CHAPTER 2. G-STRUCTURES

In this chapter, we will discuss the basic definition of G-structure of a manifold and its related concepts, examples on G-structure and the construction of transition functions on the spheres and real projective spaces. By observing the transition functions on a manifold, we can draw positive conclusions about the existence of certain structures on the manifold.

2.1. Basic definitions

Let P and M be manifolds and G a Lie group. A *principal fibre bundle* P over M with structural group G consists of a right action of G on P which fulfills the following conditions:*

(i) G acts freely on P,

(ii) There is a C^∞ surjective map from P onto M denoted by $\pi : P \to M$.

(iii) P is locally trivial, i.e. for every $x \in M$ there is a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times G$. The diffeomorphism is given by $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(u) = (\pi(u), \varphi(u))$, where φ is a mapping of $\pi^{-1}(U)$ into G satisfying $\varphi(ua) = \varphi(u)a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

From the definition of principal fibre bundle, we observed that:

(i) The map $\varphi : \pi^{-1}(x) \to G$ is bijective for all $x \in M$.

(ii) The group G acts transitively on $\pi^{-1}(x)$ since if $u_1, u_2 \in \pi^{-1}(x)$ and $g = [\varphi(u_1)]^{-1}\varphi(u_2)$ then $u_1 g = u_2$.

(iii) For each open set $V \subset U$, $\pi^{-1}(V)$ is diffeomorphic to $V \times G$.

A principal fibre bundle over M with projection π is denoted as $P(M,G,\pi)$ or $P(M,G)$ or simply P.

Example 1 (Bundle of Bases)

Let M^n be a manifold and $B(M)$ the set of $(n+1)$ tuples $(m, e_1, \ldots, e_n)$ where $(e_1, \ldots, e_n)$ is a set of basis of M_m. The projection π is defined as $\pi(m, e_1, \ldots, e_n) = m$. The general linear group $GL(n;\mathbb{R})$ acts on $B(M)$ on the right as $(m, e_1, \ldots, e_n)g = (m, \sum g_{i1}e_i, \ldots, \sum g_{in}e_i)$ where $g = (g_{ij}) \in GL(n;\mathbb{R})$.

Let $\{x_1, \ldots, x_n\}$ be the coordinate maps on an open set U of M and $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ be the standard basis of TU. For each $m \in U$ and basis $\{e_1, \ldots, e_n\}$ of M_m, there is a unique element $g \in GL(n;\mathbb{R})$ such that $(m, e_1, \ldots, e_n) = \left(m, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)g$. Therefore there is a 1-1 onto correspondence F between $\pi^{-1}(U)$ and $U \times GL(n,\mathbb{R})$ given by $F((m, e_1, \ldots, e_n)) = (m, g)$. Thus the local coordinate of $B(M)$ can be written as $\{(U_i \times GL(n;\mathbb{R}),\ x_i \times x_{ij})\}$ where x_{ij} is the standard coordinate charts of $GL(n;\mathbb{R})$. Hence $B(M)$ is a manifold and it is easy to see that the projection map is differentiable. The manifold $B(M)$ is called the bundle of bases for the manifold M. The dimension of $B(M)$ is $n^2 + n$. From the construction of local coordinates of $B(M)$, it is clear that $B(M)$ is a principal fibre bundle over M with the structural group $GL(n;\mathbb{R})$ and canonical projection π.

Bundle of Bases B(M) plays an important role in the study of G-structure of M.

Transition functions of the principal fibre bundle $P(M,G,\pi)$ with respect to a set of open covering $\{U_\alpha\}$ is a family of mappings $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ defined by $\psi_{\alpha\beta}(\pi(u)) = \varphi_\alpha(u)(\varphi_\beta(u))^{-1}$, where $u \in \pi^{-1}(U_\alpha \cap U_\beta)$ and φ_α is the mapping defined in the definition of principal fibre bundle.

The map $\psi_{\alpha\beta}$ is well defined for if $\pi(u_1) = \pi(u_2) = x$, then $u_1 = u_2 g$ for some $g \in G$ and $\psi_{\alpha\beta}(x) = \varphi_\alpha(u_1)(\varphi_\beta(u_1))^{-1} = \varphi_\alpha(u_2 g)(\varphi_\beta(u_2 g))^{-1} = \varphi_\alpha(u_2)g \cdot g^{-1}[\varphi_\beta(u_2)]^{-1} = \varphi_\alpha(u_2)[\varphi_\beta(u_2)]^{-1}$. Furthermore, for $x \in U_\alpha \cap U_\beta \cap U_\gamma$, the transition functions fulfill the following relations:

$$\psi_{\alpha\beta}(x) = \psi_{\alpha\gamma}(x)\psi_{\gamma\beta}(x) .$$

Therefore, $\psi_{\alpha\alpha}(x) = e$ and $\psi_{\alpha\beta}(x) = \psi_{\beta\alpha}^{-1}(x)$.

