CHAPTER 3. f-STRUCTURES AND RELATED STRUCTURES

The concept of f-structure was introduced by Yano [19,20] which is closely related to an almost complex structure and an almost contact structure. A manifold with an f-structure is called an f-manifold. The structural group of f-manifold is reducible to $U(\frac{r}{2}) \times O(n-r)$ where r is the rank of the f-structure [20]. Framed f-manifold was studied by Goldberg, Yano, Millman and Nakagawa [8,9,10,13]. Goldberg and Yano [9] showed that a framed f-manifold has an underlying almost complex structure if its dimension is even and an underlying almost contact structure if its dimension is odd. Ishihara and Yano [11] proved that an f-structure is integrable if and only if there exists an atlas such that f has constant components with respect to the atlas.

f-structure can be generalized to (f,k) structure, where k is a positive integer. Yano, Honh and Chen [21] have studied the structures defined by a (1,1) tensor satisfying $\varphi^4 \pm \varphi^2 = 0$. Petrakis [14,15] studied structures defined by (1,1) tensor f such that $f^{2v+3} \pm f = 0$.

In this chapter, f-structure will be defined and some basic properties of f-structure will be established. Integrability condition will also be discussed. Based on the discussion on integrability of G-structure, a necessary and sufficient condition for integrability of f-structure will be given. The reduction of structural group will then be obtained without appiying local components of the f-structure. The (f,k) structure will be defined and the reduction of structural group as well as a necessary and sufficient condition for existence of (f,k) structure will be obtained. Based on the observation of G-structures, we can define a bi-f structure on a manifold. Some basic properties of bi-f structure will also be obtained.

3.1. Definitions and preliminary results

An *f-structure* on the manifold M is a non-null (1,1) C^{∞}-tensor f such that $f^3 + f = 0$ [19].

Stong [17] proved that the rank of f is constant. Let $\ell = -f^2$ and $m = f^2 + 1$, then ℓ and m are complementary projections. Let $p = m+f = f^2 + f + 1$ and $q = m-f = f^2 - f + 1$. Yano [20] showed that ℓ, m, f, p and q satisfy the following relations:

(i) $\ell m = m\ell = 0$

(ii) $f\ell = \ell f = f$

(iii) $fm = mf = 0$

(iv) $f^2\ell = -\ell^2 = -\ell$

(v) $f^2 m = 0$

(vi) $p^2 - p + 1 = q^2$

(vii) $q^2 - q + 1 = p^2$

(viii) $p\ell = f$

(ix) $p^2\ell = -\ell$

(x) $pm = p^2m = m$

(xi) $q\ell = f$

(xii) $qm = m$

(xiii) $q^2\ell = -\ell$

(xiv) $q^2m = m$

(xv) $p^2 = q^2$

Yano [20] also proved the following propositions:

Proposition 3.1.

(i) If there is a projection operator m and a (1,1) tensor f such that $(m+f)(m-f) = 1$ and $mf = fm = 0$, then f satisfies $f^3 + f = 0$.

(ii) If there are two distinct (1,1) tensors p and q satisfying $pq = qp = 1$ and $p^2 - p + 1 = q^2$ then if we define $f = \frac{(p-q)}{2}$, it will give us an f-structure satisfying $f^3 + f = 0$.

(iii) If M is an f-manifold, there exist 2 complementary distributions D_ℓ and D_m on TM corresponding to the projections ℓ and m respectively.

By Proposition 3.1(iii), an f-manifold is an almost product manifold We obtained the following necessary and sufficient condition for the existence of f-structure.

Proposition 3.2.

There is a projection tensor ℓ of rank r on TM and a (1,1) tensor p that commutes with ℓ such that $p^2 = -2\ell+1$ if and only if there exists a f-structure of rank r on M.

Proof:

Given an f-structure, $p = f^2 + f + 1$ and $\ell = -f^2$ clearly satisfy $p^2 = -2\ell+1$. This is the result proved by Yano [20].

