CHAPTER 1. DEFINITIONS AND PRELIMINARIES

Chapter 1 consists of basic concept in differential geometry that would be used in this dissertation.

1.1. Differentiable manifolds

Let $\mathbb{R}^n$ be the n-dimensional usual Euclidean space. An $n$-dimensional (differentiable) manifold is a paracompact, second countable topological space $M$ such that

(i) $M$ is a Hausdorff space and each point of $M$ has a neighborhood that is homeomorphic to an open set of $\mathbb{R}^n$. Thus, $M$ is a locally Euclidean space.

(ii) There is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha)\}$ where $\varphi_\alpha$ is a homeomorphism of a connected set $U_\alpha \subset M$ onto an open subset of $\mathbb{R}^n$ satisfying the following three properties:

(a) $\bigcup_\alpha U_\alpha = M$.

(b) $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \to \varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ are $C^\infty$ for all $\alpha$ and $\beta$.

(c) The collection of coordinate system $\{(U_\alpha, \varphi_\alpha)\}$ is maximal with respect to (b). This collection defines the differentiable structure of the manifold $M$. The open set $U_\alpha$ is called a coordinate neighborhood for all $m \in U_\alpha$. 
An n-dimensional manifold will be denoted by $M^n$ or simply by $M$ if the underlying dimension of the manifold is not under consideration. A collection of $(U_\alpha, \varphi_\alpha)$ which fulfills the conditions ii(a) and (b) in the above definition is called an atlas of $M$. Clearly, if an atlas of the manifold $M$ is specified, the differentiable structure of $M$ is defined.

Let $M$ be a manifold with an atlas $\{(U_\alpha, \varphi_\alpha)\}$. If $N$ is an open subset of $M$, then $N$ is a manifold with an atlas $\left\{\left\{(U_\alpha \cap N, \varphi_\alpha|_{U_\alpha \cap N})\right\}\right\}$. Let $m \in M^n$. Functions $f$ and $g$ defined on an open set containing $m$ are said to be equivalent if they agree on some neighborhood of $m$. The set of equivalence classes is denoted by $\tilde{F}_m$. $\tilde{F}_m$ is an algebra. A tangent vector $v$ at the point $m$ is a linear derivation of $\tilde{F}_m$. The tangent space of $M$ at $m$, $M_m$, is the set of tangent vectors at $m$. It can be shown that $M_m$ is a vector space.

Since $M_m$ is n-dimensional, it is isomorphic to $R^n$. Its dual space will be denoted by $M^*_m$. Let $TM = \bigcup_{m \in M} M_m$ and $TM^* = \bigcup_{m \in M} M^*_m$. Let $U$ be a coordinate neighborhood of $m$ with coordinate functions $x_1, \ldots, x_n$ and coordinate map $\varphi$, namely, for each $m \in U$, $\varphi(m) = (x_1(m), x_2(m), \ldots, x_n(m))$. The coordinate function $x_i$ maps $U$ into $R$. The natural projection $\pi : TM \rightarrow M$ is defined as $\pi(v) = m$ if $v \in M_m$. For each coordinate $x_i$, the differential $dx_i$ is given by $dx_i(v)f = v(f \circ x_i)$ where $v \in M_m$ and $f$ is a function from $R$ to $R$. Define $\tilde{\varphi} : \pi^{-1}(U) \rightarrow R^{2n}$ as
for all \( v \in \pi^{-1}(U) \). Observe that

(i) If \((U, \psi)\) and \((V, \varphi)\) are coordinate neighborhoods of \( m \in M \) with coordinate charts \( \psi \) and \( \varphi \) respectively, then \( \tilde{\psi} \cdot \tilde{\varphi}^{-1} \) is \( C^\infty \).