Kobayashi and Nomizu gave a construction of principal fibre bundle based on a set of transition functions and an open covering. Proposition 5.2 of [12] is as follows:

Let M be a manifold, $\{U_\alpha\}$ an open covering of M and G a Lie group. Given a mapping $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ for every nonempty $U_\alpha \cap U_\beta$ such that $\psi_{\alpha\beta}(x) = \psi_{\alpha\gamma}(x)\psi_{\gamma\beta}(x)$ for each $x \in U_\alpha \cap U_\beta \cap U_\gamma$, then we can construct a principal fibre bundle P(M,G) with transition functions $\psi_{\alpha\beta}$.

Let $\{U_\alpha\}$ be an open covering of M and $\{\psi_{\alpha\beta}\}$ the transition functions of $P(M,G',\pi)$ corresponding to the open covering $\{U_\alpha\}$.

Let G be a subgroup of G'. If $\psi_{\alpha\beta}(x) \in G$ for all $x \in U_\alpha \cap U_\beta$ for the covering $\{U_\alpha\}$, then the structural group G' is said to be reducible to G. The existence of an open covering $\{U_\alpha\}$ and the transition functions is called a *G-structure* on M (with respect to the given principal fibre bundle $P(M,G',\pi)$).

Kobayashi and Nomizu [12] used subbundle to define G-structure. Chern [4] considered the reduction of the bundle of bases only. In general, the reduction can be done on any principal fibre bundle, but bundle of bases is commonly used. From here onwards, the reduction is restricted to bundle of bases unless specified otherwise.

Let $\{e_1,\ldots,e_n\}$ be a set of basis on U_α and $\{f_1,\ldots,f_n\}$ a set of basis of U_β such that $\varphi_\alpha\{(m,e_1,\ldots,e_n)\} = I$ and $\varphi_\beta\{(m,f_1,\ldots,f_n)\} = I$ for all $m \in U_\alpha \cap U_\beta$. If $(m,f_1,\ldots,f_n)g = (m,e_1,\ldots,e_n)$ where $g = (g_{ij})$ then $\psi_{\beta\alpha}(m) = g$. Therefore, G-structure can be reformulated as follows:

There exists a G-structure on M if and only if there is an open covering $\{U_\alpha\}$ and n-frames $\{e_\alpha\}$ such that $(m,e_\beta)g = (m,e_\alpha)$ with $g \in G$ for all $m \in U_\alpha \cap U_\beta$ and all α, β. These n-frames are called adapted (permissible) n-frames.

A homomorphism $f : P'(M',G') \to P(M,G)$ is called an *imbedding* if $f : P' \to P$ is an imbedding and if $f : G' \to G$ is a monomorphism. The induced mapping $f : M' \to M$ is also an imbedding. By identifying P' with f(P'), G' with f(G') and M' with f(M'), we say that P'(M',G') is a *subbundle* of P(M,G).

Kobayashi and Nomizu [12] proved in Proposition 1.5.3 that the existence of G-structure is equivalent to the existence of a subbundle $P(M,G)$ of $B(M)(M,GL(n;\mathbb{R}))$.

Let P be the reduced bundle with structural group G. The G-structure is *integrable* if every point of M has a coordinate system $(U,x_1,\ldots,x_n)$ such that the cross section $\left\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right\}$ of $B(M)$ over U is a cross section of P over U.

Let $\{U_\alpha\}$ be an open covering of M corresponding to a G-structure. The G-structure is integrable if the matrices of change of coordinate systems on $U_\alpha \cap U_\beta$ belong to G for all $U_\alpha, U_\beta \in \{U_\alpha\}$.

We can make use of the existence of solution for partial differential system to determine the integrability of an G-structure. The problem is "given a set of independent vector fields $\{v^1,\ldots,v^n\}$ on an open set U, is there any coordinate system $\{x_1,\ldots,x_n\}$ in which $\frac{\partial}{\partial x_i} = v^i$ for all i?".

Let $\{y_1,\ldots,y_n\}$ be a local coordinate of U. Then $\frac{\partial}{\partial y_i} = \sum_{j=1}^{n} c_{ij}v^j$. The existence of coordinate maps $\{x_1,\ldots,x_n\}$ on U is equivalent to the existence of solution for $\frac{\partial x_i}{\partial y_j} = c_{ji}$. It is well-known that the necessary and sufficient condition for the existence of the solution x_i is

$$(*) \qquad \frac{\partial c_{ji}}{\partial y_k} = \frac{\partial c_{ki}}{\partial y_j} \quad \text{for all } i,j,k = 1,\ldots,n .$$

