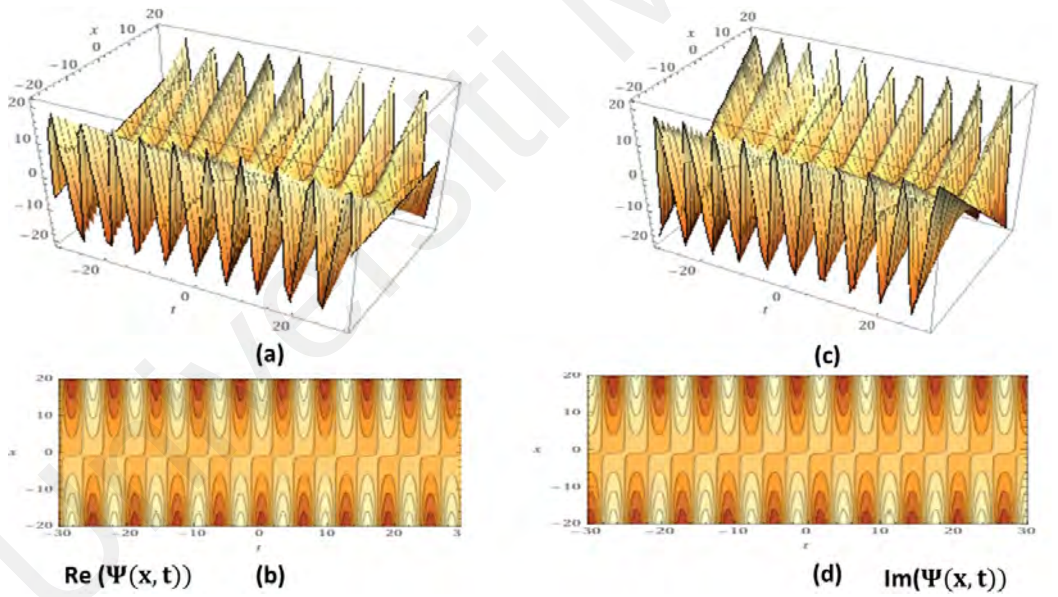

Figure 3.17: 3D Plot of the real part (a) with its contour plot (b) and imaginary part (c) with its contour plot (d) of the Approximate Analytical Solution in Eq. (3.56) **for** $\gamma = \delta = 1$

CHAPTER 4: NOVEL INVESTIGATION OF MULTIVARIABLE CONFORMABLE CALCULUS FOR MODELING SCIENTIFIC PHENOMENA

4.1 Introduction

The conformable version (ComV) of multivariable calculus (MuCL) are proposed and investigated in this chapter. A real-valued function (ReVaF) of several variables (SeVs) is studied in the context of ComD with all associated properties. We extend it to investigate the vector valued functions (VeVaFs) of several real variables (SeReVs). The ComV of chain rule (ChR) for functions of SeVs is also proposed. The ComV of implicit function theorem (ImFThm) for SeVs is proposed. All our results can be employed in several modeling scenarios arising in oceanography as a computational tool to investigate such models. The ComV of analytic functions' theory has been studied in (Khalil et al., 2018). In addition, new results on the contour conformable integral are mentioned in (Martínez et al., 2020; Uçar et al., 2019). Studying ComD and integral is essential in modeling natural sciences and engineering phenomena. While there are some recent research works that have dealt with the conformable calculus in terms of analysis such as the multivariable conformable calculus (N. Y. Gözütok & Gözütok, 2017), the ComD's behavior of functions in arbitrary Banach spaces (Kiskinov et al., 2021), the differential geometry of curves (U. Gözütok et al., 2019), and the behavioral framework for the ComV of linear differential systems' stability (Mayo-Maldonado et al., 2020) to utilize the importance of ComD in modeling scenarios of control theory and power electronics, our results provide a comprehensive investigation of $\zeta-$derivative of a function of SeVs and all related properties, the ComV of ChR for functions of SeVs, and the ComV of ImFThm with many associated numerical examples to validate our outcomes.

4.2 Fundamental Notions

From Eq. 1.3, the ComD of some functions are expressed as:

1. $\mathfrak{D}^{\zeta}\,(1) = 0,$
2. $\mathfrak{D}^{\zeta}\,(sin\,(at)\,) = at^{1-\zeta}cos\,(at),$
3. $\mathfrak{D}^{\zeta}\,(cos\,(at)\,) = -at^{1-\zeta}sin\,(at),$
4. $\mathfrak{D}^{\zeta}\,\left(e^{at}\right) = at^{1-\zeta}e^{at},\ a \in R.$

Definition 10. *(Khalil et al., 2019). The left ComD beginning from* a*, of function* $\Psi(t) : [a, \infty) \to R$ *of order* $\zeta \in (0, 1]$ *is expressed as:*

$$\left({}_{a}\mathfrak{D}^{\zeta}\Psi\right)(t) = \lim_{\Omega\to 0}\frac{\Psi\left(t + \Omega(t-a)^{1-\zeta}\right) - \Psi(t)}{\Omega}\ ; t > a. \tag{4.1}$$

For $a = 0$*, it is written as:* $\left(\mathfrak{D}^{\zeta}\Psi\right)(t)$. *If* Ψ *is* $\zeta - DF$ *in some* (a, b)*, we get:*

$$\left({}_{a}\mathfrak{D}^{\zeta}\Psi\right)(a) = \lim_{t\to a^{+}}\left({}_{a}\mathfrak{D}^{\zeta}\Psi\right)(t)\ . \tag{4.2}$$

Theorem 2. *(Abdeljawad, 2015).* ***(ChR)****. Suppose that* $\Psi, \Phi : (a, \infty) \to R$ *are left* $\zeta - DFs$*, where* $\zeta \in (0, 1]$.

Let us assume that $h\,(t) = \Psi(\Phi\,(t))$*,* $h\,(t)$ *is* $\zeta - DF\ \forall\ t \neq a$ *and* $\Phi(t) \neq 0$*, we get:*

$$\left({}_{a}\mathfrak{D}^{\zeta}h\right)(t) = \left({}_{a}\mathfrak{D}^{\zeta}\Psi\right)(\Phi\,(t))\cdot\left({}_{a}\mathfrak{D}^{\zeta}\Phi\right)(t)\cdot(\Phi(t))^{\zeta-1}. \tag{4.3}$$

If $t = a$*, then we obtain:*

$$\left({}_{a}\mathfrak{D}^{\zeta}h\right)(a) = \lim_{t\to a^{+}}\left({}_{a}\mathfrak{D}^{\zeta}\Psi\right)(\Phi\,(t))\cdot\left({}_{a}\mathfrak{D}^{\zeta}\Phi\right)(t)\cdot(\Phi(t))^{\zeta-1}\ . \tag{4.4}$$

Theorem 3. *(Khalil et al., 2014).* ***(Rolle's Theorem (RoThm))****. Suppose that* $a > 0$*,* $\zeta \in (0, 1]$*, and function* $\Psi : [a, b] \to R$ *satisfies:*

1. Ψ *is continuous function (CF) on* $[a, b]$.
2. Ψ *is* $\zeta - DF$ *on* (a, b).
3. $\Psi(a) = \Psi(b)$.

Then, $\exists\, c \in (a, b)\,,\ \ni \left(\mathfrak{D}^{\zeta}\Psi\right)(c) = 0.$

Theorem 4. *(Khalil et al., 2014).* ***(Mean Value Theorem (MeVaThm))****. Suppose that* $a > 0$*,* $\zeta \in (0, 1]$*, and function* $\Psi : [a, b] \to R$ *satisfies:*

1. Ψ *is CF on* $[a, b]$.
2. Ψ *is* $\zeta - DF$ *on* (a, b).

Then, $\exists\, c \in (a, b)\,,\ \ni$ *we have:*

$$\left(\mathfrak{D}^{\zeta}\Psi\right)(c) = \frac{\Psi(b) - \Psi(a)}{\frac{b^{\zeta}}{\zeta} - \frac{a^{\zeta}}{\zeta}}. \tag{4.5}$$

Theorem 5. *(Al Horani & Khalil, 2018).*

(Modified Mean Value Theorem (MoMeVaThm))*.*

Assume that $a > 0$*,* $\zeta \in (0, 1]$*, and function* $\Psi : [a, b] \to R$ *satisfies:*

1. Ψ *is CF on* $[a, b]$.
2. Ψ *is* $\zeta - DF$ *on* (a, b).

Then $\exists\, c \in (a, b)\,,\ \ni$ *we have:*

$$\frac{\left(\mathfrak{D}^{\zeta}\Psi\right)(c)}{\frac{c^{1-\zeta}}{\zeta}} = \frac{\Psi(b) - \Psi(a)}{\frac{b}{\zeta} - \frac{a}{\zeta}}. \tag{4.6}$$

Theorem 6. *(Iyiola & Nwaeze, 2016).Assume that* $a > 0$*,* $\zeta \in (0, 1]$*,*
and function $\Psi : [a, b] \to R$ *satisfies:*

1. Ψ *is CF on* $[a, b]$.

2. Ψ *is* $\zeta - DF$ *on* (a, b).

Then, we get:

1. *If* $\left(\mathfrak{D}^{\zeta}\Psi\right)(t) > 0\ \forall\ t \in (a, b)$, *then* Ψ *is increasing on*$[a, b]$.

2. *If* $\left(\mathfrak{D}^{\zeta}\Psi\right)(t) < 0\ \forall t \in (a, b)$, *then* Ψ *is decreasing on*$[a, b]$.

The ComV of partial derivative (PaDr) of a ReVaF with SeVs can be expressed as follows:

Definition 11. *(Atangana et al., 2015; N. Y. Gözütok & Gözütok, 2017). Suppose that* Ψ *is a ReVaF with n variables, and there is a point:* $\boldsymbol{a} = (a_1, \ldots, a_n) \in R^n$ *where its* i^{th} *component is positive. Then, the limit is written as:*

$$\lim_{\Omega \to 0} \frac{\Psi\left(a_1, .., a_i + \Omega {a_i}^{1-\zeta}, \ldots, a_n\right) - \Psi(a_1, \ldots, a_n)}{\Omega}\ . \tag{4.7}$$

If the above limit exists, the i^{th} *ComV of PaDr of* Ψ *of the order* $\zeta \in (0, 1]$ *at* $\boldsymbol{a}$, *, represented by* $\frac{\partial^{\zeta}}{\partial x_i^{\zeta}}\Psi(\boldsymbol{a})$.

4.3 The ζ-Derivative of a ReVaF of SeVs

Definition 12. *Let* Ψ *be a ReVaF with n variables* $x_1, \ldots, x_n$, *and* $\zeta \in (0, 1]$. *Then, we say that* Ψ *is* $\zeta - DF$ *at* $\boldsymbol{a} = (a_1, \ldots, a_n) \in R^n$, *each* $a_i > 0$, *if any of the three conditions which are equivalent to each other is verified:*

1. *There is a linear transformation* $L : R^n \to R$ *such that*

$$\lim_{\boldsymbol{h} \to \boldsymbol{0}} \frac{\Psi\left(a_1 + h_1 {a_1}^{1-\zeta}, .., a_n + h_n {a_n}^{1-\zeta}\right) - \Psi(a_1, \ldots, a_n) - L(h)}{\|\boldsymbol{h}\|} = 0. \tag{4.8}$$

where $\boldsymbol{h} = (h_1, \ldots, h_n)$, $\|\boldsymbol{h}\| = \sqrt{h_1^2 + \cdots + h_n^2}$ *and* $\zeta \in (0, 1]$.

2. *There is a linear transformation* $L : R^n \to R$ *and a function* $\boldsymbol{\Omega} : \boldsymbol{h} \to \boldsymbol{\Omega}(\boldsymbol{h})$ *such that*

$$\Psi\left(a_1 + h_1 {a_1}^{1-\zeta}, .., a_n + h_n {a_n}^{1-\zeta}\right) - \Psi(a_1, \ldots, a_n) = L(\boldsymbol{h}) + \boldsymbol{\Omega}(\boldsymbol{h}) \|\boldsymbol{h}\|, \quad (4.9)$$

and $lim_{h\to 0}\ \boldsymbol{\Omega}(\boldsymbol{h}) = \boldsymbol{0}$.

3. *There is a linear transformation* $L : R^n \to R$ *and n functions* $\Omega_i : \boldsymbol{h} \to \Omega_i(\boldsymbol{h})$ $\forall i = 1,\ 2,\ \ldots n, \ni$

$$\Psi\left(a_1 + h_1 {a_1}^{1-\zeta}, .., a_n + h_n {a_n}^{1-\zeta}\right) - \Psi(a_1, \ldots, a_n) = L(\boldsymbol{h}) + \sum_{i=1}^{n} \Omega_i(\boldsymbol{h}) h_i, \quad (4.10)$$

and $\lim_{h\to 0}\Omega_i\ (h) = 0$ *for* $i = 1,\ 2,\ \ldots n$.

The linear transformation $L : R^n \to R$ *is defined by* $L(\boldsymbol{h}) = \sum_{i=1}^{n} \zeta_i h_i$, *with* $\boldsymbol{h} = (h_1, \ldots, h_n)$ *and* $\zeta_1, \ldots, \zeta_n \in R$. *This linear transformation is denoted by* $\mathfrak{D}^{\zeta} f(\boldsymbol{a})$ *which is called the ComD of* Ψ *of the order* $\zeta \in (0, 1]$ *at* $\boldsymbol{a}$.

Remark 1. *The equivalence of conditions (1) and (2) is immediate, since*

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \boldsymbol{\Omega}(\boldsymbol{h}) = \boldsymbol{0} \leftrightarrow \Omega(\boldsymbol{h}) \|\boldsymbol{h}\| = o(\|\boldsymbol{h}\|).$$

To see the equivalence between conditions (2) and (3), we take: $\Omega_i = \Omega(\boldsymbol{h}) \frac{h_i}{\|\boldsymbol{h}\|}$ *and* $\boldsymbol{\Omega}(\boldsymbol{h}) = \frac{1}{\|\boldsymbol{h}\|} \sum_{i=1}^{n} \Omega_i(\boldsymbol{h}) h_i$*, As* $\left|\frac{h_i}{\|\boldsymbol{h}\|}\right| \leq 1$, *then we have the following:*

1. *If* $lim_{\boldsymbol{h}\to\boldsymbol{0}}\ \Omega(\boldsymbol{h}) = \boldsymbol{0}$, *then* $lim_{\boldsymbol{h}\to\boldsymbol{0}}\, \Omega_i(\boldsymbol{h}) = 0$.
2. *If* $lim_{\boldsymbol{h}\to\boldsymbol{0}}\, \Omega_i(h) = 0$ *for* $i = 1, \ldots,\ n$, *then we obtain:*

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \|\boldsymbol{\Omega}(\boldsymbol{h})\| \leq \lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{1}{\|\boldsymbol{h}\|} \sum_{i=1}^{n} \|\Omega_i(\boldsymbol{h})\| \leq \lim_{\boldsymbol{h}\to\boldsymbol{0}} \sum_{i=1}^{n} \|\Omega_i(\boldsymbol{h})\| = 0,$$

i.e., $lim_{\boldsymbol{h}\to\mathbf{0}}\ \boldsymbol{\Omega}(\boldsymbol{h}) = \mathbf{0}$. *Hence, the conditions (2) and (3) are equivalent.*

Example 4.3.1. *Consider a function* Ψ *defined by* $\Psi(x, y) = e^x - 2\text{cosy}$ *and a point* $(a, b) \in R^2$, *with* $a > 0$ *and* $b > 0$, *then*
$\mathfrak{D}^{\zeta}\Psi(a, b)(h_1, h_2) = h_1 a^{1-\zeta} e^a + 2h_2 b^{1-\zeta}\sin b.$

Solution: *To show this, it is noticeable that*

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\Psi\left(a + a^{1-\zeta}h_1, b + b^{1-\zeta}h_2\right) - \Psi(a, b) - L(h_1, h_2)}{\|(h_1, h_2)\|}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{e^{a+a^{1-\zeta}h_1} - 2\cos\left(b + b^{1-\zeta}h_2\right) - \left(e^a - 2cosb\right) - \left(h_1 a^{1-\zeta} e^a + 2h_2 b^{1-\zeta}\sin b\right)}{\sqrt{h_1^2 + h_2^2}}$$

$$\leq \lim_{h_1\to 0} \frac{e^{a+a^{1-\zeta}h_1} - e^a - h_1 a^{1-\zeta} e^a}{h_1} - 2\lim_{h_2\to 0} \frac{\cos\left(b + b^{1-\zeta}h_2\right) - cosb + b^{1-\zeta}\sin b}{h_2}$$

$$= \lim_{h_1\to 0}\left(\frac{e^{a+a^{1-\zeta}h_1} - e^a}{h_1} - a^{1-\zeta}e^a\right) - 2\lim_{h_2\to 0}\left(\frac{\cos\left(b + b^{1-\zeta}h_2\right) - cosb}{h_2} + b^{1-\zeta}\sin b\right)$$

$$= \left(a^{1-\zeta}e^a - a^{1-\zeta}e^a\right) - 2\left(-b^{1-\zeta}\sin b + b^{1-\zeta}\sin b\right) = 0.$$

Theorem 7. *If a ReVaF* Ψ *with n variables* $\zeta - DF$ *at* $\boldsymbol{a} = (a_1, \ldots, a_n) \in R^n$, *each* $a_i > 0$, *then* Ψ *is CF at* $\boldsymbol{a} \in R^n$.

Proof. Since Ψ is $\zeta-$DF at a, we can write the following:

$$\Psi\left(a_1 + h_1 {a_1}^{1-\zeta}, .., a_n + h_n {a_n}^{1-\zeta}\right) - \Psi(a_1, \ldots, a_n) = \sum_{i=1}^{n} \zeta_i h_i + o(\|\boldsymbol{h}\|).$$

By taking the limits of both sides of the equality as $\boldsymbol{h} \to \mathbf{0}$, we get:

$$\lim_{\boldsymbol{h}\to\mathbf{0}} \Psi\left(a_1 + h_1 {a_1}^{1-\zeta}, .., a_n + h_n {a_n}^{1-\zeta}\right) = \Psi(a_1, \ldots, a_n).$$

Hence, Ψ is CF at $\boldsymbol{a} \in R^n$.

Theorem 8. *If a ReVaF Ψ with n variables is $\zeta-DF$ at $\boldsymbol{a} = (a_1, \ldots, a_n) \in R^n$, each $a_i > 0$, then $\frac{\partial^\zeta}{\partial x_i^\zeta}\Psi(\boldsymbol{a})$ exists for $1 \le i \le n$ and the ComD of Ψ of the order $\zeta \in (0,1]$ is written as:*

$$\mathfrak{D}^\zeta \Psi(\boldsymbol{a})(\boldsymbol{h}) = \sum_{i=1}^{n} \frac{\partial^\zeta}{\partial x_i^\zeta}\Psi(\boldsymbol{a})\, h_i. \tag{4.11}$$

where $\boldsymbol{h} = (h_1, \ldots, h_n)$.

Proof. By setting $h_j = 0, \forall j \neq i$ in the formula (4.10), we get:

$$\Psi\left(a_1, .., a_i + h_i {a_i}^{1-\zeta} + \cdots + a_n\right) - \Psi(a_1, \ldots, a_n) = \zeta_i h_i + \Omega_i(\boldsymbol{h})\, h_i.$$

By multiplying by $\frac{1}{h_i}$, we can write:

$$\frac{\Psi\left(a_1, .., a_i + h_i {a_i}^{1-\zeta} + \cdots + a_n\right) - \Psi(a_1, \ldots, a_n)}{h_i} = \zeta_i + \Omega_i(\boldsymbol{h}).$$

By taking the limits of both sides of the equality as $h_i \to 0$, we have:

$$\zeta_i = \frac{\partial^\zeta}{\partial x_i^\zeta}\Psi(\boldsymbol{a}),\ \forall i = 1, 2, \ldots, n.$$

Finally, by substituting the values above h_i in the formula $\mathfrak{D}^\zeta \Psi(\boldsymbol{a})(\boldsymbol{h}) = \sum_{i=1}^{n} \zeta_i h_i$, the result is followed.

Corollary 1. *If a ReVaF Ψ with n variables is ζ−DF at $\boldsymbol{a} = (a_1, \ldots, a_n) \in R^n$, each $a_i > 0$, then $D^\zeta \Psi(\boldsymbol{a})$ is unique.*

Remark 2. *If a ReVaF Ψ with n variables is ζ−DF at $\boldsymbol{a} = (a_1, \ldots, a_n) \in R^n$, each*

$a_i > 0$*, then the ComV of gradient of* Ψ *of the order* $\zeta \in (0, 1]$ *at a is:*

$$\nabla^{\zeta}\Psi(\boldsymbol{a}) = \left(\frac{\partial^{\zeta}}{\partial x_1^{\zeta}}\Psi(\boldsymbol{a}), \ldots, \frac{\partial^{\zeta}}{\partial x_n^{\zeta}}\Psi(\boldsymbol{a})\right). \tag{4.12}$$

In addition, the matrix form (MF) of Eq. (4.11) is expressed as:

$$\mathfrak{D}^{\zeta}\Psi(\boldsymbol{a})(\boldsymbol{h}) = \nabla^{\zeta}\Psi(\boldsymbol{a}) \cdot \boldsymbol{h} = \left(\frac{\partial^{\zeta}}{\partial x_1^{\zeta}}\Psi(a), \ldots, \frac{\partial^{\zeta}}{\partial x_n^{\zeta}}\Psi(a)\right) \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}. \tag{4.13}$$

Theorem 9. *Let* $\zeta \in (0, 1]$, $\Psi, \Phi : X \to R$ *be a ReVaF defined in an open set (OS):* $X \subset R^n$, $\ni\ \forall\ \boldsymbol{x} = (x_1, \ldots, x_n) \in X$*, each* $x_i > 0$*, and a point* $\boldsymbol{a} = (a_1, \ldots, a_n) \in X$*. If* Ψ, Φ *are* $\zeta - DF$ *at* $\boldsymbol{a}$*, then we get:*

1. $\mathfrak{D}^{\zeta}(\lambda\Psi + \mu\Phi)(\boldsymbol{a}) = \lambda\mathfrak{D}^{\zeta}(\Psi)(\boldsymbol{a}) + \mu\mathfrak{D}^{\zeta}(\Phi)(\boldsymbol{a})$, $\forall\ \lambda,\ \mu \in R$.
2. $\mathfrak{D}^{\zeta}(\Psi\Phi)(\boldsymbol{a}) = \mathfrak{D}^{\zeta}(\Psi)(\boldsymbol{a}) \cdot \Phi(\boldsymbol{a}) + \Psi(\boldsymbol{a})\,\mathfrak{D}^{\zeta}(\Phi)(\boldsymbol{a})$.

Proof. (1) follows from definition (12), thus it follows the proof of (1). For (2), let $\boldsymbol{A} = \left(a_1 + h_1 a_1{}^{1-\zeta}, .., a_n + h_n a_n{}^{1-\zeta}\right)$, then we have:

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{((\Psi\Phi)(\boldsymbol{A}) - (\Psi\Phi)(\boldsymbol{a})) - \left(\mathfrak{D}^{\zeta}\Psi(\boldsymbol{a}) \cdot \Phi(\boldsymbol{a}) + \Psi(\boldsymbol{a}) \cdot \mathfrak{D}^{\zeta}\Phi(\boldsymbol{a})\right)(\boldsymbol{h})}{\|\boldsymbol{h}\|}$$

$$= \lim_{\boldsymbol{h}\to\boldsymbol{0}}\left(\frac{(\Psi(\boldsymbol{A})-\Psi(\boldsymbol{a}))-\mathfrak{D}^{\zeta}\Psi(\boldsymbol{a})(\boldsymbol{h})}{\|\boldsymbol{h}\|} \cdot \Phi(\boldsymbol{a}) + \Psi(\boldsymbol{a}) \cdot \frac{(\Phi(\boldsymbol{A})-\Phi(\boldsymbol{a}))-\mathfrak{D}^{\zeta}\Phi(\boldsymbol{a})(\boldsymbol{h})}{\|\boldsymbol{h}\|}\right)$$

$$+ \lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{(\Psi(\boldsymbol{A}) - \Psi(\boldsymbol{a})) \cdot (\Phi(\boldsymbol{A}) - \Phi(\boldsymbol{a}))}{\|\boldsymbol{h}\|}$$

$$= 0 + 0 + \lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{\left(\mathfrak{D}^{\zeta}\Psi\left(\boldsymbol{a}\right)\left(\boldsymbol{h}\right)\right)\cdot\left(\mathfrak{D}^{\zeta}\Phi\left(\boldsymbol{a}\right)\left(\boldsymbol{h}\right)\right)}{\|\boldsymbol{h}\|}$$

$$= \lim_{\boldsymbol{h}\to\boldsymbol{0}} \left(\mathfrak{D}^{\zeta}\Psi\left(\boldsymbol{a}\right)\left(\frac{\boldsymbol{h}}{\|\boldsymbol{h}\|}\right)\right)\cdot\left(\mathfrak{D}^{\zeta}\Phi\left(\boldsymbol{a}\right)\left(\frac{\boldsymbol{h}}{\|\boldsymbol{h}\|}\right)\right)\cdot\|\boldsymbol{h}\| = 0.$$

Theorem 10. *Let* $\zeta \in (0,1]$, $\Psi : X \to R$ *be a ReVaF defined in an OS:* $X \subset R^n$, $\ni \forall\, \boldsymbol{x} = (x_1, \ldots, x_n) \in X$, *each* $x_i > 0$, *and a point* $\boldsymbol{a} = (a_1, \ldots, a_n) \in X$. *If the function* Ψ *has all ComV of PaDrs of the order* ζ *at each point of a neighborhood of the point* $\boldsymbol{a}$, $U(\boldsymbol{a})$, *with* $U(\boldsymbol{a}) \subset X$, *and they are continuous at* $\boldsymbol{a}$, *then* Ψ *is* $\zeta - DF$ *at* a.

Proof. The proof of Theorem (10) is mentioned in (Al Horani & Khalil, 2018).

Remark 3. *Theorem 10 allows defining the space of ReVaFs with n variables by having continuous ComV of PaDrs of order* $\zeta \in (0,1]$ *in a domain* $X \subset R^n$, *which can be denoted by* $C_\zeta(X, R)$.

Thus, we can easily extend all of the above results to the VeVaFs of SeReVs.

Theorem 11. *Suppose that* $\zeta \in (0,1]$, $\boldsymbol{\Psi} : X \to R^m$ *be a VeVaF defined in an OS:* $X \subset R^n$, $\ni \forall \boldsymbol{x} = (x_1, \ldots, x_n) \in X$, *each* $x_i > 0$, *and the point* $\boldsymbol{a} = (a_1, \ldots, a_n) \in X$. *The function* $\boldsymbol{\Psi}$ *is* $\zeta - DF$ *at* $\boldsymbol{a}$ *iff its components are* $\zeta - DF$ *at* $\boldsymbol{a}$ *and if these components are* $\Psi_1,\ \Psi_2,\ \ldots,\ \Psi_m$, *then the components of* $\mathfrak{D}^{\zeta}\boldsymbol{\Psi}(\boldsymbol{a})$ *are the* $\zeta -$ *derivatives denoted by* $\mathfrak{D}^{\zeta}\Psi_j(\boldsymbol{a})$, *for* $j = 1, 2, \ldots, m$, *i.e.,*

$$\Psi = (\Psi_1,\ \Psi_2,\ \ldots,\ \Psi_m) \quad\Rightarrow\quad \mathfrak{D}^{\zeta}\boldsymbol{\Psi}(\boldsymbol{a}) = \left(\mathfrak{D}^{\zeta}\Psi_1(\boldsymbol{a}),\ \mathfrak{D}^{\zeta}\Psi_2(\boldsymbol{a}),\ \ldots,\ \mathfrak{D}^{\zeta}\Psi_m(\boldsymbol{a})\right). \tag{4.14}$$

Proof. This can be proven similarly by applying $\mathfrak{D}^{\zeta}$ instead of derivative.

Remark 4. *Suppose that* $\zeta \in (0,1]$, $\boldsymbol{\Psi} : X \to R^m$ *is a VeVaF defined in an OS:* $X \subset R^n$, $\ni \forall\, \boldsymbol{x} = (x_1, \ldots, x_n) \in X$, *each* $x_i > 0$, *and a point* $\boldsymbol{a} = (a_1, \ldots, a_n) \in X$.

If function $\boldsymbol{\Psi}$ *is* ζ – differentiable *at* $\boldsymbol{a}$*, then* $\frac{\partial^\zeta}{\partial x_i^\zeta}\Psi_j(\boldsymbol{a})$ *exists for* $i = 1, 2, \ldots, n$*,* $j = 1, 2, \ldots, m$*, and the ComV of* $\zeta-$ *Jacobian of* Ψ *of order* $\zeta \in (0, 1]$ *at* $\boldsymbol{a}$ *is written as:*

$$J^\zeta \boldsymbol{\Psi}(\boldsymbol{a}) = \begin{pmatrix} \frac{\partial^\zeta}{\partial x_1^\zeta}\Psi_1(\boldsymbol{a}) & \ldots & \frac{\partial^\zeta}{\partial x_n^\zeta}\Psi_1(\boldsymbol{a}) \\ \vdots & \ldots & \vdots \\ \frac{\partial^\zeta}{\partial x_1^\zeta}\Psi_m(\boldsymbol{a}) & \ldots & \frac{\partial^\zeta}{\partial x_n^\zeta}\Psi_m(\boldsymbol{a}) \end{pmatrix}. \tag{4.15}$$

4.4 The Chain Rule

The ChR will be shown for the functions of SeVs in two cases. For the proof's aim, the continuity's hypothesis of PaDrs is mentioned in (Marsden & Hoffman, 1996).

Theorem 12. *(ChR). Assume that* $t \in R$ *and* $\boldsymbol{x} = (x_1, \ldots, x_n) \in R^n$*. If* $\boldsymbol{\Psi}(t) = (\Psi_1(t), \ldots, \Psi_n(t))$ *is* $\zeta - DF$ *at* $a > 0 \ni \zeta \in (0, 1]$
and a RVF Φ *with n variables* $x_1, \ldots, x_n$*, is* $\zeta - DF$ *at* $\Psi(a) \in R^n$*, all* $\Psi_i(a) > 0 \ni \zeta \in (0, 1]$*. Then, the composition* $\Phi \circ \Psi$ *is* $\zeta - DF$ *at a and*

$$\mathfrak{D}^\zeta(\Phi \circ \boldsymbol{\Psi})(a) = \sum_{i=1}^{n} \frac{\partial^\zeta}{\partial x_i^\zeta} \Phi(\Psi(a)) \cdot (\Psi_i(a))^{\zeta-1} \cdot (\mathfrak{D}^\zeta \Psi_i)(a). \tag{4.16}$$

Proof. Assume $\Phi \in C^1(U(\boldsymbol{\Psi}(a)), R)$, $\ni U(\boldsymbol{\Psi}(a))$ is the point $\Psi(a)$ neighborhood. Suppose that $h(t) = (\Phi \circ \boldsymbol{\Psi})(t) = \Phi(\boldsymbol{\Psi}(t))$. By setting $u = a + \Omega a^{1-\zeta}$ in Eq. 1.3, we see that

$$\mathfrak{D}^\zeta(h)(a) = \lim_{t \to a} \frac{h(t) - h(a)}{t - a} a^{1-\zeta} = \lim_{t \to a} \frac{\Phi(\boldsymbol{\Psi}(t)) - \Phi(\boldsymbol{\Psi}(a))}{t - a} a^{1-\zeta}. \tag{4.17}$$

Without loss of generality (WLOG), let $U(\boldsymbol{\Psi}(a))$ be an open ball (OB), denoted by $B(\boldsymbol{\Psi}(a), r)$. Since Ψ is a CF, then along with points: $(\Psi_1(a), \ldots, \Psi_n(a))$ and $(\Psi_1(t), \ldots, \Psi_n(t))$,

the points $(\Psi_1(a), \Psi_2(t), \ldots, \Psi_n(t)), \ldots, (\Psi_1(a), \Psi_2(a), \ldots, \Psi_n(t))$

and lines connecting them must also be the ball $B(\Psi(a), r)$.

From the known MeVaThm for differentiable functions (DFs), there is one variable (Marsden & Hoffman, 1996) as follows:

$$\frac{h(t)-h(a)}{t-a}a^{1-\zeta} = \frac{\Phi(\boldsymbol{\Psi}(t)) - \Phi(\boldsymbol{\Psi}(a))}{t-a}a^{1-\zeta}$$

$$= \frac{\Phi(\Psi_1(t), \ldots \Psi_n(t)) - \Phi(\Psi_1(a), \Psi_2(t) \ldots, \Psi_n(t))}{t-a}a^{1-\zeta}$$

$$+\frac{\Phi(\Psi_1(a), \Psi_2(t) \ldots, \Psi_n(t)) - \Phi(\Psi_1(a), \Psi_2(a), \Psi_3(t), \ldots, \Psi_n(t))}{t-a}a^{1-\zeta} + \ldots$$

$$+\frac{\Phi(\Psi_1(a), \Psi_2(a), \ldots, \Psi_n(t)) - \Phi(\Psi_1(a), \ldots, \Psi_n(a))}{t-a}a^{1-\zeta}$$

$$= \frac{\partial}{\partial x_1}\Phi(c_1, \Psi_2(t) \ldots, \Psi_n(t)) \frac{\Psi_1(t) - \Psi_1(a)}{t-a}a^{1-\zeta}$$

$$+\frac{\partial}{\partial x_2}\Phi(\Psi_1(a), c_2 \ldots, \Psi_n(t)) \frac{\Psi_2(t) - \Psi_2(a)}{t-a}a^{1-\zeta} + \ldots$$

$$+\frac{\partial}{\partial x_n}\Phi(\Psi_1(a), \Psi_2(a), \ldots, c_n) \frac{\Psi_n(t) - \Psi_n(a)}{t-a}a^{1-\zeta},$$

where c_i is between $\Psi_i(a)$ and $\Psi_i(t)$ $\forall\, i = 1, 2, \ldots, n$.

