CHAPTER 1. DEFINITIONS AND PRELIMINARIES

Chapter 1 consists of basic concept in differential geometry that would be used in this dissertation.

1.1. Differentiable manifolds

Let $\mathbb{R}^n$ be the n-dimensional usual Euclidean space. An *n-dimensional (differentiable) manifold* is a paracompact, second countable topological space M such that

(i) M is a Hausdorff space and each point of M has a neighborhood that is homeomorphic to an open set of $\mathbb{R}^n$. Thus, M is a locally Euclidean space.

(ii) There is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha)\}$ where φ_α is a homeomorphism of a connected set $U_\alpha \subset M$ onto an open subset of $\mathbb{R}^n$ satisfying the following three properties:

(a) $\bigcup_\alpha U_\alpha = M$.

(b) $\varphi_\alpha \cdot \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \to \varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ are C^∞ for all α and β.

(c) The collection of coordinate system $\{(U_\alpha, \varphi_\alpha)\}$ is maximal with respect to (b). This collection defines the differentiable structure of the manifold M. The open set U_α is called a coordinate neighborhood for all $m \in U_\alpha$.

An n-dimensional manifold will be denoted by M^n or simply by M if the underlying dimension of the manifold is not under consideration. A collection of $\{(U_\alpha,\varphi_\alpha)\}$ which fulfills the conditions ii(a) and (b) in the above definition is called an *atlas* of M. Clearly, if an atlas of the manifold M is specified, the differentiable structure of M is defined.

Let M be a manifold with an atlas $\{(U_\sigma,\varphi_\alpha)\}$. If N is an open subset of M, then N is a manifold with an atlas $\left\{\left(U_\alpha \cap N, \left.\varphi_\alpha\right|_{U_\alpha \cap N}\right)\right\}$.

Let $m \in M^n$. Functions f and g defined on an open set containing m are said to be equivalent if they agree on some neighborhood of m. The set of equivalence classes is denoted by $\tilde{F}_m$. $\tilde{F}_m$ is an algebra. A tangent vector v at the point m is a linear derivation of $\tilde{F}_m$. The tangent space of M at m, M_m, is the set of tangent vectors at m. It can be shown that M_m is a vector space.

Since M_m is n-dimensional, it is isomorphic to R^n. Its dual space will be denoted by M_m^*. Let $TM = \bigcup_{m\in M} M_m$ and $TM^* = \bigcup_{m\in M} M_m^*$. Let U be a coordinate neighborhood of m with coordinate functions $x_1,\ldots,x_n$ and coordinate map φ, namely, for each $m \in U$, $\varphi(m) = (x_1(m),x_2(m),\ldots, x_n(m))$. The coordinate function x_i maps U into $\mathbb{R}$. The natural projection $\pi : TM \to M$ is defined as $\pi(v) = m$ if $v \in M_m$. For each coordinate x_i, the differential dx_i is given by $dx_i(v)f = v(f\circ x_i)$ where $v \in M_m$ and f is a function from $\mathbb{R}$ to $\mathbb{R}$. Define $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ as

$$\tilde{\varphi}(v) = (x_1(\pi(v)), x_2(\pi(v)), \ldots, x_n(\pi(v)),\ dx_1(v), \ldots, dx_n(v))$$

for all $v \in \pi^{-1}(U)$. Observe that

(i) If (U,ψ) and (V,φ) are coordinate neighborhoods of $m \in M$ with coordinate charts ψ and φ respectively, then $\tilde{\psi}\cdot\tilde{\varphi}^{-1}$ is C^{∞}.

(ii) The collection $\{\tilde{\varphi}^{-1}(W) : W$ open in $\mathbb{R}^{2n}$ and (U,φ) is a coordinate neighborhood of a point $m \in M$ and coordinate chart $\varphi\}$ forms a basis for a topology on TM which makes TM into a 2n-dimensional, second countable, locally Euclidean space.

(iii) Let τ be the maximal collection, with respect to ii(b) of definition for manifold, containing $\{(\pi^{-1}(U),\tilde{\psi})\}$. Then τ is a differentiable structure on TM.