If c_{ij} fulfills condition (*), then for another set of coordinate charts $\{z_1, \dots, z_n\}$ on U, the coefficient d_{ij}, where $\frac{\partial}{\partial z_i} = \sum_{j=1}^{n} d_{ij} v^j$ also fulfills the condition $\frac{\partial d_{ij}}{\partial z_k} = \frac{\partial d_{kj}}{\partial z_i}$ for all i, j and k. In fact, the relation between $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial y_j}$ are given as follows:

$$\frac{\partial}{\partial z_i} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial z_i} \frac{\partial}{\partial y_j} = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial y_j}{\partial z_i} c_{jk} \right) \frac{\partial}{\partial x_k} .$$

Therefore

$$d_{ik} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial z_i} c_{jk}$$

$$\frac{\partial d_{ik}}{\partial z_r} = \sum_{j=1}^{n} \frac{\partial^2 y_j}{\partial z_r \partial z_i} c_{jk} + \sum_{j=1}^{n} \sum_{s=1}^{n} \frac{\partial y_j}{\partial z_i} \frac{\partial c_{jk}}{\partial y_s} \frac{\partial y_s}{\partial z_r}$$

and

$$\frac{\partial d_{rk}}{\partial z_i} = \sum_{j=1}^{n} \frac{\partial^2 y_j}{\partial z_i \partial z_r} c_{jk} + \sum_{j=1}^{n} \sum_{s=1}^{n} \frac{\partial y_j}{\partial z_r} \frac{\partial c_{jk}}{\partial y_s} \frac{\partial y_s}{\partial z_i}$$

$$= \sum_{j=1}^{n} \frac{\partial^2 y_j}{\partial z_r \partial z_i} c_{jk} + \sum_{j=1} \sum_{s=1} \frac{\partial y_j}{\partial z_i} \frac{\partial c_{sk}}{\partial y_j} \frac{\partial y_s}{\partial z_r}$$

$$= \sum_{j=1}^{n} \frac{\partial^2 y_j}{\partial z_r \partial z_i} c_{jk} + \sum_{j=1}^{n} \sum_{s=1}^{n} \frac{\partial y_j}{\partial z_i} \frac{\partial c_{jk}}{\partial y_s} \frac{\partial y_s}{\partial z_r} = \frac{\partial d_{ik}}{\partial z_r} .$$

Therefore, a G-structure is integrable if and only if there exists a permissible frame $\{v^1, \dots, v^n\}$ on each open set U_α of coordinate covering $\{U_\alpha\}$ and c_{ij} determined by $\frac{\partial}{\partial x_i} = \sum c_{ij} v^j$ such that $\frac{\partial c_{ij}}{\partial x_k} = \frac{\partial c_{kj}}{\partial x_i}$ for all i, j and k where $\{(U_\alpha, x_\alpha)\}$ is a coordinate atlas of M.

2.2. Examples of G-structures

Example 2 (G-structures)

(i) Let M be a manifold and $\Lambda_n^*(M) = \bigcup_{m \in M} T_n(M_m)$. A manifold is orientable if $\Lambda_n^*(M) - 0$ has exactly two components, where $0 \bigcup_{m \in M} \{0 \in \Lambda_n(M_m^*)\}$, the 0-section of the exterior n-bundle $\Lambda_n^*(M)$.

If M is orientable, choose an orientation of M, that is, choose one of the two components of $\Lambda_n^*(M) - 0$, call it Λ. Then $\Lambda \cap T_n(M_m)$ is precisely one of the two components of $\Lambda_n^*(M_m) - \{0\}$. Let $\{(U_\alpha, x_\alpha)\}$ be an open covering of M with coordinate charts x_α such that the map of U_α into $\Lambda_n^*(M_m)$ defined by $m \to (dx_1 \wedge \ldots \wedge dx_n)(m)$ has ranges in Λ. Let $(U, x_1, \ldots, x_n)$ and $(V, y_1, \ldots, y_n)$ be two coordinate systems in the open covering $\{(U_\alpha, x_\alpha)\}$, then

$$(dx_1 \wedge \ldots \wedge dx_n)(m) = \det\left(\left.\frac{\partial x_i}{\partial y_j}\right|_m\right)(dy_1 \wedge \ldots \wedge dy_n)(m)$$

where $m \in U \cap V$. $\det\left(\left.\frac{\partial x_i}{\partial y_j}\right|_m\right) > 0$ since $dx_1 \wedge \ldots \wedge dx_n$ and $dy_1 \wedge \ldots \wedge dy_n$ belong to Λ. Therefore, the structural group of the orientable manifold is reducible to $GL^+(n;\mathbb{R})$, a subgroup of $GL(n;\mathbb{R})$ with positive determinant.

Conversely, if the structural group is reducible to $GL^+(n;\mathbb{R})$, then there exists a nowhere vanishing global n-form w on the manifold M. Let $\Lambda^+ = \bigcup_{m\in M} \{aw(m) : a \in \mathbb{R}^+\}$ and $\Lambda^- = \bigcup_{m\in M} \{aw(m) : a \in \mathbb{R}^-\}$, then $\Lambda_n^*(M) - 0$ is the disjoint union of two open subset Λ^+ and Λ^-. Hence M is orientable.