(ii) The collection \( \{ \tilde{\psi}^{-1}(W) : W \) open in \( \mathbb{R}^{2n} \) and \((U, \varphi)\) is a coordinate neighborhood of a point \( m \in M \) and coordinate chart \( \varphi \}\} forms a basis for a topology on \( TM \) which makes \( TM \) into a \( 2n \)-dimensional, second countable, locally Euclidean space.

(iii) Let \( \tau \) be the maximal collection, with respect to ii(b) of definition for manifold, containing \( \{(\pi^{-1}(U), \tilde{\psi})\} \). Then \( \tau \) is a differentiable structure on \( TM \).

Hence, \( TM \) is a \( 2n \)-dimensional manifold [18]. The space \( TM^* \) is also a \( 2n \)-dimensional manifold. It's differentiable structure can be constructed similar to that of \( TM \).

Let \( \psi : M^n \to N^k \) be \( C^\infty \), then \( d\psi_m \) is a mapping of \( M_m \) to \( N_{\psi(m)} \) such that \( d\psi_m (v)f = v(f \cdot \psi)_m \) where \( v \in M_m \) and \( f \) is a function on \( N \). The map \( \psi \) is an immersion if \( d\psi_m \) is non-singular at each \( m \in M \), i.e. \( d\psi_m (M_m) \subset N_{\psi(m)} \) is of n-dimensional. The pair \( (M, \psi) \) is a sub-manifold of \( N \) if \( \psi \) is a one-to-one immersion. The map \( \psi \) is an imbedding if \( \psi \) is a one-to-one immersion which is also a homeomorphism into; that is, \( \psi \) is a map into \( \varphi(M) \) with its relative topology. The map \( \psi \) is a diffeomorphism if \( \psi \) maps \( M \) one-to-one onto \( N \) and the inverse map \( \psi^{-1} \) is \( C^\infty \).
A **vector field** $X$ is an assignment of a vector $X_m$ to each point $m \in M$. A vector field $X$ is smooth if whenever $V$ is open in $M$ and $f \in C^\infty(V)$ then $Xf \in C^\infty(V)$. In this dissertation, we will only consider smooth vector fields and the adjective 'smooth' will be dropped. The set of vector fields on $M$ is denoted by $X(M)$.

A **distribution** $D$ of rank $k$ is an assignment of a $k$-dimensional subspace $D_m$ of $M_m$ to each point $m \in M$. $D$ is smooth if for each $m \in M$, there is a neighborhood $U$ of $m$ and there are $k$ vector fields $X_1, \ldots, X_k$ of class $C^\infty$ on $U$ which span $D$ at each point of $U$. A vector field $X$ on $M$ is said to belong to the distribution $D$ if $X_m \in D_m$ for each $m \in M$.

Let $U$ and $V$ be vector spaces. The set $U \times V = \{(u,v) : u \in U \text{ and } v \in V\}$ is the *Cartesian product* of $U$ and $V$. Let $N$ be the vector subspace spanned by elements of the form

$$(u+u',v) - (u,v) - (u',v), \quad (u,v+v') - (u,v) - (u,v'),$$

$$(ru,v) - r(u,v), \quad (u,rv) - r(u,v)$$

where $u,u' \in U$, $v,v' \in V$ and $r \in \mathbb{R}$. The tensor product of $U$ and $V$, denoted $U \otimes V$ is defined as $(U \times V)/N$.

The contravariant tensor space of degree $r$ for a vector space $V$, $T^r(V)$ is defined as $V \otimes V \otimes \ldots \otimes V$ ($r$ times tensor product). If $r = 1$, $T^1$ is equal to $V$. We set $T^0 = \mathbb{R}$. The covariant tensor space of degree $s$ for a vector space $V$, $T^s_s(V)$ is defined as $V^* \otimes \ldots \otimes V^*$ ($s$ times tensor product) where $V^*$ is the dual vector space of $V$. Then $T^1 = V^*$ and $T^0 = \mathbb{R}$. The tensor space of type $(r,s)$ of a vector space $V$, $T^r_s$ is defined as $V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*$ ($r$ times tensor product of $V$ and $s$ times tensor product of $V^*$).
The tangent space at \( m \) of the manifold \( M, M_m \), is a vector space. A tensor field of type \((r, s)\) (or \((r, s)\) tensor) is an assignment of a tensor \( K_m \in T^r_s(M_m) \) to each point \( m \in M \). In a coordinate neighborhood \( U \) with coordinate charts \( x_1, \ldots, x_n \), a tensor field \( K \) of type \((r, s)\) can be expressed as

