CHAPTER 4

APPROXIMATE DISTRIBUTIONS FOR THE ESTIMATORS OF THE WEIBULL PARAMETERS

4.1 Introduction

The Weibull distribution is a continuous probability distribution which is named after the Swedish engineer Waloddi Weibull (1887 – 1979) who popularized its use for reliability analysis. Now, the Weibull distribution is one of the most widely used lifetime distribution in survival analysis.

The random variable T is said to have a Weibull distribution with the scale parameter λ and shape parameter γ if its probability density function is given by

$$f(t) = \lambda \gamma t^{\gamma-1} \exp(-\lambda t^{\gamma}). \tag{4.1.1}$$

We may then write T ~ Weibull(λ, γ). The survivor function at time t is given by

$$P(T > t) = S(t) = \exp\left\{-\int_0^t \lambda \gamma u^{\gamma-1} du\right\} = \exp\left(-\lambda t^{\gamma}\right), \tag{4.1.2}$$

for $0 \le t < \infty$ and the n-th moment of a Weibull random variable T is given by

$$E(T^n) = \lambda^{-\frac{n}{\gamma}} \Gamma\left(\frac{n}{\gamma} + 1\right), \tag{4.1.3}$$

where Γ is the Gamma function.

The distributions of the maximum likelihood estimates $\hat{\lambda}$ and $\hat{\gamma}$ of the Weibull parameters λ and γ are usually intractable. In this chapter, we show that it is possible to express the estimate $\hat{\gamma}$ in terms of w_0, w_1, w_2, w_3, w_4 which are functions of n observations from the Weibull distribution. We next derive a multivariate quadratic-normal distribution for (w_0, w_1, w_2, w_3, w_4). A way to use the multivariate quadratic-normal distribution with only five dimensions is through the computation of the distribution of $\hat{\gamma}$ by means of numerical integration with respect to the five variables.

4.2 The Maximum Likelihood Estimates of the Weibull Parameters

Consider the case when the sample is made up of *n* individuals. Suppose that there are *r* deaths among the *n* individuals and *n* – *r* right censored survival times. The likelihood of the *n* survival times is then given by

$$\prod_{i=1}^{n}\left\{\lambda\gamma t_i^{\gamma-1}\exp\left(-\lambda t_i^{\gamma}\right)\right\}^{\delta_i}\left\{\exp\left(-\lambda t_i^{\gamma}\right)\right\}^{1-\delta_i}, \tag{4.2.1}$$

where δ_i is defined as

$$\delta_i = \begin{cases} 0, & \text{if survival time is censored} \\ 1, & \text{otherwise.} \end{cases}$$

Hence the corresponding log-likelihood with the unknown parameters λ and γ, denoted by $L(\lambda,\gamma)$, is given by

$$\log L(\lambda,\gamma) = r\log(\lambda\gamma) + (\gamma-1)\sum_{i=1}^{n}\delta_i \log t_i - \lambda\sum_{i=1}^{n} t_i^{\gamma}, \tag{4.2.2}$$

where have $r = \sum_{i=1}^{n}\delta_i$.

The maximum likelihood estimates $\hat{\lambda}$ and $\hat{\gamma}$ of the parameters λ and γ satisfy

$$\frac{r}{\hat{\lambda}} - \sum_{i=1}^{n} t_i^{\hat{\gamma}} = 0, \tag{4.2.3}$$

and
$$\frac{r}{\hat{\gamma}} + \sum_{i=1}^{n}\delta_i \log t_i - \hat{\lambda}\sum_{i=1}^{n} t_i^{\hat{\gamma}}\log t_i = 0. \tag{4.2.4}$$

From Equation (4.2.3), the estimate $\hat{\lambda}$ can be written as

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{n} t_i^{\hat{\gamma}}}. \tag{4.2.5}$$

By substituting Equation (4.2.5) into Equation (4.2.4), we get

$$\frac{r}{\hat{\gamma}}+\sum_{i=1}^{n}\delta_i \log t_i - \frac{r}{\sum_{i=1}^{n} t_i^{\hat{\gamma}}}\sum_{i=1}^{n} t_i^{\hat{\gamma}} \log t_i = 0 . \quad (4.2.6)$$

