# CHAPTER 5

# CONFIDENCE INTERVALS FOR THE WEIBULL PARAMETERS

## 5.1 Confidence Intervals for the Scale and Shape Parameters

The approximately 100(1-$\alpha$)% confidence interval based on the normal approximation of the distribution of $\hat{\theta}$ for the Weibull parameter $\theta$ ($\theta = \lambda$ or $\gamma$) is given by

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{\mathrm{var}\left(\hat{\theta}\right)}, \tag{5.1.1}$$

where $z_{\alpha/2}$, as defined in Section 2.2, is the (1-$\alpha/2$)-th percentile of the standard normal distribution.

Alternatively bootstrap procedure may also be used to derive a confidence interval for the parameter $\theta$. In the bootstrap procedure, we first generate $N$ values of $\tilde{\mathbf{t}} = \left(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n\right)$ based on the distributions

$$\tilde{t}_i \sim \text{Weibull } (\hat{\lambda}, \hat{\gamma}), \qquad\qquad i = 1, 2, \ldots, r,$$

and $\tilde{t}_i$ for $r+1 \le i \le n$ has a probability density function which is proportional to

$$\mathrm{P}(\tilde{T}_i > \tilde{t}_i) = \exp(-\hat{\lambda}\,\tilde{t}_i^{\,\hat{\gamma}}). \tag{5.1.2}$$

Then for the $j$-th generated value of **t**, we find the maximum likelihood estimate $\left(\tilde{\lambda}^{(j)}, \tilde{\gamma}^{(j)}\right)$ of $\left(\lambda, \gamma\right)$. We next fit the $N$ generated values of $\tilde{\theta}^{(j)}$ (Note: $\tilde{\theta}^{(j)} = \tilde{\lambda}^{(j)}$ or $\tilde{\gamma}^{(j)}$) with a quadratic-normal distribution with parameters $\tilde{\mu}$ and $\tilde{\boldsymbol{\lambda}}$ where

$$\tilde{\mu} = \frac{1}{N}\sum_{j=1}^{N}\tilde{\theta}^{(j)}, \tag{5.1.3}$$

and $\tilde{\boldsymbol{\lambda}}$ is such that the $k$-th central moment of the quadratic-normal distribution matches the estimated k-th central moment given by

$$\frac{1}{N}\sum_{j=1}^{N}[\tilde{\theta}^{(j)}-\tilde{\mu}]^{k}\,, \qquad k = 2, 3, 4. \tag{5.1.4}$$

An approximately 100(1 - $\alpha$ )% confidence interval for $\theta$ is then given by $B_L \le \theta \le B_U$ where

$$B_L = \hat{\theta} - (\tilde{\mu} - \hat{\theta}) + \tilde{\lambda}_1(-z_{\alpha/2}) + \tilde{\lambda}_2\left(\tilde{\lambda}_3(-z_{\alpha/2})^2 - \frac{1-\tilde{\lambda}_3}{2}\right), \tag{5.1.5}$$

$$B_U = \hat{\theta} - (\tilde{\mu} - \hat{\theta}) + \tilde{\lambda}_1(z_{\alpha/2}) + \tilde{\lambda}_2\left((z_{\alpha/2})^2 - \frac{1-\tilde{\lambda}_3}{2}\right), \tag{5.1.6}$$

and $\tilde{\mu} - \hat{\theta}$ is an estimate of the bias of $\hat{\theta}$ as an estimator of $\theta$.

Suppose the parameters $\lambda$ and $\gamma$ of the Weibull distribution are reparameterized to ($b$, $u$) as shown below

$$b = \frac{1}{\gamma}\,, \tag{5.1.7}$$

and

$$u = -\frac{1}{\gamma}\ln\lambda\,. \tag{5.1.8}$$

Let $\hat{u}$ and $\hat{b}$ be equivariant estimators of $u$ and $b$. Then Lawless (1978) proved that

$$z_1 = (\hat{u} - u)/\hat{b}\,, \tag{5.1.9}$$

$$z_2 = \frac{\hat{b}}{b}\,, \tag{5.1.10}$$

are pivotal quantities.