\[
K_x = \sum K_{j_1 \ldots j_s}^{i_1 \ldots i_r} \frac{\partial}{\partial x_{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}
\]

A tensor field \( K_x \) is \( C^\infty \) if all \( K_{j_1 \ldots j_s}^{i_1 \ldots i_r} \) are smooth in each of the coordinate neighborhoods of all \( m \in M \). In this dissertation, only smooth tensor field will be considered and hence the adjective 'smooth' will be dropped.

A Lie group \( G \) is a manifold endowed with a group structure such that the map \( G \times G \to G \) defined by \((\sigma, \tau) \to \sigma \tau^{-1} \) is \( C^\infty \).

\((H, \varphi)\) is a Lie subgroup of the Lie group \( G \) if \( H \) is a Lie group, \((H, \varphi)\) is a submanifold of \( G \) and \( \varphi : H \to G \) is a group homeomorphism. \((H, \varphi)\) is called a closed Lie subgroup of \( G \) if in addition, \( \varphi(H) \) is a closed subset of \( G \).

From [18], we have the following result:

Let \( G \) be a Lie group, and let \( A \) be a closed abstract subgroup of \( G \). Then \( A \) has a unique manifold structure which makes \( A \) into a Lie subgroup of \( G \). The topology in this manifold structure of \( A \) is the relative topology.

Let \( X, Y \in X(M) \); the Lie bracket \([X, Y]\) is a vector field defined by \([X, Y]f = X(Yf) - Y(Xf)\) for any function \( f \). Let \( G \) be a Lie group with the identity \( e \), the left multiplication by \( a \in G \),
1. A vector field \( X \) is called a \textit{left invariant vector field} if \( \text{d}l_{a}(X_{e}) = X_{a} \) for all \( a \in G \). The \textit{Lie algebra} \( g \) of Lie group \( G \) is defined to be the set of all left invariant vector fields on \( G \) with the usual addition, scalar multiplication and bracket operation. As a vector space, \( g \) is isomorphic with the tangent space \( G_{e} \).

1.2. Construction of manifolds

There are some techniques of constructing new manifolds.

(i) \textit{Direct product}

Direct product \( M_{1} \times M_{2} \) of manifolds \( M_{1} \) and \( M_{2} \) is a manifold of dimension \( n_{1} + n_{2} \). The set \( \{(U_{\alpha}, \varphi_{\alpha} \times \varphi_{\beta}) : \alpha \in A, \beta \in B\} \) where \( \{(U_{\alpha}, \varphi_{\alpha}) : \alpha \in A\} \) and \( \{(U_{\beta}, \varphi_{\beta}) : \beta \in B\} \) are atlas of \( M_{1} \) and \( M_{2} \) respectively, forms an atlas of \( M_{1} \times M_{2} \). If \( M_{1} \) and \( M_{2} \) are compact, the product manifold is also compact. Similarly, if \( M_{1} \) and \( M_{2} \) are orientable (refer to Chapter 2), so is \( M_{1} \times M_{2} \). If \( M_{1} \) admits \( k_{1} \) nowhere vanishing independent vector fields, then \( M_{1} \times M_{2} \) admits \( k_{1} + k_{2} \) nowhere vanishing independent vector fields. Direct product of two Lie groups is a Lie group.

