CHAPTER 4. ALMOST COTANGENT STRUCTURE AND ALMOST TANGENT STRUCTURE

The notion of an almost cotangent structure was first defined by Yano and Muto [22]. It is a natural structure on the cotangent bundle of a manifold. Clark and Goel [7] gave an alternative definition of cotangent structure. In this chapter, the definition of Clark and Goel [7] will be adopted.

We can see that an almost cotangent manifold (see definition in 4.1) admits an almost complex structure. In fact, it is well-known that the cotangent bundle is an almost complex manifold. Here, the almost complex structure will be constructed explicitly. We also gave a necessary and sufficient condition for the existence of a 2-form ω that characterizes the almost cotangent structure. The relation between an almost cotangent structure and an almost sympletic structure will also be discussed.

Clark and Goel [6] also defined an almost tangent structure. This is a natural structure on tangent bundle of a manifold. The existence of an almost tangent structure is equivalent to the existence of a particular (1,1) tensor. Similarly, an almost tangent manifold admits an almost complex structure. Finally, the relation between an almost complex structure, an almost tangent structure and an almost cotangent structure will be discussed.

4.1. Almost cotangent manifold

Let G be the Lie subgroup of $GL(2n;\mathbb{R})$ whose elements are of the form $\begin{bmatrix} A & 0 \\ B & A^{-t} \end{bmatrix}$ where $A \in GL(n;\mathbb{R})$ and $A^tB = B^tA$. The structure defined by the Lie group G is called an *almost cotangent structure*. A manifold with an almost cotangent structure is called an *almost cotangent manifold*. An almost cotangent manifold is even dimensional.

Clark and Goel [7] showed that the cotangent bundle admits an almost cotangent structure.

Clearly, an almost cotangent manifold is orientable since the determinant of any element of the group G is positive. In [7], Clark and Goel asserted that there is a 2-form ω with components $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ relative to any adapted frame of an almost cotangent manifold. Here, we illustrate explicitly the 2-form ω.

Proposition 4.1.

There exists a 2-form ω on an almost cotangent manifold with components $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ relative to any adapted frame.

Proof:

For any adapted frame $\{Y_1,\dots,Y_{2n}\}$, the 2-form ω is defined as follows:

$$\omega(Y_i,Y_j) = \begin{cases} -\delta_{i\,j-n} & \text{if } i \le n \text{ and } j > n \\ \delta_{i\,j+n} & \text{if } i > n \text{ and } j \le n \\ 0 & \text{otherwise}. \end{cases}$$

If $\{X_1,\dots,X_{2n}\}$ is another adapted frame, then

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \\ X_{n+1} \\ \vdots \\ X_{2n} \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & A^{-t} \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \\ Y_{n+1} \\ \vdots \\ Y_{2n} \end{bmatrix}$$

for some $A = (a_{ij})$, $B = (b_{ij})$ and $A^{-t} = (d_{ij})$. Since $A^tA^{-t} = I$, we have

$$\sum_{j=1}^{n} a_{ji}d_{jk} = \delta_{ik} .$$

Then

(i) If $i \le n$ and $j \le n$,

$$\omega(X_i,X_j) = \omega\left(\sum_{k=1}^{n} a_{ik}Y_k,\ \sum_{s=1}^{n} a_{js}Y_s\right) = 0 .$$

(ii) If $i > n$ and $j \le n$,

$$\omega(X_i,X_j) = \omega\left(\sum_{k=1}^{n} (b_{i-nk}Y_k + d_{i-nk}Y_{k+n}),\ \sum_{s=1}^{n} a_{js}Y_s\right)$$

$$= \sum_{s=1,k=1}^{n,n} d_{i-nk}a_{js}\ \omega(Y_{k+n},Y_s)$$

$$= \sum_{k=1}^{n} d_{i-nk}a_{jk}$$

$$= \sum_{k=1}^{n} a_{jk}d_{i-nk}$$

$$= \delta_{ji-n} = \delta_{ij+n} .$$

(iii) If $i \le n$ and $j > n$ then

$$\omega(X_i, X_j) = \omega\left(\sum_{k=1}^{n} a_{ik}Y_k, \sum_{s=1}^{n} (b_{j-ns}Y_s + d_{j-ns}Y_{s+n})\right)$$