Conversely, let $f = p\ell$.

Then $f^3 = p^3\ell$ since p commutes with ℓ and $\ell^2 = \ell$

$= p\ell(-2\ell+1)$

$= -p\ell = -f$.

Therefore $f^3 + f = 0$.

Since $\ell(x) = 0$ implies that $f(x) = 0$, we conclude that rank f $\leq$ rank ℓ. If $fX = 0$ for a vector field X, then $p^2\ell X = 0$. This implies that

$$(-2\ell+1)\ell X = 0 .$$

Then $-\ell X = 0$ and therefore

$$\ell X = 0 .$$

Thus rank f $\geq$ rank ℓ .

Therefore, rank f = rank ℓ = r .

Q.E.D.

3.2. Riemannian metric on f-manifold

Proposition 3.3.

Let M be an f-manifold. There exists a Riemannian metric g on M such that

(i) $g(\ell X, \ell Y) = g(fX, fY)$

(ii) $g(\ell X, fX) = g(\ell X, mY) = g(fX, mY) = 0$

(iii) $g(\ell X, fY) = -g(fX, \ell Y)$

Proof:

Since M is paracompact, there exists a Riemannian metric h. Metric g is defined by $g(X,Y) = h(fX,fY) + h(\ell X,\ell Y) + h(mX,mY)$. Clearly g is also a Riemannian metrix.

The properties of g are verified as follows:

(i) $g(\ell X, \ell Y) = h(f\ell X, f\ell Y) + h(\ell X, \ell Y)$ since $m\ell = 0$

$= h(fX, fY) + h(\ell X, \ell Y)$ since $f\ell = f$

$g(fX, fY) = h(f^2X, f^2Y) + h(\ell fX, \ell fY)$ since $mf = 0$

$= h(\ell X, \ell Y) + h(fX, fY)$

$= g(\ell X, \ell Y)$.

(ii) $g(\ell X, fX) = h(f\ell X, f^2X) + h(\ell X, \ell fX)$

$= -h(fX, \ell X) + h(\ell X, fX)$

$= 0$.

Since $fm = mf = m\ell = \ell m = 0$, it is easy to check that $g(\ell X, mY) = 0$ and $g(fX, mY) = 0$.

(iii) $g(\ell X, fY) = g(-f(fX), f(Y)) = g(-\ell fX, \ell Y) = -g(fX, \ell Y)$ from (i) above.

Q.E.D.

Proposition 3.3 shows that D_ℓ and D_m are orthogonal with respect to g. Hence, the structural group of the f-manifold is reducible to $O(r) \times O(n-r)$. Proposition 3.4 below basically is Theorem 2.2 of Yano [20] with an alternative proof.

Proposition 3.4

The existence of an f-structure on a manifold M with a metric g satisfying the conditions in Proposition 3.3 is equivalent to a reduction of the structural group to $U(\frac{r}{2}) \times O(n-r)$.

Proof:

Since D_ℓ is a r-dimensional distribution on M, for each point $m \in M$, there is an open set U such that $\{\bar{X}_1, \ldots, \bar{X}_r\}$ spans D_ℓ on U. Let $X_1 = \frac{1}{\|\bar{X}_1\|} \bar{X}_1$. From Proposition 3.3 above, fX_1 is orthonormal to X_1 with respect to g. Let $\{X_1, fX_1, X_2, fX_2, \ldots, X_k, fX_k\}$ be a set of orthonormal vector fields on D_ℓ in U where $k < \frac{r}{2}$. There exists a unit vector field X_{k+1} that is orthogonal to $\{X_1, fX_1, X_2, fX_2, \ldots, fX_k\}$. In fact, X_{k+1} can be obtained from the Gram-Schmidt orthonormalization of a basis $\{X_1, fX_1, \ldots, X_k, fX_k, X'_{k+1}, \ldots, X'_{r-k}\}$ of D_ℓ. From Proposition 3.3, it is easy to see that fX_{k+1} is orthonormal to $\{X_1, \ldots, fX_k, X_{k+1}\}$. Therefore, by induction there exists a set of orthonormal basis $\{X_1, fX_1, \ldots, X_{r/2}, fX_{r/2}\}$ of D_ℓ. Thus, r is even. Let $\{Y_1, fY_1, \ldots, Y_{r/2}, fY_{r/2}\}$ be another basis of D_ℓ in the coordinate neighborhood of V. Then