(ii) \textit{Quotient manifold}

The quotient manifold \( M/\rho \) of \( M \) is a manifold with which the natural map \( \mu : M/\rho \to M \) is a submersion, where \( \rho \) is an equivalence relation on \( M \). The following result ensures the existence of certain quotient manifolds.
Let $G$ be a properly discontinuous group of differentiable transformations acting on a differentiable manifold $M$. Then the quotient manifold $M/G$ has a differentiable structure such that the projection $\mu : M \to M/G$ is differentiable. The dimension of the quotient space $M/G$ is the same as the ambient space $M$ [12].

(iii) Homogeneous space

Let $G$ be a Lie group and $H$ a closed Lie subgroup. Then $G/H = \{cH : c \in G\}$ has a unique differentiable structure such that the natural projection is $C^\infty$ and there exist local sections of $G/H$ in $G$. $G/H$ is called a homogeneous manifold. The dimension of a homogeneous manifold is the difference of the dimension of $G$ and the dimension of $H$. If $H$ is a normal closed subgroup of $G$, then $G/H$ is a Lie group [12].

1.3. Examples of manifolds

(i) Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices. Clearly, $M_n(\mathbb{R})$ can be identified with $\mathbb{R}^{n^2}$. Let $\det : M_n(\mathbb{R}) \to \mathbb{R}$ be the determinant function. This is a continuous map from $\mathbb{R}^{n^2}$ to $\mathbb{R}$. Therefore $\det^{-1}(0)$ is closed in $\mathbb{R}^{n^2}$. Let $GL(n;\mathbb{R})$ be a subset of $\mathbb{R}^{n^2}$ where the determinant function is not zero. Clearly, $GL(n;\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ and hence is a manifold with the relative topology and the induced differentiable structure.

The manifold $GL(n;\mathbb{R})$ with the usual group structure is a Lie group. It is called the general linear group. Since the tangent space of $GL(n;\mathbb{R})$ at identity is equal to the
tangent space of $\mathbb{R}^{n^2}$ at this point, the Lie algebra of $\text{GL}(n;\mathbb{R})$, denoted by $\mathfrak{gl}(n;\mathbb{R})$, is isomorphic to $\mathbb{R}^{n^2}$ as a vector space. Therefore, $\mathfrak{gl}(n;\mathbb{R})$ can be considered as the set of all $n \times n$ matrices $M_n(\mathbb{R})$. The general linear group $\text{GL}(n;\mathbb{R})$ has two components corresponding to the sets 
\{x \in \text{GL}(n;\mathbb{R}) : \det x > 0\} and \{x \in \text{GL}(n;\mathbb{R}) : \det x < 0\}. The first component contains the identity element of the Lie group $\text{GL}(n;\mathbb{R})$ and thus is a Lie subgroup of $\text{GL}(n;\mathbb{R})$. This Lie subgroup is denoted by $\text{GL}^+(n;\mathbb{R})$.

The set \{x \in \text{GL}(n;\mathbb{R}) : \det x = 1\} is a closed subgroup of $\text{GL}(n;\mathbb{R})$. As discussed above, a closed subgroup of a Lie group is a Lie subgroup with a unique structure. This Lie subgroup is called special general linear group and is denoted by $\text{SL}(n;\mathbb{R})$.

The orthogonal group $O(n)$ is a subgroup of $\text{GL}(n;\mathbb{R})$ defined by $AA^t = I$ for all $A \in O(n)$. The group $O(n)$ is a closed subgroup of $\text{GL}(n;\mathbb{R})$, thus, it is a Lie subgroup of $\text{GL}(n;\mathbb{R})$. The group $O(n)$ has two components. The identity components (the component contains the identity) is a Lie subgroup of $O(n)$. This Lie group is called special orthogonal group, denoted by $\text{SO}(n)$. The Lie algebra $\mathfrak{o}(n)$ of Lie group $O(n)$ is the set of skew symmetric matrices, a subspace of $\mathfrak{gl}(n;\mathbb{R})$. Let $E_{ij}$ be the matrix with the components 1 at position $(i,j)$, -1 at position $(j,i)$ and 0 otherwise. It is clear that the set \{E_{ij}, i < j\} is a set of basis of $\mathfrak{o}(n)$. Thus, the dimension of the $\mathfrak{o}(n)$ is $\frac{n(n-1)}{2}$. The dimension of a Lie group is equal to that of
its Lie algebra (as a vector space) and therefore equals to $\frac{n(n-1)}{2}$. The dimension of the Lie group $SO(n)$ is also equal to $\frac{n(n-1)}{2}$ since it is a maximal component of $O(n)$.