Hence, TM is a 2n-dimensional manifold [18]. The space TM^* is also a 2n-dimensional manifold. It's differentiable structure can be constructed similar to that of TM.

Let $\psi : M^n \to N^k$ be C^{∞}, then $d\psi_m$ is a mapping of M_m to $N_{\psi(m)}$ such that $d\psi_m(v)f = v(f\cdot\psi)_m$ where $v \in M_m$ and f is a function on N. The map ψ is an *immersion* if $d\psi_m$ is non-singular at each $m \in M$, i.e. $d\psi_m(M_m) \subset N_{\psi(m)}$ is of n-dimensional. The pair (M,ψ) is a *sub-manifold* of N if ψ is a one-to-one immersion. The map ψ is an *imbedding* if ψ is a one-to-one immersion which is also a homeomorphism into; that is, ψ is a map into $\varphi(M)$ with its relative topology. The map ψ is a *diffeomorphism* if ψ maps M one-to-one onto N and the inverse map ψ^{-1} is C^{∞}.

A *vector field* X is an assignment of a vector X_m to each point $m \in M$. A vector field X is smooth if whenever V is open in M and $f \in C^{\infty}(V)$ then $Xf \in C^{\infty}(V)$. In this dissertation, we will only consider smooth vector fields and the adjective 'smooth' will be dropped. The set of vector fields on M is denoted by $X(M)$.

A *distribution* D of rank k is an assignment of a k-dimensional subspace D_m of M_m to each point $m \in M$. D is smooth if for each $m \in M$, there is a neighborhood U of m and there are k vector fields $X_1, \ldots, X_k$ of class C^{∞} on U which span D at each point of U. A vector field X on M is said to belong to the distribution D if $X_m \in D_m$ for each $m \in M$.

Let U and V be vector spaces. The set $U \times V = \{(u,v) : u \in U \text{ and } v \in V\}$ is the Cartesian product of U and V. Let N be the vector subspace spanned by elements of the form

$$(u+u',v) - (u,v) - (u',v), \quad (u,v+v') - (u,v) - (u,v'),$$
$$(ru,v) - r(u,v), \quad (u,rv) - r(u,v)$$

where $u,u' \in U$, $v,v' \in V$ and $r \in \mathbb{R}$. The tensor product of U and V, denoted $U \otimes V$ is defined as $(U \times V)/N$.

The contravariant tensor space of degree r for a vector space V, $T^r(V)$ is defined as $V \otimes V \otimes \ldots \otimes V$ (r times tensor product). If $r = 1$, T^1 is equal to V. We set $T^o = \mathbb{R}$. The covariant tensor space of degree s for a vector space V, $T_s(V)$ is defined as $V^* \otimes \ldots \otimes V^*$ (s times tensor product) where V^* is the dual vector space of V. Then $T_1 = V^*$ and $T_o = \mathbb{R}$. The tensor space of type (r,s) of a vector space V, T^r_s is defined as $V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*$ (r times tensor product of V and s times tensor product of V^*).

The tangent space at m of the manifold M, M_m, is a vector space. A tensor field of type (r,s) (or (r,s) tensor) is an assignment of a tensor $K_m \in T^r_s(M_m)$ to each point $m \in M$. In a coordinate neighborhood U with coordinate charts $x_1, \ldots, x_n$, a tensor field K of type (r,s) can be expressed as

$$K_x = \sum K^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial x_{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \ldots \otimes dx_{j_s} .$$

A tensor field K_x is C^∞ if all $K^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ are smooth in each of the coordinate neighborhoods of all $m \in M$. In this dissertation, only smooth tensor field will be considered and hence the adjective 'smooth' will be dropped.

A *Lie group* G is a manifold endowed with a group structure such that the map $G \times G \to G$ defined by $(\sigma, \tau) \to \sigma\tau^{-1}$ is C^∞.

(H, φ) is a *Lie subgroup* of the Lie group G if H is a Lie group, (H, φ) is a submanifold of G and $\varphi : H \to G$ is a group homeomorphism. (H, φ) is called a closed Lie subgroup of G if in addition, $\varphi(H)$ is a closed subset of G.