$$\begin{pmatrix} Y \\ fY \\ Y' \end{pmatrix} = \left(\begin{array}{cc|c} A & B & 0 \\ C & D & 0 \\ \hline 0 & 0 & E \end{array}\right) \begin{pmatrix} X \\ fX \\ X' \end{pmatrix}$$

where

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_{r/2} \end{pmatrix}, \quad fY = \begin{pmatrix} fY_1 \\ \vdots \\ fY_{r/2} \end{pmatrix},$$

Y' is an orthonormal basis of D_m in V,

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_{r/2} \end{pmatrix}, \quad fX = \begin{pmatrix} fX_1 \\ \vdots \\ fX_{r/2} \end{pmatrix}$$

and X' is an orthonormal basis of D_m in U.

Therefore $A = D$ and $B = -C$. With the metric g, the structural group is reducible to $U(\frac{r}{2}) \times O(n-r)$.

Conversely, if the structural group is reducible to $U(\frac{r}{2}) \times O(n-r)$, then f and g can be defined as

$$f = \begin{pmatrix} 0 & -I_{r/2} & 0 \\ I_{r/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} I_{r/2} & 0 & 0 \\ 0 & I_{r/2} & 0 \\ 0 & 0 & I_{n-r} \end{pmatrix}.$$

Obviously the (1,1) tensor f is an f-structure on the manifold M.

Q.E.D.

By observing the structural group of f-structure, we obtained the following conclusions:

(i) If $r = n$, then an f-structure is an almost hermitian structure since the structural group is reducible to $U(\frac{n}{2})$.

(ii) If M is orientable and $r = n-1$, then an f-structure is an almost contact structure since the structural group is reducible to $U\left(\frac{n-1}{2}\right) \times 1$.

3.3. **Integrability condition of f-structure**

If D_ℓ is integrable, then the integrable submanifold $\mathcal{L}$ of D_ℓ inherits naturally an almost complex structure from the f-structure on the manifold M. An f-structure is *integrable* if

(i) D_ℓ is integrable and the induced almost complex structure on the submanifold $\mathcal{L}$ is also integrable.

(ii) D_m is integrable.

(iii) There exists a coordinate atlas such that $x_i = c_i$ defines submanifold $\mathcal{L}$ for $i > r$ and components of f with respect to this coordinate atlas are constant.

Ishihara and Yano [11] proved that a necessary and sufficient condition for an f-structure to be integrable is that there exists a coordinate atlas in which f has constant components

$$f = \begin{bmatrix} 0 & -I_{r/2} & 0 \\ I_{r/2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where r is the rank of f.

Based on the discussion in Chapter 2, we can reformulate the integrability condition on f-structure as follows:

An f-structure is integrable if and only if there exists a frame $\{X_1,\ldots,X_{r/2},fX_1,\ldots,fX_{r/2},X_{r+1},\ldots,X_n\}$ on each open set U_α and C_{ij} determined by

$$\frac{\partial}{\partial x_i} = \sum C_{ij}V_j \text{ where } V_j = X_j \qquad \text{for } j = 1,\ldots,\tfrac{r}{2},r+1,\ldots,n \text{ and}$$

$$V_j = fX_{j-(r/2)} \text{ for } j = \tfrac{r}{2}+1,\ldots,r$$

such that

$$\frac{\partial C_{ij}}{\partial x^k} = \frac{\partial C_{kj}}{\partial x^i}$$

where $\{(U_\alpha,x_\alpha)\}$ is an atlas of M.