After obtaining the value of $\hat{\gamma}$ from Equation (4.2.6) we can obtain the value of $\hat{\lambda}$ by using Equation (4.2.5). We can use numerical procedures to find $\hat{\gamma}$ from Equation (4.2.6). Alternatively, we can also find the value of $\hat{\gamma}$ by using a polynomial of degree 2 which is derived from Equation (4.2.6). To find this polynomial, we first let

$$F(\gamma_0)=\frac{r}{\gamma_0}+\sum_{i=1}^{n}\delta_i \log t_i - \frac{r}{\sum_{i=1}^{n} t_i^{\gamma_0}}\sum_{i=1}^{n} t_i^{\gamma_0} \log t_i \ , \quad (4.2.7)$$

where γ_0 is some initial value for $\hat{\gamma}$. We next derive the derivatives of order less than or equal to 2 of $F(\gamma_0)$. The results are

$$F^{(1)}(\gamma_0) = \frac{\partial F(\gamma_0)}{\partial \gamma}$$

$$= -\frac{r}{\gamma_0^{\,2}} + \frac{r}{(\sum_{i=1}^{n} t_i^{\gamma_0})^2}\left(\sum_{i=1}^{n} t_i^{\gamma_0} \log t_i\right)^2 - \frac{r}{\sum_{i=1}^{n} t_i^{\gamma_0}}\sum_{i=1}^{n} t_i^{\gamma_0} (\log t_i)^2 , \quad (4.2.8)$$

and $$F^{(2)}(\gamma_0) = \frac{\partial^2 F(\gamma_0)}{\partial \gamma^2}$$

$$= \frac{2r}{\gamma_0^{\,3}} + \frac{3r}{(\sum_{i=1}^{n} t_i^{\gamma_0})^2}\sum_{i=1}^{n} t_i^{\gamma_0} (\log t_i)^2 \sum_{i=1}^{n} t_i^{\gamma_0} \log t_i$$

$$- \frac{2r}{(\sum_{i=1}^{n} t_i^{\gamma_0})^3}\left(\sum_{i=1}^{n} t_i^{\gamma_0} \log t_i\right)^3 - \frac{r}{\sum_{i=1}^{n} t_i^{\gamma_0}}\sum_{i=1}^{n} t_i^{\gamma_0} (\log t_i)^3 . \quad (4.2.9)$$

The quadratic approximation of $F(\hat{\gamma})$ is then given by

$$F(\hat{\gamma}) \approx F(\gamma_0) + F^{(1)}(\gamma_0)(\hat{\gamma}-\gamma_0) + \frac{1}{2!}F^{(2)}(\gamma_0)(\hat{\gamma}-\gamma_0)^2 = 0 . \quad (4.2.10)$$

From Equation (4.2.10) we can show that the second order approximation of $\hat{\gamma}$ is given by

$$\hat{\gamma} \cong \gamma_0 - \frac{F(\gamma_0)}{F^{(1)}(\gamma_0)} - \frac{1}{2!}\frac{F^{(2)}(\gamma_0)(F(\gamma_0))^2}{\left(F^{(1)}(\gamma_0)\right)^3}. \tag{4.2.11}$$

For a chosen value of (λ_0, γ_0), we generate 5 values of $\mathbf{t} = (t_1, t_2, ..., t_n)$ using the Weibull distribution. Tables 4.2.1 and 4.2.2 show that the values of the estimate $\hat{\gamma}$ obtained by solving Equation (4.2.6) and the second order approximations given by Equation (4.2.11) are fairly close to each other.

Table 4.2.1 The estimate $\hat{\gamma}$ calculated using Equations (4.2.6) and (4.2.11) for the *j*-th generated value of t, *r* = *n* (Median of the Weibull distribution is 20. The values in parentheses are the values based on Equation (4.2.11)).