From [18], we have the following result:

Let G be a Lie group, and let A be a closed abstract subgroup of G. Then A has a unique manifold structure which makes A into a Lie subgroup of G. The topology in this manifold structure of A is the relative topology.

Let $X, Y \in X(M)$; the *Lie bracket* [X,Y] is a vector field defined by $[X,Y]f = X(Yf) - Y(Xf)$ for any function f. Let G be a Lie group with the identity e, the left multiplication by $a \in G$,

L_a, is defined as $L_a(x) = ax$ for all $x \in G$. A vector field X is called a *left invariant vector field* if $dL_a(X_e) = X_a$ for all $a \in G$. The *Lie algebra* g of Lie group G is defined to be the set of all left invariant vector fields on G with the usual addition, scalar multiplication and bracket operation. As a vector space, g is isomorphic with the tangent space G_e.

1.2. Construction of manifolds

There are some techniques of constructing new manifolds.

(i) *Direct product*

Direct product $M_1 \times M_2$ of manifolds $M_1^{n_1}$ and $M_2^{n_2}$ is a manifold of dimension $n_1 + n_2$. The set $\{(U_\alpha \times U_\beta, \varphi_\alpha \times \varphi_\beta) : \alpha \in A, \beta \in B\}$ where $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ and $\{(U_\beta, \varphi_\beta) : \beta \in B\}$ are atlas of M_1 and M_2 respectively, forms an atlas of $M_1 \times M_2$. If M_1 and M_2 are compact, the product manifold is also compact. Similarly, if M_1 and M_2 are orientable (refer to Chapter 2), so is $M_1 \times M_2$. If M_i admits k_i nowhere vanishing independent vector fields, then $M_1 \times M_2$ admits $k_1 + k_2$ nowhere vanishing independent vector fields. Direct product of two Lie groups is a Lie group.

(ii) *Quotient manifold*

The quotient manifold M/ρ of M is a manifold with which the natural map $\mu : M/\rho \to M$ is a submersion, where ρ is an equivalence relation on M. The following result ensures the existence of certain quotient manifolds.

Let G be a properly discontinuous group of differentiable transformations acting on a differentiable manifold M. Then the quotient manifold M/G has a differentiable structure such that the projection $\mu : M \to M/G$ is differentiable. The dimension of the quotient space M/G is the same as the ambient space M [12].

(iii) *Homogeneous space*

Let G be a Lie group and H a closed Lie subgroup. Then $G/H = \{\sigma H : \sigma \in G\}$ has a unique differentiable structure such that the natural projection is C^∞ and there exist local sections of G/H in G. G/H is called a homogeneous manifold. The dimension of a homogeneous manifold is the difference of the dimension of G and the dimension of H. If H is a normal closed subgroup of G, then G/H is a Lie group [12].

1.3. Examples of manifolds

(i) Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices. Clearly, $M_n(\mathbb{R})$ can be identified with $\mathbb{R}^{n^2}$. Let $\det : M_n(\mathbb{R}) \to \mathbb{R}$ be the determinant function. This is a continuous map from $\mathbb{R}^{n^2}$ to $\mathbb{R}$. Therefore $\det^{-1}(0)$ is closed in $\mathbb{R}^{n^2}$. Let $GL(n;\mathbb{R})$ be a subset of $\mathbb{R}^{n^2}$ where the determinant function is not zero. Clearly, $GL(n;\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ and hence is a manifold with the relative topology and the induced differentiable structure.

The manifold $GL(n;\mathbb{R})$ with the usual group structure is a Lie group. It is called the general linear group. Since the tangent space of $GL(n;\mathbb{R})$ at identity is equal to the tangent space of $\mathbb{R}^{n^2}$ at this point, the Lie algebra of $GL(n;\mathbb{R})$, denoted by $gl(n;\mathbb{R})$, is isomorphic to $\mathbb{R}^{n^2}$ as a vector space. Therefore, $gl(n;\mathbb{R})$ can be considered as the set of all $n \times n$ matrices $M_n(\mathbb{R})$. The general linear group $GL(n;\mathbb{R})$ has two components corresponding to the sets $\{x \in GL(n;\mathbb{R}) : \det x > 0\}$ and $\{x \in GL(n;\mathbb{R}) : \det x < 0\}$. The first component contains the identity element of the Lie group $GL(n;\mathbb{R})$ and thus is a Lie subgroup of $GL(n;\mathbb{R})$. This Lie subgroup is denoted by $GL^{+}(n;\mathbb{R})$.