The unconditional method for finding confidence intervals for $u$ and $b$ is as follows:

We find the points $\ell_1, \ell_2, \ell_1', \ell_2'$ such that

$$\mathrm{P}\{z_1 \le \ell_1\} = \Pr\{z_1 \ge \ell_2\} = \alpha/2\,,$$

$$\mathrm{P}\left\{z_2 \le \ell_1'\right\} = \Pr\left\{z_2 \ge \ell_2'\right\} = \alpha/2\,.$$

Then $\left(\hat{u}-\ell_2\hat{b}, \hat{u}-\ell_1\hat{b}\right)$ and $\left(\hat{b}/\ell_2', \hat{b}/\ell_1'\right)$ are $1-\alpha$ confidence intervals for $u$ and $b$ respectively. Tables of approximate percentage points have been constructed in the literature by Monte Carlo simulation (see Mann et al. (1971), Billman et al. (1972), McCool (1970, 1974) and Thoman et al. (1970)).

Chao and Hwang (1986) compared the above unconditional method with another method known as the conditional method. They found that the confidence interval found by using unconditional method based on the maximum likelihood estimator generally has a mean length which is shorter than that of the confidence interval based on the conditional method.

Presently we consider an alternative method based on hypothesis testing for finding a confidence interval for the shape parameter $\gamma$. To implement the method based on hypothesis testing, we may first consider the problem of testing the null hypothesis

$$H_0: \quad \gamma = \gamma_0$$

against the alternative hypothesis

$$H_0: \quad \gamma \neq \gamma_0$$

at the $\alpha$ level.

Let $A_{\gamma,L}(\alpha')$ and $A_{\gamma,U}(\alpha'')$ be two constants which satisfy

$$\mathrm{P}(\hat{\gamma} < A_{\gamma,L}(\alpha') \mid \hat{\lambda}, \gamma_0) = \alpha'$$

and

$$\mathrm{P}(\hat{\gamma} > A_{\gamma,U}(\alpha'') \mid \hat{\lambda}, \gamma_0) = \alpha''$$

where $\alpha'$ and $\alpha''$ are two given constants which sum up to $\alpha$, i.e.

$$\alpha' + \alpha'' = \alpha.$$

An acceptance region for testing $H_0$ may be taken to be

$$\left\{\hat{\gamma}: \quad A_{\gamma,L}(\alpha') \le \hat{\gamma} \le A_{\gamma,U}(\alpha'')\right\}.$$

Then an approximately 100(1 - $\alpha$ )% confidence interval for $\gamma$ is given by

$$\{\gamma: \quad H_0 \text{ that } \gamma = \gamma_0 \text{ is accepted}\}.$$

Similarly a confidence interval for $\lambda$ can also be obtained by means of hypothesis testing as outlined below:

Let $A_{\lambda,L}(\alpha')$ and $A_{\lambda,U}(\alpha'')$ be two constants which satisfy

$$\mathrm{P}(\hat{\lambda} < A_{\lambda,L}(\alpha') \mid \lambda_0, \hat{\gamma}) = \alpha'$$

and $$\mathrm{P}(\hat{\lambda} > A_{\lambda,U}(\alpha'') \mid \lambda_0, \hat{\gamma}) = \alpha''.$$

In testing $H_0: \quad \lambda = \lambda_0$, we accept the null hypothesis if

$$A_{\lambda,L}(\alpha') \le \hat{\lambda} \le A_{\lambda,U}(\alpha'').$$

A confidence interval for $\lambda$ is then given by

$$\{\lambda: \quad H_0 \text{ that } \lambda = \lambda_0 \text{ is accepted}\}.$$

We may compare the following four methods of constructing confidence intervals for the Weibull parameters:

Method 1 : Method based on normal distribution (see Equation (5.1.1)).

Method 2 : Method based on bootstrap (see Equations (5.1.5) and (5.1.6)).

Method 3 : Method based on pivots (see Equations (5.1.9) and (5.1.10)).

Method 4 : Method based on hypothesis testing.