γ_0	λ_0	*j*				
		1	2	3	4	5
0.1	0.514	0.074(0.076)	0.116(0.118)	0.103(0.099)	0.078(0.079)	0.139(0.136)
0.2	0.381	0.224(0.224)	0.234(0.239)	0.196(0.205)	0.228(0.236)	0.178(0.168)
0.3	0.282	0.288(0.309)	0.343(0.325)	0.324(0.331)	0.368(0.368)	0.276(0.293)
0.4	0.209	0.398(0.409)	0.451(0.468)	0.515(0.530)	0.425(0.436)	0.425(0.443)
0.5	0.155	0.502(0.488)	0.536(0.532)	0.485(0.498)	0.632(0.573)	0.644(0.627)
0.6	0.115	0.556(0.546)	0.684(0.661)	0.742(0.735)	0.574(0.560)	0.533(0.543)
0.7	0.085	0.738(0.712)	0.790(0.807)	0.932(0.924)	0.696(0.674)	0.635(0.636)
0.8	0.063	0.808(0.826)	0.834(0.794)	0.714(0.722)	1.076(1.005)	0.759(0.769)
0.9	0.047	0.994(1.037)	0.856(0.834)	1.273(1.250)	0.749(0.789)	0.793(0.841)
1.0	0.035	1.017(1.070)	0.887(0.924)	0.939(0.955)	1.342(1.232)	1.597(1.613)
1.1	0.026	1.070(1.100)	1.437(1.330)	1.598(1.511)	1.692(1.565)	0.961(0.959)
1.2	0.019	1.021(1.028)	0.989(1.048)	1.440(1.492)	1.161(1.210)	1.275(1.353)
1.3	0.014	1.374(1.429)	1.194(1.192)	1.408(1.459)	1.038(1.066)	1.870(1.698)
1.4	0.010	1.381(1.385)	1.376(1.438)	1.607(1.576)	1.164(1.166)	1.315(1.355)
1.5	7.750E-03	1.182(1.232)	1.265(1.291)	2.179(2.210)	1.819(1.819)	1.624(1.711)
1.6	5.744E-03	1.719(1.643)	1.537(1.567)	1.755(1.796)	1.825(1.884)	2.125(2.087)
1.7	4.257E-03	1.652(1.682)	1.926(2.013)	1.527(1.545)	2.277(2.318)	2.655(2.515)
1.8	3.155E-03	1.582(1.635)	1.753(1.850)	2.343(2.089)	1.471(1.545)	2.098(2.054)
1.9	2.338E-03	2.060(2.024)	1.995(2.027)	1.774(1.824)	2.317(2.408)	1.898(1.873)
2.0	1.733E-03	2.054(2.079)	1.840(1.952)	1.580(1.693)	2.573(2.559)	2.898(2.907)
2.1	1.284E-03	2.145(2.207)	2.363(2.362)	2.795(2.770)	2.606(2.508)	3.480(3.182)
2.2	9.518E-04	2.287(2.430)	3.126(3.044)	3.470(3.301)	2.299(2.394)	2.370(2.456)
2.3	7.054E-04	2.063(2.149)	2.343(2.282)	2.410(2.389)	1.855(1.823)	1.830(1.826)
2.4	5.228E-04	2.466(2.551)	2.576(2.605)	2.571(2.587)	2.154(2.228)	2.474(2.509)
2.5	3.875E-04	2.871(3.041)	2.928(3.036)	3.434(3.140)	1.862(1.936)	3.367(3.332)
2.6	2.872E-04	2.875(2.882)	2.069(2.112)	2.179(2.291)	3.001(2.671)	2.741(2.711)
2.7	2.128E-04	2.436(2.443)	2.988(3.142)	2.068(2.202)	2.428(2.392)	3.241(3.349)
2.8	1.577E-04	2.735(2.716)	2.535(2.535)	3.253(3.375)	2.305(2.351)	2.922(2.999)
2.9	1.169E-04	2.310(2.311)	2.684(2.742)	2.771(2.857)	2.536(2.538)	2.547(2.634)
3.0	8.664E-05	3.887(3.675)	3.090(3.023)	4.511(4.358)	4.046(3.992)	3.557(3.506)

Table 4.2.2 The Estimate $\hat{\gamma}$ Calculated using Equations (4.2.6) and (4.2.11) for the *j*-th generated value of t, *r* = (*n* – 1) (Median of the Weibull distribution is 20. The values in parentheses are the values based on Equation (4.2.11)).