(ii) An $n$-dimensional sphere, $S^n$, is defined as a subset of $\mathbb{R}^{n+1}$ such that $\sum_{i=1}^{n+1} x_i^2 = 1$, where $x_i$ are coordinate charts of $\mathbb{R}^{n+1}$. Let $U = S^n\backslash\{(1,0,\ldots,0)\}$ and $V = S^n\backslash\{(-1,0,\ldots,0)\}$; we observed that $U \cup V = S^n$. Let $\phi_u$ be the stereographic projection of $U$ onto $\mathbb{R}^n$ defined by

$$\phi_u(x_1, \ldots, x_{n+1}) = \left( \frac{x_2}{1-x_1}, \frac{x_3}{1-x_1}, \ldots, \frac{x_n}{1-x_1} \right) = (u_1, \ldots, u_n)$$

and $\phi_v$ be the stereographic projection of $V$ onto $\mathbb{R}^n$ defined by

$$\phi_v(x_1, \ldots, x_{n+1}) = \left( \frac{x_2}{1+x_1}, \ldots, \frac{x_n}{1+x_1} \right) = (v_1, \ldots, v_n).$$

We have

$$\phi_u^{-1}(u_1, \ldots, u_n) = \left( \frac{1 - \sum_{i=1}^{n} u_i^2}{1 + \sum_{i=1}^{n} u_i^2}, \frac{2u_1}{1 + \sum_{i=1}^{n} u_i^2}, \ldots, \frac{2u_n}{1 + \sum_{i=1}^{n} u_i^2} \right)$$

and

$$\phi_v^{-1}(v_1, \ldots, v_n) = \left( \frac{1 - \sum_{i=1}^{n} v_i^2}{1 + \sum_{i=1}^{n} v_i^2}, \frac{2v_1}{1 + \sum_{i=1}^{n} v_i^2}, \ldots, \frac{2v_n}{1 + \sum_{i=1}^{n} v_i^2} \right).$$

The map $\phi_u \cdot \phi_v^{-1}$ is a mapping from $\mathbb{R}^n \to \mathbb{R}^n$ defined as follows:
\[ \phi_u \cdot \phi_v^{-1}(v_1, \ldots, v_n) = \phi_u \left( \begin{array}{c} 1 - \sum_{i=1}^{n} \frac{v_i^2}{1 + \sum_{i=1}^{n} v_i^2} \\ \frac{2v_1}{1 + \sum_{i=1}^{n} v_i^2} \\ \cdots \\ \frac{2v_n}{1 + \sum_{i=1}^{n} v_i^2} \end{array} \right) \]

\[ = \left( \begin{array}{c} \frac{2v_1}{1 + \sum_{i=1}^{n} v_i^2} \\ \frac{2v_n}{1 + \sum_{i=1}^{n} v_i^2} \\ \cdots \\ \frac{2v_1}{1 + \sum_{i=1}^{n} v_i^2} \end{array} \right) \]

Since \( U \cap V = S^n \setminus \{(1,0,\ldots,0), (-1,0,\ldots,0)\}, \) \( \sum_{i=1}^{n} v_i^2 \neq 0, \) thus, \( \phi_u \cdot \phi_v^{-1} \) is \( C^\infty. \) With the same argument, we can show that \( \phi_v \cdot \phi_u^{-1} \) is \( C^\infty \) on \( \phi_u(U \cap V). \) Therefore \( \{(U, \phi_u), (V, \phi_v)\} \) is an atlas on \( S^n. \) The differentiable structure on \( S^n \) is the maximal collection of coordinate charts (a mapping of a subset of \( S^n \) into \( \mathbb{R}^n \)) compatible with \( \phi_u \) and \( \phi_v. \) So, we conclude that \( S^n \) is an \( n \)-dimensional manifold.