Let $P^{(i,\theta)}(\lambda^*, \gamma^*)$ be the coverage probability (evaluated at $(\lambda, \gamma) = (\lambda^*, \gamma^*)$) of the confidence interval based on Method *i* for $\theta$. For example,

$P^{(4,\gamma)}(0.00119, 1.5)$ = P(Confidence interval based on hypothesis testing for $\gamma$ covers 1.5 | $\lambda = 0.00119, \gamma = 1.5$).

Next let $L^{(i,\theta)}(\lambda^*, \gamma^*)$ be the expected length (evaluated at $(\lambda^*, \gamma^*)$ ) of the confidence interval based on Method *i* for $\theta$. For example,

$L^{(4,\gamma)}(0.00119, 1.5)$ = E(Length of the confidence interval based on hypothesis testing for $\gamma \mid \lambda = 0.00119, \gamma = 1.5$).

## 5.2 Confidence Interval for the Quantile

Since the Weibull distribution is skewed, a more appropriate and more tractable summary of the location of the distribution is the median survival time. From Equation (4.1.2), the median $Q_{0.5}$ can be obtained such that $S\{Q_{0.5}\} = 0.5$. Therefore

$$Q_{0.5} = [\frac{1}{\lambda}\log(2)]^{\frac{1}{\gamma}}. \qquad (5.2.1)$$

In general, the *p*-quantile of the Weibull distribution, $Q_p$, is given by

$$Q_p = [\frac{1}{\lambda}\log(\frac{1}{1-p})]^{\frac{1}{\gamma}}. \qquad (5.2.2)$$

Once the maximum likelihood estimate $(\hat{\lambda}, \hat{\gamma})$ of the parameter vector $(\lambda, \gamma)$ has been found, the *p*-quantile of the survival time distribution can be estimated using Equation (5.2.2):

$$\hat{Q}_p = \left[\frac{1}{\hat{\lambda}}\log\left(\frac{1}{1-p}\right)\right]^{\frac{1}{\hat{\gamma}}}. \qquad (5.2.3)$$

The standard error of the estimated *p*-quantile can be found by using the approximate formula

$$s.e.\left\{\hat{Q}_p\right\} = \frac{\hat{Q}_p}{\hat{\lambda}\hat{\gamma}^2}\left\{\hat{\gamma}^2 \operatorname{var}\left(\hat{\lambda}\right) + \hat{\lambda}^2\left(c_p - \log\hat{\lambda}\right)^2 \operatorname{var}\left(\hat{\gamma}\right)\right.$$

$$\left. +2\hat{\lambda}\hat{\gamma}\left(c_p - \log\hat{\lambda}\right)\operatorname{cov}\left(\hat{\lambda}, \hat{\gamma}\right)\right\}^{1/2}, \qquad (5.2.4)$$

where $c_p = \log\log(\frac{1}{1-p})$. The standard error of $\log\hat{Q}_p$ is

$$s.e.\left\{\log\hat{Q}_p\right\} = \frac{1}{\hat{Q}_p} s.e.\left\{\hat{Q}_p\right\}, \qquad (5.2.5)$$

and an approximately 100(1 - $\alpha$)% confidence interval for log $Q_p$ is

$$\log \hat{Q}_p \pm z_{\alpha/2} s.e.\left\{\log \hat{Q}_p\right\}. \tag{5.2.6}$$

The approximately 100(1 - $\alpha$ )% confidence interval for the *p*-quantile is then given by

$$\hat{Q}_p \exp[\pm z_{\alpha/2} s.e.\left\{\log \hat{Q}_p\right\}]. \tag{5.2.7}$$

Presently, we consider a method based on hypothesis testing for finding a confidence interval for the *p*-quantile of the Weibull distribution. First consider the problem of testing $H_0 : Q_p = Q_p^{(0)}$ against $H_0 : Q_p \neq Q_p^{(0)}$.