γ_0	λ_0	*j*				
		1	2	3	4	5
0.1	0.514	0.104(0.104)	0.108(0.108)	0.101(0.101)	0.093(0.093)	0.112(0.112)
0.2	0.381	0.218(0.218)	0.200(0.200)	0.294(0.287)	0.239(0.238)	0.180(0.180)
0.3	0.282	0.428(0.423)	0.326(0.326)	0.319(0.319)	0.297(0.297)	0.255(0.255)
0.4	0.209	0.403(0.403)	0.438(0.438)	0.386(0.386)	0.683(0.646)	0.409(0.409)
0.5	0.155	0.447(0.447)	0.363(0.369)	0.398(0.400)	0.603(0.601)	0.690(0.684)
0.6	0.115	0.585(0.585)	0.733(0.729)	0.849(0.833)	0.729(0.728)	0.608(0.608)
0.7	0.085	0.642(0.642)	0.756(0.756)	0.684(0.684)	0.861(0.859)	0.874(0.869)
0.8	0.063	0.972(0.970)	0.838(0.838)	0.650(0.654)	0.916(0.915)	0.633(0.638)
0.9	0.047	0.792(0.793)	0.931(0.931)	0.846(0.846)	0.863(0.863)	0.795(0.797)
1.0	0.035	0.938(0.938)	1.120(1.120)	0.865(0.867)	1.309(1.299)	0.996(0.996)
1.1	0.026	1.292(1.290)	1.102(1.102)	1.131(1.131)	1.293(1.290)	1.378(1.374)
1.2	0.019	1.246(1.246)	1.353(1.352)	1.390(1.388)	1.704(1.682)	1.418(1.416)
1.3	0.014	1.489(1.487)	1.313(1.313)	1.294(1.294)	1.130(1.132)	1.254(1.254)
1.4	0.010	2.028(2.001)	1.235(1.236)	1.444(1.444)	1.233(1.235)	1.574(1.572)
1.5	7.750E-03	1.891(1.884)	1.502(1.502)	1.366(1.367)	1.599(1.598)	1.324(1.325)
1.6	5.744E-03	1.320(1.324)	1.612(1.612)	1.862(1.859)	1.911(1.905)	1.835(1.833)
1.7	4.257E-03	1.516(1.518)	1.493(1.495)	1.792(1.791)	2.011(2.009)	1.719(1.719)
1.8	3.155E-03	2.644(2.601)	2.646(2.593)	2.048(2.047)	1.965(1.964)	2.144(2.139)
1.9	2.338E-03	1.686(1.688)	1.415(1.443)	1.917(1.917)	1.807(1.807)	1.734(1.735)
2.0	1.733E-03	1.956(1.956)	3.124(3.012)	2.215(2.214)	2.065(2.065)	2.260(2.258)
2.1	1.284E-03	1.959(1.960)	2.211(2.211)	2.099(2.099)	1.793(1.798)	2.056(2.056)
2.2	9.518E-04	2.296(2.296)	3.002(2.971)	1.932(1.936)	2.171(2.171)	2.437(2.436)
2.3	7.054E-04	2.547(2.545)	2.323(2.323)	3.427(3.346)	2.384(2.384)	2.570(2.569)
2.4	5.228E-04	3.229(3.199)	3.119(3.095)	2.645(2.644)	2.135(2.138)	2.551(2.551)
2.5	3.875E-04	3.021(3.014)	2.897(2.893)	2.355(2.356)	3.295(3.266)	2.593(2.593)
2.6	2.872E-04	2.539(2.540)	3.992(3.886)	3.444(3.413)	3.095(3.089)	3.158(3.151)
2.7	2.128E-04	3.138(3.134)	4.224(4.148)	4.107(4.011)	3.567(3.526)	3.178(3.175)
2.8	1.577E-04	3.566(3.540)	3.263(3.260)	2.495(2.497)	3.047(3.046)	2.404(2.412)
2.9	1.169E-04	2.599(2.601)	2.154(2.189)	2.707(2.707)	2.918(2.918)	2.659(2.660)
3.0	8.664E-05	2.740(2.742)	2.453(2.477)	2.822(2.822)	3.193(3.193)	4.056(4.016)

4.3 Approximate Distribution of $\left(\hat{\lambda}, \hat{\gamma}\right)$

The inverse of the information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood functions in Equation (4.2.2) gives the asymptotic variance-covariance matrix of $\left(\hat{\lambda}, \hat{\gamma}\right)$. The approximate variance-covariance matrix is as shown below

$$\begin{bmatrix} \left[-\dfrac{\partial^2 \ln L}{\partial \lambda^2}\right]_{(\lambda,\gamma)=(\hat{\lambda},\hat{\gamma})} & \left[-\dfrac{\partial^2 \ln L}{\partial \lambda \partial \gamma}\right]_{(\lambda,\gamma)=(\hat{\lambda},\hat{\gamma})} \\ \left[-\dfrac{\partial^2 \ln L}{\partial \lambda \partial \gamma}\right]_{(\lambda,\gamma)=(\hat{\lambda},\hat{\gamma})} & \left[-\dfrac{\partial^2 \ln L}{\partial \gamma^2}\right]_{(\lambda,\gamma)=(\hat{\lambda},\hat{\gamma})} \end{bmatrix}^{-1} \approx \begin{bmatrix} \operatorname{var}\left(\hat{\lambda}\right) & \operatorname{cov}\left(\hat{\lambda}, \hat{\gamma}\right) \\ \operatorname{cov}\left(\hat{\lambda}, \hat{\gamma}\right) & \operatorname{var}\left(\hat{\gamma}\right) \end{bmatrix}, \quad (4.3.1)$$