(ii) A Riemannian manifold is a manifold M for which is given at each $m \in M$ a positive definite symmetric bilinear form g(,) on M_m, and this assignment is smooth. Such an assignment is called a Riemannian metric g on M. A manifold together with a Riemannian metric defined on M is called a Riemannian structure on M. Therefore, if X and Y are smooth vector fields on M, then $g_m(X,Y)$ is a smooth function. It is well-known that an O(n) structure on the manifold M is equivalent to a Riemannian structure. It can be shown that any manifold admits a Riemannian structure. Thus, it has an O(n)-structure.

(iii) An almost complex structure is a (1,1) tensor J such that $J^2 = -I$ where I is the identity (1,1) tensor. The complex general linear group, $GL(\frac{n}{2},\mathbb{C})$ can be considered as a subgroup of $GL(n;\mathbb{R})$ (refer to Example 3(i)). It is well-known that an almost complex structure on a manifold M is equivalent to a $GL(\frac{n}{2},\mathbb{C})$ structure. A metric on an almost complex manifold M such that $g(JX,JY) = g(X,Y)$ is called a hermitian metric. A hermitian structure on a manifold M is equivalent to a $U(\frac{n}{2})$-structure, where $U(\frac{n}{2})$ is the unitary group. The group $U(\frac{n}{2})$ is equal to $GL(\frac{n}{2},\mathbb{C}) \cap O(n)$. Therefore, $U(\frac{n}{2})$ is a subgroup of $GL(n;\mathbb{C})$ whose elements are of the form $A\bar{A}^t = I$.

(iv) Referring to Section 3.3, an f-structure is equivalent to $U(\frac{r}{2}) \times O(n-r)$ structure where r is the rank of the f-structure and n is the dimension of the manifold. The $U(\frac{r}{2}) \times I(n-r)$ structure is equivalent to a globally framed f-structure. This will be proved in Section 3.5. Therefore, an almost contact structure is equivalent to the reduction of structural group to $U(\frac{n-1}{2}) \times 1$ (An almost contact structure is globally framed f-structure of rank n-1).

(v) A manifold M is said to be parallelizable if there exists n independent vector fields which span the tangent bundle TM globally. The structural group of a parallelizable manifold is reducible to identity.

(vi) A reduction of structural group to a group consisting all elements of the form $\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$ where $A \in GL(n;\mathbb{R})$ is equivalent to an almost tangent structure. A reduction of structure group to a group consisting all elements of the form $\begin{pmatrix} A & 0 \\ B & A^{-t} \end{pmatrix}$ where $A \in GL(n;\mathbb{R})$ and $AB^t = BA^t$ is equivalent to an almost cotangent structure. These will be discussed in Chapter 4.

(vii) A necessary and sufficient condition for an n-dimensional manifold to admit a tensor field $\varphi \neq 0$ of type (1,1) such that $\varphi^4+\varphi^2 = 0$, rank $\varphi = \frac{1}{2}(\text{rank } \varphi^2 + n) = r$ is that the group of the bundle of bases is reducible to the group $U(r-\frac{n}{2}) \times O(n-r) \times O(n-r)$ [21].

(viii) A reduction of the structural group to $U(\frac{r}{2}) \times U(\frac{s}{2}) \times O(n-r-s)$ is equivalent to a bi-f structure. This will be proved in Chapter 3.

Example 3 (Observations on G-structures)

(i) $GL(n;\mathbb{C})$ can be considered as a subgroup of $GL(2n;\mathbb{R})$. Let σ be the mapping of $GL(n;\mathbb{C})$ to $GL(2n;\mathbb{R})$ defined as follows:

$$(a_{jk}+ib_{jk}) \in GL(n;\mathbb{C}) \xrightarrow{\sigma} \begin{pmatrix} a_{jk} & b_{jk} \\ -b_{jk} & a_{jk} \end{pmatrix} \in GL(2n;\mathbb{R}) .$$

This is a group homomorphism since

$$(a_{jk}+ib_{jk})(c_{rs}+id_{rs}) = \left(\sum [(a_{jk}c_{rs}-b_{jk}d_{rs}) + i(a_{jk}d_{ks}+b_{jk}c_{ks})]\right)$$

and

$$\begin{pmatrix} a_{jk} & b_{jk} \\ -b_{jk} & a_{jk} \end{pmatrix} \cdot \begin{pmatrix} c_{rs} & d_{rs} \\ -d_{rs} & c_{rs} \end{pmatrix}$$

$$= \begin{pmatrix} \sum (a_{jk}c_{ks}-b_{jk}d_{ks}) & \sum (a_{jk}d_{ks}+b_{jk}c_{ks}) \\ - \sum (a_{jk}d_{ks}+b_{jk}c_{ks}) & \sum (a_{jk}c_{ks}-b_{jk}d_{ks}) \end{pmatrix}$$

that is $\sigma[(a_{jk}+ib_{jk})(c_{rs}+id_{rs})] = \sigma(a_{jk}+ib_{ji})\sigma(c_{rs}+id_{rs})$.