Let \( U_{2i} = \{(x_1, \ldots, x_{n+1}) \in S^n : x_i > 0\} \) and \( U_{2i-1} = \{(x_1, \ldots, x_{n+1}) \in S^n : x_i < 0\}. \) We have \( \bigcup_{j=1}^{2n+2} U_j = M. \) Let \( f_j : U_j \rightarrow \mathbb{R}^n; f_j(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, \ldots, x_m) \) where \( \hat{x}_i \) is an omission of \( x_i \) and \( j = 2i \) or \( 2i-1. \) The following are functions \( f_j \cdot \phi_u^{-1} \) and \( \phi_u \cdot f_j^{-1} \) on appropriate domains.
\[ f_j \cdot \phi_u^{-1}(u_1, \ldots, u_n) = \begin{pmatrix}
\frac{1 - \sum_{i=1}^{n} u_i^2}{1 + \sum_{i=1}^{n} u_i^2} & \frac{2u_1}{1 + \sum_{i=1}^{n} u_i^2} & \ldots & \frac{2u_{k-1}}{1 + \sum_{i=1}^{n} u_i^2} & \ldots & \frac{2u_n}{1 + \sum_{i=1}^{n} u_i^2}
\end{pmatrix}
\]

where \( j = 2k \) or \( 2k-1 \).

\[ \phi_u \cdot f_1^{-1}(y_1, \ldots, y_n) = \begin{pmatrix}
\frac{y_1}{\sqrt{1 - \sum_{i=1}^{n} y_i^2}} & \ldots & \frac{y_n}{\sqrt{1 - \sum_{i=1}^{n} y_i^2}}
\end{pmatrix}
\]

\[ \phi_u \cdot f_2^{-1}(y_1, \ldots, y_n) = \begin{pmatrix}
\frac{y_1}{1 + \sum_{i=1}^{n} y_i^2} & \ldots & \frac{y_n}{1 + \sum_{i=1}^{n} y_i^2}
\end{pmatrix}
\]

and

\[ \phi_u \cdot f_j^{-1}(y_1, \ldots, y_n) = \begin{pmatrix}
\frac{y_2}{1+y_1} & \ldots & \frac{x_k}{1+y_1} & \ldots & \frac{y_n}{1+y_1}
\end{pmatrix}
\]

for \( j > 2 \)

\[ x_k = \begin{cases}
\sqrt{\frac{n}{1 - \sum_{i=1}^{n} y_i^2}} & \text{if } j = 2k \\
-\sqrt{\frac{n}{1 - \sum_{i=1}^{n} y_i^2}} & \text{if } j = 2k-1
\end{cases}
\]

and \( x_k \) is at the kth position.

Since all these functions are restricted to appropriate domains, they are \( C^\infty \) functions and thus all \( f_j \) are compatible to \( \phi_u \). The functions \( f_j \) are coordinate charts of \( S^n \).
The tangent space of \( S^n \) at \( x = (x_1, \ldots, x_{n+1}) \) is \( \{y = (y_1, \ldots, y_{n+1}) : \mathbb{R}^{n+1} : x \cdot y = 0\} \) (\( x \cdot y \) is the inner product of \( x \) and \( y \)). It is well-known that there exist \( \rho(n)-1 \) linearly independent vector fields on \( S^{n-1} \), where \( \rho(n) \) is defined as below:

\[
\rho(n) = 2^c + 8d \text{ with } n = (2a+1)2^b \text{ and } b = c + 4d, \quad 0 \leq c \leq 3.
\]

Clearly, \( n \) is odd (i.e. \( n-1 \) is even) implies \( b = 0 \), thus \( c = d = 0 \) and \( \rho(n) = 1 \). Therefore any vector field on even dimensional sphere has at least one zero. \( S^1, S^3 \) and \( S^7 \) are the only parallelizable spheres.