where

$$-\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{r}{\lambda^2}, \quad (4.3.2)$$

$$-\frac{\partial^2 \ln L}{\partial \lambda \partial \gamma} = \sum_{i=1}^{n} t_i^{\hat{\gamma}} \log t_i, \quad (4.3.3)$$

and

$$-\frac{\partial^2 \ln L}{\partial \gamma^2} = \frac{r}{\gamma^2} + \hat{\lambda} \sum_{i=1}^{n} t_i^{\hat{\gamma}} \left(\log t_i\right)^2. \quad (4.3.4)$$

A multivariate normal distribution with mean $\left(\lambda, \gamma\right)$ and variance-covariance given by Equation (4.3.1) may then be used to approximate the distribution of $\left(\hat{\lambda}, \hat{\gamma}\right)$.

4.4 Multivariate Quadratic-Normal Distribution

In this section, we assumed that the data are not censored (i.e. $\delta_i = 1$, i = 1, 2, …, n). From Equations (4.2.7), (4.2.8), (4.2.9) and (4.2.11), we see that we may treat $\hat{\gamma}$ as a function of

$$w_0 = \sum_{i=1}^{n} \delta_i \log t_i, \quad (4.4.1)$$

$$w_1 = \sum_{i=1}^{n} t_i^{\gamma_0} \ , \tag{4.4.2}$$

$$w_2 = \sum_{i=1}^{n} t_i^{\gamma_0} \log t_i \ , \tag{4.4.3}$$

$$w_3 = \sum_{i=1}^{n} t_i^{\gamma_0} (\log t_i)^2 \ , \tag{4.4.4}$$

and

$$w_4 = \sum_{i=1}^{n} t_i^{\gamma_0} (\log t_i)^3 \ . \tag{4.4.5}$$

i.e. $\hat{\gamma} = H(w_0, w_1, w_2, w_3, w_4)$ for some function H.

In what follows, we shall derive an approximate joint distribution for $(w_0, w_1, w_2, w_3, w_4)$.

First we note that the mean $\overline{w}_i$ of w_i can be derived as shown below:

$$\overline{w}_0 = E(w_0) = \frac{n}{\gamma_0}[\int_0^\infty \log u \, e^{-u} du - \log \lambda_0 \int_0^\infty e^{-u} du] \ , \tag{4.4.6}$$

$$\overline{w}_1 = E(w_1) = \frac{n}{\lambda_0}\int_0^\infty u \, e^{-u} du \ , \tag{4.4.7}$$

$$\overline{w}_2 = E(w_2) = \frac{n}{\lambda_0 \gamma_0}[\int_0^\infty u \log u \, e^{-u} du - \log \lambda_0 \int_0^\infty u \, e^{-u} du] \ , \tag{4.4.8}$$

$$\overline{w}_3 = E(w_3) = \frac{n}{\lambda_0 {\gamma_0}^2}[\int_0^\infty u \left(\log u\right)^2 e^{-u} du - 2\log \lambda_0 \int_0^\infty u \log u \, e^{-u} du$$

$$+\left(\log \lambda_0\right)^2 \int_0^\infty u \, e^{-u} du \ ], \tag{4.4.9}$$

and

$$\overline{w}_4 = E(w_4) = \frac{n}{\lambda_0 {\gamma_0}^3}[\int_0^\infty u \left(\log u\right)^3 e^{-u} du - 3\log \lambda_0 \int_0^\infty u \left(\log u\right)^2 e^{-u} du$$

$$+3\left(\log \lambda_0\right)^2 \int_0^\infty u \log u \, e^{-u} du - \left(\log \lambda_0\right)^3 \int_0^\infty u \, e^{-u} du \ ]. \tag{4.4.10}$$

The integral of the form

$$I\left(k_1,k_2\right)=\int_0^{\infty} u^{k_1}\left(\log u\right)^{k_2} e^{-u} du\ , \tag{4.4.11}$$

appearing in Equations (4.4.8) – (4.4.11) can be evaluated by means of numerical integration.

In general, the moment $E\left(w_0^{\ l_0}, w_1^{\ l_1}, w_2^{\ l_2}, w_3^{\ l_3}, w_4^{\ l_4}\right)$ ($l_j \geq 0$ for $0 \leq j \leq$ 4; $\sum_{l_j=0}^{4} l_j \leq 4$) can be expressed in terms of $I\left(k_1,k_2\right)$ (see Appendix B).