By taking limits as $t \to a$ and using the continuity of PaDrs of Φ in addition to taking into our account that $c_i \to \Psi_i(a)$ $\forall\, i = 1, 2, \ldots, n$, Eq. (4.17) is written as:

$$\mathfrak{D}^{\zeta} h)(a) = \lim_{t\to a}\frac{h(t)-h(a)}{t-a}a^{1-\zeta} = \lim_{t\to a}\frac{\Phi(\boldsymbol{\Psi}(t)) - \Phi(\boldsymbol{\Psi}(a))}{t-a}a^{1-\zeta}$$

$$= \lim_{t\to a}\left(\frac{\partial}{\partial x_1}\Phi(c_1, \Psi_2(t) \ldots, \Psi_n(t)) \frac{\Psi_1(t) - \Psi_1(a)}{t-a}a^{1-\zeta}\right)$$

$$+\lim_{t\to a}\left(\frac{\partial}{\partial x_2}\Phi(\Psi_1(a),c_2...,\Psi_n(t))\frac{\Psi_2(t)-\Psi_2(a)}{t-a}a^{1-\zeta}+...+\frac{\partial}{\partial x_n}\Phi(\Psi_1(a),\Psi_2(a),...,c_n)\frac{\Psi_n(t)-\Psi_n(a)}{t-a}a^{1-\zeta}\right)$$

$$=\frac{\partial}{\partial x_1}\Phi\left(\Psi(a)\right)\cdot\Psi_1^{(1)}(a)\cdot a^{1-\zeta}+\frac{\partial}{\partial x_2}\Phi\left(\Psi(a)\right)\cdot\Psi_2^{(1)}(a)\cdot a^{1-\zeta}+\dots$$

$$+\frac{\partial}{\partial x_n}\Phi\left(\Psi(a)\right)\cdot\Psi_n^{(1)}(a)\cdot a^{1-\zeta}$$

$$=\frac{\partial}{\partial x_1}\Phi\left(\Psi(a)\right)\cdot\Psi_1(a)^{1-\zeta}\cdot\Psi_1(a)^{\zeta-1}\cdot\Psi_1^{(1)}(a)\cdot a^{1-\zeta}$$

$$+\frac{\partial}{\partial x_2}\Phi\left(\Psi(a)\right)\cdot\Psi_2(a)^{1-\zeta}\cdot\Psi_2(a)^{\zeta-1}\cdot\Psi_2^{(1)}(a)\cdot a^{1-\zeta}+\dots$$

$$+\frac{\partial}{\partial x_n}\Phi\left(\Psi(a)\right)\cdot\Psi_2(a)^{1-\zeta}\cdot\Psi_2(a)^{\zeta-1}\cdot\Psi_n^{(1)}(a)\cdot a^{1-\zeta}$$

$$=\frac{\partial^\zeta}{\partial x_1^\zeta}\Phi\left(\boldsymbol{\Psi}(a)\right)\cdot\Psi_1(a)^{\zeta-1}\cdot\mathfrak{D}^\zeta\Psi_1)(a)+\frac{\partial^\zeta}{\partial x_2^\zeta}\Phi\left(\boldsymbol{\Psi}(a)\right)\cdot\Psi_2(a)^{\zeta-1}\cdot\mathfrak{D}^\zeta\Psi_2)(a)+\dots$$

$$+\frac{\partial^\zeta}{\partial x_n^\zeta}\Phi\left(\boldsymbol{\Phi}(a)\right)\cdot\Psi_n(a)^{\zeta-1}\cdot\mathfrak{D}^\zeta\Psi_n)(a).$$

Thus, our proof is complete.

Remark 5. *The MF of Eq.* (4.16) *is written as:*

$$\mathfrak{D}^\zeta\left(\boldsymbol{\Phi}\circ\boldsymbol{\Psi}\right)(a)(h)$$

$$=\left(\frac{\partial^\zeta}{\partial x_1^\zeta}\Phi(\boldsymbol{\Psi}(a)),...,\frac{\partial^\zeta}{\partial x_n^\zeta}\Phi(\boldsymbol{\Psi}(a))\right)\cdot\begin{pmatrix}\Psi_1(a)^{\zeta-1} & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \Psi_n(a)^{\zeta-1}\end{pmatrix}\cdot\begin{pmatrix}\mathfrak{D}^\zeta\Psi_1(a)\\ \vdots\\ \mathfrak{D}^\zeta\Psi_n(a)\end{pmatrix}\cdot h. \quad (4.18)$$

The following is a generalized ChR Theorem for RVF with several variables:

Theorem 13. *(ChR). Let* $\boldsymbol{x}=(x_1,\dots,x_n)\in R^n$ *and* $\boldsymbol{y}=(y_1,\dots,y_m)\in R^m$. *If* $\boldsymbol{\Psi}(x_1,\dots,x_n)=(\Psi_1(x_1,\dots,x_n),\dots,\Psi_m(x_1,\dots,x_n))$ *is a* $\zeta-DF$ *at* $\boldsymbol{a}=(a_1,\dots,a_n)\in R^n$, *each* $a_i>0\ni\zeta\in(0,1]$, *and a RVF* Φ *with variables* $y_1,\dots,y_m$ *is* $\zeta-DF$ *at* $\boldsymbol{\Psi}(\boldsymbol{a})\in R^n$, *where all* $\Psi_i(a)>0\ni\zeta\in(0,1]$.

Then, the composition $\Phi \circ \mathbf{\Psi}$ *is* $\zeta - DF$ *at a, and we have:*

$$\frac{\partial^{\zeta}}{\partial x_i^{\zeta}} (\Phi \circ \mathbf{\Psi})(\boldsymbol{a}) = \sum_{j=1}^{m} \frac{\partial^{\zeta}}{\partial y_j^{\zeta}} \Phi(\mathbf{\Psi}(\boldsymbol{a})) \cdot \Psi_j(\boldsymbol{a})^{\zeta-1} \cdot \frac{\partial^{\zeta}}{\partial x_i^{\zeta}} \Psi_j(\boldsymbol{a}). \tag{4.19}$$

$$\forall\, i = 1, 2, \ldots, n.$$

Proof. From the PaDr's definition and Theorem (13), we indicate the following:

Remark 6. *The MF of Eq.* (4.19) *is expressed as:*

$$\mathfrak{D}^{\zeta}(\Phi \circ \mathbf{\Psi})(\boldsymbol{a})(\boldsymbol{h})$$

$$= \left(\frac{\partial^{\zeta}}{\partial y_1^{\zeta}} \Phi(\mathbf{\Psi}(\boldsymbol{a})), \ldots, \frac{\partial^{\zeta}}{\partial y_m^{\zeta}} \Phi(\mathbf{\Psi}(\boldsymbol{a})) \right) \cdot \begin{pmatrix} \Psi_1(\boldsymbol{a})^{\zeta-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Psi_m(\boldsymbol{a})^{\zeta-1} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^{\zeta}}{\partial x_1^{\zeta}} \Psi_1(\boldsymbol{a}) & \ldots & \frac{\partial^{\zeta}}{\partial x_n^{\zeta}} \Psi_1(\boldsymbol{a}) \\ \ldots & \ldots & \ldots \\ \frac{\partial^{\zeta}}{\partial x_1^{\zeta}} \Psi_n(\boldsymbol{a}) & \ldots & \frac{\partial^{\zeta}}{\partial x_n^{\zeta}} \Psi_n(\boldsymbol{a}) \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}. \tag{4.20}$$

4.5 The ComV of ImFThm

The ComV of ImFThm for SeVs is discussed by formulating the ComV of the implicit function for a single equation and investigating its existence and regularity.

Theorem 14. ***(Conformable ImFThm for the Case of a Single Equation)***.
Assume that $\zeta \in (0, 1]$, $F : X \rightarrow R$ *is a ReVaF defined in an OS:* $X \subset R^{n+1}$, $\ni$ $\forall\ (x_1, \ldots, x_n, y) \in X$, *each* $x_i, y > 0$, *and point* $(a_1, \ldots, a_n, b) \in X$. *Suppose that*

1. $F(a_1, \ldots, a_n, b) = 0$,
2. $F \in C_{\zeta}(X, R)$,
3. $\frac{\partial^{\zeta}}{\partial y^{\zeta}} F(a_1, \ldots, a_n, b) \neq 0$.

Then, there is a neighbourhood, $U \subset R^n$*, of* $(a_1, \ldots, a_n) \ni$ *there is a unique function (UF)* $y = \Phi(x_1, \ldots, x_n)$ *that satisfies:* $\Phi(a_1, \ldots, a_n) = b$ *and*
$F(x_1, \ldots, x_n, \Phi(x_1, \ldots, x_n)) = 0,\ \forall (x_1, \ldots, x_n) \in U.$
Finally, $y = \Phi(x_1, \ldots, x_n)$ *is* C_ζ *in U, and for every* $i = 1, 2, \ldots, n$*, we have:*

$$\frac{\partial^\zeta}{\partial x_i^\zeta}\Phi(x_1, \ldots, x_n) = -\frac{\frac{\partial^\zeta}{\partial x_i^\zeta}F((x_1, \ldots, x_n), \Phi((x_1, \ldots, x_n)))}{\frac{\partial^\zeta}{\partial y^\zeta}F((x_1, \ldots, x_n), \Phi((x_1, \ldots, x_n))) \cdot \Phi((x_1, \ldots, x_n))^{\zeta-1}}. \tag{4.21}$$

Proof. WLOG, let X be an OB, denoted by $B((a_1, \ldots, a_n, b), \Omega_0)$. Assume that $\rho \in (0, \Omega_0)$. If we say: $\delta = \sqrt{\Omega_0^2 - \rho^2}$, we get:

$$[\|(x_1,...,x_n) - (a_1,...,a_n)\| < \delta \ \ and \ \ |y-b| < \rho] \Rightarrow (x_1,...,x_n,y) \in B((a_1,...,a_n,b),\Omega_0).$$

Note that in particular if $|y - b| < \rho$, then $(a_1, \ldots, a_n, y) \in B((a_1, \ldots, a_n, b), \Omega_0)$. Since $\frac{\partial^\zeta}{\partial y^\zeta}F(a_1, \ldots, a_n, b) \neq 0$, it is assumed to be positive (otherwise, $-F$ is considered instead of F). From fact that $F(a_1, \ldots, a_n, b) = 0$, it implies that
$F(a_1, \ldots, a_n, b - \rho) > 0$ and $F(a_1, \ldots, a_n, b - \rho) < 0$.
By the continuity of F at $(a_1, \ldots, a_n, b - \rho)$ and $(a_1, \ldots, a_n, b + \rho)$, there exists $\delta' \in (0, \delta) \ni$

$$\|(x_1,...,x_n) - (a_1,...,a_n)\| < \delta' \Rightarrow [F(x_1,...,x_n,b-\rho) > 0 \ \text{ and } \ F(x_1,...,x_n,b+\rho) < 0].$$

Since the function $y \mapsto F(x_1, \ldots, x_n, y)$ is CF on the interval $[b - \rho, b + \rho]$, $\forall (x_1, \ldots, x_n) \in B((a_1, \ldots, a_n), \delta')$, and from the known Bolzano's Theorem, it indicates that $\exists\ y_x \in (b - \rho, b + \rho) \ni F(x_1, \ldots, x_n, y_x) = 0$, for each $x = (x_1, \ldots, x_n)$. Then, y's value is unique since a function whose derivative is positive have more than a zero. On the contrary, from: $U = B((a_1, \ldots, a_n), \delta')$, for each $(x_1, \ldots, x_n) \in U$, $\exists$ a unique $y \ni F(x_1, \ldots, x_n, y) = 0$, we can write $y = \Phi(x_1, \ldots, x_n)$, and then this function will be shown to be CF on $B((a_1, \ldots, a_n), \delta')$. The continuity of the function Φ at the point $(a_1, \ldots, a_n)$ is obvious since for each $\rho > 0$, $\exists$ a value $\delta' > 0 \ni$

$$\|(x_1, \ldots, x_n) - (a_1, \ldots, a_n)\| < \delta' \Rightarrow |b - y_x| < \rho \Leftrightarrow |b - g(x_1, \ldots, x_n)| < \rho.$$

The function Φ continuity will be shown at any point
$(x_1, \ldots, x_n) \in B((a_1, \ldots, a_n), \delta')$
by simply substituting $B((a_1, \ldots, a_n), \delta')$ for an OB: $B((x_1, \ldots, x))$ contained in $B((a_1, \ldots, a_n), \delta')$.
At the end, the formula (4.21) will now be shown. From using Theorem (13) in equation:
$F(x_1, \ldots, x_n, y) = 0$, we have:

$$\frac{\partial^\zeta}{\partial x_i^\zeta} F(\boldsymbol{x}, \Phi(\boldsymbol{x})) + \frac{\partial^\zeta}{\partial y^\zeta} F(\boldsymbol{x}, \Phi(\boldsymbol{x})) \cdot \Phi(\boldsymbol{x})^{\zeta - 1} \cdot \frac{\partial^\zeta}{\partial x_i^\zeta} \Phi(\boldsymbol{x}) = 0,$$

$\forall\, i = 1, 2, \ldots, n$, $\ni \boldsymbol{x} = (x_1, \ldots, x_n)$. Solving $\frac{\partial^\zeta}{\partial x_i^\zeta} \Phi(\boldsymbol{x})$, we get: formula (4.21). In addition, the formula (4.21) right side is continuous, so the continuity of the ComV of PaDrs: $\frac{\partial^\zeta}{\partial x_i^\zeta} \Phi(\boldsymbol{x})\ \forall\, i = 1, 2, \ldots, n$, follows.
Theorem (14) will help us compute the ComV of PaDrs of implicit function of SeVs.

Example 4.5.1. *Consider:*

$$F(x, y, z) = x^3 + 3y^2 + 4xz^2 - 3yz^2 - 5 = 0.$$

This equation's one solution is $(1, 1, 1)$. *F is obviously in C_ζ which is an OB,* *denoted by* $B((1,1,1),\Omega_0)$, *with* $x,y,z>0$, *for some* $\zeta\in(0,1]$ *since*

$$\frac{\partial^{\zeta}}{\partial z^{\zeta}}F(1,1,1)=\left(8xz^{2-\zeta}-6yz^{2-\zeta}\right)\Big]_{(1,1,1)}=2\neq 0,$$

Theorem (14) *implies that there is a neighbourhood,* $U\subset R^2$, *of* $(1,1)\ni\exists$ *a UF:* $z=\Phi(x,y)$ *that satisfies the following:*

$\Phi(1,1)=1$ *and* $F(x,y,\Phi(x,y))=0,\ \forall(x,y)\in U$.

Moreover, $z=\Phi(x,y)$ *is* C_ζ *in U, and we have:*

$$\frac{\partial^{\zeta}}{\partial x^{\zeta}}\Phi(x,y)=-\frac{\left(3x+4z^2\right)x^{1-\zeta}}{(8x-6y)z},$$

$$\frac{\partial^{\zeta}}{\partial y^{\zeta}}\Phi(x,y)=-\frac{\left(6y-3z^2\right)y^{1-\zeta}}{(8x-6y)z}.$$

Finally, we obtain: $\frac{\partial^{\zeta}}{\partial x^{\zeta}}\Phi(1,1)=-\frac{7}{2}$ *and* $\frac{\partial^{\zeta}}{\partial y^{\zeta}}\Phi(1,1)=-\frac{3}{2}$.

At the end, the ComV of ImFThm for a system of several equations and SeReVs is found.

Theorem 15. ***(The Conformable General ImFThm).*** *Let* $\zeta\in(0,1]$, $\boldsymbol{F}:X\to R^m$ *be a VeVaF defined in an OS:* $X\subset R^{n+m}$, $\ni\ \forall\ (\boldsymbol{x};\boldsymbol{y})=(x_1,\ldots,x_n;y_1,\ldots,y_m)\in X$, *each* $x_i,y_j>0$, *and point* $(\boldsymbol{a};\boldsymbol{b})=(a_1,\ldots,a_n;b_1,\ldots,b_m)\in X$. *Assume that*

1. $\boldsymbol{F}(\boldsymbol{a};\boldsymbol{b})=\boldsymbol{0}$,
2. $\boldsymbol{F}\in C_\zeta(X,R^m)$,
3. $\det\left[J_y^{\zeta}\boldsymbol{F}(\boldsymbol{a};\boldsymbol{b})\right]\neq 0$.

Then, there is a neighbourhood, $U\subset R^n$, *of* $\boldsymbol{a}\ni\exists$ *a UF:* $\boldsymbol{\Psi}:U\to R^m$, $\boldsymbol{x}\to\boldsymbol{y}=\boldsymbol{\Psi}(\boldsymbol{x})$ *that satisfies:*

$\boldsymbol{\Psi}(\boldsymbol{a})=\boldsymbol{b}$ *and* $\boldsymbol{F}(\boldsymbol{x};\boldsymbol{\Psi}(\boldsymbol{x}))=\boldsymbol{0}$, $\forall\boldsymbol{x}\in U$,

Finally, $\boldsymbol{y}=\boldsymbol{\Psi}(\boldsymbol{x})$ *is class* C_ζ *in U, and for every* $i=1,2,\ldots,n$, *we have:*

$$\left[\frac{\partial^{\zeta}\boldsymbol{\Psi}}{\partial x_i^{\zeta}}\right]^t = -\left(\boldsymbol{\Psi}^{\zeta-1}\right)^{-1}\cdot\left(J_y^{\zeta}\boldsymbol{F}\right)^{-1}\cdot\left[\frac{\partial^{\zeta}\boldsymbol{F}}{\partial x_i^{\zeta}}\right]^t, \tag{4.22}$$

where

$$\left[\frac{\partial^{\zeta}\boldsymbol{\Psi}}{\partial x_i^{\zeta}}\right] = \left(\frac{\partial^{\zeta}\Psi_1}{\partial x_i^{\zeta}}, \ \ldots, \frac{\partial^{\zeta}\Psi_m}{\partial x_i^{\zeta}}\right), \boldsymbol{\Psi}^{\zeta-1} = \begin{pmatrix} \Psi_1^{\zeta-1} & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & \Psi_m^{\zeta-1} \end{pmatrix},$$

$$J_y^{\zeta}\boldsymbol{F} = \begin{pmatrix} \frac{\partial^{\zeta}F_1}{\partial y_1^{\zeta}} & \ldots & \frac{\partial^{\zeta}F_1}{\partial y_m^{\zeta}} \\ \ldots & \ldots & \ldots \\ \frac{\partial^{\zeta}F_m}{\partial y_1^{\zeta}} & \ldots & \frac{\partial^{\zeta}F_m}{\partial y_m^{\zeta}} \end{pmatrix} \text{and} \left[\frac{\partial^{\zeta}\boldsymbol{F}}{\partial x_i^{\zeta}}\right] = \left(\frac{\partial^{\zeta}F_1}{\partial x_i^{\zeta}}, \ \ldots, \frac{\partial^{\zeta}F_m}{\partial x_i^{\zeta}}\right).$$

Proof. The existence and uniqueness of the implicit function can be proven same as the known MuCL via the mathematical induction on q and using the ComV oF ImFThm for several variables (Marsden & Hoffman, 1996).

To prove formula(4.22), we suppose that a system with several equations and SeReVs is expressed as:

$$F\left(\boldsymbol{x};\boldsymbol{y}\right) = \boldsymbol{0} \quad or \quad \begin{cases} F_1\left(x_1,\ldots,x_n;y_1,\ldots,y_m\right) = 0 \\ \vdots \\ F_m\left(x_1,\ldots,x_n;y_1,\ldots,y_m\right) = 0 \end{cases}, \tag{4.23}$$

satisfies hypotheses (1)-(3) of the theorem, then this system is defined in a neighbourhood, $U \subset R^n$, of $\boldsymbol{a}$ the implicit function $\boldsymbol{y} = \boldsymbol{\Psi}\left(\boldsymbol{x}\right)$ class C^{ζ} in U, such that $\boldsymbol{\Psi}\left(\boldsymbol{a}\right) = \boldsymbol{b}$ which satisfies Eq. 1.3, i.e.,

$$F\left(\boldsymbol{x};\boldsymbol{\Psi}(\boldsymbol{x})\right) = \boldsymbol{0} \quad or \quad \begin{cases} F_1\left(\boldsymbol{x};\Psi_1(\boldsymbol{x}),\ldots,\Psi_m(\boldsymbol{x})\right) = 0 \\ \ldots \\ F_m\left(\boldsymbol{x};\Psi_1(\boldsymbol{x}),\ldots,\Psi_m(\boldsymbol{x})\right) = 0 \end{cases}. \tag{4.24}$$

By employing the ComV of ChR to the above equation, we get:

$$\left.\begin{array}{c}\frac{\partial^\zeta F_1}{\partial x_i^\zeta}+\frac{\partial^\zeta F_1}{\partial y_1^\zeta}\cdot\Psi_1^{\zeta-1}\cdot\frac{\partial^\zeta \Psi_1}{\partial x_i^\zeta}+\cdots+\frac{\partial^\zeta F_1}{\partial y_m^\zeta}\cdot\Psi_m^{\zeta-1}\cdot\frac{\partial^\zeta \Psi_m}{\partial x_i^\zeta}=0\\ \vdots\\ \frac{\partial^\zeta F_m}{\partial x_i^\zeta}+\frac{\partial^\zeta F_m}{\partial y_1^\zeta}\cdot\Psi_1^{\zeta-1}\cdot\frac{\partial^\zeta \Psi_1}{\partial x_i^\zeta}+\cdots+\frac{\partial^\zeta F_m}{\partial y_m^\zeta}\cdot\Psi_m^{\zeta-1}\cdot\frac{\partial^\zeta \Psi_m}{\partial x_i^\zeta}=0\end{array}\right\}$$

or

$$\frac{\partial^\zeta \boldsymbol{F}}{\partial x_i^\zeta}+\sum_{j=1}^{m}\frac{\partial^\zeta \boldsymbol{F}}{\partial y_j^\zeta}\cdot\Psi_j^{\zeta-1}\cdot\frac{\partial^\zeta \Psi_j}{\partial x_i^\zeta}=0, \tag{4.25}$$

$\forall i=1,2,\ldots,n.$

Additionally, the MF of Eq. (4.25) is provided as follows:

$$\begin{pmatrix}\frac{\partial^\zeta F_1}{\partial x_i^\zeta}\\ \vdots\\ \frac{\partial^\zeta F_m}{\partial x_i^\zeta}\end{pmatrix}=-\begin{pmatrix}\frac{\partial^\zeta F_1}{\partial y_1^\zeta} & \cdots & \frac{\partial^\zeta F_1}{\partial y_m^\zeta}\\ \cdots & \cdots & \cdots\\ \frac{\partial^\zeta F_m}{\partial y_1^\zeta} & \cdots & \frac{\partial^\zeta F_m}{\partial y_m^\zeta}\end{pmatrix}\cdot\begin{pmatrix}\Psi_1^{\zeta-1} & \cdots & 0\\ \vdots & \cdots & \vdots\\ 0 & \cdots & \Psi_m^{\zeta-1}\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial^\zeta \Psi_1}{\partial x_i^\zeta}\\ \vdots\\ \frac{\partial^\zeta \Psi_m}{\partial x_i^\zeta}\end{pmatrix}$$

or

$$\left[\frac{\partial^\zeta \boldsymbol{F}}{\partial x_i^\zeta}\right]^t=-J_y^\zeta \boldsymbol{F}\cdot\boldsymbol{\Psi}^{\boldsymbol{\zeta-1}}\cdot\left[\frac{\partial^\zeta \boldsymbol{\Psi}}{\partial x_i^\zeta}\right]^t. \tag{4.26}$$

Since $J_y^\zeta \boldsymbol{F}$ and $\boldsymbol{\Psi}^{\boldsymbol{\zeta-1}}$ are regular matrices by hypothesis, we have:

$$\begin{pmatrix}\frac{\partial^\zeta \Psi_1}{\partial x_i^\zeta}\\ \vdots\\ \frac{\partial^\zeta F_m}{\partial x_i^\zeta}\end{pmatrix}=-\begin{pmatrix}\Psi_1^{\zeta-1} & \cdots & 0\\ \vdots & \cdots & \vdots\\ 0 & \cdots & \Psi_m^{\zeta-1}\end{pmatrix}^{-1}\cdot\begin{pmatrix}\frac{\partial^\zeta F_1}{\partial y_1^\zeta} & \cdots & \frac{\partial^\zeta F_1}{\partial y_m^\zeta}\\ \cdots & \cdots & \cdots\\ \frac{\partial^\zeta F_m}{\partial y_1^\zeta} & \cdots & \frac{\partial^\zeta F_m}{\partial y_m^\zeta}\end{pmatrix}^{\mathbf{-1}}\cdot\begin{pmatrix}\frac{\partial^\zeta F_1}{\partial x_i^\zeta}\\ \vdots\\ \frac{\partial^\zeta F_m}{\partial x_i^\zeta}\end{pmatrix}$$

or

$$\left[\frac{\partial^\zeta \boldsymbol{\Psi}}{\partial x_i^\zeta}\right]^t=-\left(\boldsymbol{\Psi}^{\boldsymbol{\zeta-1}}\right)^{-1}\cdot\left(J_y^\zeta \boldsymbol{F}\right)^{-1}\cdot\left[\frac{\partial^\zeta \boldsymbol{F}}{\partial x_i^\zeta}\right]^t,$$

which finalizes the proof.

Theorem (15) will be shown to compute the ComV of PaDrs of systems with several equations and SeReVs.

Example 4.5.2. *Consider a system of* 2 *equations and* 2 *real variables:*

$$\begin{cases} F_1(x,y,z,w) = x^2 + y^2 + z^2 + w^2 - 6 = 0 \\ F_2(x,y,z,w) = x^2 - y^2 + z^2 - w^2 = 0 \end{cases}.$$

One solution of this equation is $(x,y,z,w) = (1,1,\sqrt{2},\sqrt{2})$. *Clearly,* $\boldsymbol{F} = (F_1, F_2)$ *is in* C_ζ *which is an OB:* $B(\left(1,1,\sqrt{2},\sqrt{2}\right),\Omega_0)$, *with* $x,y,z,w > 0$, *for some* $\zeta \in (0,1]$ *since*

$$det\left[J^{\zeta}_{z,w}\boldsymbol{F}\left(\left(\sqrt{2},\sqrt{2},1,1\right)\right)\right] = det\left[\begin{pmatrix} 2z^{2-\zeta} & 2w^{2-\zeta} \\ 2z^{2-\zeta} & -2w^{2-\zeta} \end{pmatrix}\right]_{\left(1,1,\sqrt{2},\sqrt{2}\right)} = -\frac{32}{2^{\propto}} \neq 0,$$

Theorem (15) *indicates that there is a neighbourhood,* $U \subset R^2$, *of* $\left(\sqrt{2},\sqrt{2}\right) \ni \exists$ *a UF:* $\boldsymbol{\Psi} = (\Psi_1, \Psi_2)$ *given by*

$$\begin{cases} z = \Psi_1(x,y) \\ w = \Psi_2(x,y) \end{cases},$$

that satisfies:

$$\begin{cases} \Psi_1(1,1) = \sqrt{2} \\ \Psi_2(1,1) = \sqrt{2} \end{cases},$$

and

$$\begin{cases} F_1(x,y,\Psi_1(x,y),\Psi_2(x,y)) = 0 \\ F_2(x,y,\Psi_1(x,y),\Psi_2(x,y)) = 0 \end{cases}, \forall (x,y) \in U.$$

Moreover, $\boldsymbol{\Psi} = (\Psi_1, \Psi_2)$ *is class* C_ζ *in U, and we have:*

$$\begin{pmatrix} \frac{\partial^\zeta \Psi_1}{\partial x^\zeta} \\ \frac{\partial^\zeta \Psi_2}{\partial x^\zeta} \end{pmatrix} = -\begin{pmatrix} z^{\zeta-1} & 0 \\ 0 & w^{\zeta-1} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2z^{2-\zeta} & 2w^{2-\zeta} \\ 2z^{2-\zeta} & -2w^{2-\zeta} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2x^{2-\zeta} \\ 2x^{2-\zeta} \end{pmatrix}$$

$$= -\frac{1}{4} \cdot \begin{pmatrix} z^{1-\zeta} & 0 \\ 0 & w^{1-\zeta} \end{pmatrix} \cdot \begin{pmatrix} z^{\zeta-2} & z^{\zeta-2} \\ w^{\zeta-2} & -w^{\zeta-2} \end{pmatrix} \cdot \begin{pmatrix} 2x^{2-\zeta} \\ 2x^{2-\zeta} \end{pmatrix}$$

$$= -\frac{1}{4} \cdot \begin{pmatrix} z^{-1} & z^{-1} \\ w^{-1} & -w^{-1} \end{pmatrix} \cdot \begin{pmatrix} 2x^{2-\zeta} \\ 2x^{2-\zeta} \end{pmatrix} = \begin{pmatrix} -\frac{x^{2-\zeta}}{z} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial^\zeta \Psi_1}{\partial y^\zeta} \\ \frac{\partial^\zeta \Psi_2}{\partial y^\zeta} \end{pmatrix} = -\begin{pmatrix} z^{\zeta-1} & 0 \\ 0 & w^{\zeta-1} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2z^{2-\zeta} & 2w^{2-\zeta} \\ 2z^{2-\zeta} & -2w^{2-\zeta} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2y^{2-\zeta} \\ -2y^{2-\zeta} \end{pmatrix}$$

$$= -\frac{1}{4} \cdot \begin{pmatrix} z^{1-\zeta} & 0 \\ 0 & w^{1-\zeta} \end{pmatrix} \cdot \begin{pmatrix} z^{\zeta-2} & z^{\zeta-2} \\ w^{\zeta-2} & -w^{\zeta-2} \end{pmatrix} \cdot \begin{pmatrix} 2y^{2-\zeta} \\ -2y^{2-\zeta} \end{pmatrix}$$

$$= -\frac{1}{4} \cdot \begin{pmatrix} z^{-1} & z^{-1} \\ w^{-1} & -w^{-1} \end{pmatrix} \cdot \begin{pmatrix} 2y^{2-\zeta} \\ -2y^{2-\zeta} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{y^{2-\zeta}}{w} \end{pmatrix}.$$

Finally, we have:

$$\begin{pmatrix} \frac{\partial^\zeta \Psi_1}{\partial x^\zeta} \\ \frac{\partial^\zeta \Psi_2}{\partial x^\zeta} \end{pmatrix}_{(1,1,\sqrt{2},\sqrt{2})} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{\partial^\zeta \Psi_1}{\partial y^\zeta} \\ \frac{\partial^\zeta \Psi_2}{\partial y^\zeta} \end{pmatrix}_{(1,1,\sqrt{2},\sqrt{2})} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

4.6 Conclusion

The ComV of MuCL have been studied in detail. The $\zeta-$derivative of a function of SeVs and all related properties have been discussed. The ComV of ChR for functions of SeVs has also been investigated. The ComV of ImFThm has been proposed, and numerical examples have been provided to validate our theoretical analysis. All our results in the context of ComD are compatible with the integer order ones. Various scientific systems can be modelled by using our results. Further extensions or generalizations can be considered as a new direction in our future works.

CHAPTER 5: ON ABU-SHADY–KAABAR FRACTIONAL DERIVATIVE WITH APPLICATIONS

5.1 Introduction

The ComD satisfies some important properties that cannot be satisfied in RL and Cp definitions. In (Abdelhakim, 2019), the author proved that the ComD in (Khalil et al., 2014) cannot provide good results in comparison with the Cp definition for some functions. Therefore, to overcome all such issues, we have proposed in this chapter a new generalized definition of fractional derivative, known as Abu-Shady–Kaabar (ASK) fractional derivative, that has advantages in comparison with other previous definitions to obtain simple solutions of fractional differential equations.

5.2 Basic Definitions and Tools

This ASK definition is written as follows:

Definition 13. *For a function:* $\Psi : [0, \infty) \rightarrow R$*, the ASK fractional derivative of order* $0 < \zeta \leq 1$ *of* $\Psi(t)$ *at* $t > 0$ *is defined as:*

$$\mathfrak{D}^{ASK}\Psi(t) = \lim_{\Omega \to 0} \frac{\Psi(t + \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}\Omega t^{1-\zeta}) - \Psi(t)}{\Omega}; \beta > -1, \beta \in R^{+}, \tag{5.1}$$

and the fractional derivative at 0 is defined as: $\mathfrak{D}^{ASK}\Psi(0) = \lim_{\Omega \to 0^{+}} \mathfrak{D}^{ASK}\Psi(t)$.

Theorem 16. *If* $\Psi(t)$ *is a* $\zeta-$*DF, then* $\mathfrak{D}^{ASK}\Psi(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)} t^{1-\zeta} \frac{d\Psi(t)}{dt}; \beta > -1, \beta \in R^{+}$.

We formulate the ASK fractional derivative for some functions to prove that our new proposed definition is powerful in obtaining analytical solutions for solving various types of fractional differential equations, and ASK definition in Eq. 5.1

coincides with the results from the well-known classical fractional derivatives such as Caputo and Riemann-Liouville fractional derivatives.

5.3 Theoretical Investigation

Let us first prove our proposed Theorem 16 as follows:

Proof: From Eq. 5.1, we have:

$$\mathfrak{D}^{ASK}\Psi(t) = \lim_{\Omega\to 0} \frac{\Psi(t + \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}\Omega t^{1-\zeta}) - \Psi(t)}{\Omega}; \beta > -1, \beta \in R^{+}, \tag{5.2}$$

where at $\zeta = \beta = 1$, the classical limit-based derivative of a function is obtained. Now, let

$$h = \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}\Omega t^{1-\zeta}, \tag{5.3}$$

$$\Omega = \frac{\Gamma(\beta-\zeta+1)}{\Gamma(\beta)} h t^{\zeta-1}. \tag{5.4}$$

By substituting from Eq. 5.4 into Eq. 5.2, we get:

$$\mathfrak{D}^{ASK}\Psi(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)} t^{1-\zeta} \lim_{h\to 0} \frac{\Psi(t+h) - \Psi(t)}{h}, \tag{5.5}$$

thus

$$\mathfrak{D}^{ASK}\Psi(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)} t^{1-\zeta} \frac{d\Psi(t)}{dt}. \tag{5.6}$$

For a function: $\Psi(t) = t^{k}, k > -1, k \in R^{+}$, we prove that

$$\mathfrak{D}^{ASK}\Psi(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\zeta+1)} t^{\beta-\zeta}. \tag{5.7}$$

By using Eq. 5.6, we obtain:

$$\mathfrak{D}^{ASK}\Psi(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)} t^{1-\zeta} k t^{k-1}. \tag{5.8}$$

$$\mathfrak{D}^{ASK}\Psi(t) = \frac{k\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{k-\zeta}. \tag{5.9}$$

By taking $k = \beta$, we get:

$$\mathfrak{D}^{ASK}t^{\beta} = \frac{\beta\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{\beta-\zeta}, \tag{5.10}$$

then

$$\mathfrak{D}^{ASK}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\zeta+1)}t^{\beta-\zeta}. \tag{5.11}$$

Eq. 5.11 is compatible with the results of Cp and RL derivatives (Podlubny, 1998).