The set $\{x \in GL(n;\mathbb{R}) : \det x = 1\}$ is a closed subgroup of $GL(n;\mathbb{R})$. As discussed above, a closed subgroup of a Lie group is a Lie subgroup with a unique structure. This Lie subgroup is called special general linear group and is denoted by $SL(n;\mathbb{R})$.

The orthogonal group $O(n)$ is a subgroup of $GL(n;\mathbb{R})$ defined by $AA^t = I$ for all $A \in O(n)$. The group $O(n)$ is a closed subgroup of $GL(n;\mathbb{R})$, thus, it is a Lie subgroup of $GL(n;\mathbb{R})$. The group $O(n)$ has two components. The identity components (the component contains the identity) is a Lie subgroup of $O(n)$. This Lie group is called special orthogonal group, denoted by $SO(n)$. The Lie algebra $o(n)$ of Lie group $O(n)$ is the set of skew symmetric matrices, a subspace of $gl(n;\mathbb{R})$. Let E_{ij} be the matrix with the components 1 at position (i,j), -1 at position (j,i) and 0 otherwise. It is clear that the set $\{E_{ij}, i < j\}$ is a set of basis of $o(n)$. Thus, the dimension of the $o(n)$ is $\frac{n(n-1)}{2}$. The dimension of a Lie group is equal to that of its Lie algebra (as a vector space) and therefore equals to $\frac{n(n-1)}{2}$. The dimension of the Lie group SO(n) is also equal to $\frac{n(n-1)}{2}$ since it is a maximal component of O(n).

(ii) An n-dimensional sphere, S^n, is defined as a subset of $\mathbb{R}^{n+1}$ such that $\sum_{i=1}^{n+1} x_i^2 = 1$, where x_i are coordinate charts of $\mathbb{R}^{n+1}$. Let $U = S^n\setminus\{(1,0,\ldots,0)\}$ and $V = S^n\setminus\{(-1,0,\ldots,0)\}$; we observed that $U \cup V = S^n$. Let ϕ_u be the stereographic projection of U onto $\mathbb{R}^n$ defined by

$$\phi_u(x_1,\ldots,x_{n+1}) = \left(\frac{x_2}{1-x_1}, \frac{x_3}{1-x_1}, \ldots, \frac{x_n}{1-x_1}\right) = (u_1,\ldots,u_n)$$

and ϕ_v be the stereographic projection of V onto $\mathbb{R}^n$ defined by

$$\phi_v(x_1,\ldots,x_{n+1}) = \left(\frac{x_2}{1+x_1}, \ldots, \frac{x_n}{1+x_1}\right) = (v_1,\ldots,v_n) .$$

We have

$$\phi_u^{-1}(u_1,\ldots,u_n) = \left(-\frac{1-\sum_{i=1}^{n} u_i^2}{1+\sum_{i=1}^{n} u_i^2}, \frac{2u_1}{1+\sum_{i=1}^{n} u_i^2}, \ldots, \frac{2u_n}{1+\sum_{i=1}^{n} u_i^2}\right)$$

and

$$\phi_v^{-1}(v_1,\ldots,v_n) = \left(+\frac{1-\sum_{i=1}^{n} v_i^2}{1+\sum_{i=1}^{n} v_i^2}, \frac{2v_1}{1+\sum_{i=1}^{n} v_i^2}, \ldots, \frac{2v_n}{1+\sum_{i=1}^{n} v_i^2}\right) .$$