Furthermore, every unitary matrix is unitary similar to a diagonal matrix, i.e. there exists a unitary matrix A such that $A^* UA = D$ where U is the unitary matrix considered and D is a diagonal matrix. The determinant of U is equal to the determinant of the diagonal matrix D. Let $[a_j + ib_j]$ be the diagonal components of D, then $D \in GL(2n;\mathbb{R})$ is as follows:

$$\left(\begin{array}{ccc|ccc} a_1 & & 0 & b_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & a_n & 0 & & b_n \\ \hline -b_1 & & 0 & a_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & -b_n & 0 & & a_n \end{array}\right).$$

Hence, $\det D = \prod_{i=1}^{n} (a_i^2 + b_i^2) = 1$. Therefore almost hermitian manifolds are orientable.

(ii) The G-structure of a product manifold is equal to the product of the G-structures of the manifolds. This is because if P_i is the principal subbundle of bundle of bases of M_i, then $P_1 \times P_2$ is the principal subbundle of bundle of bases of $M_1 \times M_2$ with structural group $G_1 \times G_2$. Therefore, the product of two globally framed manifold is a globally framed manifold since the product manifold has an $U(\frac{r}{2}) \times U(\frac{s}{2}) \times I(n+m-r-s)$ structure and $U(\frac{r}{2}) \times U(\frac{s}{2}) \times I(n+m-r-s)$ is a subgroup of $U\left(\frac{r+s}{2}\right) \times I(n+m-r-s)$.

(iii) All orientable 2-dimensional manifold admits an almost complex structure since $SO(2) = U(1)$ [12].

(iv) There exist a nowhere vanishing vector field on compact odd-dimensional manifold [2]. Therefore, a compact orientable (2n+1)-dimensional manifold has an $SO(2n) \times 1$ structure. Since $SO(2) = U(1)$, a three dimensional compact orientable manifold admits an $U(1) \times 1$ structure which is equivalent to an almost contact structure or globally framed f-structure of rank 2. (Refer to Chapter 3 for further discussion).

(v) From Example 2(v) on page 22, we see that any parallelizable manifold admits an arbitrary chosen G-structure since the subgroup consisting of the identity is a subgroup of all subgroups of $GL(n;\mathbb{R})$.

(vi) If the dimension of M is even, then the structural group of a globally framed f-structure is reducible to $U(\frac{r}{2}) \times I(n-r)$ which is a subgroup of $U(\frac{n}{2})$. Hence, it induces an almost complex structure on M. (Refer to Chapter 3).

(vii) If the dimension of the manifold is odd, then the structural group of globally framed f-manifold is reducible to $U(\frac{r}{2}) \times I(n-r-1) \times 1$ which is a subgroup of $U\left(\frac{n-1}{2}\right) \times 1$. Therefore, it induces an almost contact structure on M. (Refer to Chapter 3).

2.3. Transition functions of certain manifolds

Example 4 (Sphere)

Let S^2 be the unit sphere imbedded in $\mathbb{R}^3$ such that $x_1^2 + x_2^2 + x_3^2 = 1$. It can be covered by 2 open sets,

$U = S^2\backslash\{(-1,0,0)\}$ and $V = S^2\backslash\{(1,0,0)\}$. Let ϕ_u be the stereographic projection of U into $\mathbb{R}^2$ and ϕ_v the stereographic projection of V into $\mathbb{R}^2$. The set $\{(U,\phi_u), (V,\phi_v)\}$ forms an atlas of S^2. In this example, we will find the transition functions corresponding to the open covering above. The stereographic projections ϕ_u and ϕ_v can be represented as follows:

$$\phi_u(x_1,x_2,x_3) = \left(\frac{x_2}{1+x_1}, \frac{x_3}{1+x_1}\right) = (u_1,u_2)$$

$$\phi_u^{-1}(u_1,u_2) = \left(\frac{1-(u_1)^2-(u_2)^2}{1+(u_1)^2+(u_2)^2}, \frac{2u_1}{1+(u_1)^2+(u_2)^2}, \frac{2u_2}{1+(u_1)^2+(u_2)^2}\right)$$

$$\phi_v(x_1,x_2,x_3) = \left(\frac{x_2}{1-x_1}, \frac{x_3}{1-x_1}\right) = (v_1,v_2)$$

$$\phi_v^{-1}(v_1,v_2) = \left(-\frac{1-(v_1)^2-(v_2)^2}{1+(v_1)^2+(v_2)^2}, \frac{2v_1}{1+(v_1)^2+(v_2)^2}, \frac{2v_2}{1+(v_1)^2+(v_2)^2}\right)$$