3.4. Globally framed f-structure

An f-structure is called a *globally framed f-structure* [9] if there exist (n-r) independent vector fields that span D_m globally. The structural group of the manifold with a globally framed f-structure is reducible to $U(\frac{r}{2}) \times I(n-r) = U(\frac{r}{2})$. If n is even, then $U(\frac{r}{2})$ is a subgroup of $U(\frac{n}{2})$. Thus, it has an underlying almost complex structure. If n is odd, $U(\frac{r}{2})$ is a subgroup of $U\left(\frac{n-1}{2}\right) \times 1$ and it has an underlying almost contact structure.

Product of globally framed f-manifolds is a globally framed f-manifold. This is because the structural group of product manifold is the product of the structural groups and $U(\frac{r}{2}) \times U(\frac{r}{2})$ is a subgroup of $U\left(\frac{r+s}{2}\right)$. Hence, as in [8] we have:

Proposition 3.5.

The direct product of two framed manifolds whose dimensions are both even or both odd (one even and one odd) has an underlying almost complex (almost contact) structure.

An almost contact structure is an example of a globally framed f-structure of rank n-1. The sphere S^{2n+1} admits an almost contact structure. Thus $S^{2p+1} \times S^{2q+1}$ has an underlying almost complex structure.

3.5. (f,k) structures

An *(f,k) structure* on the manifold M is a (1,1) tensor f such that $f^k + f = 0$ and $f^m + f \neq 0$ for all $1 \leq m < k$.

Proposition 3.6.

If there is a (1,1) tensor f on M and integer k such that $f^{2^k+1} + f = 0$ then $f^{2^{k+1}m+2^k+1} + f = 0$ for $\forall\, m \in \mathbb{N}$. Conversely, if there is a (1,1) tensor f such that $f^{2^{k+1}m+2^k+1} + f = 0$ for some natural numbers k and m, then there exists a (1,1) tensor ν such that $\nu^{2^k+1} + \nu = 0$.

Proof:

From the definition of f, we have $f^{2^k+1} = -f$. We will derive a recurrence formula for $f^{2^{k+1}m}$.

$$f^{2^{k+1}m} = \left(f^{2^{k+1}}\right)^2 \left(f^{2^{k+1}}\right)^{m-2} \qquad \text{for } m \geq 2$$

$$= \left(f^{2^k+1} \cdot f^{2^k-1}\right)^2 \left(f^{2^{k+1}}\right)^{m-2}$$

$$= \left(f \cdot f^{2^k-1}\right)^2 \left(f^{2^{k+1}}\right)^{m-2}$$

$$= \left(f^{2^{k+1}}\right)\left(f^{2^{k+1}}\right)^{m-2}$$

$$= f^{2^{k+1}(m-1)} \quad .$$

Therefore $\qquad f^{2^{k+1}m} = f^{2^{k+1}} \qquad$ for $m \geq 1$.

Furthermore $\qquad f^{2^{k+1}} = f^{2^k-1}\ f^{2^k+1} = f^{2^k-1}(-f) = -f^{2^k}$.

Hence $\qquad f^{2^{k+1}m} = -f^{2^k}$.

Now, we can simplify $f^{2^{k+1}m+2^k+1}$ as follows:

$$f^{2^{k+1}m+2^k+1} = -f^{2^k+2^k+1}$$

$$= -f^{2^{k+1}+1}$$

$$= f^{2^k+1}$$

$$= -f \quad .$$

Therefore $\qquad f^{2^{k+1}m+2^k+1} + f = 0$.

Conversely, if $f^{2^{k+1}m+2^k+1} + f = 0$ for some $k,m \in \mathbb{N}$, let $\nu = f^{2m+1}$, then it is easy to check that $\nu^{2^k+1} + \nu = 0$.

Q.E.D.

With a similar argument, we would be able to obtain the following proposition.

Proposition 3.7.