Theorem 17. *For a function derivative of* $\Psi(t) = t^k$, $k \in R^+$, *we obtain:* $\mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}t^k = \mathfrak{D}^{\zeta+\beta}t^k$.

Proof: By using Eq. 5.11, we get:

$$\mathfrak{D}^{\beta}t^k = \frac{\Gamma(k+1)}{\Gamma(k-\beta+1)}t^{k-\beta}. \tag{5.12}$$

$$\mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}t^k = \frac{\Gamma(k+1)}{\Gamma(k-\beta+1)}\mathfrak{D}^{\zeta}t^{k-\beta}. \tag{5.13}$$

$$\mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}t^k = \frac{\Gamma(k+1)}{\Gamma(k-\beta+1)}\frac{\Gamma(k-\beta+1)}{\Gamma(k-\beta-\zeta+1)}t^{k-\beta-\zeta}. \tag{5.14}$$

$$L.H.S = \mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}t^k = \frac{\Gamma(k+1)}{\Gamma(k-\beta-\zeta+1)}t^{k-\beta-\zeta}. \tag{5.15}$$

Also, we have:

$$R.H.S = \mathfrak{D}^{\zeta+\beta}t^k = \frac{\Gamma(k+1)}{\Gamma(k-\beta-\zeta+1)}t^{k-\beta-\zeta}. \tag{5.16}$$

thus by Eq. 5.15 and Eq. 5.16, we get:

$$\mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}t^{k} = \mathfrak{D}^{\zeta+\beta}t^{k}. \tag{5.17}$$

This property is not satisfied in the ComD (Khalil et al., 2014).

Theorem 18. *For a differentiable function: $\Psi(t)$ that expands about a point such as $\Psi(t) = \sum_{k=0}^{\infty}\frac{\Psi^{k}(0)}{k!}t^{k}$, we have: $\mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}\Psi(t) = \mathfrak{D}^{\zeta+\beta}\Psi(t)$.*

Proof: The expanded function by Taylor theory is given by: $\Psi(t) = \sum_{k=0}^{\infty}\frac{\Psi^{k}(0)}{k!}t^{k}$,

$$\mathfrak{D}^{\beta}\Psi(t) = \sum_{k=0}^{\infty}\frac{\Psi^{k}(0)}{k!}\mathfrak{D}^{\beta}t^{k}, \tag{5.18}$$

$$\mathfrak{D}^{\beta}\Psi(t) = \sum_{k=0}^{\infty}\frac{\Psi^{k}(0)}{k!}\frac{\Gamma(k+1)}{\Gamma(k-\beta+1)}t^{k-\beta}, \tag{5.19}$$

$$\mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}\Psi(t) = \sum_{k=0}^{\infty}\frac{\Psi^{k}(0)}{k!}\frac{\Gamma(k+1)}{\Gamma(k-\beta+1)}\mathfrak{D}^{\zeta}t^{k-\beta}, \tag{5.20}$$

$$\mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}\Psi(t) = \sum_{k=0}^{\infty}\frac{\Psi^{k}(0)}{k!}\frac{\Gamma(k+1)}{\Gamma(k-\beta+1)}\frac{\Gamma(k-\beta+1)}{\Gamma(k-\beta-\zeta+1)}t^{k-\beta-\zeta}, \tag{5.21}$$

$$L.H.S = \mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}\Psi(t) = \sum_{k=0}^{\infty}\frac{\Psi^{k}(0)}{k!}\frac{\Gamma(k+1)}{\Gamma(k-\beta-\zeta+1)}t^{k-\beta-\zeta}, \tag{5.22}$$

$$R.H.S = \mathfrak{D}^{\zeta+\beta}\Psi(t) = \sum_{k=0}^{\infty}\frac{\Psi^{k}(0)}{k!}\mathfrak{D}^{\zeta+\beta}t^{k}, \tag{5.23}$$

$$R.H.S = \mathfrak{D}^{\zeta+\beta}\Psi(t) = \sum_{k=0}^{\infty}\frac{\Psi^{k}(0)}{k!}\frac{\Gamma(k+1)}{\Gamma(k-\beta-\zeta+1)}t^{k-\beta-\zeta}. \tag{5.24}$$

Thus by 5.22 and 5.24, we have:

$$\mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}\Psi(t) = \mathfrak{D}^{\zeta+\beta}\Psi(t). \tag{5.25}$$

This property is not satisfied in the ComD (Khalil et al., 2014).

Theorem 19. *Let $\zeta \in (0,1]$ and Ψ, Φ be $\zeta-$DFs, then*

$$(i)\ \mathfrak{D}^{ASK}(\Psi\Phi) = \Psi\mathfrak{D}^{ASK}(\Phi) + \Phi\mathfrak{D}^{ASK}(\Psi), \tag{5.26}$$

$$(ii)\ \mathfrak{D}^{ASK}\left(\frac{\Psi}{\Phi}\right) = \frac{\Phi\mathfrak{D}^{ASK}(\Psi) - \Psi\mathfrak{D}^{ASK}(\Phi)}{\Phi^2}. \tag{5.27}$$

Proof: By using Eq. 5.6, we have:

$$L.H.S = \mathfrak{D}^{ASK}(\Psi\Phi), \tag{5.28}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{1-\zeta}\frac{d(\Psi\Phi)}{dt}, \tag{5.29}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{1-\zeta}\left[\Psi\frac{d\Phi}{dt} + \Phi\frac{d\Psi}{dt}\right], \tag{5.30}$$

$$= \Psi\frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{1-\zeta}\frac{d\Phi}{dt} + \Phi\frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{1-\zeta}\frac{d\Psi}{dt}, \tag{5.31}$$

$$= \Psi\mathfrak{D}^{ASK}(\Phi) + \Phi\mathfrak{D}^{ASK}(\Psi) = R.H.S. \tag{5.32}$$

This proves (i)

Now, to prove (ii), we use Eq. 5.6 as follows:

$$L.H.S = \mathfrak{D}^{ASK}\left(\frac{\Psi}{\Phi}\right), \tag{5.33}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{1-\zeta}\frac{d}{dt}\left(\frac{\Psi}{\Phi}\right), \tag{5.34}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{1-\zeta}\left[\frac{\Phi\frac{df}{dt}-\Psi\frac{d\Phi}{dt}}{\Phi^2}\right], \tag{5.35}$$

$$= \frac{\Phi\left[\frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{1-\zeta}\frac{d\Psi}{dt}\right]-\Psi\left[\frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}t^{1-\zeta}\frac{d\Phi}{dt}\right]}{\Phi^2}, \tag{5.36}$$

$$= \frac{\Phi\mathfrak{D}^{ASK}(\Psi)-\Psi\mathfrak{D}^{ASK}(\Phi)}{\Phi^2} = R.H.S. \tag{5.37}$$

The rules (i) and (ii) are not satisfied in the Cp and RL definitions.

Theorem 20. *(Rolle's Theorem for the Generalized Fractional Differential Function). Let* $a > 0$ *and* $\Psi : [a, b] \to R$ *be a given function that satisfies the following:*

i) Ψ *is continuous on [a, b].*

ii)Ψ *is* $\zeta-DF$ *for some* $\zeta \in (0, 1]$.

iii)$\Psi(a) = \Psi(b)$.

then, there exists $c \in [a, b]$, *such that* $\Psi^{(\zeta)}(c) = 0$.

Proof: Since Ψ is continuous on $[a, b]$,and $\Psi(a) = \Psi(b)$, there is a $c \in (a, b)$, which is a point of local extrema, and c is assumed to be a point of local minimum. So, we have:

$$\mathfrak{D}^{ASK}\Psi(c^+) = \lim_{\Omega\to 0^+}\frac{\Psi(c+\frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}\Omega c^{1-\zeta})-\Psi(c)}{\Omega}; \beta > -1, \beta \in R^+, \tag{5.38}$$

$$\mathfrak{D}^{ASK}\Psi(c^{-}) = \lim_{\Omega\to 0^{-}} \frac{\Psi(c + \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta+1)}\Omega c^{1-\zeta}) - \Psi(c)}{\Omega}; \beta > -1, \beta \in R^{+}, \quad (5.39)$$

But, $\mathfrak{D}^{ASK}\Psi(c^{+})$ and $\mathfrak{D}^{ASK}\Psi(c^{-})$ have opposite signs. Hence, $\mathfrak{D}^{ASK}\Psi(c) = 0$.

Theorem 21. *(Mean Value Theorem for the Generalized Fractional Differential Function) Let $a > 0$ and $\Psi : [a, b] \to R$ be a given function that satisfies the following:*

i) Ψ is continuous on [a, b]

ii) Ψ is $\zeta-$DF for some $\zeta \in (0, 1)$.

then, there exists $c \in [a, b]$, such that

$$\mathfrak{D}^{ASK}\Psi(c) = \left[\frac{\Psi(b) - \Psi(a)}{h\left(b^{\zeta} - a^{\zeta}\right)}\right], \quad (5.40)$$

where $h = \frac{1}{\Gamma(\zeta)}$

Proof: Consider a function such as in (Khalil et al., 2014)

$$\Phi(t) = \Psi(t) - \Psi(a) - \left[\frac{\Psi(b) - \Psi(a)}{h\left(b^{\zeta} - a^{\zeta}\right)}\right]\left(ht^{\zeta} - ha^{\zeta}\right), \quad (5.41)$$

where $h = \frac{1}{\Gamma(\zeta)}$.

$$\mathfrak{D}^{ASK}\Phi(t) = \mathfrak{D}^{ASK}\Psi(t) - \mathfrak{D}^{ASK}\Psi(a) - \left[\frac{\Psi(b) - \Psi(a)}{h\left(b^{\zeta} - a^{\zeta}\right)}\right]\left(h\mathfrak{D}^{ASK}t^{\zeta} - h\mathfrak{D}^{ASK}a^{\zeta}\right), \quad (5.42)$$

By using Eq. 5.6, we get:

$$\mathfrak{D}^{ASK}\Phi(t) = \mathfrak{D}^{ASK}\Psi(t) - \left[\frac{\Psi(b) - \Psi(a)}{h\left(b^{\zeta} - a^{\zeta}\right)}\right], \quad (5.43)$$

at $c \in [a, b]$

$$\mathfrak{D}^{ASK}\Phi(c) = \mathfrak{D}^{ASK}\Psi(c) - \left[\frac{\Psi(b) - \Psi(a)}{h\left(b^{\zeta} - a^{\zeta}\right)}\right], \tag{5.44}$$

the auxiliary function: $\Phi(c)$ satisfies all conditions of the Rolle's theorem. Therefore, there exists a $c \in [a, b]$ such that $\mathfrak{D}^{ASK}\Phi(c) = 0$. Then, we have:

$$\mathfrak{D}^{ASK}\Psi(c) = \left[\frac{\Psi(b) - \Psi(a)}{h\left(b^{\zeta} - a^{\zeta}\right)}\right], \tag{5.45}$$

Definition 14. $I_{\zeta}^{a}(\Psi)(t) = I_{1}^{0}\left(t^{\zeta-1}\Psi(x)\right) = \frac{\Gamma(\beta-\zeta+1)}{\Gamma(\beta)}\int_{0}^{t}\frac{\Psi(x)}{x^{1-\zeta}}dx$ *and* $\zeta \in (0, 1)$

Theorem 22. $\mathfrak{D}^{\zeta} I_{\zeta}(\Psi)(t) = \Psi(t)$ *for* $t \geq 0$ *where* Ψ *is any continuous function in the domain.*

Proof: Since Ψ is continuous, then $I_{\zeta}^{a}(\Psi)(t)$ *is* differentiable. Hence,

$$\mathfrak{D}^{\zeta} I_{\zeta}(\Psi)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta - \zeta + 1)} t^{1-\zeta}\frac{d}{dt} I_{\zeta}(\Psi)(t), \tag{5.46}$$

$$\mathfrak{D}^{\zeta} I_{\zeta}(\Psi)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta - \zeta + 1)} t^{1-\zeta}\frac{d}{dt}\frac{\Gamma(\beta - \zeta + 1)}{\Gamma(\beta)}\int_{0}^{\infty}\frac{\Psi(x)}{x^{1-\zeta}}dx, \tag{5.47}$$

$$\mathfrak{D}^{\zeta} I_{\zeta}(\Psi)(t) = t^{1-\zeta}\frac{d}{dt}\int_{0}^{t}\frac{\Psi(x)}{x^{1-\zeta}}dx, \tag{5.48}$$

$$\mathfrak{D}^{\zeta} I_{\zeta}(\Psi)(t) = t^{1-\zeta}\frac{\Psi(t)}{t^{1-\zeta}}, \tag{5.49}$$

$$\mathfrak{D}^{\zeta} I_{\zeta}(\Psi)(t) = \Psi(t). \tag{5.50}$$

5.4 Computation

The fractional derivative of the exponential function: $\Psi(t) = e^{\lambda t}, \lambda \in c$

$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^k, \tag{5.51}$$

$$\mathfrak{D}^{ASK} e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathfrak{D}^{ASK} t^k. \tag{5.52}$$

From Eq. 5.11, we get:

$$\mathfrak{D}^{ASK} t^k = \mathfrak{D}^C t^k. \tag{5.53}$$

Let us now write Eq. 5.52 as:

$$\mathfrak{D}^{ASK} e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathfrak{D}^C t^k, \tag{5.54}$$

$$\mathfrak{D}^{ASK} e^{\lambda t} = \mathfrak{D}^C e^{\lambda t}. \tag{5.55}$$

Fractional Derivative of Sine and Cosine Functions: For sine function, we define: $\Psi(t) = \sin \omega t$ as:

$$\sin \omega t = \frac{1}{2i} \left(e^{i\omega t} - e^{-i\omega t} \right), \tag{5.56}$$

$$\mathfrak{D}^{ASK} \sin \omega t = \frac{1}{2i} \left(\mathfrak{D}^{ASK} e^{i\omega t} - \mathfrak{D}^{ASK} e^{-i\omega t} \right). \tag{5.57}$$

From Eq. 5.57, we obtain:

$$\mathfrak{D}^{ASK} \sin \omega t = \frac{1}{2i} \left(\mathfrak{D}^C e^{i^{i\omega t}} - \mathfrak{D}^C e^{-i^{i\omega t}} \right), \tag{5.58}$$

$$^{ASK}\mathfrak{D}^{\zeta} \sin \omega t =^{C} \mathfrak{D}^{\zeta} \frac{1}{2i} \left(e^{i\omega t} - e^{-i\omega t}\right), \tag{5.59}$$

$$^{ASK}\mathfrak{D}^{\zeta} \sin \omega t =^{C} \mathfrak{D}^{\zeta} \sin \omega t. \tag{5.60}$$

Similarly, we can prove the following for $\Psi(t) = \cos \omega t$:

$$^{ASK}\mathfrak{D}^{\zeta} \cos \omega t =^{C} \mathfrak{D}^{\zeta} \cos \omega t. \tag{5.61}$$

5.5 Numerical Validation

To validate our obtained results, we provide the following illustrative example:

Example 5.5.1. *Consider the following Riccati fractional differential equation (Yüzbaşı, 2013):*

$$\mathfrak{D}^{\zeta} y\,(x) + y^2(x) = 1, y\,(0) = 0, 0 < \zeta \leq 1. \tag{5.62}$$

Solution: *By applying Eq. 5.6, we obtain:*

$$\frac{\Gamma(\beta)}{\Gamma(\beta - \zeta + 1)} x^{1-\zeta} \frac{dy}{dx} + y^2(x) = 1, y\,(0) = 0, 0 < \zeta \leq 1. \tag{5.63}$$

To solve this equation at $\zeta = \frac{3}{4}$ and $\zeta = \frac{9}{10}$, the package of Wolfram Mathematica has been used to obtain the following:

$$y(x) = \frac{-1 + e^{\frac{8x^{\frac{3}{4}}}{3A}}}{1 + e^{\frac{8x^{\frac{3}{4}}}{3A}}}, \tag{5.64}$$

where $A = \frac{\Gamma(\beta)}{\Gamma(\beta+\frac{1}{4})}$ and $\beta = \zeta = \frac{3}{4}$ as in (Krishnasamy et al., 2017).

Table 5.1: Comparison of the results of the ASK with other works at $\zeta = \frac{3}{4}$

t	ASK	BPM	EHPM	IABMM	ComD
0	0	0	0	0	0
0.2	0.31439	0.30996891	0.3214	0.3117	0.37889
0.4	0.49848	0.48162749	0.5077	0.4855	0.58539
0.6	0.63022	0.59777979	0.6259	0.6045	0.72064
0.8	0.72609	0.67884745	0.7028	0.6880	0.81029
1.0	0.79618	0.73684181	0.7542	0.7478	0.87006

$$y(x) = \frac{-1 + e^{\frac{20x^{\frac{9}{10}}}{9A}}}{1 + e^{\frac{20x^{\frac{9}{10}}}{9A}}}, \tag{5.65}$$

where $A = \frac{\Gamma(\beta)}{\Gamma(\beta+\frac{1}{4})}$ *and* $\beta = \zeta = \frac{9}{10}$ *as in (Krishnasamy et al., 2017).*

5.6 Comparative Study

Some results for Riccati fractional differential equation are provided in Table (5.1) and Table (5.2) for different values of ζ, where parameters are taken as $\beta = \zeta$ (Krishnasamy et al., 2017). In Table 1, our results are compared with previous results using other methods and approaches such as Bernstein Polynomials Method (BPM) (Yüzbaşı, 2013), Enhanced Homotopy Perturbation Method (EHPM) (HosseinNia et al., 2008), Improved Adams-Bashforth-Moulton Method (IABMM) (Yüzbaşı, 2013), and ComD (Khalil et al., 2014) at $\zeta = \frac{3}{4}$.

In Table (5.2), our results are compared with previous results using other methods and approaches such as Bernstein Polynomials Method (BPM) (Yüzbaşı, 2013), Modified Homotopy Perturbation Method (MHPM) (HosseinNia et al., 2008), Improved Adams-Bashforth-Moulton Method (IABMM) (Yüzbaşı, 2013), and ComD (Khalil et al., 2014) at $\zeta = \frac{9}{10}$.

Table 5.2: Comparison of the results of the ASK with other works at $\zeta = \frac{9}{10}$

t	ASK	BPM	MHPM	IABMM	ComD
0	0	0	0	0	0
0.2	0.23952	0.23878798	0.2391	0.2393	0.25526
0.4	0.42667	0.42258214	0.4229	0.4234	0.45191
0.6	0.57607	0.56617082	0.5653	0.5679	0.60539
0.8	0.69138	0.67462642	0.6740	0.6774	0.72063
1.0	0.77780	0.75460256	0.7569	0.7584	0.80445

5.7 Discussion of Results

It is noticeable from the above Table (5.1) and Table (5.2) that our results are in a good agreement with BPM, MHPM, EHPM, and IABMM results. In addition, the ComD (Khalil et al., 2014) has been used to solve fractional Riccati differential equation. However, the results of conformable derivative do not coincide with other works and our present results. Therefore, the obtained results that have been calculated analytically via ASK are in good agreement with other methods. However, in comparison with ComD, the present results are better than ComD's results as suggested in (Khalil et al., 2014). In Fig. (5.1), the absolute relative error shows that the present result of Riccati fractional differential equation is exactly obtained at $\alpha = 1$ in (Yüzbaşı, 2013), by comparing it with $\alpha = \frac{3}{4}$ using the proposed definition and the conformable one. The figure shows a good accuracy for the results of the proposed definition in comparison with the conformable one. A similar situation is provided in Fig. (5.2) at $\alpha = \frac{9}{10}$.

5.8 Conclusion

In this chapter, ASK derivative has been suggested to provide more advantages than other classical Cp and RL definitions such as the derivative of two functions,

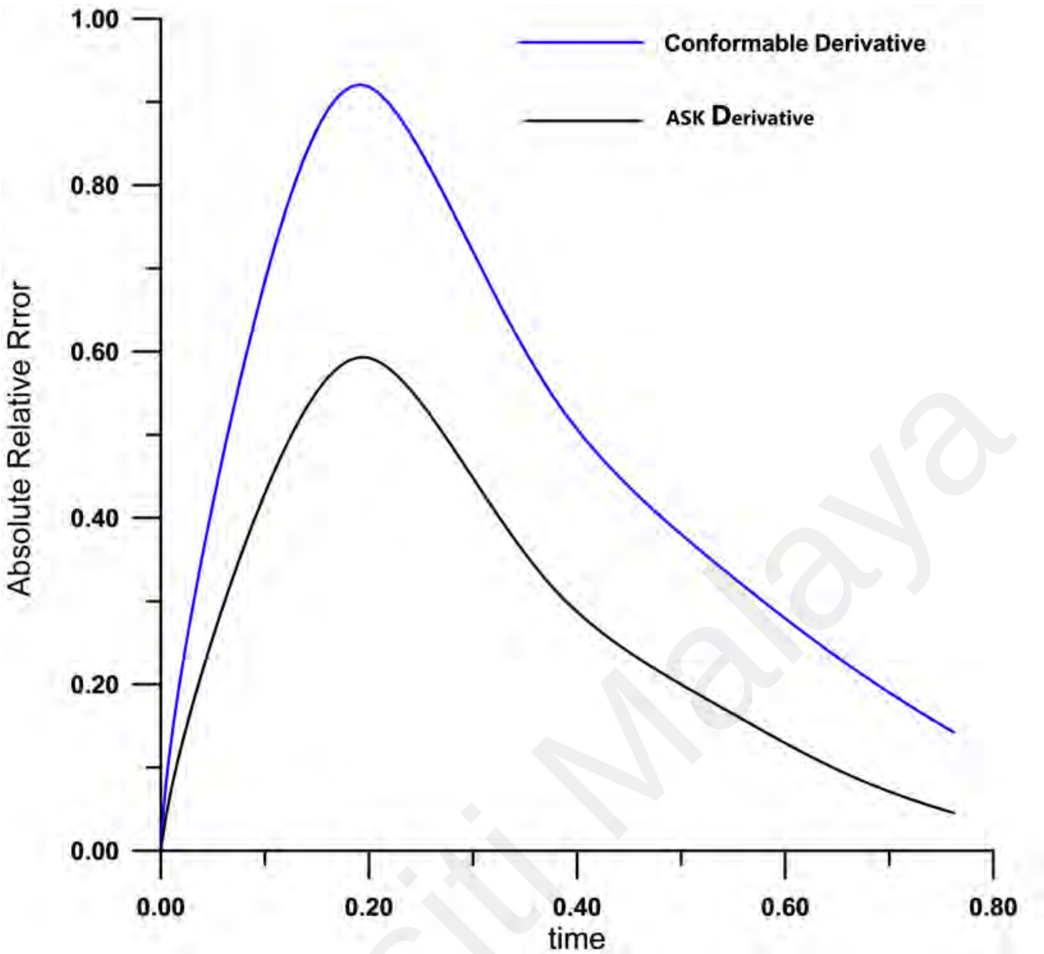

Figure 5.1: The absolute relative error is plotted for Riccati fractional differential equation for the conformable derivative and ASK derivative at $\zeta = 0.75$

the derivative of the quotient of two functions, the Rolle's theorem, and the mean value theorem which have been satisfied in ASK. The present definition satisfies: $\mathfrak{D}^{\zeta}\mathfrak{D}^{\beta}\Psi(t) = \mathfrak{D}^{\zeta+\beta}\Psi(t)$ for a differentiable function: $\Psi(t)$ expanded by Taylor series. The fractional integral is introduced. Compatible results with Cp and RL results have been obtained for functions that are given in sections 5.3 and 5.4. Also, a comparison with ComD is studied. We conclude that our proposed ASK definition gives a new direction for solving fractional differential equations in a simple manner in which the results of the Cp and RL definitions are exactly deduced. In future

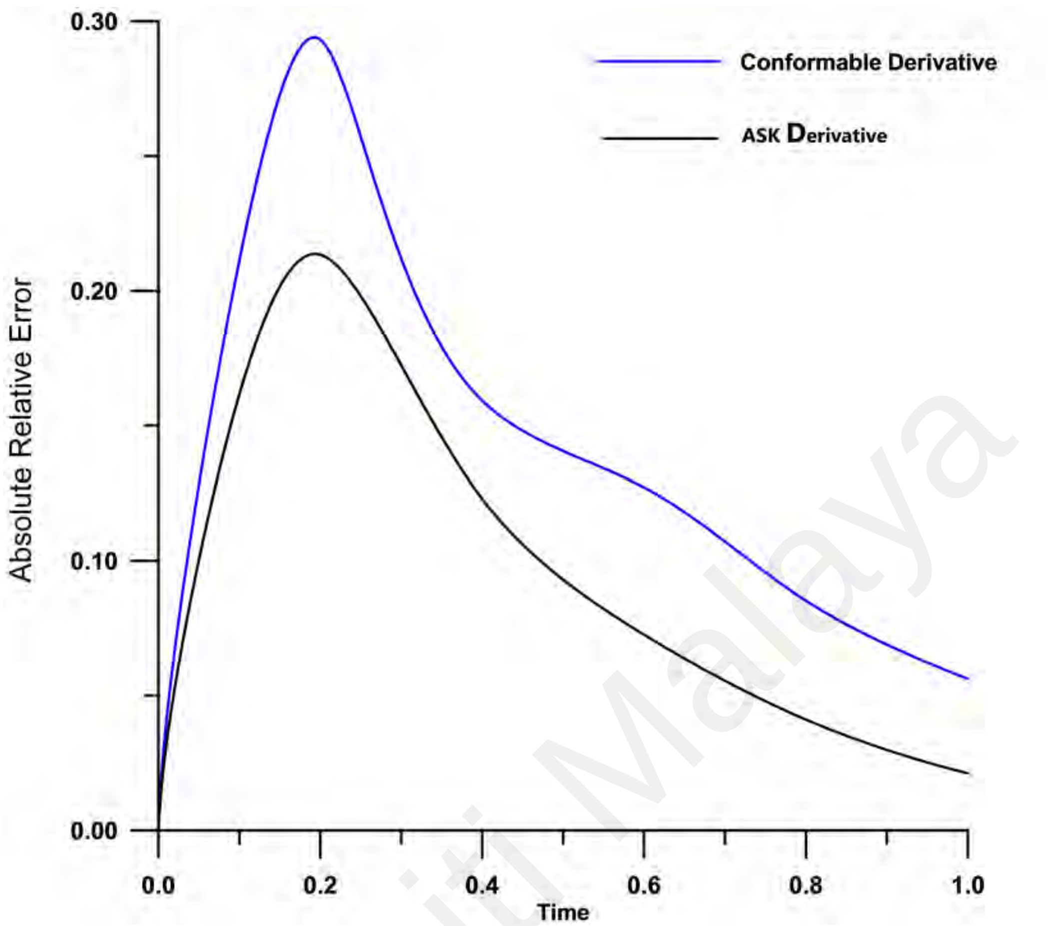

Figure 5.2: The absolute relative error is plotted for Riccati fractional differential equation for the conformable derivative and ASK derivative at $\zeta = 0.90$

study, a full application example of the Chebyshev differential equation of first kind will be studied in the context of ASK definition by establishing the drivability and integrability results of the sum function of functional power series, introducing the generalized fractional power series technique, studying solution's existence around an ordinary point of a homogeneous sequential linear generalized fractional differential equation of order 2ζ, and applying the proposed technique to study the series solutions including the properties of the generalized fractional Chebyshev polynomials.

CHAPTER 6: INVESTIGATION OF THE EXISTENCE AND ULAM–HYERS–RASSIAS STABILITY OF SOLUTIONS TO THE IMPLICIT NONLINEAR FBVP IN THE VARIABLE ORDER SETTINGS

6.1 Introduction

In this chapter, the solutions' existence and its stability to the fractional boundary value problem (FBVP) are studied for an implicit nonlinear variable order fractional differential equation (VOFDfEq). The FrCL of variable order extends the theory of the constant order one. The order of a system varies continuously to describe the changes of memory property with space or time (Baleanu et al., 2011). Bouazza et al. (2021) studied the multi-term variable order fractional boundary value problem (VOFBVP) by showing that there exists exactly one solution to such a system under some conditions. In (X. Li et al., 2020), by proposing a novel kernel function via polynomial form, a general structure of Atangana-Baleanu VOFBVPs was studied. Derakhshan (2021) solved a Cp linear time-fractional VOFDfEq arising in fluid mechanics and displayed the existence-uniqueness-stability. Refice et al. (2021) carefully studied the Hadamard VOFBVP and derived solutions via the Kuratowski measure of noncompactness (KMNC) technique. Recently, a few contributions to the solutions of fractional constant order BVPs have been previously provided. However, the solutions' existence to FBVPs of variable order have been rarely studied (see (Sousa & de Oliveira, 2018; Tavares et al., 2016; Yang et al., 2018)).

Inspired by all mentioned works along with the paper (Benchohra & Lazreg, 2014), we investigate the solutions to the following FBVP for implicit nonlinear VOFDfEq:

$$\begin{cases} \mathfrak{D}_{0^+}^{u(t)} x(t) = m(t, x(t), \mathfrak{D}_{0^+}^{u(t)} x(t)), \\ \\ x(0) = 0, \ \ x(\Omega) = 0, \ \ t \in \mathfrak{J} := [0, \Omega], \end{cases} \tag{6.1}$$

where $0 < \Omega < +\infty$, $u(t) : \mathfrak{J} \to (1,2]$, $m : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a CF, and $\mathfrak{J}_{0^+}^{u(t)}$, $\mathfrak{D}_{0^+}^{u(t)}$ are the Riemann-Liouville fractional (RLFr) integral and derivative in the context of variable order $u(t)$.

All existence criteria in our investigation are derived via Krasnoselskii's fixed point theorem (KFPThm), and then its Ulam–Hyers–Rassias (U-H-R) stability is also verified.

6.2 Essential Notions

Some important notions are presented to be used later in our results.

By $\mathfrak{C}(\mathfrak{J}, \mathbb{R})$, we illustrate the Banach space (BS) of CF from $\mathfrak{J}$ into $\mathbb{R}$ via

$$\|x\| = \sup\{|x(t)| : \ t \in \mathfrak{J}\}.$$

Definition 15. *(Samko, 1995; Samko & Ross, 1993; Valério & Da Costa, 2011) Let $-\infty < c < d < +\infty$, and $u(t) : [c,d] \to (0,+\infty)$, the left RLFr integral in the context of variable order $u(t)$ for $h(t)$ is expressed as:*

$$\mathfrak{J}_{c^+}^{u(t)} h(t) = \int_c^t \frac{(t-w)^{u(t)-1}}{\Gamma(u(t))} h(w)dw, \ \ t > c, \tag{6.2}$$

where the gamma function is denoted by $\Gamma(.)$.

Definition 16. *(Samko, 1995; Samko & Ross, 1993; Valério & Da Costa, 2011) Let $-\infty < c < d < +\infty, \ r \in \mathbb{N}$ and $u(t) : [c,d] \to (r-1, r)$; the left RLFr derivative in the context of variable order $u(t)$ for $h(t)$ is expressed as:*

$$\mathfrak{D}_{c^+}^{u(t)} h(t) = \left(\frac{d}{dt}\right)^r \mathfrak{J}_{c^+}^{r-u(t)} h(t) = \left(\frac{d}{dt}\right)^r \int_c^t \frac{(t-w)^{r-u(t)-1}}{\Gamma(r-u(t))} h(w)dw, \ \ t > c. \tag{6.3}$$

Obviously, if $u(t)$ is a constant function $u \in \mathbb{R}$, then the variable order Riemann-Liouville fractional (VORLFr) derivative (6.3) and integral (6.2) are the usual RLFr derivative and integral, respectively (see (Kilbas et al., 2006; Samko, 1995; Samko & Ross, 1993)). Some essential properties are provided as follows:

Lemma 1. *(Kilbas et al., 2006) Assume that* $\delta > 0$*, then*

$$\mathfrak{D}^{\delta}_{c^+} h = 0$$

has a unique solution

$$h(t) = \omega_1 (t-c)^{\delta-1} + \omega_2 (t-c)^{\delta-2} + \ldots + \omega_r (t-c)^{\delta-r}$$

$\omega_j \in \mathbb{R}$, $j = 1, 2, \ldots, r$, *here* $r - 1 < \delta \leq r$.

Lemma 2. *(Kilbas et al., 2006) Let* $c > 0$, $h \in L(c, d)$, $\mathfrak{D}^{\delta}_{c^+} h \in L(c, d)$, *then*

$$\mathfrak{I}^{\delta}_{c^+} \mathfrak{D}^{\delta}_{c^+} h(t) = h(t) + \omega_1 (t-c)^{\delta-1} + \omega_2 (t-c)^{\delta-2} + \ldots + \omega_r (t-c)^{\delta-r}$$

$\omega_j \in \mathbb{R}$, $j = 1, 2, \ldots, r$, *here* $r - 1 < \delta \leq r$.