The map $\phi_u \cdot \phi_v^{-1}$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined as follows:

$$\phi_u \cdot \phi_v^{-1}(v_1,\ldots,v_n) = \phi_u\left(-\frac{1-\sum_{i=1}^{n} v_i^2}{1+\sum_{i=1}^{n} v_i^2}, \frac{2v_1}{1+\sum_{i=1}^{n} v_i^2}, \ldots, \frac{2v_n}{1+\sum_{i=1}^{n} v_i^2}\right)$$

$$= \left(\frac{\dfrac{2v_1}{1+\sum_{i=1}^{n} v_i^2}}{1-\dfrac{1-\sum_{i=1}^{n} v_i^2}{1+\sum_{i=1}^{n} v_i^2}}, \ldots, \frac{\dfrac{2v_n}{1+\sum_{i=1}^{n} v_i^2}}{1-\dfrac{1-\sum_{i=1}^{n} v_i^2}{1+\sum_{i=1}^{n} v_i^2}}\right)$$

$$= \left(\frac{v_1}{\sum_{i=1}^{n} v_i^2}, \ldots, \frac{v_n}{\sum_{i=1}^{n} v_i^2}\right)$$

Since $U \cap V = S^n \setminus \{(1,0,\ldots,0), (-1,0,\ldots,0)\}$, $\sum_{i=1}^{n} v_i^2 \neq 0$, thus, $\phi_u \cdot \phi_v^{-1}$ is C^∞. With the same argument, we can show that $\phi_v \cdot \phi_u^{-1}$ is C^∞ on $\phi_u(U \cap V)$. Therefore $\{(U,\phi_u), (V,\phi_v)\}$ is an atlas on S^n. The differentiable structure on S^n is the maximal collection of coordinate charts (a mapping of a subset of S^n into $\mathbb{R}^n$) compatible with ϕ_u and ϕ_v. So, we conclude that S^n is an n-dimensional manifold.

Let $U_{2i} = \{(x_1,\ldots,x_{n+1}) \in S^n : x_i > 0\}$ and $U_{2i-1} = \{(x_1,\ldots,x_{n+1}) \in S^n : x_i < 0\}$. We have $\bigcup_{j=1}^{2n+2} U_j = M$. Let $f_j : U_j \to \mathbb{R}^n$; $f_j(x_1,\ldots,x_{n+1}) = (x_1,\ldots,\hat{x}_i,\ldots,x_m)$ where $\hat{x}_i$ is an omission of x_i and $j = 2i$ or $2i-1$. The following are functions $f_j \cdot \phi_u^{-1}$ and $\phi_u \cdot f_j^{-1}$ on appropriate domains.

$$f_j \cdot \phi_u^{-1}(u_1, \ldots, u_n)$$

$$= \left(\frac{1 - \sum_{i=1}^{n} u_i^2}{1 + \sum_{i=1}^{n} u_i^2}, \frac{2u_1}{1 + \sum_{i=1}^{n} u_i^2}, \ldots, \frac{2\hat{u}_{k-1}}{1 + \sum_{i=1}^{n} u_i^2}, \ldots, \frac{2u_n}{1 + \sum_{i=1}^{n} u_i^2} \right)$$

where $j = 2k$ or $2k-1$.

$$\phi_u \cdot f_1^{-1}(y_1, \ldots, y_n) = \left(\frac{y_1}{1 - \sqrt{1 - \sum_{i=1}^{n} y_i^2}}, \ldots, \frac{y_n}{1 - \sqrt{1 - \sum_{i=1}^{n} y_i^2}} \right),$$

$$\phi_u \cdot f_2^{-1}(y_1, \ldots, y_n) = \left(\frac{y_1}{1 + \sqrt{ + \sum_{i=1}^{n} y_i^2}}, \ldots, \frac{y_n}{1 + \sqrt{1 - \sum_{i=1}^{n} y_i^2}} \right) \text{ and}$$

$$\phi_u \cdot f_j^{-1}(y_1, \ldots, y_n) = \left(\frac{y_2}{1+y_1}, \ldots, \frac{x_k}{1+y_1}, \ldots, \frac{y_n}{1+y_1} \right) \quad \text{for } j > 2$$

$$\text{where } x_k = \begin{cases} \sqrt{1 - \sum_{i=1}^{n} y_i^2} & \text{if } j = 2k \\ -\sqrt{1 - \sum_{i=1}^{n} y_i^2} & \text{if } j = 2k-1 . \end{cases}$$

and x_k is at the kth position.