If there is a (1,1) tensor f such that $f^2 + f = 0$, then $f^{2k} + f = 0$ for $\forall\ k \in \mathbb{N}$. Conversely, if there is a (1,1) tensor f such that $f^{2k} + f = 0$, then there exists a (1,1) tensor ν such that $\nu^2 + \nu = 0$.

Proposition 3.8.

A necessary and sufficient condition for the existence of (f,2k+1) structure is

(i) The structural group is reducible to $U(\frac{r}{2}) \times O(n-r)$;

(ii) The rank r is a multiple of 2k.

Proof of Proposition 3.8 is similar to that of Proposition 3.4. First, we obtain a metric g with the properties below:

(i) $g(f^iX, f^iY) = g(fX, fY)$ for all natural number $i > 0$.

(ii) $g(f^iX, mY) = 0$ where $m = f^{2k} + 1$.

(iii) $g(f^iX, f^jX) = 0$ whenever $i \neq j \pmod{2k}$

Therefore, TM has two orthogonal complementary distributions with respect to g. Let D_ℓ and D_m denote these two complementary distributions corresponding to $\ell = -f^{2k}$ and $m = -\ell+1$. By similar argument as that of Proposition 3.4, there exists a set of orthonormal basis in D_ℓ on coordinate neighborhood U. Finally it can be shown that the structural group is reducible to $U(\frac{r}{2}) \times O(n-r)$ and r is a multiple of 2k since the set of orthonormal basis in D_ℓ on each U_α of the covering $\{U_\alpha\}$ can be written as $\left\{X_1, fX_1, \ldots, f^{2k-1}X_1, \ldots, X_{r/2k}, fX_{r/2k}, \ldots, f^{2k-1}X_{r/2k}\right\}$.

Conversely, if the structural group is reducible to $U(\frac{r}{2}) \times O(n-r)$ and r is a multiple of 2k, then f and g can be defined as

$$f = \begin{pmatrix} 0 & I_{r/2k} & & 0 \\ 0 & & \ddots & I_{r/2k} & \\ -I_{r/2k} & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and}$$

$$g = \begin{pmatrix} I_{r/2k} & & 0 \\ & \ddots & \\ & & I_{r/2k} & \\ 0 & & & I_{n-r} \end{pmatrix}.$$

Obviously, the (1,1) tensor f is an (f,2k+1) structure.

Q.E.D.

The above proposition was obtained by Petrakis [15]. Here, we gave an alternative proof similar to our proof in Proposition 3.4. We observed that (f,2k+1) structure is also an almost product structure, namely $TM = D_\ell \oplus D_m$, where D_ℓ and D_m are two complementary distributions corresponding to $\ell = -f^{2k}$ and $m = 1-\ell$.

3.6. bi-f structure

Let M be an n-dimensional manifold. A *bi-f structure* on M is a pair of (1,1) tensors $(f,\bar{f})$ of rank r and s respectively such that $f^3 + f = 0$, $\bar{f}^3 + \bar{f} = 0$, $\bar{f}f = 0$ and $f\bar{f} = 0$. A *bi-f manifold* is a manifold with a bi-f structure.

Proposition 3.9.

The structural group of a bi-f manifold is reducible to $U(\frac{r}{2}) \times U(\frac{s}{2}) \times O(n-r-s)$.

Proof:

Let M be the manifold with the bi-f structure $(f,\bar{f})$. There are three complementary projections on M. Let $\ell = -f^2$, $\bar{\ell} = -\bar{f}^2$ and $m = 1 + f^2 + \bar{f}^2$. These projections define three complementary distributions D_ℓ, $D_{\bar{\ell}}$ and D_m on TM.

We can obtain a Riemannian metric satisfying the following conditions:

(i) $g(f^2X,f^2Y) = g(fX,fY)$.

(ii) $g(\bar{f}^2X,\bar{f}^2Y) = g(\bar{f}X,\bar{f}Y)$.

(iii) $g(fX,\bar{f}Y) = g(\bar{f}X,fY) = g(fX,mY) = g(\bar{f}X,mY) = 0$.