Lemma 3. *(Kilbas et al., 2006) Let* $\delta > 0$*, then we get:*

$$\mathfrak{D}^{\delta}_{c^+} \mathfrak{I}^{\delta}_{c^+} h(t) = h(t).$$

Lemma 4. *(Kilbas et al., 2006) Let* $\delta,\ \beta > 0$*, then we get:*

$$\mathfrak{I}^{\delta}_{c^+} \mathfrak{I}^{\beta}_{c^+} h(t) = \mathfrak{I}^{\beta}_{c^+} \mathfrak{I}^{\delta}_{c^+} h(t) = \mathfrak{I}^{\delta+\beta}_{c^+} h(t).$$

Remark 7. *(S. Zhang, 2013; S. Zhang & Hu, 2019; S. Zhang et al., 2019) For* *general functions $u(t)$, $v(t)$, it is noticeable that the semigroup property is invalid, i.e:*

$$\mathfrak{I}_{c^+}^{u(t)}\mathfrak{I}_{c^+}^{v(t)}h(t) \neq \mathfrak{I}_{c^+}^{u(t)+v(t)}h(t).$$

Example 6.2.1. *Let*

$$u(t) = \frac{t^2}{3},\ \ t \in [0,4], \quad v(t) = \begin{cases} 3,\ t \in [0,1] \\ 2,\ t \in]1,4]. \end{cases} \quad h(t) = 2,\ \ t \in [0,4].$$

Then

$$\begin{aligned}
\mathfrak{I}_{0^+}^{u(t)}\mathfrak{I}_{0^+}^{v(t)}h(t) &= \int_0^t \frac{(t-w)^{u(t)-1}}{\Gamma(u(t))}\int_0^w \frac{(w-\tau)^{v(w)-1}}{\Gamma(v(w))}h(\tau)d\tau dw \\
&= \int_0^t \frac{(t-w)^{u(t)-1}}{\Gamma(u(t))}[\int_0^1 \frac{(w-\tau)^2}{\Gamma(3)}2d\tau + \int_1^w \frac{(w-\tau)}{\Gamma(2)}2d\tau]dw \\
&= \int_0^t \frac{(t-w)^{u(t)-1}}{\Gamma(u(t))}[\frac{(w-1)^3}{3} + 2w - 1]dw,
\end{aligned}$$

and

$$\mathfrak{I}_{0^+}^{u(t)+v(t)}h(t) = \int_0^t \frac{(t-w)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))}h(w)dw.$$

It is clear that

$$\begin{aligned}
\mathfrak{I}_{0^+}^{u(t)}\mathfrak{I}_{0^+}^{v(t)}h(t)|_{t=3} &= \int_0^3 \frac{(3-w)^2}{\Gamma(3)}[\frac{(w-1)^3}{3} + 2w - 1]dw \\
&= \frac{1}{2}\int_0^3 (\frac{w^5}{3} - 3w^4 + 12w^3 - \frac{85}{3}w^2 + 35w - 12)dw \\
&= \frac{21}{10},
\end{aligned}$$

and

$$
\begin{aligned}
\mathfrak{I}_{0^+}^{u(t)+v(t)} h(t)|_{t=3} &= \int_0^3 \frac{(3-w)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))} h(w)dw \\
&= \int_0^1 \frac{(3-w)^5}{\Gamma(6)} 2dw + \int_1^3 \frac{(3-w)^4}{\Gamma(5)} 2dw \\
&= \frac{1}{60}\int_0^1 (-w^5+15w^4-90w^3+270w^2-405w+243)dw \\
&+ \frac{1}{12}\int_1^3 (w^4-12w^3+54w^2-108w+81)dw \\
&= \frac{665}{360}+\frac{32}{60}=\frac{857}{360}.
\end{aligned}
$$

Therefore, we obtain

$$
\mathfrak{I}_{0^+}^{u(t)}\mathfrak{I}_{0^+}^{v(t)} h(t)|_{t=3} \neq I_{0^+}^{u(t)+v(t)} h(t)|_{t=3}.
$$

Lemma 5. *(S. Zhang et al., 2018) Let* $u : \mathfrak{J} \to (1,2]$ *be a CF. Then for*

$$
y \in \mathfrak{C}_\zeta(\mathfrak{J},\mathbb{R}) = \{y(t) \in \mathfrak{C}(\mathfrak{J},\mathbb{R}),\ t^\zeta y(t) \in \mathfrak{C}(\mathfrak{J},\mathbb{R})\},\ (0 \le \zeta \le \min_{t\in\mathfrak{J}} |u(t)|)
$$

the variable order (VO) fractional integral $\mathfrak{I}_{0^+}^{u(t)} y(t)$ *exists for any points on* $\mathfrak{J}$.

Lemma 6. *(S. Zhang et al., 2018) Assume that* $u : \mathfrak{J} \to (1,2]$ *be a CF, then* $\mathfrak{I}_{0^+}^{u(t)} y(t) \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$ *for any* $y \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$.

Definition 17. *(An & Chen, 2019; S. Zhang, 2018; S. Zhang & Hu, 2020) The set* $\mathfrak{J}$ *in* $\mathbb{R}$ *is named as a generalized interval (G-interval)if either it is an standard interval, a point* $\{c_1\}$*, or the empty set* $\emptyset$.

Definition 18. *(An & Chen, 2019; S. Zhang, 2018; S. Zhang & Hu, 2020) If* $\mathfrak{J}$ *is a*

G-interval, then the finite set $\mathcal{P}$ of G-intervals belonging to $\mathfrak{I}$ is a partition of $\mathfrak{I}$ whenever each x contained in $\mathfrak{I}$ lies in exactly one of the G-intervals.

Let E be a BS as follows:

Definition 19. *(An & Chen, 2019; S. Zhang, 2018; S. Zhang & Hu, 2020) Assume that $\mathfrak{I}$ is a G-interval, $g : \mathfrak{I} \to \mathbb{R}$ a mapping, $\mathcal{P}$ a partition of $\mathfrak{I}$. In this case, g is a piecewise constant by terms of $\mathcal{P}$ if for every $E \in \mathcal{P}$, g is constant on E.*

Theorem 23. *(Kilbas et al., 2006) (KFPThm) Suppose that S is a closed, convex, bounded subset of E and suppose that W_1 and W_2 are operators on S satisfy:*
(i) $W_1(S) + W_2(S) \subset S$,
(ii) W_1 is continuous on S and $W_1(S)$ is relatively compact in E,
(iii) W_2 is a strict contraction on S, that is; $\exists\, k \in [0, 1)$ s.t.

$$\|W_2(x) - W_2(y)\| \leq k\|x - y\|$$

for every $x, y \in S$.
Then, $\exists\, x \in S$ s.t. $W_1(x) + W_2(x) = x$.

Definition 20. *(Rus, 2010) (U-H-R stability) The equation of (6.1) is U-H-R stable w.r.t $\varphi \in \mathfrak{C}(\mathfrak{J}, \mathbb{R}_+)$ if $\exists\, a_m > 0$ s.t. $\forall\, \epsilon > 0$ and $\forall\, z \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ satisfying*

$$|\mathfrak{D}_{0^+}^{u(t)} z(t) - m(t, z(t), \mathfrak{I}_{0^+}^{u(t)} z(t))| \leq \epsilon\varphi(t), \;\; t \in \mathfrak{J},$$

$\exists\, x \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ as a solution of equation (6.1) with

$$|z(t) - x(t)| \leq a_m \epsilon\varphi(t), \;\; t \in \mathfrak{J}.$$

6.3 Existence of Solutions

Some assumptions are presented as follows:

(H1) Let $r \in \mathbb{N}$, $\mathcal{P} = \{\mathfrak{J}_1 := [0, \Omega_1], \mathfrak{J}_2 := (\Omega_1, \Omega_2], \mathfrak{J}_3 := (\Omega_2, \Omega_3], ...\mathfrak{J}_r := (\Omega_{r-1}, \Omega_r]\}$ be a partition of $\mathfrak{J}$, and $u(t) : \mathfrak{J} \to (1, 2]$ be a piecewise constant mapping by terms of $\mathcal{P}$, i.e.,

$$u(t) = \sum_{j=1}^{r} u_j \mathfrak{J}_j(t) = \begin{cases} u_1, & if\ t \in \mathfrak{J}_1, \\ u_2, & if\ t \in \mathfrak{J}_2, \\ . \\ . \\ . \\ u_r, & if\ t \in \mathfrak{J}_r, \end{cases}$$

in which $1 < u_j \leq 2$ belong to $\mathbb{R}$, and $\mathfrak{J}_j$ is the indicator of $\mathfrak{J}_j := (\Omega_{j-1}, \Omega_j],\ j = 1, 2, ..., r$, $(\Omega_0 = 0,\ \Omega_r = T)$ s.t.

$$\mathfrak{J}_j(t) = \begin{cases} 1, & for\ t \in \mathfrak{J}_j, \\ 0, & for\ elsewhere. \end{cases}$$

(H2) Assume that $t^{\zeta} m : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a CF: $(0 \leq \zeta \leq \min_{t \in \mathfrak{J}} |(u(t))|)$, there exist constants, $K, L > 0$, s.t. $t^{\zeta} |m(t, y_1, z_1) - m(t, y_2, z_2)| \leq K|y_1 - y_2| + L|z_1 - z_2|$, for any $y_j, z_j \in \mathbb{R}$ and $t \in \mathfrak{J}$.

By $E_j = \mathfrak{C}(\mathfrak{J}_j, \mathbb{R})$, we denote the BS of CFs from $\mathfrak{J}_j$ into $\mathbb{R}$ with the norm

$$\|x\|_{E_j} = \sup_{t \in \mathfrak{J}_j} |x(t)|,$$

where $j \in \{1, 2, ..., r\}$.

We firstly do the analysis of the FBVP (6.1) to obtain novel results.

By (6.3), FDfEq of FBVP (6.1) can be written as:

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-w)^{1-u(t)}}{\Gamma(2-u(t))}x(w)dw = m(t, x(t), \mathfrak{D}_{0^+}^{u(t)}x(t)), \quad t \in \mathfrak{J}. \tag{6.4}$$

According to $(H1)$, the equation(6.4) on $\mathfrak{J}_j$ can be represented by

$$\frac{d^2}{dt^2}\left(\int_0^{\Omega_1} \frac{(t-w)^{1-u_1}}{\Gamma(2-u_1)}x(w)dw+...+\int_{\Omega_{j-1}}^t \frac{(t-w)^{1-u_j}}{\Gamma(2-u_j)}x(w)dw\right)=m(t,x(t),\mathfrak{D}_{0^+}^{u_j}x(t)), \quad t\in\mathfrak{J}_j, \tag{6.5}$$

for , $j = 1, 2, ..., r$. The solution of the supposed FBVP (6.1) is presented due to its essential role in our results as follows:

Definition 21. *The FBVP (6.1) is said to have a solution, if* $\exists x_j \in \mathfrak{C}([0, \Omega_j], \mathbb{R})$ *satisfying equation (6.5) and* $x_j(0) = 0 = x_j(\Omega_j)$.

From the above, the FDfEq of FBVP (6.1) can be indicated as the FDfEq (6.4), which can be formulated on $\mathfrak{J}_j$, $j \in \{1, 2, ..., r\}$ as (6.5). For $0 \le t \le \Omega_{j-1}$, we set $x(t) \equiv 0$, then (6.5) is illustrated as follows:

$$\mathfrak{D}_{\Omega_{j-1}^+}^{u_j}x(t) = m(t, x(t), \mathfrak{D}_{\Omega_{j-1}^+}^{u_j}x(t)), \quad t \in \mathfrak{J}_j.$$

Let us now regard the following equivalent standard FBVP:

$$\begin{cases} \mathfrak{D}_{\Omega_{j-1}^+}^{u_j}x(t) = m(t, x(t), \mathfrak{D}_{\Omega_{j-1}^+}^{u_j}x(t)), \\ x(\Omega_{j-1}) = 0, x(\Omega_j) = 0, \quad t \in \mathfrak{J}_j. \end{cases} \tag{6.6}$$

For the existence of solutions to the equivalent standard FBVP (6.6), an auxiliary lemma is indicated by follows:

Lemma 7. *$x \in E_i$ is the solution to the equivalent standard FBVP (6.6) iff it satisfies*

$$x(t) = -(\Omega_j - \Omega_{j-1})^{1-u_j}(t - \Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(t), \quad t \in \mathfrak{J}_j. \quad (6.7)$$

where

$$y(t) \;=\; m\Big(t, -(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(\Omega_j))+\mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(t), y(t)\Big), \quad t\in\mathfrak{J}_j.$$

Proof. Let $x \in E_i$ be a solution to the equivalent standard FBVP (6.6). Now, we take $\mathfrak{D}^{u_j}_{\Omega^+_{j-1}} x(t) = y(t)$ and apply $\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}$ to both sides of the FDfEq of the equivalent standard FBVP (6.6). By Lemma (2), we have:

$$x(t) = \omega_1(t - \Omega_{j-1})^{u_j-1} + \omega_2(t - \Omega_{j-1})^{u_j-2} + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(t), \quad t \in \mathfrak{J}_j.$$

By $x(\Omega_{j-1}) = 0$ and the given assumption for the mapping m, we obtain $\omega_2 = 0$. Assume that $x(t)$ satisfy $x(\Omega_j) = 0$, thus we get: $\omega_1 = -(\Omega_j - \Omega_{j-1})^{1-u_j}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)$. Then, we have:

$$x(t) \;=\; -(\Omega_j - \Omega_{j-1})^{1-u_j}(t - \Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(\Omega_j) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(t), \quad t \in \mathfrak{J}_j,$$

where

$$y(t) \;=\; m\Big(t, -(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)+\mathfrak{I}^{u_j}_{\Omega^+_{j-1}} y(t), y(t)\Big), \quad t\in\mathfrak{J}_j.$$

Conversely, assume that $x \in E_j$ satisfies the integral Eq. (6.7) as its solution. Then, according to the continuity of $t^{\zeta} m$ and Lemma (3), x is a solution to the equivalent standard FBVP (6.6).

Our existence result is derived with the help of Theorem 23.

Theorem 24. *Suppose that* $(H1)$ *and* $(H2)$ *are fulfilled and by assuming*

$$\frac{(\Omega_j - \Omega_{j-1})^{u_j-1}({\Omega_j}^{1-\zeta} - {\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)}\Big(\frac{K(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} + \frac{L}{2}\Big) < \frac{1}{4}, \tag{6.8}$$

the FBVP (6.1) admits a solution on E.

Proof. In the first place, convert the equivalent standard FBVP (6.6) to a fixed point problem. Consider the operators:

$$W_1, W_2 : E_j \longrightarrow E_j$$

defined by:

$$W_1 y(t) = -(\Omega_j - \Omega_{j-1})^{1-u_j}(t - \Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j), \quad W_2 y(t) = \Im^{u_j}_{\Omega^+_{j-1}} y(t), \tag{6.9}$$

in which

$$y(t) \;=\; m(t, x(t), y(t)).$$

It is followed, from the specifications of fractional operators and in view of the continuity of $t^{\zeta}m$, that the operators $W_1, W_2 : E_j \to E_j$ illustrated in (6.9) are well–defined. Let

$$R_j \;\geq\; \frac{\frac{2m^*(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j)}}{1 - \frac{2(\Omega_j-\Omega_{j-1})^{u_j-1}({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)}\Big(\frac{2K(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} + L\Big)},$$

where

$$m^* = \sup_{t \in \mathfrak{J}_j} |m(t, 0, 0)|.$$

We consider the set:

$$B_{R_j} = \{x \in E_j,\ \|x\|_{E_j} \leq R_j\}.$$

Obviously, B_{R_j} is nonempty, bounded, convex and closed.

Let us prove that W_1, W_2 satisfy Theorem's (23) assumption. The proof is divided into 4 steps.

Step 1: $W_1(B_{R_j}) + W_2(B_{R_j}) \subseteq (B_{R_j})$.

Let $y \in B_{R_j}$, we show that $W_1(y) + W_2(y) \in B_{R_j}$.

For $t \in \mathfrak{J}_j$, we have:

$$
\begin{aligned}
&|(W_1 y)(t)+(W_2 y)(t)| \leq \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \\
&\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}\left|m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y_r(\Omega_j))+\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(w),y(w)\right)\right|dw \\
+ \quad & \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}\left|m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j))+\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(w),y(w)\right)\right|dw \\
\leq \quad & \frac{2}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}\left|m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j))+\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(w),y(w)\right)\right|dw \\
\leq \quad & \frac{2}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}\left|m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j))+\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(w),y(w)\right)\right. \\
- \quad & \left. m(w,0,0)\right|dw+\frac{2}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}|m(w,0,0)|dw \\
\leq \quad & \frac{2(\Omega_j-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{\Omega_j}w^{-\zeta}(K|-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j))+\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(w)| \\
+ \quad & L|y(w)|)dw+\frac{2m^*(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j)} \\
\leq \quad & \frac{2(\Omega_j-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{\Omega_j}w^{-\zeta}\left(K(|\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j))|+|\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(w)|)+L|y(w)|\right)dw+\frac{2m^*(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j)} \\
\leq \quad & \frac{2(\Omega_j-\Omega_{j-1})^{u_j-1}({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)}\left(2K\|\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y\|_{E_j}+L\|y\|_{E_j}\right)+\frac{2m^*(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j)} \\
\leq \quad & \frac{2(\Omega_j-\Omega_{j-1})^{u_j-1}({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)}\left(\frac{2K(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)}+L\right)R_j+\frac{2m^*(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j)} \\
\leq \quad & R_j,
\end{aligned}
$$

which means that $W_1(B_{R_j}) + W_2(B_{R_j}) \subseteq B_{R_j}$.

Step 2: W_1 is continuous

Let (y_r) be a sequence s.t. $y_r \to y$ in E_j. Then, $\forall\, t \in \mathfrak{J}_j$, we obtain:

$$
\begin{aligned}
&|(W_1 y_r)(t)-(W_1 y)(t)| \le \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \\
&\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}\left|m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y_r(\Omega_j))+\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y_r(w),y_r(w)\right)\right. \\
- \quad & \left. m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j))+\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}y(w),y(w)\right)\right|dw \\
\le \quad & \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}w^{-\zeta} \\
\times \quad & \left(K(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}|y_r(\Omega_j)-y(\Omega_j)|+K\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}|y_r(w)-y(w)|+L|y_r(w)-y(w)|\right)dw \\
\le \quad & \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}w^{-\zeta}\left(2K\mathfrak{J}^{u_j}_{\Omega^+_{j-1}}\|y_r-y\|_{E_j}+L\|y_r-y\|_{E_j}\right)dw \\
\le \quad & \frac{(\Omega_j{}^{1-\zeta}-\Omega_{j-1}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)}\left(\frac{2K(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)}\|y_r-y\|_{E_j}+L\|y_r-y\|_{E_j}\right) \\
\le \quad & \frac{(\Omega_j{}^{1-\zeta}-\Omega_{j-1}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)}\left(\frac{2K(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)}+L\right)\|y_r-y\|_{E_j}.
\end{aligned}
$$

Thus

$$\|(W_1 y_r)-(W_1 y)\|_{E_j} \to 0 \ \ as \ \ r \to \infty.$$

As a result, W_1 is continuous on E_j.

Step 3: $W_1(B_{R_j})$ is relatively compact

Let us now prove that $W_1(B_{R_j})$ is relatively compact. Obviously, $W_1(B_{R_j})$ has the uniform boundedness, since by Step 2, $W_1(B_{R_j}) = \{W_1(x) : x \in B_{R_j}\} \subset W_1(B_{R_j}) + W_2(B_{R_j}) \subseteq B_{R_j}$. Thus, for each $x \in B_{R_j}$, we have: $\|W_1(x)\|_{E_j} \le R_j$ which means that $W_1(B_{R_j})$is uniformly bounded. Lastly, it is necessary that we verify that $W_1(B_{R_j})$ is equicontinuous. For $t_1, t_2 \in \mathfrak{J}_j$ and $y \in B_{R_j}$, we estimate

$(t_1 < t_2)$:

$$\begin{aligned}
&|(W_1y)(t_2)-(W_1y)(t_1)|\leq \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}}{\Gamma(u_j)}\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-T_{j-1})^{u_j-1}\right)\\
&\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}\left|m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j))+\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(w),y(w)\right)\right|dw\\
\leq\ & \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}}{\Gamma(u_j)}\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-\Omega_{j-1})^{u_j-1}\right)\\
&\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}\left|m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j))+\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(w),y(w)\right)-m(w,0,0)\right|dw\\
+\ & \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}}{\Gamma(u_j)}\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-\Omega_{j-1})^{u_j-1}\right)\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}|m(w,0,0)|dw\\
\leq\ & \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}}{\Gamma(u_j)}\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-\Omega_{j-1})^{u_j-1}\right)\\
&\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}w^{-\zeta}\left(K(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}|\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j))+\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(w)|+L|y(w)|\right)dw\\
+\ & \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}m^*}{\Gamma(u_j)}\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-\Omega_{j-1})^{u_j-1}\right)\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}w^{-\zeta}dw\\
\leq\ & \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}}{\Gamma(u_j)}\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-\Omega_{j-1})^{u_j-1}\right)\\
&\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}w^{-\zeta}\left(2K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\|y\|_{E_j}+L\|y\|_{E_j}\right)dw\\
+\ & \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}m^*}{\Gamma(u_j)}\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-\Omega_{j-1})^{u_j-1}\right)\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}w^{-\zeta}dw\\
\leq\ & \frac{({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)}\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-\Omega_{j-1})^{u_j-1}\right)\left(2K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\|y\|_{E_j}+L\|y\|_{E_j}\right)\\
+\ & \frac{({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})m^*}{(1-\zeta)\Gamma(u_j)}\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-\Omega_{j-1})^{u_j-1}\right)\\
\leq\ & \frac{({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)}\left(2K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\|y\|_{E_j}+L\|y\|_{E_j}+m^*\right)\left((t_2-\Omega_{j-1})^{u_j-1}-(t_1-\Omega_{j-1})^{u_j-1}\right).
\end{aligned}$$

Hence, $|(W_1y)(t_2)-(W_1y)(t_1)| \to 0$ as $|t_2-t_1| \to 0$. It implies that $W_1(B_{R_j})$ is equicontinuous.

Step 4: W_2 is a strict contraction on B_{R_j}

For $y_1(t), y_2(t) \in B_{R_i}$, we obtain that

$$
\begin{aligned}
& |(W_2y_2)(t)-(W_2y_1)(t)| \\
\leq\ & \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}\Bigg|m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y_2(\Omega_j))+\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y_2(w),y_2(w)\right) \\
-\ & m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y_1(\Omega_j))+\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y_1(w),y_1(w)\right)\Bigg|dw \\
\leq\ & \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}w^{-\zeta}\Bigg(K(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}|y_2(\Omega_j)-y_1(\Omega_j)| \\
+\ & K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}|y_2(w)-y_1(w)|+L|y_2(w)-y_1(w)|\Bigg)dw \\
\leq\ & \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}w^{-\zeta}\left(2K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\|y_2-y_1\|_{E_j}+L\|y_2-y_1\|_{E_j}\right)dw \\
\leq\ & \frac{({\Omega_j}^{1-\zeta}-\Omega_{j-1}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)}\left(\frac{2K(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)}\|y_2-y_1\|_{E_j}+L\|y_2-y_1\|_{E_j}\right) \\
\leq\ & \frac{({\Omega_j}^{1-\zeta}-\Omega_{j-1}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)}\left(\frac{2K(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)}+L\right)\|y_2-y_1\|_{E_j}.
\end{aligned}
$$

Consequently, by (6.8), W_2 is a strict contraction. Hence, by KFPThm, $\exists$ $\widetilde{x_j} \in B_{R_j}$ s.t. $W_1(x) + W_2(x) = x$, which is the equivalent standard problem's (6.6) solution.

We let

$$
x_j = \begin{cases} 0, & t \in [0, \Omega_{j-1}], \\ \widetilde{x}_j, & t \in \mathfrak{I}_j. \end{cases} \tag{6.10}
$$

On the other side, it is known that $x_j \in \mathfrak{C}([0, \Omega_j], \mathbb{R})$ given by (6.10) fulfills

$$
\frac{d^2}{dt^2}\left(\int_0^{\Omega_1}\frac{(t-w)^{1-u_1}}{\Gamma(2-u_1)}x_j(w)dw+\cdots+\int_{\Omega_{j-1}}^{t}\frac{(t-w)^{1-u_j}}{\Gamma(2-u_j)}x_j(w)dw\right)=m(w,x_j(w),\mathfrak{D}^{u_j}_{0^+}x_j(w)),
$$

for $t \in \mathfrak{J}_j$, which indicates that x_j will be a solution to equation (6.5) furnished with $x_j(0) = 0,\ x_j(\Omega_j) = \widetilde{x}_j(\Omega_j) = 0$.

Then,

$$x(t) = \begin{cases} x_1(t), \quad t \in \mathfrak{J}_1, \\ x_2(t) = \begin{cases} 0, \quad t \in \mathfrak{J}_1, \\ \widetilde{x}_2, \quad t \in \mathfrak{J}_2 \end{cases} \\ . \\ . \\ . \\ x_r(t) = \begin{cases} 0, \quad t \in [0, \Omega_{j-1}], \\ \widetilde{x}_j, \quad t \in \mathfrak{J}_j \end{cases} \end{cases} .$$

is the solution for the main variable order FBVP (6.1).

6.4 U-H-R Stability

We study a general form of such a notion in sense of U-H-R.

Theorem 25. *Consider (H1), (H2), (6.8) and assume:*

(H3) $\varphi \in \mathfrak{C}(\mathfrak{J}_j, \mathbb{R}_+)$ *is increasing and* $\exists\ \lambda_\varphi > 0$ *s.t.* $\forall\, t \in \mathfrak{J}_j$*, we get:*

$$\mathfrak{I}^{u_j}_{\Omega_{j-1}^+}\varphi(t) \leq \lambda_{\varphi(t)}\varphi(t).$$

then, the given implicit nonlinear VOFBVP (6.1) *is U-H-R stable w.r.t* φ*.*

Proof. Assume that $z \in \mathfrak{C}(\mathfrak{J}_j, \mathbb{R})$ is an inequality's solution as follows:

$$|\mathfrak{D}^{u_j}_{\Omega_{j-1}^+}z(t) - m(t, z(t), \mathfrak{D}^{u_j}_{\Omega_{j-1}^+}z(t))| \leq \epsilon\varphi(t), t \in \mathfrak{J}_j. \tag{6.11}$$

For any $j \in \{1, 2, ..., n\}$ we define the functions $z_1(t) \equiv z(t), t \in [1, \Omega_1]$ and for $j = 2, 3, ..., n$:

$$z_j(t) = \begin{cases} 0, \quad t \in [0, \Omega_{j-1}], \\ z(t), \quad t \in \mathfrak{I}_j. \end{cases}$$

By considering $\mathfrak{I}^{u_j}_{\Omega_{j-1}{}^+}$ on both sides of the inequality (6.11), we obtain for $t \in \mathfrak{I}_j$

$$\begin{aligned} \Bigg| z_j(t) \quad &+ \quad \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \\ &\int_{\Omega_{j-1}}^{\Omega_j} (\Omega_i-w)^{u_j-1} m\Big(w, -(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}} z_j(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}} z_j(w), z_j(w)\Big) dw \\ - \quad &\frac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1} m\Big(w, -(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}} z_j(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}} z_j(w), z_j(w)\Big) dw \Bigg| \\ \leq \quad &\epsilon \int_{\Omega_{j-1}}^{t} \frac{(t-w)^{u(j)-1}}{\Gamma(u(j))} \varphi(w) dw \\ \leq \quad &\lambda_{\varphi(t)} \epsilon \varphi(t). \end{aligned}$$

In accordance with above argument, VOFBVP (6.1) involves a solution y which is defined by $y(t) = y_j(t)$ for $t \in \mathfrak{I}_j, \; j = 1, 2, ..., n$, where

$$y_j(t) = \begin{cases} 0, \quad t \in [0, \Omega_{j-1}], \\ \widetilde{y_j}, \quad t \in \mathfrak{I}_j, \end{cases}$$

and $\widetilde{y_j} \in E_i$ is a solution of FBVP (6.6). By Lemma (7) the integral equation

$$(\widetilde{y}_j)(t) = -\frac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}$$

$$\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1} m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_J}_{\Omega^+_{J-1}}\widetilde{y}_j(\Omega_J))+\Im^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_j(w),\widetilde{y}_j(w)\right)dw$$

$$+\quad \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1} m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_j(\Omega_j))+\Im^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_j(w),\widetilde{y}_j(w)\right)dw$$

holds. Then, we arrive at, for each $t \in \Im_j$

$$|z(t)-y(t)|=|z(t)-y_i(t)|=|z_i(t)-\widetilde{y}_i(t)|$$

$$= \quad \Bigg|(z_j)(t)+\frac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}$$

$$\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1} m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(\Omega_j))+\Im^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(w),\widetilde{y}_i(w)\right)dw$$

$$-\quad \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1} m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(\Omega_j))+\Im^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(w),\widetilde{y}_i(w)\right)dw\Bigg|$$

$$\leq \quad \Bigg|(z_j)(t)+\frac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}$$

$$\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1} m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}}z_j(\Omega_j))+\Im^{u_j}_{\Omega^+_{j-1}}z_j(w),z_j(w)\right)dw$$

$$-\quad \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1} m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}}z_j(\Omega_j))+\Im^{u_j}_{\Omega^+_{j-1}}z_j(w),z_j(w)\right)dw\Bigg|$$

$$+\quad \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}$$

$$\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}\Bigg|m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}}z_j(\Omega_j)+\Im^{u_j}_{\Omega^+_{j-1}}z_j(w),z_j(w)\right)dw$$

$$-\quad m\left(w,-(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(\Omega_j)+\Im^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(w),\widetilde{y}_i(w)\right)dw\Bigg|$$

$$
\begin{aligned}
+ & \ \tfrac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1} \Big| m\Big(w, -(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} z_j(\Omega_j) + \Im^{u_j}_{\Omega^+_{j-1}} z_j(w), z_j(w)\Big) dw \\
- & \ m\Big(w, -(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} \widetilde{y}_i(\Omega_j) + \Im^{u_j}_{\Omega^+_{j-1}} \widetilde{y}_i(w), \widetilde{y}_i(w)\Big) dw \Big| \\
\leq & \ \lambda_{\varphi(t)} \epsilon \varphi(t) + \tfrac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \\
& \ \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j-w)^{u_j-1} w^{-\zeta} \Big(K \Big[(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} |(z_j(\Omega_j)-\widetilde{y}_i(\Omega_j)| \\
+ & \ \Im^{u_j}_{\Omega^+_{j-1}} |z_j(w)-\widetilde{y}_i(w)| \Big] + L|z_j(w)-\widetilde{y}_i(w)| \Big) dw \\
+ & \ \tfrac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1} w^{-\zeta} \Big(K(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} |z_j(\Omega_j)-\widetilde{y}_i(\Omega_j)| \\
+ & \ \Im^{u_j}_{T^+_{j-1}} |z_j(w)-\widetilde{y}_i(w)| + L|z_j(w)-\widetilde{y}_i(w)| \Big) dw \\
\leq & \ \lambda_{\varphi(t)} \epsilon \varphi(t) + \tfrac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j-w)^{u_j-1} w^{-\zeta} \Big(K\Im^{u_j}_{\Omega^+_{j-1}} |z_j(\Omega_j)-\widetilde{y}_i(\Omega_j)| + K\Im^{u_j}_{\Omega^+_{j-1}} |z_j(w)-\widetilde{y}_i(w)| \\
+ & \ L|z_j(w)-\widetilde{y}_i(w)| \Big) dw + \tfrac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1} w^{-\zeta} \Big(K\Im^{u_j}_{\Omega^+_{j-1}} |z_j(\Omega_j)-\widetilde{y}_i(\Omega_j)| \\
+ & \ K\Im^{u_j}_{\Omega^+_{j-1}} |z_j(w)-\widetilde{y}_i(w)| + L|z_j(w)-\widetilde{y}_i(w)| \Big) dw + \tfrac{m^*}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1} dw \\
\leq & \ \lambda_{\varphi(t)} \epsilon \varphi(t) + \tfrac{(\Omega_j-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \Big(2K\Im^{u_j}_{\Omega^+_{j-1}} \|z_j-\widetilde{y}_i\|_{E_j} + L\|z_j-\widetilde{y}_i\|_{E_j} \Big) \int_{\Omega_{j-1}}^{\Omega_j} w^{-\zeta} dw \\
+ & \ \tfrac{(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \Big(2K\Im^{u_j}_{\Omega^+_{j-1}} \|z_j-\widetilde{y}_i\|_{E_j} + L\|z_j-\widetilde{y}_i\|_{E_j} \Big) \int_{\Omega_{j-1}}^{t} w^{-\zeta} dw \\
\leq & \ \lambda_{\varphi(t)} \epsilon \varphi(t) + \tfrac{({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)} \Big(2K \tfrac{(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} \|z_j-\widetilde{y}_i\|_{E_j} + L\|z_j-\widetilde{y}_i\|_{E_j} \Big) \\
+ & \ \tfrac{(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \Big(2K \tfrac{(t^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j}}{(1-\zeta)\Gamma(u_j+1)} \|z_j-\widetilde{y}_i\|_{E_j} + L\|z_j-\widetilde{y}_i\|_{E_j} \Big) \\
\leq & \ \lambda_{\varphi(t)} \epsilon \varphi(t) + \tfrac{2({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)} \Big(2K \tfrac{(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} + L \Big) \|z_j-\widetilde{y}_i\|_{E_j}. \\
\leq & \ \lambda_{\varphi(t)} \epsilon \varphi(t) + \mu\|z-y\|.
\end{aligned}
$$

where

$$\mu = \max_{i=1,2,\ldots,n} \frac{2({\Omega_j}^{1-\zeta} - {\Omega_{j-1}}^{1-\zeta})(\Omega_j - \Omega_{j-1})^{u_j - 1}}{(1-\zeta)\Gamma(u_j)} \Big(2K \frac{(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j + 1)} + L \Big).$$

Then

$$\|z - y\|(1 - \mu) \leq \lambda_{\varphi(t)} \epsilon \varphi(t).$$

It gives, for each $t \in \mathfrak{J}$, that

$$|z(t) - y(t)| \leq \|z - y\| \leq \frac{\lambda_{\varphi(t)}}{1 - \mu} \epsilon \varphi(t) := a_m \epsilon \varphi(t).$$

Then, the given implicit nonlinear VOFBVP (6.1) is U-H-R stable w.r.t φ.