Since all these functions are restricted to appropriate domains, they are C^∞ functions and thus all f_j are compatible to ϕ_u. The functions f_j are coordinate charts of S^n.

The tangent space of S^n at $x = (x_1, \ldots, x_{n+1})$ is $\{y = (y_1, \ldots, y_{n+1}) : \mathbb{R}^{n+1} : x \cdot y = 0\}$ ($x \cdot y$ is the inner product of x and y). It is well-known that there exist $\rho(n)-1$ linearly independent vector fields on S^{n-1}, where $\rho(n)$ is defined as below:

$$\rho(n) = 2^c + 8d \text{ with } n = (2a+1)2^b \text{ and } b = c+4d, \ 0 \le c \le 3 .$$

Clearly n is odd (i.e. n-1 is even) implies $b = 0$, thus $c = d = 0$ and $\rho(n) = 1$. Therefore any vector field on even dimensional sphere has at least one zero. S^1, S^3 and S^7 are the only parallelizable spheres.

(iii) Let $\mathbb{R}^*$ be $\mathbb{R} - \{0\}$. The n-dimensional real projective space $\mathbb{R}P^n$ is defined to be $(\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^*$, namely, if $x, y \in \mathbb{R}^{n+1} - \{0\}$, then they define the same point in $\mathbb{R}P^n$ iff $x = cy$ for some $c \in \mathbb{R}^*$. Let $U_i = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} - \{0\} : x_i = 1\}$ and $\phi_i : U_i \to \mathbb{R}^{n+1}$; $\phi_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$ where $\hat{x}_i$ is the omission of x_i. Each point of U_i defines a unique point in $\mathbb{R}P^n$ and hence U_i can be considered as an open set in $\mathbb{R}P^n$. Clearly, $\bigcup_{i=1}^{n+1} U_i = \mathbb{R}P^n$. Now, we derive the function $\phi_i \cdot \phi_j^{-1}$ on appropriate domains.

$$\phi_i \cdot \phi_j^{-1}(x_1, \ldots, x_n) = \phi_i(x_1, \ldots, 1, \ldots, x_n)$$

where 1 is at position j

$$= (x_1, \ldots, \hat{x}_i, \ldots, 1, \ldots, x_n) .$$

Clearly, these functions are C^∞. Therefore $\{(U_i, \phi_i) : i = 1, \ldots, n+1\}$ is an atlas on $\mathbb{R}P^n$.

(iv) A complex manifold can be defined similar to the real manifolds. We substitute $\mathbb{R}^n$ with $\mathbb{C}^n$ and instead of C^∞, $\varphi_\alpha \cdot \varphi_\beta^{-1}$ are required to be analytic for all α and β.

Clearly $\mathbb{C}^n$ is an n-dimensional complex manifold. As in (i), we let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with complex components. Let $\det : M_n(\mathbb{C}) \to \mathbb{C}$ be the determinant function. This is a continuous map from $\mathbb{C}^{n^2}$ into $\mathbb{C}$. $\det^{-1}(0)$ is closed in $\mathbb{C}^{n^2}$. Let $GL(n;\mathbb{C})$ be a subset of $\mathbb{C}^{n^2}$ defined by $GL(n;\mathbb{C}) = \{x \in M_n(\mathbb{C}) : \det x \neq 0\}$. This is an open subset of $\mathbb{C}^n$ and hence is an n-dimensional complex manifold.

$GL(n;C)$ with the usual group structure is a Lie group. In Chapter 2, we will show that $GL(n;\mathbb{C})$ can be considered as a Lie subgroup of $GL(n;\mathbb{R})$; namely, $GL(n;\mathbb{C})$ is a $2n^2$-dimensional real manifold.

The set $\{x \in GL(n;\mathbb{C}) : \det x = 1\}$ is a closed subgroup of $GL(n;C)$ and hence is a Lie subgroup of $GL(n;\mathbb{C})$ with relative topology. This is the special linear complex group and denoted by $SL(n;\mathbb{C})$.

Unitary group, $U(n)$, is a subgroup of $GL(n;\mathbb{C})$ such that $A\bar{A}^t = I$ for all $A \in U(n)$. Unitary group is a Lie subgroup of $GL(n;C)$.