Condition (iii) above implies D_ℓ, $D_{\bar{\ell}}$ and D_m are orthogonal to each other. Hence, the structural group is reducible to $O(r) \times O(s) \times O(n-r-s)$.

With a similar argument on Proposition 3.4, we obtain a set of orthonormal basis $\{X_1,fX_1,\ldots,X_{r/2},fX_{r/2},Y_1,\bar{f}Y_1,\ldots,Y_{s/2},\bar{f}Y_{s/2},\ldots,Y_{s/2},\bar{f}Y_{s/2},Z_1,\ldots,Z_{n-r-s}\}$ on an open set U where $\{X_1,fX_1,$

$\ldots, X_{r/2}, fX_{r/2}\}$, $\{Y_1, \bar{f}Y_1, \ldots, Y_{s/2}, \bar{f}Y_{s/2}\}$ and $\{Z_1, \ldots, Z_{n-r-s}\}$ are orthonormal basis of D_ℓ, $D_{\bar{\ell}}$ and D_m respectively. This implies that the structural group of the bi-f manifold is reducible to $U(\frac{r}{2}) \times U(\frac{s}{2}) \times O(n-r-s)$. Therefore, r and s are even.

Conversely, if the structural group of a manifold is reducible to $U(\frac{r}{2}) \times U(\frac{s}{2}) \times O(n-r-s)$ then f, $\bar{f}$ and Riemannian metric g can be defined as follows:

$$f = \left(\begin{array}{cc|c|c} 0 & -I_{r/2} & 0 & 0 \\ I_{r/2} & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & 0 \end{array}\right) \qquad \bar{f} = \left(\begin{array}{c|cc|c} 0 & \multicolumn{2}{c|}{0} & 0 \\ \hline & 0 & -I_{s/2} & \\ 0 & & & 0 \\ & I_{s/2} & 0 & \\ \hline 0 & \multicolumn{2}{c|}{0} & 0 \end{array}\right)$$

$$g = \left(\begin{array}{c|c|c} I_r & 0 & 0 \\ \hline 0 & I_s & 0 \\ \hline 0 & 0 & I_{n-r-s} \end{array}\right).$$

Obviously, this defines a bi-f structure on M.

Q.E.D.

In fact, $U(\frac{r}{2}) \times U(\frac{s}{2}) \times O(n-r-s)$ can be considered as a subgroup of $U(\frac{r}{2}) \times O(n-r)$, $U(\frac{s}{2}) \times O(n-s)$ and also $U\left(\frac{r+s}{2}\right) \times O(n-r-s)$. Therefore, a bi-f structure can be considered as an f-structure of rank r, rank s and rank r+s. Thus, we obtain the following:

Proposition 3.10.

Let M be a manifold with a bi-f structure $(f,\bar{f})$ of ranks r and s respectively. An f-structure of rank r+s can be defined as $f' = f + \bar{f}$.

Proposition 3.11.

Let M be a manifold with an almost complex structure J. If there is a projection tensor ℓ that commutes with J, then there exists a bi-f structure on M with rank r and n-r where r is the rank of ℓ.

Proof:

Let $f = \ell J$. We have $f^2 = \ell^2 J^2 = \ell(-1) = -\ell$ and $f^3 = \ell J(-\ell) = -\ell J = -f$. Hence, we get $f^3 + f = 0$. Similarly, we can define $\bar{f}$ as $\bar{f} = (1-\ell)J$. It is easy to verify that $\bar{f}^3 + \bar{f} = 0$ and $f\bar{f} = \bar{f}f = 0$. Therefore $(f,\bar{f})$ is a bi-f structure on manifold M.

Since J is of maximum rank, rank f = rank $\ell = r$ and rank $\bar{f} = n-r$.

Q.E.D.