6.5 Numerical Example

Example 6.5.1. *Let us consider the implicit nonlinear VOFBVP by assuming* $\Omega = 2$*, as follows:*

$$\begin{cases} \mathfrak{D}_{0^+}^{u(t)} x(t) = \Big(\frac{|x^{\frac{1}{2}}(t)|}{10} + \frac{2}{15} |\mathfrak{D}_{0^+}^{u(t)} x(t)| + \frac{1}{3} \Big) t^{-\frac{1}{4}}, \quad t \in \mathfrak{J} := [0, 2], \\ x(0) = 0, \;\; x(2) = 0. \end{cases} \tag{6.12}$$

Let

$$m(t, y, z) = (\frac{1}{10} y^{\frac{1}{2}} + \frac{2}{15} z + \frac{1}{3}) t^{-\frac{1}{4}}, \;\; (t, y, z) \in [0, 2] \times [1, +\infty) \times [1, +\infty),$$

and

$$u(t) = \begin{cases} \frac{8}{5}, & t \in \mathfrak{J}_1 := [0,1], \\ \frac{9}{5}, & t \in \mathfrak{J}_2 :=]1,2]. \end{cases} \tag{6.13}$$

Thus, we obtain:

$$\begin{aligned} t^{\frac{1}{4}}|m(t,y_1,z_1) - m(t,y_2,z_2)| &= |\frac{1}{10}(y_1^{\frac{1}{2}} - y_2^{\frac{1}{2}}) + \frac{2}{15}(z_1 - z_2)| \\ &\leq \frac{1}{10}|y_1 - y_2| + \frac{2}{15}|z_1 - z_2|. \end{aligned}$$

Therefore, (H2) holds with $\zeta = \frac{1}{4}$ *and* $K = \frac{1}{10}$, $L = \frac{2}{15}$.

By (6.13)*, the implicit nonlinear VOFBVP (6.12) is divided into two expressions as follows:*

$$\begin{cases} \mathfrak{D}_{0^+}^{\frac{8}{5}}x(t) = (\frac{1}{10}|x^{\frac{1}{2}}(t)| + \frac{2}{15}|\mathfrak{D}_{0^+}^{\frac{8}{5}}x(t)| + \frac{1}{3})t^{-\frac{1}{4}}, & t \in \mathfrak{J}_1, \\ \mathfrak{D}_{1^+}^{\frac{9}{5}}x(t) = (\frac{1}{10}|x^{\frac{1}{2}}(t)| + \frac{2}{15}|\mathfrak{D}_{1^+}^{\frac{9}{5}}x(t)| + \frac{1}{3})t^{-\frac{1}{4}}, & t \in \mathfrak{J}_2. \end{cases}$$

For $t \in \mathfrak{J}_1$*, the implicit nonlinear VOFBVP (6.12) is corresponding to the following FBVP:*

$$\begin{cases} \mathfrak{D}_{0^+}^{\frac{8}{5}}x(t) = (\frac{1}{10}|x^{\frac{1}{2}}(t)| + \frac{2}{15}|\mathfrak{D}_{0^+}^{\frac{8}{5}}x(t)| + \frac{1}{3})t^{-\frac{1}{4}}, & t \in \mathfrak{J}_1, \\ x(0) = 0, \ \ x(1) = 0. \end{cases} \tag{6.14}$$

We can immediately check that (6.8) holds

$$\frac{({\Omega_1}^{1-\zeta} - \Omega_0^{1-\zeta})(\Omega_1 - \Omega_0)^{u_1 - 1}}{(1-\zeta)\Gamma(u_1)}\left(\frac{2K(\Omega_1 - \Omega_0)^{u_1}}{\Gamma(u_1+1)} + L\right) = \frac{1}{\frac{3\Gamma(\frac{8}{5})}{4}}\left(\frac{\frac{1}{5}}{\Gamma(\frac{13}{5})} + \frac{2}{15}\right) \simeq 0.4076 < 1.$$

Let $\varphi(t) = t^{\frac{1}{2}}$. *Then*

$$
\begin{aligned}
I_{0^+}^{u_1}\varphi(t) &= \frac{1}{\Gamma(\frac{8}{5})}\int_0^t (t-w)^{\frac{3}{5}} w^{\frac{1}{2}} dw \\
&\le \frac{1}{\Gamma(\frac{8}{5})}\int_0^t (t-w)^{\frac{3}{5}} dw \\
&\le \frac{5}{8\Gamma(\frac{8}{5})}\varphi(t) := \lambda_{\varphi(t)}\varphi(t).
\end{aligned}
$$

Hence, (H3) holds with $\varphi(t) = t^{\frac{1}{2}}$ *and* $\lambda_{\varphi(t)} = \dfrac{5}{8\Gamma(\frac{8}{5})}$.

By Theorem (24), the equivalent standard implicit nonlinear FBVP (6.14) has a solution $x_1 \in E_1$*, and from Theorem (25), the same FBVP (6.14) is U-H-R stable.*

For $t \in \mathfrak{I}_2$*, the implicit nonlinear VOFBVP (6.12) can be converted to the equivalent standard implicit nonlinear FBVP as follows:*

$$
\begin{cases}
\mathfrak{D}_{1^+}^{\frac{9}{5}} x(t) = (\frac{1}{10}|x^{\frac{1}{2}}(t)| + \frac{2}{15}|\mathfrak{D}_{1^+}^{\frac{9}{5}} x(t)| + \frac{1}{3})t^{-\frac{1}{4}}, \quad t \in \mathfrak{I}_2, \\
x(1) = 0, \;\; x(2) = 0.
\end{cases}
\tag{6.15}
$$

We simply see that

$$
\frac{(\Omega_2{}^{1-\zeta} - \Omega_1^{1-\zeta})(\Omega_2-\Omega_1)^{u_2-1}}{(1-\zeta)\Gamma(u_2)}\left(\frac{2K(\Omega_2-\Omega_1)^{u_2}}{\Gamma(u_2+1)} + L\right) = \frac{2^{\frac{3}{4}}-1}{\frac{3\Gamma(\frac{9}{5})}{4}}\left(\frac{\frac{1}{5}}{\Gamma(2.8)} + \frac{2}{15}\right) \simeq 0.2465 < 1.
$$

Thus, the condition (6.8) is satisfied. Also

$$\begin{aligned}
\mathfrak{J}_{1^+}^{u_2}\varphi(t) &= \frac{1}{\Gamma(\frac{9}{5})}\int_1^t (t-w)^{\frac{4}{5}} w^{\frac{1}{2}} dw \\
&\leq \frac{1}{\Gamma(\frac{9}{5})}\int_1^t (t-w)^{\frac{4}{5}} dw \\
&\leq \frac{5}{9\Gamma(\frac{9}{5})}\varphi(t) := \lambda_{\varphi(t)}\varphi(t).
\end{aligned}$$

Hence, (H3) fulfills with $\varphi(t) = t^{\frac{1}{2}}$ *and* $\lambda_{\varphi(t)} = \dfrac{5}{9\Gamma(\frac{9}{5})}$.
By Theorem (24), the equivalent standard implicit nonlinear FBVP (6.15) has a solution $\widetilde{x}_2 \in E_2$*, and from Theorem (25), the same implicit nonlinear FBVP (6.15) is U-H-R stable.*
Clearly, we have:

$$x_2(t) = \begin{cases} 0, & t \in \mathfrak{J}_1 \\ \widetilde{x}_2(t), & t \in \mathfrak{J}_2. \end{cases}$$

Accordingly, by Definition 21, the solution of the implicit nonlinear VOFBVP (6.12) admits a form as

$$x(t) = \begin{cases} x_1(t), & t \in \mathfrak{J}_1, \\ x_2(t) = \begin{cases} 0, & t \in \mathfrak{J}_1, \\ \widetilde{x}_2(t), & t \in \mathfrak{J}_2, \end{cases} \end{cases}$$

and, by Theorem (25), the implicit nonlinear VOFBVP (6.12) *is U-H-R stable w.r.t* φ.

6.6 Conclusion

New results concerning the solutions' existence and stability of our proposed FBVP as an implicit nonlinear FDfEq in the variable order settings have been carefully studied in this work. With the aid of both KFPThm and the criterion of U-H-R stability, our results have been successfully obtained. A numerical example has been presented to show the applicability of our theoretical analysis. In future work, our results can be extended or generalized include various classes of implicit nonlinear FDfEq in the variable order settings.

CHAPTER 7: A GENERALIZED MITTAG–LEFFLER–HYERS–ULAM STABILITY OF QUADRATIC FRACTIONAL INTEGRAL EQUATION

7.1 Introduction

Various integral equations (IEs) types are essential in functional analysis because of their diverse applications in science and engineering. Many real-life applications can be well modelled via quadratic fractional IEs. Quadratic integral equations (QIEs) are encountered in kinetic molecular, radiative, neutron transport, traffic, and queuing theories (Argyros, 1985; Busbridge, 1960). While QIEs have several applications, studying QIEs in the context of FrCL offers a powerful computational tool in many modelling scenarios, particularly queuing theory and biology (Darwish, 2005).

This work investigates the following quadratic fractional integral equation's (FIE) stability:

$$y(t) = \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}\left((t-\xi)^q\right) \mathcal{W}\left(\xi, y(\xi)\right)\, \mathrm{d}\xi \right], \quad (7.1)$$

where $\mathcal{V}, \mathcal{W} : J \times \mathbf{R} \to \mathbf{R}$ are CFs, $q \in [1, 2)$, Γ represents Gamma function(GF), and $Q^{\gamma,q,r}_{\alpha,\beta,\delta}$ is the generalized Mittag–Leffler (ML) function. The quadratic operator equations' existence can be proven under the conditions of mixed Lipschitz and compactness along with a certain growth condition on the nonlinearities included in the quadratic operator.

In this chapter, a proposed quadratic fractional IE is studied via a generalized ML function. The generalized ML–Hyers–Ulam (ML-H-U) stability is obtained. Hyers–Ulam (H-U) stability and ML–Hyers–Ulam–Rassias (ML-H-U-R) stability are investigated.

7.2 Essential Concepts

For a nonempty set $\mathcal{Y}$, the generalized metric on $\mathcal{Y}$ is initiated in this section. Given a function: $\hat{\rho} : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty]$, known as a generalized metric on $\mathcal{Y}$ iff the given assumptions are satisfied as follows:

(A1) $\hat{\rho}(y_1, y_2) = 0$ iff $y_1 = y_2$.

(A2) $\hat{\rho}(y_1, y_2) = \hat{\rho}(y_2, y_1)\ \forall\, y_1, y_2 \in \mathcal{Y}$.

(A3) $\hat{\rho}(y_1, y_2) \leq \hat{\rho}(y_1, y_3) + \hat{\rho}(y_3, y_2)\ \forall\, y_i \in \mathcal{Y}$ with $i = 1, 2, 3$.

Obviously, the above definition differs from the known complete metric space where not every 2 points in $\mathcal{Y}$ have necessarily a finite distance. Therefore, this space can be called as a generalized complete metric space (GCMSp).

Banach's fixed point theorem (BFPThm) in a GCMSp is expressed as:

Theorem 26. *Assume that $(\mathcal{Y}, \hat{\rho})$ is a GCMSp. Let $\mathcal{O} : \mathcal{Y} \to \mathcal{Y}$ be a strictly contractive operator with the Lipschitz constant $\ell < 1$. If $\exists$ a nonnegative integer k $\ni$*

$$\hat{\rho}\left(\mathcal{O}^{k+1}(y), \mathcal{O}^{k}(y)\right) < \infty,$$

for some $y \in \mathcal{Y}$, then the following are true:

(I) The sequence $\mathcal{O}^n(y)$ converges to a fixed point y^* of $\mathcal{O}$.

(II) y^* is the unique fixed point of $\mathcal{O}$ in

$$\mathcal{Y}^* = \left\{ y \in \mathcal{Y} \,\middle|\, \hat{\rho}\left(\mathcal{O}^{k}\left(y^*\right), y\right) < \infty \right\}.$$

(III) If $y \in \mathcal{Y}^*$, then we have:

$$\hat{\rho}\left(y, y^*\right) \leq \frac{1}{1-\ell} \hat{\rho}\left(\mathcal{O}(y), y\right).$$

Definition 22. *(Mittag-Leffler, 1903) (ML function) The one-parameter ML function, represented by $\mathfrak{E}_\alpha(z)$, is expressed as:*

$$\mathcal{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+\alpha k)} z^k, \tag{7.2}$$

where z, $\alpha \in \mathbf{C}$, $Re(\alpha) > 0$. If we substitute $\alpha = 1$ in the above equation, then we get:

$$\mathcal{E}_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

In 1905, the generalized form of $\mathcal{E}_\alpha(z)$ was proposed by (Wiman, 1905). Then, both Agarwal (1953), and Humbert and Agarwal (1953) introduced a function as follows:

Definition 23.

$$\mathcal{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\beta+\alpha k)} z^k, \tag{7.3}$$

where z, α, $\beta \in \mathbf{C}$, $Re(\alpha) > 0$, $Re(\beta) > 0$.

Prabhakar generalized in 1971 this function in the following form:

$$\mathcal{E}^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\beta+\alpha k)} z^k.$$

where $z, \alpha, \beta, \gamma \in \mathbf{C}$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\gamma) > 0$, such that $\gamma \neq 0$,

$$(\gamma)_k = \prod_{i=0}^{k-1} (\gamma + i), \tag{7.4}$$

which is named as the Pochhammer symbol (A. Shukla & Prajapati, 2007) $\ni$ $(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$. Further generalization of this function was initiated by (A. Shukla & Prajapati, 2007) as:

$$\mathcal{E}^{\gamma,q}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}}{k!\,\Gamma(\beta+\alpha k)} z^k. \tag{7.5}$$

where $z, \alpha, \beta, \gamma \in \mathbf{C}$,

$$\min\left\{Re(\alpha), Re(\beta), Re(\gamma)\right\} > 0, \tag{7.6}$$

and $q \in (0,1) \cup \mathbf{N}$. In 2009, again A. K. Shukla and Prajapati (2009) introduced a generalized ML function. In 2012, a novel generalized form of ML function was proposed by both (Salim & Faraj, 2012) and (Chouhan & Saraswat, 2011) as:

$$\mathcal{E}^{\gamma,q}_{\alpha,\beta,\delta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}}{(\delta)_{qk}\,\Gamma(\beta+\alpha k)} z^k. \tag{7.7}$$

where $z, \alpha, \beta, \gamma \in \mathbf{C}$, Eq. (7.6) holds, $q \in (0,1) \cup \mathbf{N}$ and

$$(\gamma)_{qk} = \frac{\Gamma(\gamma+qk)}{\Gamma(\gamma)}, \qquad (\delta)_{qk} = \frac{\Gamma(\delta+qk)}{\Gamma(\delta)}, \tag{7.8}$$

denote the generalized Pochhammer symbol (A. Shukla & Prajapati, 2007). After them, Desai et al. (2016) proposed another definition of generalized ML function.

Definition 24. *(Mazhar-ul Haque & Holambe, 2015). The generalized ML function,* *denoted by* $Q^{\gamma,q,r}_{\alpha,\beta,\delta}(y)$*, can be expressed as:*

$$\begin{aligned} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(z) &= Q^{\gamma,q,r}_{\alpha,\beta,\delta}\big(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r, z\big) \\ &= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)\, (\gamma)_{qs}}{\prod_{n=1}^{r} \beta(b_n, s)\, (\delta)_{qs}\, \Gamma(\beta + \alpha s)} z^s, \end{aligned} \tag{7.9}$$

where y*,* α*,* β*,* γ*,* δ*,* a_i*,* $b_i \in \mathbf{C}$*, Equation* (7.6) *holds,* $q \in (0,1) \cup \mathbf{N}$*,* $(\gamma)_{qk}$ *and* $(\delta)_{qk}$ *are defined in* (7.8)*.*

7.3 H-U-R Stability

The H-U-R stability and H-U stability of equation (7.1) are studied on a compact interval $[0, a]$.

Definition 25. *If for each given function y satisfies*

$$\left| y(t) - \mathcal{V}\left(t, y(t)\right) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1}\, Q^{\gamma,q,r}_{\alpha,\beta,\delta}\left((t-\xi)^q\right) \mathcal{W}\left(\xi, y(\xi)\right)\, \mathrm{d}\xi \right] \right| \le \varepsilon\varphi(t),$$

$\exists$ *an equation's* (7.1) *solution* u_0 *and a constant,* $c > 0$*, which is independent of both* y *and* u_0 $\ni$

$$|y(t) - u_0(t)| \le c\varepsilon\varphi(t),$$

for $t \in [a, b]$*, then equation* (7.1) *is named as H-U-R stable. On the other hand, when* φ *is formed as a constant function, equation* (7.1) *is known as H-U stable.*

Theorem 27. *For a closed and bounded interval* $J = [0, a]$ *of the real line* $\mathbf{R}$ *for some* $a > 0$*, suppose that* $\mathcal{V}$ *and* $\mathcal{W} : J \times \mathbf{R} \to \mathbf{R}$ *are CFs,* $q \in [1, 2)$ *and a gamma function, denoted by* Γ*, the following are satisfied:*

$$|\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \le M_v |y(t) - u(t)| \tag{7.10}$$

and

$$|\mathcal{W}(t, y(t)) - \mathcal{W}(t, u(t))| \leq M_w |y(t) - u(t)|, \tag{7.11}$$

for each $t \in J$, $y, u \in \mathbf{R}$, *and suppose that*

$$\left| y(t) - \mathcal{V}\left(t, y(t)\right) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}\left((t-\xi)^q\right) g\left(\xi, y(\xi)\right)\, \mathrm{d}\xi \right] \right| \leq \varepsilon, \tag{7.12}$$

and we also assume that

$$\begin{aligned} 0 &< \left(M_v M_w K\varepsilon + M_v \|\mathcal{W}\| + M_w \|\mathcal{V}\|\right) \\ &\quad \times \left[\frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\, (\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\, \Gamma(\beta + \alpha m)} \cdot \left(\frac{t^{q(m+1)}}{q(m+1)} \right) \right] \\ &= K' < 1. \end{aligned}$$

Then, the quadratic FIE is H-U stable.

Proof. Let us consider the CFs' space: $\mathcal{Y} = C([0, a], \mathbf{R})$ with a generalized metric (GMr), expressed as:

$$\hat{\rho}(g, h) = \inf \left\{ K \in [0, \infty] \ : \ |g(x) - h(x)| \leq K\varepsilon, \ \forall\, t \in J \right\}.$$

From Sec. 8.2, $(\mathcal{Y}, \hat{\rho})$ is a GCMSp (see Theorem 26). Let us now formulate an operator: $\mathcal{O} : \mathcal{Y} \to \mathcal{Y}$ as:

$$\mathcal{O}(y(t)) = \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q \mathcal{W}\left(\xi, y(\xi)\right)\, \mathrm{d}\xi \right].$$

From the definition of $\mathcal{O}$ and equations (7.10) and (7.11), we get:

$$
\begin{aligned}
|\mathcal{O}(y(t))-\mathcal{O}(u(t))| = &\left|\mathcal{V}(t,y(t))\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\mathcal{W}(\xi,y(\xi))\ \mathrm{d}\xi\right]\right.\\
&\left.-\mathcal{V}(t,u(t))\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\mathcal{W}(\xi,u(\xi))\ \mathrm{d}\xi\right]\right|\\
\leq\ & |\mathcal{V}(t,y(t))-\mathcal{V}(t,u(t))|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q|\mathcal{W}(\xi,y(\xi))-\mathcal{W}(\xi,u(\xi))|\ \mathrm{d}\xi\right]\\
&+|\mathcal{V}(t,y(t))-\mathcal{V}(t,u(t))|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q|\mathcal{W}(\xi,u(\xi))|\ \mathrm{d}\xi\right]\\
&+|\mathcal{V}(t,u(t))|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q|\mathcal{W}(\xi,y(\xi))-\mathcal{W}(\xi,u(\xi))|\ \mathrm{d}\xi\right]\\
\leq\ & M_v\,|y(t)-u(t)|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q M_w\,|y(\xi)-u(\xi)|\ \mathrm{d}\xi\right]\\
&+M_v\,|y(t)-u(t)|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\ \|\mathcal{W}\|\ \mathrm{d}\xi\right]\\
&+\|\mathcal{V}\|\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q M_w\,|y(\xi)-u(\xi)|\ \mathrm{d}\xi\right]\\
\leq\ & \left(M_vM_wK^2\varepsilon^2+M_vK\varepsilon\|\mathcal{W}\|+M_vK\varepsilon\|\mathcal{V}\|\right)\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\ \mathrm{d}\xi\right]\\
\leq\ & \left(M_vM_wK^2\varepsilon^2+M_vK\varepsilon\|\mathcal{W}\|+M_wK\varepsilon\|\mathcal{V}\|\right)\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} (t-\xi)^{mq}\, \mathrm{d}\xi \Bigg] \\
&= \Big(M_v M_w K^2 \varepsilon^2 + M_v K \varepsilon \|\mathcal{W}\| + M_w K \varepsilon \|\mathcal{V}\| \Big) \\
&\quad \times \left[\frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} \right. \\
&\quad \left. \times \int_0^t (t-\xi)^{q-1} (t-\xi)^{mq}\, \mathrm{d}\xi \right] \\
&\le K\varepsilon \left(M_v M_w + M_v \|\mathcal{W}\| + M_w \|\mathcal{V}\| \right) \\
&\quad \times \left[\frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} . \left(\frac{t^{q(m+1)}}{q(m+1)} \right) \right].
\end{aligned}
$$

Because $0 < K' < 1$, we conclude that $\mathcal{O}$ is contraction mapping. Let us take $y_0' \in \mathcal{Y}$, from the continuous property of $y_0' \in \mathcal{Y}$ and $\mathcal{O}(y_0') \in \mathcal{Y}$, $\exists$ a constant $0 < C_1 < \infty$ with

$$
\left|(\mathcal{O}y_0')(t) - y_0'(t)\right| \left| \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta} (t-\xi)^q \mathcal{W}(\xi, y(\xi))\, \mathrm{d}\xi \right] - y_0'(t) \right| \le C_1 \varepsilon, \tag{7.13}
$$

$\forall\, t \in [0, a]$. so $\hat{\rho}\left(\mathcal{O}(y_0'), y_0'\right) < \infty$. Therefore, Theorem 26(I) indicates that $\exists$ a CF: $y_0' : [0, a] \to \mathbf{R}$ such that $\mathcal{O}^n y_0' \to y_0'$ in $(\mathcal{Y}, \hat{\rho})$ as $n \to \infty$, $y_0' = \mathcal{O}(y_0')$ where y_0' satisfies equation (7.1) for any $t \in J$. If $y \in \mathcal{Y}$, then y_0' and y are CFs defined on a compact interval $[0, a]$. Thus, $\exists$ a constant $C_y > 0$ with

$$
|y_0'(t) - y(t)| \le C_x \varepsilon,
$$

$\forall\, t \in [0, a]$. This indicates that $\hat{\rho}(y_0', y) < \infty$ for every $y \in \mathcal{Y}$ or equivalently

$$
\Big\{ y \in \mathcal{Y} \;:\; \hat{\rho}(y_0', y) < \infty \Big\} = \mathcal{Y}.
$$

Hence, from Theorem 26(II) y_0' is a unique continuous function (UqCF) with property (7.1). Also, it implies from (7.10)

$$\hat{\rho}\left(O(y(t)), y(t)\right) \leq \varepsilon,$$

$\forall\, t \in [0, a]$. At last,

$$\hat{\rho}\left(y, y_0'\right) \leq \frac{1}{1-K'}\hat{\rho}\left(Oy, y\right) \leq \frac{1}{1-K'}\varepsilon.$$

Thus, the quadratic FIE is H-U stable.

The H-U-R stability of equation (7.1) is studied as follows:

Theorem 28. *For a closed and bounded interval $J = [0, a]$ of the real line $\mathbb{R}$ for some $a > 0$, suppose that $\mathcal{V}, \mathcal{W} : J \times \mathbb{R} \to \mathbb{R}$ are CFs, $q \in [1, 2)$ and a gamma function, represented by Γ, the following are satisfied:*

$$|\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \leq M_v |y(t) - u(t)| \tag{7.14}$$

and

$$|\mathcal{W}(t, y(t)) - \mathcal{W}(t, u(t))| \leq M_w |y(t) - u(t)| \tag{7.15}$$

and

$$\left[\int_0^t (\varphi(\xi))^{1/p}\, \mathrm{d}\xi\right]^p \leq C\varphi(t),$$

for any $t \in J$, $y, u \in \mathbb{R}$ and suppose that

$$\left| y(t) - \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q \mathcal{W}(\xi, y(\xi))\, \mathrm{d}\xi\right]\right| \leq \varepsilon\varphi(t), \tag{7.16}$$

and we also suppose that

$$
\begin{aligned}
0 &< \big(M_v M_w K C \varepsilon \varphi(t) + M_w C \|\mathcal{V}\|\big) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^{r} \beta(b_n, m)\, (\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m) (\delta)_{qm}\, \Gamma(\beta + \alpha m)} \right]^{1/w} \left(\frac{w t^{\frac{mq+q+w-1}{w}}}{mq + q + w - 1} \right) \\
&\quad + M_v \|\mathcal{W}\| \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\, (\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m) (\delta)_{qm}\, \Gamma(\beta + \alpha m)} . \left(\frac{t^{q(m+1)}}{q(m+1)} \right) \\
&= K' < 1.
\end{aligned}
$$

Then, the quadratic FIE is H-U stable.

Proof. Consider the CFs' space: $\mathcal{Y} = C([0, a], \mathbb{R})$ and $g \in \mathcal{Y}$, with a GMr, written as:

$$
\hat{\rho}(g, h) = \inf \Big\{ K \in [0, \infty] \ : \ |g(t) - h(t)| \le K \varepsilon \varphi(t), \ \forall\, t \in J \Big\}.
$$

Clearly, $(\mathcal{Y}, \hat{\rho})$ is a GCMSp. Let us now formulate an operator: $\mathcal{O} : \mathcal{Y} \to \mathcal{Y}$ as:

$$
\mathcal{O}(y(t)) = \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q^{\gamma, q, r}_{\alpha, \beta, \delta}(t - \xi)^q \mathcal{W}(\xi, y(\xi))\, \mathrm{d}\xi \right]. \quad (7.17)
$$

From the definition O and equations (7.14) and (7.15), we get:

$$
\begin{aligned}
|O(y(t))-O(u(t))| = &\left|\mathcal{V}(t,y(t))\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\mathcal{W}(\xi,y(\xi))\,\mathrm{d}\xi\right]\right.\\
&\left.-\mathcal{V}(t,u(t))\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\mathcal{W}(\xi,u(\xi))\,\mathrm{d}\xi\right]\right|\\
\leq\ &|\mathcal{V}(t,y(t))-\mathcal{V}(t,u(t))|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q|\mathcal{W}(\xi,y(\xi))-\mathcal{W}(\xi,u(\xi))|\,\mathrm{d}\xi\right]\\
&+|\mathcal{V}(t,y(t))-\mathcal{V}(t,u(t))|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\,|\mathcal{W}(\xi,u(\xi))|\,\mathrm{d}\xi\right]\\
&+|\mathcal{V}(t,u(t))|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q|\mathcal{W}(\xi,y(\xi))-\mathcal{W}(\xi,u(\xi))|\,\mathrm{d}\xi\right]\\
\leq\ &M_v\,|y(t)-u(t)|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q M_w\,|y(\xi)-u(\xi)|\,\mathrm{d}\xi\right]\\
&+M_v\,|y(t)-u(t)|\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\,\|\mathcal{W}\|\,\mathrm{d}\xi\right]\\
&+\|\mathcal{V}\|\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q M_v\,|y(\xi)-u(\xi)|\,\mathrm{d}\xi\right]\\
\leq\ &M_vM_wK^2\varepsilon^2\varphi(t)\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\varphi(\xi)\,\mathrm{d}\xi\right]\\
&+M_vK\varepsilon\,\|\mathcal{W}\|\,\varphi(t)\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\,\mathrm{d}\xi\right]\\
&+M_vK\varepsilon\,\|\mathcal{V}\|\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\varphi(\xi)\,\mathrm{d}\xi\right]\\
\leq\ &\Big(M_vM_wK^2\varepsilon^2\varphi(t)+M_vK\varepsilon\,\|\mathcal{V}\|\Big)\\
&\times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\varphi(\xi)\,\mathrm{d}\xi\right]\\
&+M_vK\varepsilon\,\|\mathcal{W}\|\,\varphi(t)\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\,\mathrm{d}\xi\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(M_v M_w K^2 \varepsilon^2 \varphi(t) + M_v K \varepsilon \left\|\mathcal{V}\right\|\right) \\
&\quad \times \frac{1}{\Gamma(q)} \left[\int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta} \left((t-\xi)^q\right)^{1/w} \, \mathrm{d}\xi \right]^w \\
&\quad \times \left[\int_0^t (\varphi(\xi))^{1/p} \, \mathrm{d}\xi \right]^p \\
&\quad + M_v K \varepsilon \left\|\mathcal{W}\right\| \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta} (t-\xi)^q \, \mathrm{d}\xi \right] \\
&\leq \left(M_v M_w K^2 \varepsilon^2 \varphi(t) + M_v K \varepsilon \left\|\mathcal{V}\right\|\right) C \varphi(t) \\
&\quad \times \frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{(q-1)/w} \\
&\quad \times \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^r \beta(b_n, m) \, (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \, \Gamma(\beta + \alpha m)} \right]^{1/w} \left((t-\xi)^q\right)^{mq/w} \, \mathrm{d}\xi \\
&\quad + M_v K \varepsilon \left\|\mathcal{W}\right\| \varphi(t) \frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} \\
&\quad \times \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^r \beta(b_n, m) \, (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \, \Gamma(\beta + \alpha m)} \right]^{1/w} (t-\xi)^{mq} \, \mathrm{d}\xi \\
&\leq \left(M_v M_w K^2 \varepsilon^2 \varphi(t) + M_v K \varepsilon \left\|\mathcal{V}\right\|\right) C \varphi(t) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^r \beta(b_n, m) \, (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \, \Gamma(\beta + \alpha m)} \right]^{1/w} \\
&\quad \times \int_0^t (t-\xi)^{(mq+q-1)/w} \, \mathrm{d}\xi \\
&\quad + M_v K \varepsilon \left\|\mathcal{W}\right\| \varphi(t) \frac{1}{\Gamma(q)} \\
&\quad \times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) \, (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \, \Gamma(\beta + \alpha m)} \int_0^t (t-\xi)^{mq+q-1} \, \mathrm{d}\xi \\
&\leq \left(M_v M_w K^2 \varepsilon^2 \varphi(t) + M_v K \varepsilon \left\|\mathcal{V}\right\|\right) C \varphi(t) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^r \beta(b_n, m) \, (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \, \Gamma(\beta + \alpha m)} \right]^{1/w} \\
&\quad \times \frac{w}{mq + q + q - 1} t^{(mq+q+w-1)/w} \\
&\quad + M_v K \varepsilon \left\|\mathcal{W}\right\| \varphi(t) \frac{1}{\Gamma(q)} \\
&\quad \times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) \, (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \, \Gamma(\beta + \alpha m)} \frac{t^{(m+1)q}}{(m+1)q} \\
&\leq K \varepsilon \varphi(t) K'.
\end{aligned}
$$

Note that $0 < K' < 1$. We conclude that $\mathcal{O}$ is contraction mapping (CoMp). Let us take $y_0' \in \mathcal{Y}$, from the continuous property of $y_0' \in \mathcal{Y}$ and $\mathcal{O}(y_0') \in \mathcal{Y}$ $\exists$ a constant $0 < C_1 < \infty$ with

$$\left|\mathcal{O}(y_0')(t)-y_0'(t)\right| = \left|\mathcal{V}(t,y(t))\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}(t-\xi)^q \mathcal{W}(\xi,y(\xi))\,\mathrm{d}\xi\right]-y_0'(t)\right|$$
$$\leq C_1 \varepsilon \varphi(t),$$

$\forall\ t \in [0,a]$. So, $\hat{\rho}(\mathcal{O}(y_0), y_0') < \infty$. Thus, Theorem 26(I) indicates that $\exists$ a CF: $u_0' : [0,a] \to \mathbb{R}$ such that $\mathcal{O}^n(y_0) \to u_0'$ in $(\mathcal{Y}, \hat{\rho})$ as $n \to \infty$, $u_0' = \mathcal{O}(u_0')$; therefore, u_0' satisfies equation (7.1) for any $t \in J$. If $y \in \mathcal{Y}$, then y_0' and y are CFs defined on a compact interval $[0,a]$. Thus, $\exists$ a constant $C_x > 0$ with

$$|y_0'(t) - y(t)| \leq C_y \varepsilon \varphi(t), \qquad \forall t \in [0,a].$$

This indicates that $\hat{\rho}(y_0', y) < \infty$ for every $y \in \mathcal{Y}$ or equivalently $\{y \in \mathcal{Y} : \hat{\rho}(y_0', y) < \infty\} = \mathcal{Y}$. Hence, from Theorem (26)(II) u_0' is a UqCF with property (7.1). As a result, from (7.16), it implies that

$$\hat{\rho}\left(\mathcal{O}(u(t)), u(t)\right) \leq \varepsilon \varphi(t),$$

$\forall\ t \in [0,a]$. At last,

$$\hat{\rho}(u, u_0') \leq \frac{1}{1-K'} \hat{\rho}(\mathcal{O}(u), u) \leq \frac{1}{1-K'} \varepsilon \varphi(t).$$

Thus, the quadratic FIE is H-U-R stable.

7.4 ML-H-U Stability

The ML-H-U stability of equation (7.1) is investigated.