Therefore

$$\frac{\partial v_1}{\partial x_1} = \frac{x_2}{(1-x_1)^2} = \frac{u_1(1+(u_1)^2+(u_2)^2)}{2((u_1)^2+(u_2)^2)^2}$$

$$\frac{\partial v_1}{\partial x_2} = \frac{1+(u_1)^2+(u_2)^2}{2((u_1)^2+(u_2)^2)}$$

$$\frac{\partial v_1}{\partial x_3} = 0$$

$$\frac{\partial v_2}{\partial x_1} = \frac{u_2(1+(u_1)^2+(u_2)^2)}{2((u_1)^2+(u_2)^2)^2}$$

$$\frac{\partial v_2}{\partial x_2} = 0$$

$$\frac{\partial v_2}{\partial x_3} = \frac{1+(u_1)^2+(u_2)^2}{2((u_1)^2+(u_2)^2)}$$

$$\frac{\partial x_1}{\partial u_1} = -\frac{4u_1}{(1+(u_1)^2+(u_2)^2)^2}$$

$$\frac{\partial x_1}{\partial u_2} = -\frac{4u_2}{(1+(u_1)^2+(u_2)^2)^2}$$

$$\frac{\partial x_2}{\partial u_1} = \frac{2(1-(u_1)^2+(u_2)^2)}{(1+(u_1)^2+(u_2)^2)^2}$$

$$\frac{\partial x_2}{\partial u_2} = -\frac{4u_1u_2}{(1+(u_1)^2+(u_2)^2)^2}$$

$$\frac{\partial x_3}{\partial u_1} = -\frac{4u_1u_2}{(1+(u_1)^2+(u_2)^2)^2}$$

$$\frac{\partial x_3}{\partial u_2} = \frac{2(1+(u_1)^2-(u_2)^2)}{(1+(u_1)^2+(u_2)^2)^2} .$$

Then

$$\frac{\partial v_1}{\partial u_1} = \frac{\partial v_1}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial v_1}{\partial x_2}\frac{\partial x_2}{\partial u_1} + \frac{\partial v_1}{\partial x_3}\frac{\partial x_3}{\partial u_1}$$

$$= - \frac{(u_2)^2-(u_1)^2}{(u_1)^2+(u_2)^2} .$$

Similarly

$$\frac{\partial v_2}{\partial u_1} = - \frac{2u_1u_2}{((u_1)^2+(u_2)^2)^2}$$

$$\frac{\partial v_1}{\partial u_2} = - \frac{2u_1u_2}{((u_1)^2+(u_2)^2)^2}$$

and

$$\frac{\partial v_2}{\partial u_2} = \frac{(u_2)^2-(u_1)^2}{(u_1)^2+(u_2)^2} .$$

We have

$$\begin{pmatrix} \frac{\partial}{\partial u_1} \\ \frac{\partial}{\partial u_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_1}{\partial u_1} & \frac{\partial v_2}{\partial u_1} \\ \frac{\partial v_1}{\partial u_2} & \frac{\partial v_2}{\partial u_2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial v_1} \\ \frac{\partial}{\partial v_2} \end{pmatrix}$$

$$= \begin{pmatrix} - \frac{(u_2)^2-(u_1)^2}{(u_1)^2+(u_2)^2} & - \frac{2u_1u_2}{((u_1)^2+(u_2)^2)^2} \\ - \frac{2u_1u_2}{((u_1)^2+(u_2)^2)^2} & \frac{(u_2)^2-(u_1)^2}{(u_1)^2+(u_2)^2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial v_1} \\ \frac{\partial}{\partial v_2} \end{pmatrix} .$$

Let $\frac{\partial}{\partial w_1}$ and $\frac{\partial}{\partial w_2}$ be defined as

$$\begin{pmatrix} \frac{\partial}{\partial w_1} \\ \frac{\partial}{\partial w_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial v_1} \\ \frac{\partial}{\partial v_2} \end{pmatrix} .$$

Therefore

$$\begin{pmatrix} \frac{\partial}{\partial u_1} \\ \frac{\partial}{\partial u_2} \end{pmatrix} = \begin{pmatrix} -\frac{2u_1u_2}{((u_1)^2+(u_2)^2)^2} & \frac{(u_2)^2-(u_1)^2}{(u_1)^2+(u_2)^2} \\ -\frac{(u_2)^2-(u_1)^2}{(u_1)^2+(u_2)^2} & -\frac{2u_1u_2}{((u_1)^2+(u_2)^2)^2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial w_1} \\ \frac{\partial}{\partial w_2} \end{pmatrix}$$

$$= g \begin{pmatrix} \frac{\partial}{\partial w_1} \\ \frac{\partial}{\partial w_2} \end{pmatrix}$$

where $g \in GL(1;\mathbb{C})$. Thus S^2 admits an almost complex structure with

$$J\left(\frac{\partial}{\partial u_1}\right) = \frac{\partial}{\partial u_2},\ J\left(\frac{\partial}{\partial u_2}\right) = -\frac{\partial}{\partial u_1},\ J\left(\frac{\partial}{\partial w_1}\right) = \frac{\partial}{\partial w_2} \text{ and } J\left(\frac{\partial}{\partial w_2}\right) = -\frac{\partial}{\partial w_1}.$$