$S^{2p+1} \times S^{2q+1}$ has an almost complex structure with a projection of rank 2p that commutes with J. Let (ψ,ξ,η) and $(\bar{\psi},\bar{\xi},\bar{\eta})$ be the contact structures on S^{2p+1} and S^{2q+1} respectively. An almost complex structure on $S^{2p+1} \times S^{2q+1}$ is defined by $J(X,\bar{X}) = (\varphi X-\bar{\eta}(\bar{X})\xi,\ \overline{\varphi X}+\eta(X)\bar{\xi})$.

The projection tensor ℓ on $S^{2p+1} \times S^{2q+1}$ is defined by $\ell(X,\bar{X}) = (-\varphi^2 X,0)$. We have $\ell^2(X,\bar{X}) = (-\varphi^4 X,0) = (\varphi^2 X,0) = \ell(X,\bar{X})$.

Thus, $\ell J(X,\bar{X}) = (-\varphi^3 X, 0)$ and $J\ell(X,\bar{X}) = J(-\varphi^2 X, 0) = (-\varphi^3 X, 0)$. Obviously, the projection tensor ℓ has rank 2p and commutes with J. Hence it has a bi-f structure of rank 2p and 2(q+1).

The converse of Proposition 3.11 also holds, namely, if $(f,\bar{f})$ is a bi-f structure on M of rank r and n-r respectively, then M admits an almost complex structure J and a projection tensor ℓ that commutes with each other. The proof is as follows:

Let $J = f + \bar{f}$. There are two complementary distributions of TM such that $TM = D_f \oplus D_{\bar{f}}$. Every vector field X in TM can be represented as $X = X_f + X_{\bar{f}}$. Since $J^2 = f^2 + \bar{f}^2$, we get $J^2X = f^2X_f + \bar{f}^2X_{\bar{f}}$, and therefore $J^2X = -X$. Thus, J is an almost complex structure. Let $\ell = -f^2$, then J commutes with ℓ.

A similar result holds for an almost contact structure. Let M be an almost contact manifold with (φ,ξ,η) structure. If there is a projection tensor ℓ of rank r that commutes with φ and $\ell\xi = 0$, then there exists a bi-f structure on M with rank r and n-r-1.

Similarly, if there is a projection tensor $\tilde{\ell}$ of rank $s < r$ on a f-manifold M and $\tilde{\ell}$ commutes with f such that $\tilde{\ell}m = 0$, then M admits a bi-f structure of rank s and r-s.

Proposition 3.12.

If there exists a projection tensor ℓ of rank r and a pair of (1,1) tensors (t,s), both commuting with ℓ and such that $t^2 = -2\ell+1$ and $s^2 = -2\ell-1$, then there is a bi-f structure on M of total rank n, where n is the dimension of the manifold. Hence, n and r must be even.

Proof:

Let $f = t\ell$ and $\tilde{f} = s(-\ell+1)$. Obviously, $(f,\tilde{f})$ defines a bi-f structure. The rank of f is equal to the rank of ℓ as shown below:

$$\text{rank } f \leq \text{rank } \ell \quad \text{since } f = t\ell$$

if $$fX = 0 \Rightarrow t\ell(X) = 0$$

$$\Rightarrow t^2\ell(X) = 0$$

$$\Rightarrow (-2\ell+1)\ell(X) = 0$$

then $$\ell(X) = 0 .$$

Therefore, $\text{rank } f \geq \text{rank } \ell$.

Q.E.D.

Proof:

Let $f = t\ell$ and $\bar{f} = s(-\ell+1)$. Obviously, $(f,\bar{f})$ defines a bi-f structure. The rank of f is equal to the rank of ℓ as shown below:

$$\text{rank } f \leq \text{rank } \ell \quad \text{since } f = t\ell$$

if $$fX = 0 \Rightarrow t\ell(X) = 0$$

$$\Rightarrow t^2\ell(X) = 0$$

$$\Rightarrow (-2\ell+1)\ell(X) = 0$$

then $$\ell(X) = 0 \, .$$

Therefore, $\text{rank } f \geq \text{rank } \ell$.

Q.E.D.