Definition 26. *If for each function y satisfies*

$$|y(t) - \mathcal{V}(t, y(t))\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q \mathcal{W}(\xi, y(\xi))\,\mathrm{d}\xi\right]| \leq \varepsilon E_q(t^q),$$

$\exists$ *an equation's* (7.1) *solution* y_0*, and a constant,* $c > 0$*, which is an independent of both* y *and* y_0 *such that*

$$|y(t) - y_0(t)| \leq c\varepsilon E_q(t^q),$$

for each $t \in [0, a]$*, then equation* (7.1) *is named as ML-H-U stable.*

Theorem 29. *For a closed and bounded interval* $J = [0, a]$ *of the real line* $\mathbb{R}$ *for some* $a > 0$*, suppose that* $\mathcal{V}, \mathcal{W} : J \times \mathbf{R} \to \mathbb{R}$ *are CFs,* $q \in [1, 2)$ *and a gamma function, represented by* Γ*, the following are satisfied:*

$$|\mathcal{V}(t, y(t)) - \mathcal{W}(t, u(t))| \leq M_v |y(t) - u(t)| \tag{7.18}$$

and

$$|\mathcal{W}(t, y(t)) - \mathcal{W}(t, u(t))| \leq M_w |y(t) - u(t)|, \tag{7.19}$$

for any $t \in J$*,* $y, u \in \mathbb{R}$*, and suppose that*

$$\left|y(t) - \mathcal{V}(t, y(t))\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q \mathcal{W}(\xi, y(\xi))\,\mathrm{d}\xi\right]\right| \leq \varepsilon E_q(t^q). \tag{7.20}$$

Also, suppose that

$$
\begin{aligned}
0 &< \left(M_v M_w K\varepsilon E_q(t^q) + M_w K\varepsilon \|\mathcal{V}\|\right) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} \Gamma((m+1)q) \\
&\quad + M_v \|\mathcal{W}\| \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} \left(\frac{t^{q(m+1)}}{q(m+1)}\right) \\
&= K' < 1,
\end{aligned}
$$

then quadratic FIE is H-U stable.

Proof. Consider CFs' space: $\mathcal{Y} = C([0, a], \mathbb{R})$ and $g \in \mathcal{Y}$, with a GMr, expressed as:

$$\hat{\rho}(g, h) = \inf \left\{ K \in [0, \infty] \ : \ |g(x) - h(x)| \leq K\varepsilon E_q(t^q), \ \forall\, t \in J \right\}.$$

Clearly, $(\mathcal{Y}, \hat{\rho})$ is a GCMSp. Let us formulate an operator: $\mathcal{O} : \mathcal{Y} \to \mathcal{Y}$ by

$$\mathcal{O}(y(t)) = \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q \mathcal{W}(\xi, y(\xi))\, \mathrm{d}\xi \right]. \quad (7.21)$$

From the definition O and equations (7.18) and (7.19), we get:

$$
\begin{aligned}
|O(y(t))-O(u(t))| = & \Bigg|\mathcal{V}(t,y(t))\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\mathcal{W}(\xi,y(\xi))\,\mathrm{d}\xi\right] \\
& - \mathcal{V}(t,u(t))\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\mathcal{W}(\xi,u(\xi))\,\mathrm{d}\xi\right]\Bigg| \\
\le\ & |\mathcal{V}(t,y(t))-\mathcal{V}(t,u(t))| \\
& \times\left[\tfrac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q|\mathcal{W}(\xi,y(\xi))-\mathcal{W}(\xi,u(\xi))|\,\mathrm{d}\xi\right] \\
& + |\mathcal{V}(t,y(t))-\mathcal{V}(t,u(t))| \\
& \times\left[\tfrac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q|\mathcal{W}(\xi,u(\xi))|\,\mathrm{d}\xi\right] \\
& + |\mathcal{V}(t,u(t))| \\
& \times\left[\tfrac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q|\mathcal{W}(\xi,y(\xi))-\mathcal{W}(\xi,u(\xi))|\,\mathrm{d}\xi\right] \\
\le\ & M_v\,|y(t)-u(t)| \\
& \times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q M_w\,|y(\xi)-u(\xi)|\,\mathrm{d}\xi\right] \\
& + M_v\,|y(t)-u(t)| \\
& \times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\,\|\mathcal{W}\|\,\mathrm{d}\xi\right] \\
& + \|\mathcal{V}\|\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q M_v\,|y(\xi)-u(\xi)|\,\mathrm{d}\xi\right] \\
\le\ & M_vM_wK^2\varepsilon^2E_q(t^q) \\
& \times\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q E_q(\xi^q)\,\mathrm{d}\xi\right] \\
& + M_vK\varepsilon\,\|\mathcal{W}\|\,E_q(t^q)\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\,\mathrm{d}\xi\right] \\
& + M_vK\varepsilon\,\|\mathcal{V}\|\left[\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q E_q(\xi^q)\,\mathrm{d}\xi\right]
\end{aligned}
$$

$$\leq \Big(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_v K\varepsilon \left\|\mathcal{V}\right\|\Big)$$

$$\times \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q E_q(\xi^q)\, \mathrm{d}\xi\right]$$

$$+ M_v K\varepsilon \left\|\mathcal{W}\right\| E_q(t^q) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t-\xi)^q\, \mathrm{d}\xi\right]$$

$$\leq \Big(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_v K\varepsilon \left\|\mathcal{V}\right\|\Big)$$

$$\times \frac{1}{\Gamma(q)} \Bigg[\int_0^t (t-\xi)^{q-1} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\, (\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\, \Gamma(\beta+\alpha m)}$$

$$\times (t-\xi)^{mq} \sum_{n=0}^{\infty} \frac{s^{nq}}{\Gamma(qn+1)}\, \mathrm{d}\xi$$

$$+ M_v K\varepsilon \left\|\mathcal{W}\right\| E_q(t^q) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1}\right.$$

$$\times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\, (\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\, \Gamma(\beta+\alpha m)}\, \mathrm{d}\xi$$

$$\leq \Big(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_v K\varepsilon \left\|\mathcal{V}\right\|\Big)$$

$$\times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\, (\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\, \Gamma(\beta+\alpha m)}$$

$$\times \sum_{n=0}^{\infty} \frac{s^{nq}}{\Gamma(qn+1)} \int_0^t (t-\xi)^{q-1} (t-\xi)^{mq} \xi^{nq}\, \mathrm{d}\xi$$

$$+ M_v K\varepsilon \left\|\mathcal{W}\right\| E_q(t^q) \frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1}$$

$$\times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\, (\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\, \Gamma(\beta+\alpha m)} (t-\xi)^{mq}\, \mathrm{d}\xi$$

$$\leq \Big(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_v K\varepsilon \left\|\mathcal{V}\right\|\Big)$$

$$
\begin{aligned}
&\times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} \\
&\times \sum_{n=0}^{\infty} \frac{s^{nq}}{\Gamma(qn+1)} \int_0^t (t-\xi)^{mq+q-1} s^{nq}\,\mathrm{d}\xi \\
&+ M_v K\varepsilon \,\|\mathcal{W}\|\, E_q(t^q) \frac{1}{\Gamma(q)} \\
&\times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} \\
&\times \int_0^t (t-\xi)^{mq+q-1}\,\mathrm{d}\xi \\
\leq &\Big(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_w K\varepsilon \,\|\mathcal{V}\| \Big) \\
&\times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} \\
&\times \Gamma((m+1)q) \sum_{s=0}^{\infty} \frac{1}{\Gamma(qs+1)} t^{sq} \Bigg) \\
&+ M_v K\varepsilon \,\|\mathcal{W}\|\, E_q(t^q) \\
&\times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} \left(\frac{t^{(m+1)q}}{(m+1)q} \right)
\end{aligned}
$$

$$
\begin{aligned}
\leq &\Big(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_w K\varepsilon \,\|\mathcal{V}\| \Big) \\
&\times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} \Gamma((m+1)q) E_q(t^q) \\
&+ M_v K\varepsilon \,\|\mathcal{W}\|\, E_q(t^q) \\
&\times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, m)\,(\gamma)_{qm}}{\prod_{n=1}^{r} \beta(a_n, m)(\delta)_{qm}\,\Gamma(\beta+\alpha m)} \left(\frac{t^{(m+1)q}}{(m+1)q} \right) \\
\leq &K\varepsilon E_q(t^q) K'.
\end{aligned}
$$

We note that $0 < K' < 1$. We conclude that $\mathcal{O}$ is contraction mapping. Let us take $y_0' \in \mathcal{Y}$, from the continuous property of $y_0' \in \mathcal{Y}$ and $\mathcal{O}(y_0') \in \mathcal{Y}$, $\exists$ a constant

$0 < C_1 < \infty$ with

$$|\mathcal{O}(y_0')(t) - y_0'(t)| = \left| \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q \mathcal{W}(\xi, y(\xi)) \, \mathrm{d}\xi \right] - y_0'(t) \right|$$
$$\leq C_1 \varepsilon E_q(t^q),$$

$\forall\, t \in [0, a]$. So, $\hat{\rho}(\mathcal{O}(y_0'), y_0') < \infty$. Thus, Theorem 26(I) indicates that $\exists$ a CF: $u_0' : [0, a] \to \mathbb{R} \ni \mathcal{O}^n y_0' \to u_0'$ in $(\mathcal{Y}, \hat{\rho})$ as $n \to \infty$, $u_0' = \mathcal{O}(u_0')$; therefore, u_0' satisfies equation (7.1) for any $t \in J$. If $y \in \mathcal{Y}$, then y_0' and y are CFs defined on a compact interval $[0, a]$. Thus, $\exists$ a constant $C_y > 0$ with

$$|y_0'(t) - y(t)| \leq C_y \varepsilon \mathcal{E}_q(t^q),$$

$\forall\, t \in [0, a]$. This indicates that $\hat{\rho}(y_0', y) < \infty$ for every $y \in \mathcal{Y}$ or equivalently

$$\left\{ y \in \mathcal{Y} \, : \, \hat{\rho}(y_0', y) < \infty \right\} = \mathcal{Y}.$$

Hence, from Theorem (26)(II) u_0' is a UqCF with property (7.1). From (7.20), it implies that

$$\hat{\rho}\left(\mathcal{O}(u(t)), u(t) \right) \leq \varepsilon \mathcal{E}_q(t^q),$$

$\forall\, t \in [0, a]$. At last,

$$\hat{\rho}(u, u_0') \leq \frac{1}{1 - K'} \hat{\rho}(\mathcal{O}(u), u) \leq \frac{1}{1 - K'} \varepsilon \mathcal{E}_q(t^q).$$

Thus, the quadratic FIE is ML-H-U stable.

7.5 Conclusion

Quadratic fractional IEs have been employed in inner product spaces' characterization.

$$\|y + z\|^2 + \|y + z\|^2 = 2\left(\|y\|^2 + \|z\|^2\right),$$

which is a parallelogram equality that is satisfied by a square norm on an inner product space. H-U stability and ML-H-U-R stability have been studied in this chapter. ML is an essential tool in showing differential equation's stability. Various differential equations' classes can been unified via our new proposed procedure which can inspire interested engineers and scientists to work on future research studies.

CHAPTER 8: MONOTONE ITERATIVE METHOD FOR ψ-CAPUTO FRACTIONAL DIFFERENTIAL EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

8.1 Introduction

In this chapter, the extremal solutions' existence is proven for a novel class of fractional differential equation (FDfEq) in the context of ψ-Caputo formulation with nonlinear boundary conditions (NLBCs). The monotone iterative technique is employed along with the technique of upper solution (USo) and lower solution (LSo). We investigate the ψ-Caputo fractional differential equation (CpFDfEq) with NLBCs as follows:

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\tau;\psi}\left({}^c\mathbb{D}_{a^+}^{\lambda;\psi} - \sigma\right)\mathfrak{z}(\vartheta) = \mathbb{F}\left(\vartheta, \mathfrak{z}(\vartheta), {}^c\mathbb{D}_{a^+}^{\lambda;\psi}(\vartheta)\right), \\ \mathbb{H}\left({}^c\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}(a), {}^c\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}(b)\right) = 0, \qquad \mathbb{G}(\mathfrak{z}(a), \mathfrak{z}(b)) = 0, \end{cases} \tag{8.1}$$

for $\vartheta \in \Omega := [a, b]$, where ${}^c\mathbb{D}_{a^+}^{\tau;\psi}$ and ${}^c\mathbb{D}_{a^+}^{\lambda;\psi}$ represent the ψ-Caputo fractional derivatives of order τ and λ, respectively, $\ni \tau, \lambda \in (0, 1]$, $\sigma > 0$, $\mathbb{F} \in C(\Omega \times \mathbb{R}^2, \mathbb{R})$, $\mathbb{G}$, $\mathbb{H} \in C(\mathbb{R}^2, \mathbb{R})$. The CpFDfEq (8.1) is subject to NLBCs. Equation (8.1) is the deterministic FDfEq where the FDfEq with its deterministic solution is only investigated in this chapter without including any random processes.

8.2 Preliminaries

Some fundamental definitions and tools of FrCL that will be used later in this chapter. Assume that $\Omega = [a, b]$, $0 \leq a < b < \infty$ is a finite interval and $\psi : \Omega \to \mathbb{R}$ is an increasing differentiable function $\ni \psi'(\vartheta) \neq 0$, $\forall\, \vartheta \in \Omega$.

Definition 27. *(Almeida, 2017) The Riemann–Lebesgue (RLb) fractional integral* *of order $\tau > 0$ for an integrable function $\mathfrak{z} : \Omega \to \mathbb{R}$ w.r.t. ψ is expressed as:*

$$\mathbb{I}_{a^+}^{\tau;\psi} \mathfrak{z}(\vartheta) = \frac{1}{\Gamma(\tau)} \int_a^{\vartheta} \psi'(\eta) \left(\psi(\vartheta) - \psi(\eta)\right)^{\tau-1} \mathfrak{z}(\eta) \, \mathrm{d}\eta, \tag{8.2}$$

where $\Gamma(\tau) = \int_0^{+\infty} \vartheta^{\tau-1} e^{-\vartheta} \, \mathrm{d}\vartheta$, $\tau > 0$ is the Gamma function.

Definition 28. *(Almeida, 2017) Let ψ, $\mathfrak{z} \in C^n(\Omega, \mathbb{R})$. The RLb fractional derivative of a function $\mathfrak{z}$ of order $n - 1 < \tau < n$ w.r.t. ψ is given as follows:*

$$\begin{aligned} \mathbb{D}_{a^+}^{\tau;\psi} \mathfrak{z}(\vartheta) &= \left(\frac{D_\vartheta}{\psi'(\vartheta)} \right)^n \mathbb{I}_{a^+}^{n-\tau;\psi} \mathfrak{z}(\vartheta) \\ &= \frac{1}{\Gamma(n-\tau)} \left(\frac{D_\vartheta}{\psi'(\vartheta)} \right)^n \int_a^{\vartheta} \psi'(\eta) \left(\psi(\vartheta) - \psi(\eta)\right)^{n-\tau-1} \mathfrak{z}(\eta) \, \mathrm{d}\eta, \end{aligned}$$

where $n = [\tau] + 1$, $n \in \mathbb{N}$ and $D_\vartheta = \frac{\mathrm{d}}{\mathrm{d}\vartheta}$.

Definition 29. *(Almeida, 2017) Let ψ, $\mathfrak{z} \in C^n(\Omega, \mathbb{R})$. The Cp fractional derivative of $\mathfrak{z}$ of order $n - 1 < \tau < n$ w.r.t. ψ is defined as:*

$${}^c\mathbb{D}_{a^+}^{\tau;\psi} \mathfrak{z}(\vartheta) = \mathbb{I}_{a^+}^{n-\tau;\psi} \mathfrak{z}_{\psi}^{[n]}(\vartheta),$$

where $n = [\tau] + 1$ for $\tau \notin \mathbb{N}$, $n = \tau$ for $\tau \in \mathbb{N}$ and $\mathfrak{z}_{\psi}^{[n]}(\vartheta) = \left(\frac{D_\vartheta}{\psi'(\vartheta)} \right)^n \mathfrak{z}(\vartheta)$. From the definition, we get:

$${}^c\mathbb{D}_{a^+}^{\tau;\psi} \mathfrak{z}(\vartheta) = \begin{cases} \displaystyle\int_a^{\vartheta} \frac{\psi'(\eta) \left(\psi(\vartheta) - \psi(\eta)\right)^{n-\tau-1}}{\Gamma(n-\tau)} \mathfrak{z}_{\psi}^{[n]}(\eta) \, \mathrm{d}\eta, & \tau \notin \mathbb{N}, \\ \mathfrak{z}_{\psi}^{[n]}(\vartheta), & \tau \in \mathbb{N}. \end{cases} \tag{8.3}$$

The following Lemma lists some essential properties of the ψ-fractional operators:

Lemma 8 ((Almeida, 2017)). *Let τ, $\lambda > 0$, and $\mathfrak{z} \in C(\Omega, \mathbb{R})$. Then, for each $\vartheta \in \Omega$, we have:*

1. ${}^{c}\mathbb{D}_{a^+}^{\tau;\psi}\mathbb{I}_{a^+}^{\tau;\psi}\mathfrak{z}(\vartheta) = \mathfrak{z}(\vartheta)$,

2. $\mathbb{I}_{a^+}^{\tau;\psi}{}^{c}\mathbb{D}_{a^+}^{\tau;\psi}\mathfrak{z}(\vartheta) = \mathfrak{z}(\vartheta) - \mathfrak{z}(a)$ *for* $0 < \tau \leq 1$,

3. $\mathbb{I}_{a^+}^{\tau;\psi}(\psi(\vartheta) - \psi(a))^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\tau)}(\psi(\vartheta) - \psi(a))^{\lambda+\tau-1}$,

4. ${}^{c}\mathbb{D}_{a^+}^{\tau;\psi}(\psi(\vartheta) - \psi(a))^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\tau)}(\psi(\vartheta) - \psi(a))^{\lambda-\tau-1}$,

5. ${}^{c}\mathbb{D}_{a^+}^{\tau;\psi}(\psi(\vartheta) - \psi(a))^{k} = 0$, *for all* $k \in \{0, \ldots, n-1\}$, $n \geq 1$.

Definition 30. *(Gorenflo et al., 2014) The Mittag–Leffler functions (MLFs) of* 1 *and* 2 *parameters are written as:*

$$\mathbb{E}_{\nu}(\varpi) = \sum_{k=0}^{\infty} \frac{\varpi^k}{\Gamma(\nu k + 1)}, \qquad (\varpi \in \mathbb{R}, \nu > 0), \tag{8.4}$$

and

$$\mathbb{E}_{\nu,\lambda}(\varpi) = \sum_{k=0}^{\infty} \frac{\varpi^k}{\Gamma(\nu k + \lambda)}, \qquad (\nu, \lambda > 0, \varpi \in \mathbb{R}), \tag{8.5}$$

respectively. It is obvious that $\mathbb{E}_{1,1}(\varpi) = \mathbb{E}_{1}(\varpi) = e^{\varpi}$.

We represent the set $\mathbb{X}$ by:

$$\mathbb{X} = C^{\lambda}(\Omega) = \left\{ x : {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}x(\xi) \in C(\Omega) \right\}.$$

Equipped with the norm, we get:

$$\|x\|_{\mathbb{X}} = \|x\|_{\infty} + \left\| {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}x \right\|_{\infty},$$

where $\|x\|_\infty = \max_{\xi\in\Omega} |x(\xi)|$ and one can conclude that $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a BS.

Lemma 9. *For a given $\ell \in C(\Omega, \mathbb{R})$, $\lambda, \tau \in (0, 1]$ and $\sigma > 0$, the linear fractional initial value problem is as follows:*

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\tau;\psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\psi} - \sigma\right)\mathfrak{z}(\vartheta) = \ell(\vartheta), \\ \\ {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}(a) = \mathfrak{z}_\lambda, \qquad \mathfrak{z}(a) = \mathfrak{z}_a, \end{cases} \tag{8.6}$$

for $\vartheta \in \Omega$ is equivalent to the following Volterra integral equation.

$$\mathfrak{z}(\vartheta) = \mathfrak{z}_a + \frac{\mathfrak{z}_\lambda - \sigma\mathfrak{z}_a}{\Gamma(\lambda+1)}(\psi(\vartheta) - \psi(a))^\lambda + \sigma\mathbb{I}_{a^+}^{\lambda;\psi}\mathfrak{z}(\vartheta) + \mathbb{I}_{a^+}^{\tau+\lambda;\psi}\ell(\vartheta). \tag{8.7}$$

In addition, the Volterra integral Equation's (8.7) *explicit solution can be expressed as:*

$$\begin{aligned} \mathfrak{z}(\vartheta) =& \mathfrak{z}_a + \mathfrak{z}_\lambda\left(\psi(\vartheta) - \psi(a)\right)^\lambda \mathbb{E}_{\lambda,\lambda+1}\left(\sigma\left(\psi(\vartheta) - \psi(a)\right)^\lambda\right) \\ &+ \int_a^\vartheta \psi'(\eta)\left(\psi(\vartheta) - \psi(\eta)\right)^{\lambda+\tau-1} \\ &\times \mathbb{E}_{\lambda,\lambda+\tau}\left(\sigma\left(\psi(\vartheta) - \psi(\eta)\right)^\lambda\right)\ell(\eta)\,\mathrm{d}\eta. \end{aligned} \tag{8.8}$$

Proof. Employing the ψ-RL fractional integral of order τ to both sides of (8.6) and using Lemma 8, we get:

$${}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}(\vartheta) = \mathfrak{z}_\lambda + \sigma(\mathfrak{z}(\vartheta) - \mathfrak{z}_a) + \mathbb{I}_{a^+}^{\tau;\psi}\ell(\vartheta). \tag{8.9}$$

Hence, we have:

$$\mathfrak{z}(\vartheta) = \mathfrak{z}_a + \frac{\mathfrak{z}_\lambda - \sigma\mathfrak{z}_a}{\Gamma(\lambda+1)}(\psi(\vartheta) - \psi(a))^\lambda + \sigma\mathbb{I}_{a^+}^{\lambda;\psi}\mathfrak{z}(\vartheta) + \mathbb{I}_{a^+}^{\tau+\lambda;\psi}\ell(\vartheta). \tag{8.10}$$

The converse can be proven by direct computation. The technique of successive approximations is now applied to show that the Equation (8.7) can be expressed as:

$$\begin{aligned}\mathfrak{z}(\vartheta) =& \mathfrak{z}_a + \mathfrak{z}_\lambda(\psi(\vartheta)-\psi(a))^\lambda \mathbb{E}_{\lambda,\lambda+1}\left(\sigma(\psi(\vartheta)-\psi(a))^\lambda\right)\\ &+\int_a^\vartheta \psi'(\eta)(\psi(\vartheta)-\psi(\eta))^{\lambda+\tau-1}\\ &\times \mathbb{E}_{\lambda,\lambda+\tau}\left(\sigma(\psi(\vartheta)-\psi(\eta))^\lambda\right)\ell(\eta)\,\mathrm{d}\eta.\end{aligned}$$

For this, we set the following:

$$\begin{cases} \mathfrak{z}_0(\vartheta) = \mathfrak{z}_a + \dfrac{\mathfrak{z}_\lambda - \sigma\mathfrak{z}_a}{\Gamma(\lambda+1)}(\psi(\vartheta)-\psi(a))^\lambda, \\ \mathfrak{z}_m(\vartheta) = \mathfrak{z}_0(\vartheta) + \sigma\mathbb{I}_{a^+}^{\lambda;\psi}\mathfrak{z}_{m-1}(\vartheta) + \mathbb{I}_{a^+}^{\tau+\lambda;\psi}\ell(\vartheta). \end{cases} \tag{8.11}$$

It implied from Equation (8.11) and Lemma 8 that we get the following case:

$$\begin{aligned}\mathfrak{z}_1(\vartheta) &= \mathfrak{z}_0(\vartheta) + \sigma\mathbb{I}_{a^+}^{\lambda;\psi}\mathfrak{z}_0(\vartheta) + \mathbb{I}_{a^+}^{\lambda+\tau;\psi}\ell(\vartheta)\\ &= \mathfrak{z}_a + \frac{\mathfrak{z}_\lambda-\sigma\mathfrak{z}_a}{\Gamma(\lambda+1)}(\psi(\vartheta)-\psi(a))^\lambda + \sigma\frac{\mathfrak{z}_a}{\Gamma(\lambda+1)}\left[\psi(\vartheta)-\psi(a)\right]^\lambda \\ &\quad+\sigma\frac{\mathfrak{z}_\lambda-\sigma\mathfrak{z}_a}{\Gamma(2\lambda+1)}(\psi(\vartheta)-\psi(a))^{2\lambda} + \mathbb{I}_{a^+}^{\lambda+\tau;\psi}\ell(\vartheta).\end{aligned} \tag{8.12}$$

Similarly, Equations (8.11) and (8.12) and Lemma 8 yield the following:

$$\begin{aligned}
\mathfrak{z}_2(\vartheta) &= \mathfrak{z}_0(\vartheta) + \sigma \mathbb{I}_{a^+}^{\lambda;\psi} \mathfrak{z}_1(\vartheta) + \mathbb{I}_{a^+}^{\lambda+\tau;\psi} \ell(\vartheta) \\
&= \mathfrak{z}_a + \frac{\mathfrak{z}_\lambda - \sigma \mathfrak{z}_a}{\Gamma(\lambda+1)} (\psi(\vartheta) - \psi(a))^\lambda \\
&\quad + \sigma \mathbb{I}_{a^+}^{\lambda;\psi} \Bigg(\mathfrak{z}_a + \frac{\mathfrak{z}_\lambda - \sigma \mathfrak{z}_a}{\Gamma(\lambda+1)} (\psi(\vartheta) - \psi(a))^\lambda \\
&\quad + \sigma \frac{\mathfrak{z}_a}{\Gamma(\lambda+1)} (\psi(\vartheta) - \psi(a))^\lambda + \sigma \frac{\mathfrak{z}_\lambda - \sigma \mathfrak{z}_a}{\Gamma(2\lambda+1)} (\psi(\vartheta) \\
&\quad - \psi(a))^{2\lambda} + \mathbb{I}_{a^+}^{\lambda+\tau;\psi} \ell(\vartheta) \Bigg) + \mathbb{I}_{a^+}^{\lambda+\tau;\psi} \ell(\vartheta) \\
&= \mathfrak{z}_a + \sum_{k=0}^{2} \frac{\sigma^k}{\Gamma(k\lambda+1)} (\psi(\vartheta) - \psi(a))^{k\lambda} \\
&\quad + (\mathfrak{z}_\lambda - \sigma \mathfrak{z}_a) \sum_{k=0}^{2} \frac{\sigma^k}{\Gamma(k\lambda+\lambda+1)} (\psi(\vartheta) - \psi\psi(a))^{k\lambda+\lambda} \\
&\quad + \sigma \mathbb{I}_{a^+}^{2\lambda+\tau;\psi} \ell(\vartheta) + I_{a^+}^{\lambda+\tau;\psi} \ell(\vartheta) \\
&= \mathfrak{z}_a + \sum_{k=0}^{2} \frac{\sigma^k}{\Gamma(k\lambda+1)} (\psi(\vartheta) - \psi(a))^{k\lambda} \\
&\quad + (\mathfrak{z}_\lambda - \sigma \mathfrak{z}_a) \sum_{k=0}^{2} \frac{\sigma^k}{\Gamma(k\lambda+\lambda+1)} (\psi(\vartheta) - \psi(a))^{k\lambda+\lambda} \\
&\quad + \int_a^\vartheta \psi'(\eta) \sum_{k=1}^{2} \frac{\sigma^{k-1} (\psi(\vartheta) - \psi(\eta))^{k\lambda+\tau-1}}{\Gamma(k\lambda+\tau)} \ell(\eta) \, \mathrm{d}\eta.
\end{aligned}$$

Similarity, we derive the following:

$$
\begin{aligned}
\mathfrak{z}_m(\vartheta) &= \mathfrak{z}_a + \sum_{k=0}^{m} \frac{\sigma^k}{\Gamma(k\lambda+1)}(\psi(\vartheta)-\psi(a))^{k\lambda} \\
&\quad + (\mathfrak{z}_\lambda - \sigma\mathfrak{z}_a)\sum_{k=0}^{m} \frac{\sigma^k}{\Gamma(k\lambda+\lambda+1)}(\psi(\vartheta)-\psi(a))^{k\lambda+\lambda} \\
&\quad + \int_a^\vartheta \psi'(\eta)\sum_{k=1}^{m} \frac{\sigma^{k-1}(\psi(\vartheta)-\psi(\eta))^{k\lambda+\tau-1}}{\Gamma(k\lambda+\tau)}\ell(\eta)\mathrm{d}\eta \\
&= \mathfrak{z}_a + \sum_{k=0}^{m} \frac{\sigma^k}{\Gamma(k\lambda+1)}(\psi(\vartheta)-\psi(a))^{k\lambda} \\
&\quad + \frac{(\mathfrak{z}_\lambda - \sigma\mathfrak{z}_a)}{\sigma}\sum_{k=1}^{m} \frac{\sigma^k}{\Gamma(k\lambda+1)}(\psi(\vartheta)-\psi(a))^{k\lambda} \\
&\quad + \int_a^\vartheta \psi'(\eta)(\psi(\vartheta)-\psi(\eta))^{\lambda+\tau-1}\sum_{k=1}^{m} \frac{\sigma^{k-1}(\psi(\vartheta)-\psi(\eta))^{k\lambda-\lambda}}{\Gamma(k\lambda+\tau)}\ell(\eta)\,\mathrm{d}\eta.
\end{aligned}
$$

Taking the limit as $m \to \infty$, we get the explicit solution $\mathfrak{z}(\vartheta)$ of the Equation (8.7) as follows:

$$
\begin{aligned}
\mathfrak{z}(\vartheta) &= \mathfrak{z}_a + \sum_{k=0}^{\infty} \frac{\sigma^k}{\Gamma(k\lambda+1)}(\psi(\vartheta)-\psi(a))^{k\lambda} \\
&\quad + \frac{(\mathfrak{z}_\lambda - \sigma\mathfrak{z}_a)}{\sigma}\sum_{k=1}^{\infty} \frac{\sigma^k}{\Gamma(k\lambda+1)}(\psi(\vartheta)-\psi(a))^{k\lambda} \\
&\quad + \int_a^\vartheta \psi'(\eta)(\psi(\vartheta)-\psi(\eta))^{\lambda+\tau-1}\sum_{k=0}^{\infty} \frac{\sigma^{k}(\psi(\vartheta)-\psi(\eta))^{k\lambda}}{\Gamma(k\lambda+\lambda+\tau)}\ell(\eta)\,\mathrm{d}\eta \\
&= \mathfrak{z}_a + \frac{\mathfrak{z}_\lambda}{\sigma}\sum_{k=1}^{\infty} \frac{\sigma^k}{\Gamma(k\lambda+1)}(\psi(\vartheta)-\psi(a))^{k\lambda} \\
&\quad + \int_a^\vartheta \psi'(\eta)(\psi(\vartheta)-\psi(\eta))^{\lambda+\tau-1}\mathbb{E}_{\lambda,\lambda+\tau}\big(\sigma(\psi(\vartheta)-\psi(\eta))^{\lambda}\big)\ell(\eta)\,\mathrm{d}\eta \\
&= \mathfrak{z}_a + \mathfrak{z}_\lambda(\psi(\vartheta)-\psi(a))^{\lambda}\mathbb{E}_{\lambda,\lambda+1}\left(\sigma(\psi(\vartheta)-\psi(a))^{\lambda}\right) \\
&\quad + \int_a^\vartheta \psi'(\eta)(\psi(\vartheta)-\psi(\eta))^{\lambda+\tau-1}\mathbb{E}_{\lambda,\lambda+\tau}\left(\sigma(\psi(\vartheta)-\psi(\eta))^{\lambda}\right)\ell(\eta)\,\mathrm{d}\eta.
\end{aligned}
$$

Thus, this finalizes this proof.

Lemma 10 (Comparison Result)**.** *Let* λ*,* $\tau \in (0,1]$*, and* $\sigma > 0$*. If* $\Delta \in C(\Omega, \mathbb{R})$ *satisfies:*

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\Delta(\vartheta) \geq 0, & \vartheta \in (a,b], \\ \Delta(a) \geq 0, \qquad {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\Delta(a) \geq 0, \end{cases}$$

then $\Delta(\vartheta) \geq 0$ *and* ${}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\Delta(\vartheta) \geq 0$ *for all* $\vartheta \in \Omega$*.*

Proof. Since $\mathbb{E}_{\rho_1,\rho_2}(x) \geq 0$ for $\rho_1 \in (0,1]$, $\rho_2 \geq \rho_1$, $x \in \mathbb{R}$, we allow the following:

$$\ell(\vartheta) = {}^{c}\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\Delta(\vartheta) \geq 0,$$

$\Delta(a) = \mathfrak{z}_a \geq 0$ and ${}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\Delta(a) = \mathfrak{z}_\lambda \geq 0$ in Lemma 9. Then, it implies by Equations (8.8) and (8.9) that Lemma 10 holds.

Let $(\mathcal{X}, \mathcal{T})$ be a topological Hausdorff space and $g_1, g_2 : \mathcal{X} \to \mathbb{R}$ be a lower semi-CF and an upper semi-CF, respectively. Thus, for every $r \in \mathbb{R}$, the subsets of the following:

$$\left\{g_1 > r\right\} := \left\{x \in \mathcal{X} \,:\, g_1(x) > r\right\}, \qquad \left\{g_2 < r\right\} := \left\{x \in \mathcal{X} \,:\, g_2(x) < r\right\},$$

are open in $\mathcal{X}$. Suppose that $g_1(x) \leq g_2(x)$ for all $x \in \mathcal{X}$, and we allow the interval $[g_1, g_2]$ that contains those upper or lower semi-CFs $h : \mathcal{X} \to \mathbb{R} \ni g_1(x) \leq h(x) \leq g_2(x)$ for all $x \in \mathcal{X}$. Let $\Omega : [g_1, g_2] \to [g_1, g_2]$ be a monotone mapping in the sense that $g_1 \leq h_1 \leq h_2 \leq g_2$ implies $g_1 \leq \Omega(h_1) \leq \Omega(h_2) \leq g_2$. Additionally, we assume that the sequence $\{\Omega(h_n)\}_{n\in\mathbb{N}} \subset [g_1, g_2]$ contains lower semi-CFs that increase pointwise to $\Omega(h)$ whenever the sequence $\{h_n\}_{n\in\mathbb{N}} \subset [g_1, g_2]$ contains lower semi-CFs that increase pointwise to h. A similar assumption is done when the sequence $\{h_n\}_{n\in\mathbb{N}} \subset [g_1, g_2]$ contains upper semi-CFs, which decreases pointwise to $h \in [g_1, g_2]$. In particular, we suppose that $\Omega(h)$ is lower semi-CF whenever h is lower semi-CF and that $\Omega(h)$ is upper semi-CF whenever h is so. Then, for every $n \in \mathbb{N}$, we get:

$$g_1 \leq \Omega^n(g_1) \leq \Omega^{n+1}(g_1) \leq \Omega^{n+1}(g_2) \leq \Omega^n(g_2) \leq g_2.$$

Substitute $\omega_1 = \sup_{n\in\mathbb{N}} \Omega^n(g_1)$ and $\omega_2 = \sup_{n\in\mathbb{N}} \Omega^n(g_2)$. Then, ω_1 and ω_2 belong to the interval $[g_1, g_2]$, the function ω_1 is lower semi-CF, the function ω_2 is upper semi-CF, and the equalities $\Omega(\omega_1) = \omega_1$ and $\Omega(\omega_2) = \omega_2$ are valid. If the monotone mapping Ω has at most one fixed point, then $\omega_1 = \omega_2 = \Omega(\omega_1) = \Omega(\omega_2)$ is a CF. When the mapping $\Omega : [g_1, g_2] \to [g_1, g_2]$ does not posses this sequential continuity property, then one needs a more subtle version of the Tarski–Knaster fixed point theorem. We carefully define the functions h_{prefix} and $h_{postfix}$, respectively, by the following:

$$h_{\text{prefix}} = \inf \left\{ h \in [g_1, g_2]\, \Omega(h) \leq h,\ h \text{ is upper semi-CF} \right\},$$

and the following:

$$h_{\text{postfix}} = \sup \left\{ h \in [g_1, g_2]\, \Omega(h) \geq h,\ h \text{ is lower semi-CF} \right\}.$$

Then, we get:

$$\Omega(h_{\text{prefix}}) = h_{\text{prefix}} \leq \inf_{n\in\mathbb{N}} \Omega^n(g_2), \qquad \Omega(h_{\text{postfix}}) = h_{\text{postfix}} \geq \sup_{n\in\mathbb{N}} \Omega^n(g_1).$$

Moreover, the function h_{prefix} is upper semi-CF and the function h_{postfix} is lower semi-CF. Consequently, if Ω has at most one fixed point, then

$$\Omega(h_{\text{prefix}}) = h_{\text{prefix}} = \Omega(h_{\text{postfix}}) = h_{\text{postfix}},$$

and, therefore, this unique fixed point is a CF.