(iii) Let \( \mathbb{R}^n \) be \( \mathbb{R} - \{0\} \). The n-dimensional real projective space \( \mathbb{R}P^n \) is defined to be \( (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^* \), namely, if \( x, y \in \mathbb{R}^{n+1} - \{0\} \), then they define the same point in \( \mathbb{R}P^n \) iff \( x = cy \) for some \( c \in \mathbb{R}^* \). Let \( U_1 = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} - \{0\} : x_1 = 1\} \) and \( \phi_1 : U_1 \to \mathbb{R}^{n+1} ; \phi_1(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_1, \ldots, x_{n+1}) \) where \( \hat{x}_1 \) is the omission of \( x_1 \). Each point of \( U_1 \) defines a unique point in \( \mathbb{R}P^n \) and hence \( U_1 \) can be considered as an open set in \( \mathbb{R}P^n \). Clearly, \( \bigcup_{i=1}^{n+1} U_i = \mathbb{R}P^n \). Now, we derive the function \( \phi_1 \cdot \phi_j ^{-1} \) on appropriate domains.

\[
\phi_1 \cdot \phi_j ^{-1}(x_1, \ldots, x_n) = \phi_1(x_1, \ldots, 1, \ldots, x_n)
\]

where 1 is at position \( j \)

\[
= (x_1, \ldots, \hat{x}_1, \ldots, 1, \ldots, x_n)
\]

Clearly, these functions are \( C^\infty \). Therefore \( \{(U_i, \phi_i) : i = 1, \ldots, n+1\} \) is an atlas on \( \mathbb{R}P^n \).
(iv) A complex manifold can be defined similar to the real manifolds. We substitute $\mathbb{R}^n$ with $\mathbb{C}^n$ and instead of $C^\infty$, $\varphi^{-1}_\alpha \varphi^{-1}_\beta$ are required to be analytic for all $\alpha$ and $\beta$.

Clearly $\mathbb{C}^n$ is an $n$-dimensional complex manifold. As in (i), we let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with complex components. Let $\det : M_n(\mathbb{C}) \to \mathbb{C}$ be the determinant function. This is a continuous map from $\mathbb{C}^{n^2}$ into $\mathbb{C}$. $\det^{-1}(0)$ is closed in $\mathbb{C}^{n^2}$. Let $GL(n;\mathbb{C})$ be a subset of $\mathbb{C}^{n^2}$ defined by $GL(n;\mathbb{C}) = \{ x \in M_n(\mathbb{C}) : \det x \neq 0 \}$. This is an open subset of $\mathbb{C}^n$ and hence is an $n$-dimensional complex manifold.

$GL(n;\mathbb{C})$ with the usual group structure is a Lie group. In Chapter 2, we will show that $GL(n;\mathbb{C})$ can be considered as a Lie subgroup of $GL(n;\mathbb{R})$; namely, $GL(n;\mathbb{C})$ is a $2n^2$-dimensional real manifold.

The set $\{ x \in GL(n;\mathbb{C}) : \det x = 1 \}$ is a closed subgroup of $GL(n;\mathbb{C})$ and hence is a Lie subgroup of $GL(n;\mathbb{C})$ with relative topology. This is the special linear complex group and denoted by $SL(n;\mathbb{C})$.

Unitary group, $U(n)$, is a subgroup of $GL(n;\mathbb{C})$ such that $A^* A = I$ for all $A \in U(n)$. Unitary group is a Lie subgroup of $GL(n;\mathbb{C})$. 