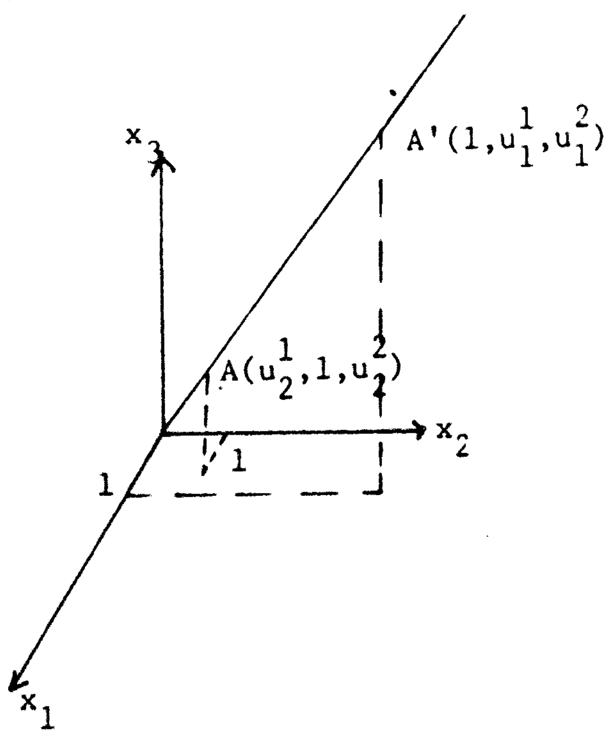
The structural group of S^n with $\{(U_i,\phi_i)\}$ can be constructed as above. The structural group of S^n with respect to $\{(U_i,\phi_i)\}$ is

$$\begin{pmatrix} \frac{s-2(u_1)^2}{s^2} & -\frac{2u_1u_2}{s} & \cdots & -\frac{2u_1u_n}{s} \\ \vdots & \ddots & & \vdots \\ -\frac{2u_1u_n}{s} & \cdots & & \frac{s-2(u_n)^2}{s^2} \end{pmatrix}$$

where $s = (u_1)^2 + \ldots + (u_n)^2$.

Example 5 (Real projective space $\mathbb{R}P^n$)

Real projective space can be covered by (n+1) open sets. Consider $\mathbb{R}P^2$. Let $U_1 = \{(x_1,x_2,x_3) \in \mathbb{R}P^2 : x_1 \neq 0\}$, $U_2 = \{(x_1,x_2,x_3) \in \mathbb{R}P^2 : x_2 \neq 0\}$ and $U_3 = \{(x_1,x_2,x_3) \in \mathbb{R}P^2 : x_3 \neq 0\}$. $\{U_1,U_2,U_3\}$ is an open covering of $\mathbb{R}P^2$. If $A'(1,u_1^1,u_1^2)$ and $A(u_2^1,1,u_2^2)$ as in the diagram below represent a point in $U_1 \cap U_2$ of $\mathbb{R}P^2$, then there exists a $k \in \mathbb{R}$ and $k \neq 0$ such that $k(u_2^1,1,u_2^2) = (1,u_1^1,u_1^2)$.



Therefore $ku_2^1 = 1$, $k = u_1^1$ and $ku_2^2 = u_1^2$.

Thus, $$u_1^1 = \frac{1}{u_2^1}$$

$$u_1^2 = \frac{u_2^2}{u_2^1}.$$

Therefore, $\frac{\partial u_1^1}{\partial u_2^1} = -\frac{1}{(u_2^1)^2}$, $\frac{\partial u_1^1}{\partial u_2^2} = 0$, $\frac{\partial u_1^2}{\partial u_2^1} = -\frac{u_2^2}{(u_2^1)^2}$ and $\frac{\partial u_1^2}{\partial u_2^2} = \frac{1}{u_2^1}$.

Thus, for $U_1 \cap U_2$

$$\begin{pmatrix} \frac{\partial}{\partial u_2^1} \\ \frac{\partial}{\partial u_2^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1^1}{\partial u_2^1} & \frac{\partial u_1^2}{\partial u_2^1} \\ \frac{\partial u_1^1}{\partial u_2^2} & \frac{\partial u_1^2}{\partial u_2^2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial u_1^1} \\ \frac{\partial}{\partial u_1^2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{(u_2^1)^2} & -\frac{u_2^2}{(u_2^1)^2} \\ 0 & \frac{1}{u_2^1} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial u_1^1} \\ \frac{\partial}{\partial u_1^2} \end{pmatrix}.$$