8.3 Main Results

The extremal solutions' existence for problem (8.1) is shown in this section. First, the definitions of LSo and USo of the problem (8.1) are provided.

Definition 31. *A function $\mathfrak{z}_0 \in \mathbb{X}$ is named as a LSo of Equation* (8.1)*, if it satisfies:*

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\mathfrak{z}_0(\vartheta) \leq \mathbb{F}\left(\vartheta, \mathfrak{z}_0(\vartheta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(\vartheta)\right), \\ \mathbb{H}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(a), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(b)\right) \leq 0, \qquad \mathbb{G}\left(\mathfrak{z}_0(a), \mathfrak{z}_0(b)\right) \leq 0, \end{cases}$$

for $\vartheta \in \Omega$.

Definition 32. *A function $\tilde{\mathfrak{z}}_0 \in \mathbb{X}$ is named as an USo of Equation* (8.1)*, if it satisfies:*

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\tilde{\mathfrak{z}}_0(\vartheta) \geq \mathbb{F}\left(\vartheta, \tilde{\mathfrak{z}}_0(\vartheta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(\vartheta)\right), \\ \mathbb{H}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(a), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(b)\right) \geq 0, \qquad \mathbb{G}\left(\tilde{\mathfrak{z}}_0(a), \tilde{\mathfrak{z}}_0(b)\right) \geq 0, \end{cases}$$

for each $\vartheta \in \Omega$.

Theorem 30. *Assume that $\mathbb{F} : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a CF $\ni$ the following assumptions hold:*

(H1) *$\exists$ $\mathfrak{z}_0$ and $\tilde{\mathfrak{z}}_0$ as LSo and USo of* (8.1) *in $\mathbb{X}$, respectively, with $\mathfrak{z}_0(\vartheta) \leq$*

$\tilde{\mathfrak{z}}_0(\vartheta)$ *and;*

$$^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(\vartheta), \qquad \vartheta \in \Omega.$$

(H2) $\mathbb{F}$ *satisfies the following condition:*

$$\mathbb{F}(\vartheta, y(\vartheta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}y(\vartheta)) \leq \mathbb{F}(\vartheta, \mathfrak{z}(\vartheta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(\vartheta)),$$

for $y_0(\vartheta) \leq y(\vartheta) \leq \mathfrak{z}(\vartheta) \leq \mathfrak{z}_0(\vartheta)$ *and the following:*

$$^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}y_0(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}y(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(\vartheta),$$

for each $\vartheta \in \Omega$.

(H3) $\exists$ *constants* $c > 0$ *and* $d \geq 0$, $\ni$ *for* $\mathfrak{z}_0(a) \leq \xi_1 \leq \xi_2 \leq \tilde{\mathfrak{z}}_0(a)$ *and* $\mathfrak{z}_0(b) \leq \zeta_1 \leq \zeta_2 \leq \tilde{\mathfrak{z}}_0(b)$,

$$\mathbb{G}(\xi_2, \zeta_2) - \mathbb{G}(\xi_1, \zeta_1) \leq c(\xi_2 - \xi_1) - d(\zeta_2 - \zeta_1).$$

(H4) $\exists$ *constants* $e > 0$ *and* $f \geq 0$, $\ni$ *for the following:*

$$^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(a) \leq \xi_1 \leq \xi_2 \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(a),$$

$$^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(b) \leq \zeta_1 \leq \zeta_2 \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(b),$$

and the following is obtained:

$$\mathbb{H}(\xi_2, \zeta_2) - \mathbb{H}(\xi_1, \zeta_1) \leq e(\xi_2 - \xi_1) - f(\zeta_2 - \zeta_1).$$

Hence, $\exists$ *monotone iterative sequences* $\{\mathfrak{z}_n\}$ *and* $\{\tilde{\mathfrak{z}}_n\}$*, which converge uniformly* *on* Ω *to the extremal solutions of* (8.1) *in the sector* $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$*, where*

$$[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0] = \Big\{\mathfrak{z} \in \mathbb{X} \,:\, \mathfrak{z}_0(\vartheta) \leq \mathfrak{z}(\vartheta) \leq \tilde{\mathfrak{z}}_0(\vartheta), \quad \vartheta \in \Omega\Big\}.$$

Proof. For any $\mathfrak{z}_0, \tilde{\mathfrak{z}}_0 \in \mathbb{X}$, we define:

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\mathfrak{z}_{n+1}(\vartheta) = \mathbb{F}\left(\vartheta, \mathfrak{z}_n(\vartheta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_n(\vartheta)\right), & \vartheta \in \Omega, \\ {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_{n+1}(a) = {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_n(a) - \frac{1}{e}\mathbb{H}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_n(a), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_n(b)\right), \\ \mathfrak{z}_{n+1}(a) = \mathfrak{z}_n(a) - \frac{1}{c}\mathbb{G}\left(\mathfrak{z}_n(a), \mathfrak{z}_n(b)\right), \end{cases} \tag{8.13}$$

and the following as well.

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\tilde{\mathfrak{z}}_{n+1}(\vartheta) = \mathbb{F}\left(\vartheta, \tilde{\mathfrak{z}}_n(\vartheta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_n(\vartheta)\right), & \vartheta \in \Omega, \\ {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_{n+1}(a) = {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_n(a) - \frac{1}{e}\mathbb{H}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_n(a), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_n(b)\right), \\ \tilde{\mathfrak{z}}_{n+1}(a) = \tilde{\mathfrak{z}}_n(a) - \frac{1}{c}\mathbb{G}\left(\tilde{\mathfrak{z}}_n(a), \tilde{\mathfrak{z}}_n(b)\right). \end{cases} \tag{8.14}$$

By Lemma 9, we know that (8.13) and (8.14) have unique solutions in $\mathbb{X}$ that are the following:

$$
\begin{aligned}
\mathfrak{z}_{n+1}(\vartheta) = {} & \mathfrak{z}_n(a) - \frac{1}{c}\mathbb{G}\left(\mathfrak{z}_n(a), \mathfrak{z}_n(b)\right) \\
& + \left({}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}_n(a) - \frac{1}{e}\mathbb{H}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}_n(a), {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}_n(b)\right)\right) \\
& \times \left(\psi(\vartheta) - \psi(a)\right)^{\lambda}\mathbb{E}_{\lambda,\lambda+1}\left(\sigma(\psi(\vartheta) - \psi(a))^{\lambda}\right) \\
& + \int_a^{\vartheta} \psi'(\eta)\left(\psi(\vartheta) - \psi(\eta)\right)^{\lambda+\tau-1} \\
& \times \mathbb{E}_{\lambda,\lambda+\tau}\left(\sigma\left(\psi(\vartheta) - \psi(\eta)\right)^{\lambda}\right)\mathbb{F}\left(\eta, \mathfrak{z}_n(\eta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}_n(\eta)\right) \,\mathrm{d}\eta, \\
\tilde{\mathfrak{z}}_{n+1}(\vartheta) = {} & \tilde{\mathfrak{z}}_n(a) - \frac{1}{c}\mathbb{G}\left(\tilde{\mathfrak{z}}_n(a), \tilde{\mathfrak{z}}_n(b)\right) \\
& + \left({}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\tilde{\mathfrak{z}}_n(a) - \frac{1}{e}\mathbb{H}({}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\tilde{\mathfrak{z}}_n(a), {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\tilde{\mathfrak{z}}_n(b))\right) \\
& \times \left(\psi(\vartheta) - \psi(a)\right)^{\lambda}\mathbb{E}_{\lambda,\lambda+1}\left(\sigma(\psi(\vartheta) - \psi(a))^{\lambda}\right) \\
& + \int_a^{\vartheta} \psi'(\eta)(\psi(\vartheta) - \psi(\eta))^{\lambda+\tau-1} \\
& \times \mathbb{E}_{\lambda,\lambda+\tau}\left(\sigma(\psi(\vartheta) - \psi(\eta))^{\lambda}\right)\mathbb{F}\left(\eta, \tilde{\mathfrak{z}}_n(\eta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\tilde{\mathfrak{z}}_n(\eta)\right) \,\mathrm{d}\eta.
\end{aligned}
$$

First, we show that the sequences $\mathfrak{z}_n(\vartheta)$ and $\tilde{\mathfrak{z}}_n(\vartheta)(n \geq 1)$ are LSo and USo of Equation (8.1), respectively, and $\mathfrak{z}_n(\vartheta)$ and $\tilde{\mathfrak{z}}_n(\vartheta)(n \geq 1)$ satisfy the following relations:

$$
\mathfrak{z}_0(\vartheta) \leq \mathfrak{z}_1(\vartheta) \leq \cdots \leq \mathfrak{z}_n(\vartheta) \leq \cdots \leq \tilde{\mathfrak{z}}_n(\vartheta) \leq \cdots \leq \tilde{\mathfrak{z}}_1(\vartheta) \leq \tilde{\mathfrak{z}}_0(\vartheta), \tag{8.15}
$$

for $\vartheta \in \Omega$, and we get:

$$
\begin{aligned}
{}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}_0(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}_1(\vartheta) \leq \cdots \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}_n(\vartheta) \leq \cdots \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\tilde{\mathfrak{z}}_n(\vartheta) \leq \cdots \\
\leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\tilde{\mathfrak{z}}_1(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\psi}\tilde{\mathfrak{z}}_0(\vartheta),
\end{aligned} \tag{8.16}
$$

for $\vartheta \in \Omega$, respectively. Now, we show that $\mathfrak{z}_0(\vartheta) \leq \mathfrak{z}_1(\vartheta) \leq \tilde{\mathfrak{z}}_1(\vartheta) \leq \tilde{\mathfrak{z}}_0(\vartheta)$, for $\vartheta \in \Omega$ and

$$ {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(\vartheta) \leq {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_1(\vartheta) \leq {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_1(\vartheta) \leq {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(\vartheta), $$

for each $\vartheta \in \Omega$. For this end, set $\Delta(\vartheta) = \mathfrak{z}_1(\vartheta) - \mathfrak{z}_0(\vartheta)$. From (8.13) and Definition 31, we get:

$$ \begin{aligned} {}^c\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^c\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\Delta(\vartheta) &= {}^c\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^c\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\mathfrak{z}_1(\vartheta) - {}^c\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^c\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\mathfrak{z}_0(\vartheta) \\ &= \mathbb{F}\left(\vartheta, \mathfrak{z}_0(\vartheta), {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(\vartheta)\right) - {}^c\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^c\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\mathfrak{z}_0(\vartheta) \geq 0. \end{aligned} $$

Again, we have:

$$ \begin{cases} \Delta(a) = -\frac{1}{c}\mathbb{G}(\mathfrak{z}_0(a), \mathfrak{z}_0(b)) \geq 0, \\ \\ {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\Delta(a) = -\frac{1}{e}\mathbb{H}({}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(a), {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(b)) \geq 0. \end{cases} $$

Invoking Lemma 10, we obtain $\Delta(\vartheta) \geq 0$ and ${}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\Delta(\vartheta) \geq 0$ for $\vartheta \in \Omega$. Thus, $\mathfrak{z}_0(\vartheta) \leq \mathfrak{z}_1(\vartheta)$ and the following is the case:

$$ {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(\vartheta) \leq {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_1(\vartheta), $$

$\vartheta \in \Omega$. Similarity, $\tilde{\mathfrak{z}}_1(\vartheta) \leq \tilde{\mathfrak{z}}_0(\vartheta)$ and the following:

$$ {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_1(\vartheta) \leq {}^c\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(\vartheta), $$

$\vartheta \in \Omega$. Now, let $\Delta(\vartheta) = \tilde{\mathfrak{z}}_1(\vartheta) - \mathfrak{z}_1(\vartheta)$. Using (8.13) and (8.14) together with assumptions (H_1)–(H_3) we obtain the following.

$$ {}^{c}\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}-\sigma\right)\Delta(\vartheta)=\mathbb{F}\left(\vartheta,\tilde{\mathfrak{z}}_0(\vartheta),{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(\vartheta)\right)-\mathbb{F}\left(\vartheta,\mathfrak{z}_0(\vartheta),{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(\vartheta)\right)\geq 0. $$

Notice the following inequalities:

$$ \begin{aligned} \Delta(a) &= \tilde{\mathfrak{z}}_0(a)-\mathfrak{z}_0(a)-\frac{1}{c}\left[\mathbb{G}\left(\tilde{\mathfrak{z}}_0(a),\tilde{\mathfrak{z}}_0(b)\right)-\mathbb{G}\left(\mathfrak{z}_0(a),\mathfrak{z}_0(b)\right)\right] \\ &\geq \frac{d}{c}\left(\tilde{\mathfrak{z}}_0(b)-\mathfrak{z}_0(b)\right)\geq 0, \end{aligned} $$

and the following is the case.

$$ \begin{aligned} {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\Delta(a) &= {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(a)-{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(a) \\ &\quad -\frac{1}{e}\left[\mathbb{H}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(a),{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(b)\right)-\mathbb{H}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(a),{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(b)\right)\right] \\ &\geq \frac{f}{e}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_0(b)-{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(b)\right)\geq 0. \end{aligned} $$

According to Lemma 10, we obtain$\mathfrak{z}_1(\vartheta)\leq\tilde{\mathfrak{z}}_1(\vartheta)$ and

$$ {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_1(\vartheta)\leq{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_1(\vartheta), $$

$\vartheta\in\Omega$. Next, we show that the functions $\mathfrak{z}_1(\vartheta),\tilde{\mathfrak{z}}_1(\vartheta)$ are a LSo and an USo of the Equation (8.1), respectively. Since $\mathfrak{z}_0$ and $\tilde{\mathfrak{z}}_0$ are lower and upper solutions of (8.1), by (H_2) and (H_3), it implies that the following:

$$ \begin{aligned} {}^{c}\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}-\sigma\right)\mathfrak{z}_1(\vartheta) &= \mathbb{F}\left(\vartheta,\mathfrak{z}_0(\vartheta),{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(\vartheta)\right)\leq\mathbb{F}\left(\vartheta,\mathfrak{z}_1(\vartheta),{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_1(\vartheta)\right) \\ \mathbb{G}(\mathfrak{z}_1(a),\mathfrak{z}_1(b)) &\leq \mathbb{G}(\mathfrak{z}_0(a),\mathfrak{z}_0(b))+c\left(\mathfrak{z}_1(a)-\mathfrak{z}_0(a)\right)-d(\mathfrak{z}_1(b)-\mathfrak{z}_0(b)) \\ &= -d(\mathfrak{z}_1(b)-\mathfrak{z}_0(b))\leq 0, \end{aligned} $$

and the following is obtained:

$$\mathbb{H}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_1(a), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_1(b)\right) \leq \mathbb{H}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(a), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(b)\right)$$
$$+ e\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_1(a) - {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(a)\right)$$
$$- f({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_1(b) - {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(b))$$
$$= -f({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_1(b) - {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_0(b)) \leq 0.$$

Therefore, $\mathfrak{z}_1(\vartheta)$ is a LSo of Equation (8.1). Analogously, it can be obtained that $\tilde{\mathfrak{z}}_1(\vartheta)$ is an USo of Equation (8.1). By the above arguments and mathematical induction, we can show that the sequences $\mathfrak{z}_n(\vartheta), \tilde{\mathfrak{z}}_n(\vartheta)$, $(n \geq 1)$ are LSo and USo of Equation (8.1), respectively, and the relations (8.15) and (8.16) are true. On the contrary, by employing the earlier arguments, together with Ascoli–Arzela's Theorem, we can show:

$$\|\mathfrak{z}_n - \mathfrak{z}^*\|_{\mathbb{X}} \to 0, \qquad \|\tilde{\mathfrak{z}}_n - \tilde{\mathfrak{z}}^*\|_{\mathbb{X}} \to 0,$$

when $n \to \infty$. Finally, it remains to show that $\mathfrak{z}^*$ and $\tilde{\mathfrak{z}}^*$ are extremal solutions of (8.1) in $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$. To conduct this, let $\mathfrak{z} \in [\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$ be any solution of (8.1). Suppose for some $n \in \mathbb{N}^*$ that the following is the case:

$$\mathfrak{z}_n(\vartheta) \leq \mathfrak{z}(\vartheta) \leq \tilde{\mathfrak{z}}_n(\vartheta), \qquad {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_n(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_n(\vartheta), \quad (8.17)$$

for $\vartheta \in \Omega$. Setting $\Delta(\vartheta) = \mathfrak{z}(\vartheta) - \mathfrak{z}_{n+1}(\vartheta)$. It implies that the following is obtained:

$${}^{c}\mathbb{D}_{a^+}^{\tau;\Psi}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi} - \sigma\right)\Delta(\vartheta) = \mathbb{F}\left(\vartheta, \mathfrak{z}(\vartheta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(\vartheta)\right) - \mathbb{F}\left(\vartheta, \mathfrak{z}_n(\vartheta), {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_n(\vartheta)\right) \geq 0.$$

Notice the inequalities in the following:

$$\begin{aligned}\mathfrak{z}_{n+1}(a) &= \mathfrak{z}_n(a) + \frac{1}{c}\big[\mathbb{G}(\mathfrak{z}(a), \mathfrak{z}(b)) - \mathbb{G}(\mathfrak{z}_n(a), \mathfrak{z}_n(b))\big] \\ &\leq \mathfrak{z}(a) - \frac{d}{c}\big(\mathfrak{z}(b) - \mathfrak{z}_n(b)\big) \\ &\leq \mathfrak{z}(a),\end{aligned}$$

and the following is obtained:

$$\begin{aligned}{}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_{n+1}(a) &= {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_n(a) + \frac{1}{e}\left[\mathbb{H}\left(\mathfrak{z}(a), \mathfrak{z}\mathfrak{z}(b)\right) - \mathbb{H}\left(\mathfrak{z}_n(a), \mathfrak{z}\mathfrak{z}_n(b)\right)\right] \\ &\leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(a) - \frac{f}{e}\left({}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(b) - {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_n(b)\right) \\ &\leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(a).\end{aligned}$$

By Lemma 10, we obtain $\Delta(\vartheta) \geq 0$, $\vartheta \in \Omega$, which implies $\mathfrak{z}_{n+1}(\vartheta) \leq \mathfrak{z}(\vartheta)$ and the following:

$${}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_{n+1}(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(\vartheta),$$

for almost all $\vartheta \in \Omega$.

By using the same method, we can show that $\mathfrak{z}(\vartheta) \leq \tilde{\mathfrak{z}}_{n+1}(\vartheta)$ and

$${}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_{n+1}(\vartheta),$$

for each $\vartheta \in \Omega$. Hence, $\mathfrak{z}_{n+1}(\vartheta) \leq \mathfrak{z}(\vartheta) \leq \tilde{\mathfrak{z}}_{n+1}(\vartheta)$, for $\vartheta \in \Omega$, and the following is the case:

$${}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}_{n+1}(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\mathfrak{z}(\vartheta) \leq {}^{c}\mathbb{D}_{a^+}^{\lambda;\Psi}\tilde{\mathfrak{z}}_{n+1}(\vartheta),$$

for $\vartheta \in \Omega$. Therefore, (8.17) holds on Ω for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$

on both sides of (8.17), we get: $\mathfrak{z}^*(\vartheta) \leq \mathfrak{z}(\vartheta) \leq \tilde{\mathfrak{z}}^*(\vartheta)$ and the following:

$$ {}^c\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}^*(\vartheta) \leq {}^c\mathbb{D}_{a^+}^{\lambda;\psi}\mathfrak{z}(\vartheta) \leq {}^c\mathbb{D}_{a^+}^{\lambda;\psi}\tilde{\mathfrak{z}}^*(\vartheta), $$

for each $\vartheta \in \Omega$. Then, $\mathfrak{z}^*$, $\tilde{\mathfrak{z}}^*$ are the extremal solutions of (8.1) in $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$. Thus, the proof of Theorem 30 is finalized.

8.4 Numerical Experiments

Our numerical experiments have been done with the help of MATLAB.

Example 8.4.1. *Consider Problem* (8.1) *with:*

$$ \tau = \lambda = 0.5, \sigma = \frac{\sqrt{\pi}}{2}, a = 0, b = 1, \psi(\vartheta) = \vartheta. \tag{8.18} $$

In order to validate Theorem 30, we set:

$$ \begin{aligned} \mathbb{F}\left(\vartheta, \mathfrak{z}(\vartheta), {}^c\mathbb{D}_{0^+}^{\lambda;\psi}\mathfrak{z}(\vartheta)\right) = (1 - \sqrt{\vartheta}) \\ \times \exp\left(\mathfrak{z}(\vartheta) + {}^c\mathbb{D}_{0^+}^{\lambda;\psi}\mathfrak{z}(\vartheta) - \frac{2}{\sqrt{\pi}} - 2\right), \end{aligned} \tag{8.19} $$

for $\vartheta \in [0, 1]$, *and we get:*

$$ \begin{aligned} &\mathbb{H}\left({}^c\mathbb{D}_{0^+}^{\lambda;\psi}\mathfrak{z}(0), {}^c\mathbb{D}_{0^+}^{\lambda;\psi}\mathfrak{z}(1)\right) = {}^c\mathbb{D}_{0^+}^{\lambda;\psi}\mathfrak{z}(0), \\ &\mathbb{G}\left(\mathfrak{z}(0), \mathfrak{z}(1)\right) = \mathfrak{z}(0) - 1. \end{aligned} \tag{8.20} $$

Clearly, $\mathbb{F}$, $\mathbb{G}$ *and* $\mathbb{H}$ *are CFs. In addition, we can easily verify that* $\mathfrak{z}_0(\vartheta) = 1$ *and* $\tilde{\mathfrak{z}}_0(\vartheta) = 1 + \vartheta$ *are LOs and USo of Equation* (8.1)*, respectively. In addition,* *we get:* $\mathfrak{z}_0(\vartheta) \leq \tilde{\mathfrak{z}}_0(\vartheta)$ *and the following:*

$$ {}^{c}\mathbb{D}_{0^+}^{\lambda;\Psi} \mathfrak{z}_0(\vartheta) \leq {}^{c}\mathbb{D}_{0^+}^{\lambda;\Psi} \tilde{\mathfrak{z}}_0(\vartheta), $$

for all $\vartheta \in [0, 1]$. *On the contrary,the assumptions* ***(H2)–(H4)*** *of Theorem 30 are satisfied. So, An application of Theorem 30 shows that the problem* (8.1) *with the data* (8.18) *and* (8.19) *has extremal solutions in* $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$*, which can be approximated by the following iterative sequences:*

$$ \mathfrak{z}_{n+1}(\vartheta) = 1 + \int_0^{\vartheta} \mathbb{E}_{0.5}\left(\frac{\sqrt{\pi(\vartheta - \eta)}}{2}\right)(1 - \sqrt{\eta}) \times \exp\left(\mathfrak{z}_n(\eta) + {}^{c}\mathbb{D}_{0^+}^{\lambda;\Psi} \mathfrak{z}_n(\eta) - \frac{2}{\sqrt{\pi}} - 2\right) \mathrm{d}\eta, \tag{8.21} $$

with $\mathfrak{z}_0(\vartheta) = 1$ *and the following:*

$$ \tilde{\mathfrak{z}}_{n+1}(\vartheta) = 1 + \int_0^{\vartheta} \mathbb{E}_{0.5}\left(\frac{\sqrt{\pi(\vartheta - \eta)}}{2}\right)(1 - \sqrt{\eta}) \times \exp\left(\tilde{\mathfrak{z}}_n(\eta) + {}^{c}\mathbb{D}_{0^+}^{\lambda;\Psi} \tilde{\mathfrak{z}}_n(\eta) - \frac{2}{\sqrt{\pi}} - 2\right) \mathrm{d}\eta, \tag{8.22} $$

with $\tilde{\mathfrak{z}}_0(\vartheta) = 1 + \vartheta$. *Tables 8.1 and 8.2 show the numerical results of the iterative sequences of* $\mathfrak{z}_{n+1}(\vartheta)$ *for* $\vartheta = 0$, 0.1, 0.2, 0.3, 0.4 *and* 0.5, 0.6, 0.7, 0.8, 0.9*, respectively. Our results are graphically represented in Figure 8.1.*

[$\vartheta = 0, 0.1, 0.2, 0.3, 0.4$]

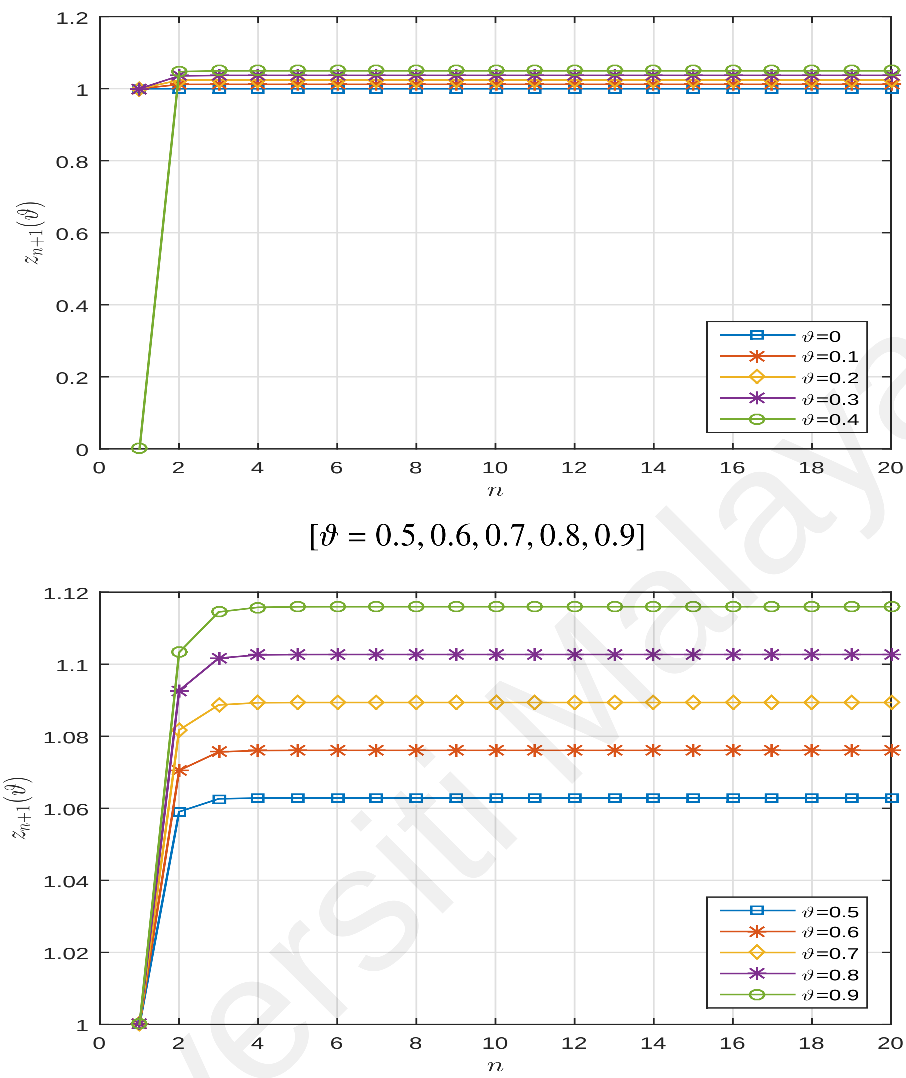

Figure 8.1: Graphical representation of $\mathfrak{z}_{n+1}(\vartheta)$ in Example 8.4.1.

Tables 8.3 and 8.4 show the numerical results of the iterative sequences of $\tilde{\mathfrak{z}}_{n+1}(\vartheta)$ for $\vartheta = 0, 0.1, 0.2, 0.3, 0.4$ and $0.5, 0.6, 0.7, 0.8, 0.9$, respectively. We plot these results in Figure 8.2a,b.

8.5 Conclusion

A new type of ψ-CpFDfEq has been studied in this chapter. The addressed problem is considered in the framework of nonlinear boundary value conditions.

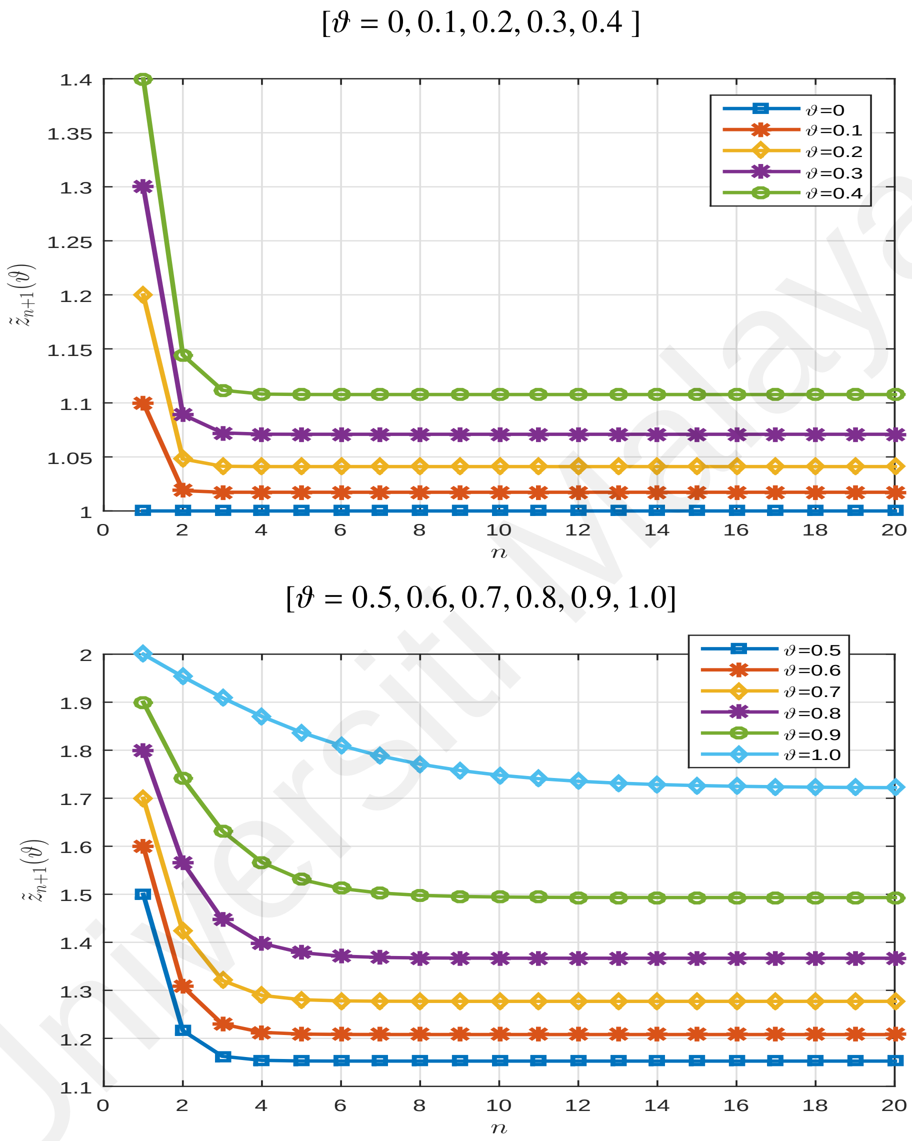

Figure 8.2: Graphical representation of $\tilde{\mathfrak{z}}_{n+1}(\vartheta)$ in Example 8.4.1.

Table 8.1: Numerical Experiment of $\mathfrak{z}_{n+1}(\vartheta)$ for $n = 1, 2, \ldots, 20$ and $\vartheta = 0, 0.1,$ 0.2, 0.3, 0.4 by using (8.21) in Example 8.4.1.

n	ϑ				
	0	**0.1**	**0.2**	**0.3**	**0.4**
1	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.011901	1.023786	1.035626	1.047386
3	1.000000	1.012044	1.024359	1.036918	1.049685
4	1.000000	1.012045	1.024373	1.036966	1.049799
5	1.000000	1.012045	1.024373	1.036968	1.049805
6	1.000000	1.012045	1.024373	1.036968	1.049805
7	1.000000	1.012045	1.024373	1.036968	1.049805
8	1.000000	1.012045	1.024373	1.036968	1.049805
9	1.000000	1.012045	1.024373	1.036968	1.049805
10	1.000000	1.012045	1.024373	1.036968	1.049805
11	1.000000	1.012045	1.024373	1.036968	1.049805
12	1.000000	1.012045	1.024373	1.036968	1.049805
13	1.000000	1.012045	1.024373	1.036968	1.049805
14	1.000000	1.012045	1.024373	1.036968	1.049805
15	1.000000	1.012045	1.024373	1.036968	1.049805
16	1.000000	1.012045	1.024373	1.036968	1.049805
17	1.000000	1.012045	1.024373	1.036968	1.049805
18	1.000000	1.012045	1.024373	1.036968	1.049805
19	1.000000	1.012045	1.024373	1.036968	1.049805
20	1.000000	1.012045	1.024373	1.036968	1.049805

Our results are unique and novel due to the employment of monotone iterative technique along with the technqiue of USo and LSo in comparison to the previous works which were based on fixed point techniques. The applied techniques are closely similar to the Tarski–Knaster theorem which also gets fixed point results. Our results have been illustrated through numerical experiments using explicit numerical values.