Let $(\lambda^+, \gamma^+)$ be the value of $(\lambda, \gamma)$ which minimizes

$$D^2 = \begin{bmatrix} \lambda - \hat{\lambda} & \gamma - \hat{\gamma} \end{bmatrix} \begin{bmatrix} \operatorname{var}(\hat{\lambda}) & \operatorname{cov}(\hat{\lambda}, \hat{\gamma}) \\ \operatorname{cov}(\hat{\lambda}, \hat{\gamma}) & \operatorname{var}(\hat{\gamma}) \end{bmatrix}^{-1} \begin{bmatrix} \lambda - \hat{\lambda} \\ \gamma - \hat{\gamma} \end{bmatrix}, \tag{5.2.8}$$

subject to the condition that the *p*-quantile which corresponds to $(\lambda, \gamma)$ is $Q_p^{(0)}$, i.e.

$$Q_p^{(0)} = \left[\frac{1}{\lambda} \log \frac{1}{1-p}\right]^{\frac{1}{\gamma}}. \tag{5.2.9}$$

Let $A_{Q_p,L}(\alpha')$ and $A_{Q_p,U}(\alpha'')$ be two constants which satisfy

$$\mathrm{P}(\hat{Q}_p < A_{Q_p,L}(\alpha') \mid \lambda^+, \gamma^+) = \alpha'$$

and

$$\mathrm{P}(\hat{Q}_p > A_{Q_p,U}(\alpha'') \mid \lambda^+, \gamma^+) = \alpha''.$$

An acceptance region for testing $H_0 : \quad Q_p = Q_p^{(0)}$ may be taken to be

$$\left\{\hat{Q}_p : \quad A_{Q_p,L}(\alpha') \leq \hat{Q}_p \leq A_{Q_p,U}(\alpha'')\right\}.$$

Then an approximately 100(1 - $\alpha$ )% confidence interval for $Q_p$ is given by

$$\{ Q_p : \quad H_0 \text{ that } Q_p = Q_p^{(0)} \text{ is accepted}\}.$$

The above method for finding confidence interval is modified from the method which has been used in Ng and Pooi (2008) and Goh (2008). The major variation lies in Equation (5.2.8). Instead of using an identity matrix in forming $D^2$, we used the inverse of the variance-covariance matrix.

The coverage probability and expected length (evaluated at $(\lambda,\gamma)=(\lambda^*,\gamma^*)$) of the *i*-th type of confidence interval ($i$ = 1, 2, 4) for $Q_p$ may be denoted respectively as $P^{(i,Q_p)}(\lambda^*,\gamma^*)$ and $L^{(i,Q_p)}(\lambda^*,\gamma^*)$.

## 5.3 Numerical Results

Tables 5.3.1 and 5.3.2 show some results on the coverage probabilities and expected lengths of the confidence intervals for $\gamma$ and $\lambda$. From Table 5.3.1 we see that, as expected, the coverage probability and expected length of the confidence interval based on Method 3 are more satisfactory than those of the confidence intervals based on Methods 1, 2 and 4. We also note that Method 4 would be comparable to Method 3 if $\alpha' = \alpha'' = 0.025$.

Method 3 cannot be used to construct a confidence interval for $\lambda$ as we cannot as yet find the corresponding pivotal quantity. The results regarding the performance of the methods (Methods 1, 2 and 4) which can be used to construct a confidence interval for $\lambda$ are shown in Table 5.3.2. This table shows that when $\lambda^*$ is small, the coverage probabilities of the confidence intervals based on Methods 1, 2 and 4 are all lower than the target value 0.95. The table also reveals that when $\lambda^*$ is relatively larger, Method 4 with $\alpha'$ = 0.0499 and $\alpha''$ = 0.0001 seems to produce the best coverage probability.

Table 5.3.3 shows that when $\gamma^* < 1.0$, Method 4 with ($\alpha'$ = 0.049 and $\alpha''$ = 0.001) or ($\alpha'$ = 0.0499 and $\alpha''$ = 0.0001) tends to produce shorter confidence intervals for the Weibull quantiles while still achieving satisfactory levels of coverage probability. The table also shows that when $\gamma^* \geq 1.0$, Method 4 with "large" value of $\alpha'$ and Method 2 produce comparable results. Furthermore as seen from the table, the increase of $\alpha'$ from 0.049 to 0.0499 only gives slight effects on the performance of the confidence interval based on Method 4.