Similarly for $U_1 \cup U_3$, we have $k(u_3^1, u_3^2, 1) = (1, u_1^1, u_1^2)$ and

$$\begin{pmatrix} \frac{\partial}{\partial u_3^1} \\ \frac{\partial}{\partial u_3^2} \end{pmatrix} = \begin{pmatrix} -\frac{u_3^2}{(u_3^1)^2} & -\frac{1}{(u_3^1)^2} \\ \frac{1}{u_3^1} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial u_1^1} \\ \frac{\partial}{\partial u_1^2} \end{pmatrix}.$$

Similarly for $U_2 \cap U_3$, we have $k(u_3^1, u_3^2, 1) = (u_2^1, 1, u_2^2)$ and

$$\begin{pmatrix} \frac{\partial}{\partial u_3^1} \\ \frac{\partial}{\partial u_3^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{u_3^2} & 0 \\ -\frac{u_3^1}{(u_3^2)^2} & -\frac{1}{(u_3^2)^2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial u_2^1} \\ \frac{\partial}{\partial u_2^2} \end{pmatrix} .$$

The transition functions of $\mathbb{R}P^n$ can be constructed as before. For $U_1 \cap U_2$, $k(u_2^1, 1, u_2^2, \ldots, u_2^n) = (1, u_1^1, \ldots, u_1^n)$ and the matrix for coordinate transformation is as follows:

$$G_{12} = \begin{pmatrix} -\frac{1}{(u_2^1)^2} & -\frac{u_2^1}{(u_2^1)^2} & \cdots & -\frac{u_2^n}{(u_2^1)^2} \\ & \frac{1}{u_2^1} & 0 & 0 \\ & & \ddots & \vdots \\ & & & 0 \\ & 0 & & \frac{1}{u_2^1} \end{pmatrix}$$

For $U_1 \cap U_3$, $k(1, u_1^1, \ldots, u_1) = (u_3^1, u_3^2, 1, \ldots, u_3^n)$. The matrix of the coordinate transformation is as follows:

$$G_{13} = \begin{bmatrix} -\frac{u_3^2}{(u_3^1)^2} & -\frac{1}{(u_3^2)^2} & -\frac{u_3^3}{(u_3^1)^2} & \cdots & -\frac{u_3^n}{(u_3^1)^2} \\ \frac{1}{u_3^1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \frac{1}{u_3^1} & 0 & \\ & & & \ddots & \\ & 0 & & & \frac{1}{u_3^1} \end{bmatrix}$$

If n is odd, then n = 2k-1 for some natural number k. Let G_{ij} be the matrix of the coordinate transformation on $U_i \cap U_j$. From the calculation above,

$$\det G_{12} = -\frac{1}{(u_2^1)^{2k}} \quad \text{and} \quad \det G_{13} = \frac{1}{(u_3^1)^{2k}}$$

It can be proved that det $G_{ij} > 0$ if both i and j are even or odd and det $G_{ij} < 0$ if either i or j is even.

On U_{2i}, let $v_{2i}^1 = -u_{2i}^1$ and $v_{2i}^j = u_{2i}^j$ for all $2 \le j \le 2k-1$ The collection of the maps $\left(v_{2i}^1, \ldots, v_{2i}^{2k-1}\right)$ is a coordinate chart for U_{2i}.

For $U_1 \cap U_2$,

$$\begin{bmatrix} \dfrac{\partial}{\partial u_2^1} \\ \vdots \\ \dfrac{\partial}{\partial u_2^{2k-1}} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{(v_2^1)^2} & -\dfrac{v_2^2}{(v_2^1)^2} & \cdots & -\dfrac{v_2^{2k-1}}{(v_2^1)^2} \\ & \dfrac{1}{v_2^1} & & 0 \\ & & \ddots & \\ 0 & & & \dfrac{1}{v_2^1} \end{bmatrix} \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial v_1^1} \\ \vdots \\ \dfrac{\partial}{\partial v_1^{2k-1}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{(v_2^1)^2} & -\dfrac{v_2^1}{(v_2^1)^2} & \cdots & -\dfrac{v_2^{2k-1}}{(v_2^1)^2} \\ & \dfrac{1}{v_2^1} & & 0 \\ & & \ddots & \\ 0 & & & \dfrac{1}{v_2^1} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial v_1^1} \\ \vdots \\ \dfrac{\partial}{\partial v_1^{2k-1}} \end{bmatrix}.$$

Therefore, determinant of the matrix of the coordinate transformation is given as $\frac{1}{(v_2^1)^{2k}} > 0.$

With the same argument, it can be shown that for $U_i \cap U_j$, the determinant of the matrix of the coordinate transformation is positive for each U_i and U_j in the open covering $\left\{\left(U_1, u_1^1, \ldots, u_1^{2k-1}\right), \left(U_2, v_2^1, \ldots, v_2^{2k-1}\right), \ldots, \left(U_{2k}, v_{2k}^1, \ldots, v_{2k}^{2k-1}\right)\right\}$.