Table 8.2: Numerical Experiment of $\mathfrak{z}_{n+1}(\vartheta)$ for $n = 1, 2, \ldots, 20$ and $\vartheta = 0.5$, 0.6, 0.7, 0.8, 0.9 by using (8.21) in Example 8.4.1.

n	$\boldsymbol{\vartheta}$				
	0.5	**0.6**	**0.7**	**0.8**	**0.9**
1	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.059021	1.070482	1.081716	1.092662	1.103258
3	1.062609	1.075629	1.088674	1.101659	1.114490
4	1.062834	1.076019	1.089293	1.102578	1.115784
5	1.062848	1.076049	1.089348	1.102672	1.115933
6	1.062849	1.076051	1.089353	1.102682	1.115951
7	1.062849	1.076051	1.089353	1.102683	1.115953
8	1.062849	1.076051	1.089353	1.102683	1.115953
9	1.062849	1.076051	1.089353	1.102683	1.115953
10	1.062849	1.076051	1.089353	1.102683	1.115953
11	1.062849	1.076051	1.089353	1.102683	1.115953
12	1.062849	1.076051	1.089353	1.102683	1.115953
13	1.062849	1.076051	1.089353	1.102683	1.115953
14	1.062849	1.076051	1.089353	1.102683	1.115953
15	1.062849	1.076051	1.089353	1.102683	1.115953
16	1.062849	1.076051	1.089353	1.102683	1.115953
17	1.062849	1.076051	1.089353	1.102683	1.115953
18	1.062849	1.076051	1.089353	1.102683	1.115953
19	1.062849	1.076051	1.089353	1.102683	1.115953
20	1.062849	1.076051	1.089353	1.102683	1.115953

Table 8.3: Numerical results of $\tilde{\mathfrak{z}}_{n+1}(\vartheta)$ for $n = 1, 2, \ldots, 20$ and $\vartheta = 0$, 0.1, 0.2, 0.3, 0.4 by using (8.22) in Example 8.4.1.

n	ϑ				
	0	**0.1**	**0.2**	**0.3**	**0.4**
1	1.000000	1.100000	1.200000	1.300000	1.400000
2	1.000000	1.018793	1.048121	1.089222	1.144311
3	1.000000	1.017327	1.041341	1.072265	1.111752
4	1.000000	1.017302	1.041061	1.071050	1.108172
5	1.000000	1.017301	1.041050	1.070964	1.107785
6	1.000000	1.017301	1.041049	1.070958	1.107743
7	1.000000	1.017301	1.041049	1.070957	1.107739
8	1.000000	1.017301	1.041049	1.070957	1.107738
9	1.000000	1.017301	1.041049	1.070957	1.107738
10	1.000000	1.017301	1.041049	1.070957	1.107738
11	1.000000	1.017301	1.041049	1.070957	1.107738
12	1.000000	1.017301	1.041049	1.070957	1.107738
13	1.000000	1.017301	1.041049	1.070957	1.107738
14	1.000000	1.017301	1.041049	1.070957	1.107738
15	1.000000	1.017301	1.041049	1.070957	1.107738
16	1.000000	1.017301	1.041049	1.070957	1.107738
17	1.000000	1.017301	1.041049	1.070957	1.107738
18	1.000000	1.017301	1.041049	1.070957	1.107738
19	1.000000	1.017301	1.041049	1.070957	1.107738
20	1.000000	1.017301	1.041049	1.070957	1.107738

Table 8.4: Numerical Experiment of $\tilde{\mathfrak{z}}_{n+1}(\vartheta)$ for $n = 1, 2, \ldots, 20$ and $\vartheta = 0.5$, 0.6, 0.7, 0.8, 0.9, 1 by using (8.22) in Example 8.4.1.

n	$\boldsymbol{\vartheta}$					
	0.5	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
1	1.500000	1.600000	1.700000	1.800000	1.900000	2.000000
2	1.216106	1.307784	1.422976	1.565785	1.740782	1.953002
3	1.162695	1.229793	1.320631	1.447644	1.631747	1.909249
4	1.154233	1.212553	1.289440	1.397764	1.566487	1.870324
5	1.152934	1.208920	1.280551	1.378410	1.530698	1.837098
6	1.152735	1.208162	1.278068	1.371156	1.512041	1.809741
7	1.152705	1.208004	1.277379	1.368474	1.502577	1.787890
8	1.152700	1.207972	1.277187	1.367487	1.497842	1.770860
9	1.152699	1.207965	1.277134	1.367124	1.495491	1.757843
10	1.152699	1.207963	1.277120	1.366991	1.494327	1.748043
11	1.152699	1.207963	1.277116	1.366942	1.493752	1.740747
12	1.152699	1.207963	1.277115	1.366925	1.493469	1.735363
13	1.152699	1.207963	1.277114	1.366918	1.493329	1.731414
14	1.152699	1.207963	1.277114	1.366916	1.493260	1.728532
15	1.152699	1.207963	1.277114	1.366915	1.493225	1.726435
16	1.152699	1.207963	1.277114	1.366914	1.493209	1.724913
17	1.152699	1.207963	1.277114	1.366914	1.493200	1.723811
18	1.152699	1.207963	1.277114	1.366914	1.493196	1.723013
19	1.152699	1.207963	1.277114	1.366914	1.493194	1.722437
20	1.152699	1.207963	1.277114	1.366914	1.493193	1.722021

CHAPTER 9: ON THE OSCILLATION OF EVEN-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH MIXED NEUTRAL TERMS

9.1 Introduction

In this chapter, the even-order nonlinear differential equations' (NLDfEq) oscillation with mixed nonlinear neutral terms (MNLNTs) is studied. Novel oscillation criteria are also proposed.

Differential equations with a sub-linear neutral term have been studied in (Agarwal et al., 2014). Grace et al. (2019) initiated differential equations consisting of both sub-linear and super-linear neutral terms, where a 2nd-order half-linear differential equation has been studied as:

$$\left(r(t)\left[y^{(n-1)}(t)\right]^{\alpha}\right)' + q(t)x^{\gamma}(\tau_1(t)) = 0, \tag{9.1}$$

where $n > 0$ is an even integer, and

$$y(t) = x(t) + p_1(t)x^{\beta}(\tau_2(t)) - p_2(t)x^{\delta}(\tau_2(t)). \tag{9.2}$$

From Eq. (9.1) and Eq. (9.2), some assumptions are:

(i) α, β, γ and δ are the two positive odd integers' ratios with $\alpha \geq 1$;

(ii) $p_1, p_2, q : [t_0, \infty) \to \mathbb{R}^+$ are CFs;

(iii) $\tau_k : [t_0, \infty) \to \mathbb{R}$ are CFs; $\tau_k(t) \leq t$ and $\tau_k(t) \to \infty$ as $t \to \infty$ for $k = 1, 2$;

(iv) $h(t) = \tau_2^{-1}(\tau_1(t)) \leq t$ and $h(t) \to \infty$ as $t \to \infty$.

Let us suppose that

$$A^*(t, t_0) := \int_{t_0}^{t} (t - s)^{(n-2)} A(s, t_0) ds \to \infty \quad \text{as} \quad t \to \infty \tag{9.3}$$

for which

$$A(t, t_0) := \int_{t_0}^{t} r^{-1/\alpha}(s)ds \to \infty \quad \text{as} \quad t \to \infty.$$

A CF x satisfying Eq. (9.1) on $[t_*, \infty)$, $t_* \geq t_0$, is supposed to be a solution of Eq. (9.1) on $[t_*, \infty)$ where $y(t)$ is defined in Eq. (9.2). We only consider those solutions x of Eq. (9.1) which satisfy

$$\sup\{|x(t)| : t \geq t^*\} > 0 \quad \text{for all} \quad t^* \geq t_*.$$

On one hand, a solution x of Eq. (9.1) is said to be oscillatory if $\exists$ a sequence $\{\xi_n\}$ $\ni x(\xi_n) = 0$ and

$$\lim_{n\to\infty} \xi_n = \infty.$$

On the other hand, it is non-oscillatory. Eq. (9.1) is oscillatory (or non-oscillatory) equation if all its solutions are oscillatory (or non-oscillatory).

The higher-order differential equations with nonlinear neutral terms have not been investigated yet in other research works. As a result, the proposed differential equation's oscillation in Equation (9.1) is studied in detail. Novel oscillation results for Equation (9.1) are established via the comparison with the $1st$-order delay differential equations whose oscillatory characters are known via an the integral criterion.

9.2 Main Results

Equation's (9.1) oscillation criteria are investigated when $\beta < 1$ and $\delta > 1$.

The following lemma is essential for our results:

Lemma 11. *(Grace et al., 2012) Assume that $\mathcal{X}$ and $\mathcal{Y}$ are two non-negative real* *numbers. Then, we get:*

$$\mathcal{X}^{\lambda} + (\lambda - 1)\mathcal{Y}^{\lambda} - \lambda \mathcal{X}\mathcal{Y}^{\lambda-1} \begin{cases} \geq 0 & for \quad \lambda > 1, \\ \leq 0 & for \quad 0 < \lambda < 1, \end{cases} \tag{9.4}$$

where equality holds iff $\mathcal{X} = \mathcal{Y}$.

As a result, we assume:

$$g_1(t) := (1-\beta)\beta^{\beta/(1-\beta)} p^{\beta/(\beta-1)}(t) p_1^{1/(1-\beta)}(t),$$
$$g_2(t) := (\delta-1)\delta^{\delta/(1-\delta)} p^{\delta/(\delta-1)}(t) p_2^{1/(1-\delta)}(t),$$

and

$$Q(t) := q(t)[p_2(h(t))]^{-\gamma/\delta}$$

for $t \geq t_1$ for some $t_1 \geq t_0$, where $p : [t_0, \infty) \to (0, \infty)$ is a CF.

Theorem 31. *Assume that* $\beta < 1$ *and* $\delta > 1$*, conditions (i)-(iv) and* (9.3) *hold, and suppose that* $p \in C([t_0, \infty), (0, \infty)) \ni$

$$p_2(t) \neq 0 \textit{ is bounded and } \lim_{t\to\infty}[g_1(t) + g_2(t)] = 0, \tag{9.5}$$

and the equation:

$$z'(t) + Cq(t)A^{\gamma}(\tau_1(t))z^{\gamma/\alpha}(\tau_1(t)) = 0 \tag{9.6}$$

is oscillatory for all constant $C > 0$*. Assume that* $\exists$ *constants* μ_i*,* $i = 1, 2, 3$*, and* $\varphi \in (0, 1) \ni$

$$1 \leq \mu_1 \leq \mu_2 \leq \mu_3 \quad and \quad \mu_3 h(t) \leq t, \tag{9.7}$$

and the equations:

$$Z'(t) + Q(t)\left\{\frac{(\mu_2-\mu_1)^{n-2}}{(n-2)!}h^{n-2}(t)A\big(\mu_3 h(t), \mu_2 h(t)\big)\right\}^{\gamma/\delta} Z^{\gamma/(\alpha\delta)}(\mu_3 h(t)) = 0, \tag{9.8}$$

and

$$X'(t) + Q(t)\left\{\frac{\varphi(\mu_2-\mu_1)}{(n-2)!}h^{n-1}(t)\right\}^{\gamma/\delta} X^{\gamma/(\alpha\delta)}(\mu_2 h(t)) = 0 \tag{9.9}$$

are oscillatory and

$$\int_{t_0}^{\infty} Q(s)[A^*(h(s), t_0)]^{\gamma/\delta} ds = \infty, \tag{9.10}$$

then every solution $x(t)$ of Eq. (9.1) *is oscillatory, or*

$$\lim_{t\to\infty} x(t) = \infty.$$

Proof. WLOG, the solution $x(t)$ of Equation (9.1) is supposed to be positive and $x(\tau_1(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$ (i.e. a non-oscillatory solution). From Eq. (9.1), we have: $x(\tau_2(t)) > 0$ and

$$\left(r(t)\left[y^{(n-1)}(t)\right]^{\alpha}\right)' = -q(t)x^{\gamma}(\tau_1(t)) \leq 0. \tag{9.11}$$

Hence, $r(t)\left[y^{(n-1)}(t)\right]^{\alpha}$ is non-increasing with a constant sign. Namely, $y^{(n-1)}(t) > 0$ or $y^{(n-1)}(t) < 0$ for $t \geq t_2$ for some $t_2 \geq t_1$, so the following 4 cases are examined separately:

(a) $y(t) > 0$ and $y^{(n-1)}(t) < 0$;

(b) $y(t) > 0$ and $y^{(n-1)}(t) > 0$;

(c) $y(t) < 0$ and $y^{(n-1)}(t) > 0$;

(d) $y(t) < 0$ and $y^{(n-1)}(t) < 0$.

Let us first consider the case (a). Since $y^{(n-1)}(t) < 0$ for $t \geq t_2$, we obtain:

$$r(t)\left[y^{(n-1)}(t)\right]^{\alpha} \leq -c,$$

for some positive constant c, i.e.,

$$y^{(n-1)}(t) \leq \left(-\frac{c}{r(t)}\right)^{1/\alpha},$$

for $t \geq t_2$. Integrating the last inequality $(n1)$-times and by condition (9.3), we conclude that

$$\lim_{t\to\infty} y^{(n-1)}(t) = -\infty,$$

which is a contradiction.

From the case (b), we obviously note that

$$\begin{aligned} y(t) = x(t) &+ \left[p(t)x(\tau_2(t)) - p_2(t)x^{\delta}(\tau_2(t))\right] \\ &+ \left[p_1(t)x^{\beta}(\tau_2(t)) - p(t)x(\tau_2(t))\right]. \end{aligned}$$

From Definition 9.2 of $y(t)$, i.e., we obtain:

$$\begin{aligned} x(t) = y(t) &- \left[p(t)x(\tau_2(t)) - p_2(t)x^{\delta}(\tau_2(t))\right] \\ &- \left[p_1(t)x^{\beta}(\tau_2(t)) - p(t)x(\tau_2(t))\right]. \end{aligned} \quad (9.12)$$

If we apply the first inequality in (9.4) with $\lambda = \delta > 1$, $\mathcal{X} = p_2^{1/\delta}(t)x(\tau_2(t))$ and

$$\mathcal{Y} = \left[\frac{1}{\delta}p(t)p_2^{-1/\delta}(t)\right]^{1/(\delta-1)},$$

then we have:

$$p(t)x(\tau_2(t)) - p_2(t)x^{\delta}(\tau_2(t))$$
$$\leq (\delta - 1)\delta^{\delta/(1-\delta)} p^{\delta/(\delta-1)}(t) p_2^{1/(1-\delta)}(t) =: g_2(t). \tag{9.13}$$

Similarity, by employing the 2nd inequality in (9.4) with $\lambda = \beta < 1$, $\mathcal{X} = p_1^{1/\beta}(t)x(\tau_2(t))$, and

$$\mathcal{Y} = \left[\frac{1}{\beta} p(t) p_1^{-1/\beta}(t)\right]^{1/(\beta-1)},$$

we obtain:

$$p_1(t)x^{\beta}(\tau_2(t)) - p(t)x(\tau_2(t))$$
$$\leq (1 - \beta)\beta^{\beta/(1-\beta)} p^{\beta/(\beta-1)}(t) p_1^{1/(1-\beta)}(t) =: g_1(t). \tag{9.14}$$

By using (9.12) and (9.13), (9.14) turns out that

$$x(t) \geq y(t) - g_1(t) - g_2(t) = \left\{1 - \frac{g_1(t) + g_2(t)}{y(t)}\right\} y(t). \tag{9.15}$$

Since $y(t)$ in non-decreasing, we have: $y(t) \geq c_0$ for some $c_0 > 0$. Hence, (9.15) turns that

$$x(t) \geq \left\{1 - \frac{g_1(t) + g_2(t)}{c_0}\right\} y(t). \tag{9.16}$$

Now, we see

$$x(t) \geq c_1 y(t), \tag{9.17}$$

from (9.5) and (9.16) for some $c_1 \in (0, 1)$. (9.17) implies that Eq. (5.9) turns to be

$$\left(r(t)\left[y^{(n-1)}(t)\right]^{\alpha}\right)' + c_1^{\gamma} q(t) y^{\gamma}(\tau_1(t)) \leq 0. \tag{9.18}$$

$\exists$ a constant $\theta_0 \in (0,1) \ni$

$$y(\tau_1(t)) \geq \frac{\theta_0}{(n-1)!}\tau_1^{n-1}(t)y^{(n-1)}(\tau_1(t)),$$

for $t \geq t_1$ (see (Agarwal et al., 2000)). By setting $w(t) = r(t)\left[y^{(n-1)}(t)\right]^{\alpha}$, we obtain:

$$y(\tau_1(t)) \geq \frac{\theta_0}{(n-1)!}\tau_1^{n-1}(t)r^{-1/\alpha}(\tau_1(t))w^{1/\alpha}(\tau_1(t)). \tag{9.19}$$

By using (9.19), (9.18) turns that

$$w'(t) \leq -K\left(\tau_1^{n-1}(t)r^{-1/\alpha}(\tau_1(t))\right)^{\gamma} q(t)w^{\gamma/\alpha}(\tau_1(t)),$$

where

$$K = \left(\frac{c_1\theta_0}{(n-1)!}\right)^{\gamma}.$$

From Corollary 2 in (Philos, 1981), we conclude that $\exists$ a positive solution $w(t)$ of Eq. (9.6) with $\lim_{t\to\infty} w(t) = 0$, which contradicts the fact that Eq. (9.6) is oscillatory.

From the cases when $y(t) < 0$ for $t \geq t_2$, we assume that

$$\begin{aligned} v(t) = -y(t) &= -x(t) - p_1(t)x^{\beta}(\tau_2(t)) + p_2(t)x^{\delta}(\tau_2(t)) \\ &\leq p_2(t)x^{\delta}(\tau_2(t)), \end{aligned}$$

which implies

$$x(\tau_2(t)) \geq \left[\frac{v(t)}{p_2(t)}\right]^{1/\delta},$$

or

$$x(t) \geq \left[\frac{v(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))}\right]^{1/\delta}.$$

On the other hand, we obtain:

$$\begin{aligned}\left(r(t)\left[v^{(n-1)}(t)\right]^{\alpha}\right)' &= q(t)x^{\gamma}(\tau_1(t)) \\ &\geq q(t)\left[\frac{v(\tau_2^{-1}(\tau_1(t)))}{p_2(\tau_2^{-1}(\tau_1(t)))}\right]^{\gamma/\delta} \\ &= Q(t)v^{\gamma/\delta}(h(t)). \end{aligned} \tag{9.20}$$

From case (c), it is obvious that $v^{(n-1)}(t) \leq 0$, and either $v'(t) < 0$ or $v'(t) > 0$ for $t \geq t_1$. First, we suppose that $v'(t) < 0$ for $t \geq t_1$. We get:

$$v(\mu_3 h(t)) \geq \frac{(\mu_2 - \mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) v^{(n-2)}(\mu_2 h(t)), \tag{9.21}$$

(refer to (Agarwal et al., 2000)). Now, we may express

$$\begin{aligned} v^{(n-2)}(u_1) - v^{(n-2)}(u_2) &= -\int_{u_1}^{u_2} r^{-1/\alpha}(s)\left[r(s)\left[z^{(n-1)}(s)\right]^{\alpha}\right]^{1/\alpha} ds \\ &\geq A(u_2, u_1)\left[-r^{-1/\alpha}(u_2)v^{(n-1)}(u_2)\right], \end{aligned} \tag{9.22}$$

for $t_1 \leq u_1 \leq u_2$. By taking $u_1 = \mu_2 h(t)$ and $u_2 = \mu_3 h(t)$ for $t \geq t_1$ in inequality (9.22), we see that

$$v^{(n-2)}(\mu_2 h(t)) \geq A\big(\mu_3 h(t), \mu_2 h(t)\big)\left[-r^{-1/\alpha}(\mu_3 h(t))v^{(n-1)}(\mu_3 h(t))\right]. \tag{9.23}$$

By using Eq. (9.23), (9.21) turns out to be

$$\begin{aligned} v(\mu_3 h(t)) \geq{}& \frac{(\mu_2 - \mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) A\big(\mu_3 h(t), \mu_2 h(t)\big) \\ &\times \left[-r^{-1/\alpha}(\mu_3 h(t))v^{(n-1)}(\mu_3 h(t))\right]. \end{aligned} \tag{9.24}$$

By setting $V(t) := -r(t)\left[v^{(n-1)}(t)\right]^{\alpha}$ for $t \geq t_1$, (9.24) turns that

$$\begin{aligned} v(h(t)) &\geq v(\mu_3 h(t)) \\ &\geq \frac{(\mu_2-\mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) A\big(\mu_3 h(t), \mu_2 h(t)\big) \\ &\quad \times \left[-V^{1/\alpha}(\mu_3 h(t))\right]. \end{aligned} \tag{9.25}$$

From (9.25) and (9.19), we obtain:

$$\begin{aligned} -V'(t) &\geq Q(t) v^{\gamma/\delta}(h(t)) \\ &\geq Q(t)\left\{\frac{(\mu_2-\mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) A\big(\mu_3 h(t), \mu_2 h(t)\big)\right\}^{\gamma/\delta} \\ &\quad \times \left[V^{\gamma/(\alpha\delta)}(\mu_3 h(t))\right], \end{aligned}$$

which implies

$$V'(t) + Q(t)\left\{\frac{(\mu_2-\mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) A\big(\mu_3 h(t), \mu_2 h(t)\big)\right\}^{\gamma/\delta} \left[V^{\gamma/(\alpha\delta)}(\mu_3 h(t))\right] \leq 0.$$

The proof can be similarity done same as the one in case (a).

Let $v'(t) > 0$ for $t \geq t_1$. We clearly get:

$$v^{(n-2)}(\mu_3 h(t)) \geq -(\mu_2-\mu_1) h(t) v^{(n-1)}(\mu_2 h(t)).$$

$\exists$ a constant $\theta_1 \in (0,1)$ such that

$$\begin{aligned} v(h(t)) &\geq \frac{\theta_1}{(n-2)!} h^{n-2}(t) v^{(n-2)}(h(t)) \\ &\geq \frac{\theta_1}{(n-2)!} h^{n-2}(t) v^{(n-2)}(\mu_1 h(t)), \end{aligned}$$

for $t \geq t_1$. Now, we see that

$$v(h(t)) \geq \frac{\theta_1}{(n-2)!} h^{n-2}(t) v^{(n-2)}(\mu_1 h(t))$$
$$\geq \frac{\theta_1}{(n-2)!} h^{n-2}(t)(\mu_2 - \mu_1) h(t)[-v^{(n-1)}(\mu_2 h(t))].$$

The remaining part of the proof is similar to the above case.

Finally, from case (d). we clearly get: $r(t)\left[v'(t)\right]^{\alpha} > 0$ and so

$$r(t)\left[v^{(n-1)}(t)\right]^{\alpha} \geq c_2,$$

or that

$$v^{(n-1)}(t) \geq \left(\frac{c_2}{r(t)}\right)^{1/\alpha},$$

for some $c_2 > 0$. Thus, we obtain:

$$v(t) \geq c_2^{1/\alpha} A^*(t, t_2) \tag{9.26}$$

for $t \geq t_3 \geq t_2$. By using (9.26), (9.20) turns out

$$\left(r(t)\left[v^{(n-1)}(t)\right]^{\alpha}\right)' \geq Q(t) v^{\gamma/\delta}(h(t))$$
$$\geq Q(t)\left[c_2^{1/\alpha} A^*(h(t), t_2)\right]^{\gamma/\delta}.$$

The remaining part of the proof is trivial. This finalizes the proof.

Corollary 2. *Assume that* $\beta < 1$ *and* $\delta > 1$*, conditions (i)-(iv) and* (9.3) *hold and let* $p \in C([t_0, \infty), (0, \infty)) \ni$ (9.5) *holds. Suppose that* $\exists$ *real numbers* μ_i*,* $i = 1, 2, 3$

$\ni$ (9.7) *is satisfied. If we have condition* (9.10)*, then*

$$\lim_{t\to\infty}\int_{\tau_1(t)}^{t} q(s)A^{\gamma}(\tau_1(s))ds = \infty \quad when \quad \gamma \le \alpha,$$

$$\liminf_{t\to\infty}\int_{\mu_3 h(t)}^{t} Q(s)\left\{h^{n-2}(s)A\big(\mu_3 h(s),\mu_2 h(s)\big)\right\}^{\gamma/\delta} ds > \frac{1}{e}\left(\frac{(n-2)!}{(\mu_2-\mu_1)^{n-2}}\right)^{\gamma/\delta} \quad when \quad \gamma = \alpha\delta,$$

$$\lim_{t\to\infty}\int_{\mu_3 h(t)}^{t} Q(s)\left\{h^{n-2}(s)A\big(\mu_3 h(s),\mu_2 h(s)\big)\right\}^{\gamma/\delta} ds = \infty \quad when \quad \gamma < \alpha\delta,$$

and

$$\lim_{t\to\infty}\int_{\mu_2 h(t)}^{t} \left[h^{n-1}(s)(\mu_2-\mu_1)\right]^{\gamma/\delta} Q(s)ds = \infty \quad when \quad \gamma \le \alpha,$$

then Eq. (9.1) *is oscillatory.*

9.3 Illustrative Examples

Two numerical examples are illustrated as follows:

Example 9.3.1. *Consider the following* 2*nd-order equation:*

$$\left(e^{-t}\left(x(t)+\frac{1}{t}x^{1/3}(t/2)-x^{3}(t/2)\right)'\right)' + \left(\frac{3}{4}-\left(\frac{5}{36t}+\frac{1}{2t^2}+\frac{2}{t^3}\right)e^{-4t/3}\right)x(t/2) = 0. \qquad (9.27)$$

Clearly, $r(t) = e^{-t}$*,* $p_1(t) = p(t) = t^{-1}$ *and* $p_2(t) = 1$*, and hence there exists a*

$t_* \geq 3$ *such that*

$$\frac{3}{4} - \left(\frac{5}{36t} + \frac{1}{2t^2} + \frac{2}{t^3}\right) e^{-4t/3} > 0,$$

for $t \geq t_*$. *Verifying all conditions of the Theorem 31 indicates that every solution* x *of Eq.* (9.27) *is oscillatory, otherwise* $\lim_{t\to\infty} x(t) = \infty$. *It is worth mentioning that* $x_1(t) = e^t$ *is such a solution of Eq.* (9.27).

Example 9.3.2. *Consider the following even-order equation:*

$$\left(e^{-t}\left(x(t) + \frac{1}{t}x^{1/3}(t/2) - x^3(t/2)\right)^{(n-1)}\right)' + \left(\frac{1}{t}e^{-t/2}\right)x(t/2) = 0. \qquad (9.28)$$

By noting that $r(t) = e^{-t}$, $p_1(t) = p(t) = t^{-1}$, $p_2(t) = 1$ *and* $q(t) = e^{-t/2}/t$ *and letting* $\mu_1 = 1/8$, $\mu_2 = 1/4$ *and* $\mu_3 = 3/8$, *it can be easily seen that all the conditions of Corollary 2 hold, and hence Eq.* (9.28) *is oscillatory.*

9.4 Conclusion

The NLDfEq's oscillation with MNLNTs have been studied via the basic inequality and some comparison results to show our main theorem. Two numerical examples have been provided to validate our theoretical analysis. The oscillation of Equations (9.27) and (9.28) has never been previously studied.

CHAPTER 10: SUMMARY AND CONCLUSION

10.1 Summary

Fractional calculus is an essential field of research for modeling many scientific phenomena arising in physics and engineering. Various definitions, generalizations, or extensions have been proposed to formulate various classes of differential equations in the context of fractional calculus. This field of research is considered as an open research problem due to the fact that many research works are still under study, and investigations are needed for many arising research problems in this field of research. One of the most challenges that face the researchers in this research field is that the difficulty of proposing one universal definition that can be employed for all systems and cases. Each definition has both advantages and disadvantages when it is applied for various models. In Physics and engineering, analytical solutions are very important and highly needed. Another challenging part of this research field is to find analytical solutions for the fractional-order differential equations, especially partial differential equations. Some definitions can not be employed for finding solutions analytically; therefore, new definitions, techniques, or numerical solutions via generalized numerical methods are needed to overcome this challenge. Due to all these challenges, it is nearly impossible to find a university/college curriculum that teaches this field of research although this research field is very powerful for many applications in science and engineering. As a result, this thesis provides a comprehensive research work on fractional calculus in terms of computational methods and analysis. Various definitions, techniques, theorems, generalizations, extensions, and numerical experiments are investigated in detail in this thesis. Three interesting techniques have been successfully employed in solving two essential nonlinear partial differential equations, constructed in the context of conformable and fractional calculus, in Chapter 3. A full investigation of the conformable version of multivariable calculus is studied in Chapter 4. While each definition in this research field has a limited applicability, a newly proposed definition, named as Abu-Shady–Kaabar fractional derivative, is presented in Chapter 5, to provide a new possible direction for analytical solutions for various differential equations in the context of fractional calculus. Then, the implicit nonlinear variable order fractional differential equation is discussed via the Krasnoselskii's fixed point theorem in Chapter 6. The fractional formulated quadratic integral equation is investigated by employing the generalized Mittag-Leffler function to provide a detailed study on the stability of this equation in Chapter 7. A novel technique is presented in Chapter 8 to investigate the ψ-Caputo fractional differential equation with nonlinear boundary conditions. In the last chapter of our main results, a unique investigation of the oscillation of even-order nonlinear differential equations with mixed nonlinear neutral terms is discussed in Chapter 9 to provide a new direction for further research works and extensions related to the presented equation. The unique research results of the problems presented in Chapter 3 are published in Q1, Science Citation Index Expanded (SCIE), journals, namely: Journal of Function Spaces and Mathematical Methods in the Applied Sciences. Chapter 4 research results are published in Q2, SCIE, journal, namely: Journal of Mathematics. Chapter 5 research results are published in Q3, SCIE, journal, namely: Mathematical Problems in Engineering. Chapter 6 research results are published in Q1, SCIE, journal, namely: Mathematics. The results of the research problems presented in Chapter 7 are published in Q1 Scopus/Web of Science journal, namely: Nonlinear Engineering. Chapter 8 research results are published in Q1, SCIE, journal, namely: Fractal and Fractional. Chapter 9 research results are published in Q1, SCIE, journal, namely: Journal of Function Spaces. All results of our research problems have been cited in many recent published research works in prestigious journals.

10.2 Conclusion

Various classes of differential equations have been constructed in the context of fractional calculus. All obtained results have been investigated theoretically and numerically via several techniques. The following is a list of conclusions based on our findings in this thesis:

1. The study of the Wazwaz–Benjamin–Bona–Mahony and modified nonlinear Schrödinger equation with spatio-temporal dispersion in the context of fractional calculus provides a good understanding to many scientific phenomena arising in oceanography, optics, electromagnetism, and optical communication.
2. The investigation of multivariable conformable calculus offers a unique mathematical tool for modeling phenomena in physics and engineering.
3. The newly proposed definition, Abu-Shady–Kaabar fractional derivative, solves many issues associated with other previously proposed definitions and offers a simple direction to obtain analytical solutions efficiently for many classes of differential equations, formulated in the context of fractional calculus.
4. The applicability of variable-order spaces of fractional type needs a series of systematic approaches to investigate fractional differential equation's solutions such as existence-uniqueness-stability.
5. The investigation of quadratic fractional integral equations provides a significant tool in modeling scientific scenarios due to the essential properties of fractional calculus in investigating systems' dynamics and behavior.
6. The monotone iterative technique, along with upper and lower solutions'

technique provides a full investigation of the ψ-Caputo fractional differential equation with nonlinear boundary conditions.

7. The investigation of differential equations' oscillation with nonlinear neutral terms which has been rarely mentioned in other research works can provide a new path for more related future works in the context of fractional calculus.

10.3 Future Work

This thesis has provided several novel contributions to the field of fractional calculus. Various definitions have been proposed and investigated theoretically and numerically. Several illustrative examples have been provided to validate the applicability of all theoretical results. One of the limitation of the ASK definition is that ASK is a local definition, therefore, a nonlocal version of ASK will have more advantages than the local version of ASK due to the helpful nonlocality property in modeling some complex scientific phenomena. This thesis provides many new directions for many related future research works. Some examples of possible future research works that can be done based on the results of this thesis are listed as follows:

1. The Abu-Shady–Kaabar fractional definition can be extended further to include chain rule and special functions.
2. The multivariable version of the Abu-Shady–Kaabar fractional definition can be investigated.
3. The Abu-Shady–Kaabar fractional definition's version of vector-valued function of several real variables can also be studied.
4. A nonlocal version of Abu-Shady-Kaabar fractional definition can be proposed in the near future.

5. Many applications in science and engineering can be studied in the context of the Abu-Shady-Kaabar fractional definition.
6. A universal definition can be defined based on the ASK, NComD, and ψ-Caputo fractional definitions by proposing a suitable functional kernel.
7. The results in Chapter 3 about wave transformation and double Laplace transform can be studied in the context of ASK. The same applied to Chapters 6,7,8, and 9.
8. The nonlinear variable order differential equation can be formulated in the context of ASK to model various phenomena in vscoelasticity, mechanics, and fluid dynamics.
9. The NComD results in (Lugo et al., 2022; Valdés, 2022; Valdés et al., 2020) can be extended further to be studied in the context of ASK.
10. The mixed Morrey spaces results in (Guliyev, 2009; Ragusa & Scapellato, 2017) can be generalized further to be investigated in the context of ASK for fractional partial differential equations.

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