Let **V** be the variance-covariance matrix of (w_0, w_1, w_2, w_3, w_4). The (*i*, *j*) entry of **V** is then given by

$$v_{ij} = E(w_i\, w_j) - E(w_i)E(w_j)\,. \tag{4.4.12}$$

Let **B** be the matrix of which its *j*-th column represents the *j*-th eigenvector of **V**, and

$$\mathbf{u} = \mathbf{B}^{\mathrm{T}}(\mathbf{w} - \bar{\mathbf{w}})\,, \tag{4.4.13}$$

where **w** and $\bar{\mathbf{w}}$ are vectors of which the *i*-th component are respectively w_i and $\bar{w}_i$.

We may obtain a quadratic-normal distribution with parameter 0 and $\boldsymbol{\lambda}^{(i)}$ such that its first four moments match those of u_i , $0 \leq i \leq 4$. We may then approximate the joint distribution of $\bar{\mathbf{w}}$ by a distribution called the multivariate quadratic-normal distribution (see Fam and Pooi (2004)) which is defined below.

y is said to have a *k*-dimensional multivariate quadratic-normal distribution with parameters $\underset{\sim}{\mu}$, **E** and $\boldsymbol{\lambda}^{(i)}$, $1 \leq i \leq k$ if

$$\mathbf{y} = \underset{\sim}{\mu} + \mathbf{Es}, \tag{4.4.14}$$

where **E** is an orthogonal matrix, $s_i \sim \mathrm{QN}(0, \boldsymbol{\lambda}^{(i)})$, $1 \leq i \leq k$ and $s_1, s_2, \ldots, s_k$ are uncorrelated.

Thus a multivariate quadratic-normal approximation for the joint distribution of **w** may be specified as follows:

$$\mathbf{w} = \bar{\mathbf{w}} + \mathbf{Bu}, \qquad (4.4.15)$$

where $u_i \sim \mathrm{QN}(0, \boldsymbol{\lambda}^{(i)})$, $0 \le i \le 4$, and u_0, u_1, u_2, u_3, u_4 are uncorrelated.

We shall investigate the adequacy of the approximations given by the multivariate quadratic-normal distribution in the next section.

4.5 Adequacy of Approximations given by the Multivariate Quadratic- Normal Distribution

For a given value of (λ_0, γ_0) , we obtain N values of $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ where $t_1,\ t_2,\ t_3,\ ...,\ t_n$ are independent and t_i is generated from the Weibull (λ_0, γ_0) distribution. For the *j*-th generated value of **t** we find the corresponding estimate $\hat{\gamma}^{(j)}$ of γ . Based on $\hat{\gamma}^{(1)}, \hat{\gamma}^{(2)}, \ldots, \hat{\gamma}^{(N)}$, a histogram is then formed.

Next we use the results in Section 4.4 to find a multivariate quadratic-normal distribution for $\mathbf{w} = (w_0\ w_1\ w_2\ w_3\ w_4)$ when the parameters of the underlying Weibull distribution are λ_0 and γ_0. We now generate N values of **w**. For the *j*-th generated value of **w**, we find the corresponding maximum likelihood estimate $\tilde{\gamma}^{(j)}$ of γ . Based on $\tilde{\gamma}^{(1)}, \tilde{\gamma}^{(2)}, \ldots, \tilde{\gamma}^{(N)}$, a histogram is then formed.

Figures (4.5.1) – (4.5.6) show some histograms based on $\hat{\gamma}^{(1)}, \hat{\gamma}^{(2)}, \ldots, \hat{\gamma}^{(N)}$ and $\tilde{\gamma}^{(1)}, \tilde{\gamma}^{(2)}, \ldots, \tilde{\gamma}^{(N)}$. The figures show that the histograms obtained by different types of simulation are fairly similar. However there are some minor differences between the two types of histograms. For example, the peak of the histogram based on $\tilde{\gamma}^{(1)}, \tilde{\gamma}^{(2)}, \ldots, \tilde{\gamma}^{(N)}$ is slightly higher than that based on $\hat{\gamma}^{(1)}, \hat{\gamma}^{(2)}, \ldots, \hat{\gamma}^{(N)}$. These differences cannot be attributed to random errors alone. To improve the approximations, we may consider imposing a nonlinear dependence structure on w_0, w_1, w_2, w_3, w_4.