$$= -w\left(\sum_{s=1}^{n} (b_{j-ns}Y_s + d_{j-ns}Y_{s+n}), \sum_{k=1}^{n} a_{ik}Y_k\right)$$

$$= -\delta_{j-ni} \qquad \text{(from (ii) above)}$$

$$= -\delta_{ij-n} .$$

(iv) If $i > n$ and $j > n$ then

$$\omega(X_i, X_j)$$

$$= \omega\left(\sum_{k=1}^{n} (b_{i-nk}Y_k + d_{i-nk}Y_{k+n}), \sum_{s=1}^{n} (b_{j-ns}Y_s + d_{j-ns}Y_{s+n})\right)$$

$$= \omega\left(\sum_{k=1}^{n} b_{i-nk}Y_k, \sum_{s=1}^{n} d_{j-ns}Y_{s+n}\right)$$

$$+ \omega\left(\sum_{k=1}^{n} d_{i-nk}Y_{k+n}, \sum_{s=1}^{n} b_{j-ns}Y_s\right)$$

$$= -\sum_{k=1}^{n} b_{i-nk}d_{j-nk} + \sum_{k=1}^{n} d_{i-nk}b_{j-nk}$$

$$= 0 \quad \text{since } A^tB = B^tA \left(\text{equivalent to } BA^{-1} = A^{-t}B^t \text{ or } \sum_{s=1}^{n} b_{sk}d_{jk} = \sum_{s=1}^{n} d_{sk}b_{jk}\right) .$$

Thus ω is a well defined 2-form with components $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ with respect to any adapted frames on an almost cotangent manifold.

Q.E.D.

A necessary and sufficient condition for the existence of a 2-form ω with components $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ with respect to a family of 2n-frames that covered the manifold is given below.

Proposition 4.2.

There exists a 2-form ω with components $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ relative to a family of local 2n-frames that covered M if and only if the structural group is reducible to a Lie group consists of elements of the form $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$ where A, B, C and D are $n \times n$ matrices satisfying the following conditions:

$$CA^t = AC^t$$

$$AD^t - CB^t = I$$

$$BD^t = DB^t \ .$$

Proof:

Assume that the 2-form ω exists. Let $\{X_1, \ldots, X_{2n}\}$ and $\{Y_1, \ldots, Y_{2n}\}$ be two 2n-frames from the family define on U_α and U_β respectively such that on $U_\alpha \cap U_\beta$, they are related by

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \\ X_{n+1} \\ \vdots \\ X_{2n} \end{bmatrix} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \\ Y_{n+1} \\ \vdots \\ Y_{2n} \end{bmatrix}$$

where $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $D = (d_{ij})$ are matrix functions on $U_\alpha \cap U_\beta$. Thus

(i) if $i \le n$ and $j \le n$, then

$$0 = \omega(X_i, X_j)$$

$$= \omega\left(\sum_{k=1}^{n} (a_{ik}Y_k + c_{ik}Y_{k+n}),\ \sum_{s=1}^{n} (a_{js}Y_s + c_{js}Y_{s+n})\right)$$

$$= \sum_{k=1}^{n} (-a_{ik}c_{jk} + c_{ik}a_{jk})$$

i.e. $-AC^t + CA^t = 0$ or $AC^t = CA^t$

(ii) if $i \le n$ and $j > n$, then

$$-\delta_{ij-n} = \omega(X_i, X_j)$$

$$= \omega\left(\sum_{k=1}^{n} (a_{ik}Y_k + c_{ik}Y_{k+n}),\ \sum_{s=1}^{n} (b_{j-ns}Y_s + d_{j-ns}Y_{s+n})\right)$$

$$= \sum_{k=1}^{n} (-a_{ik}d_{j-nk} + c_{ik}b_{j-nk})$$

or in the matrix form

$$-I = -AD^t + CB^t .$$

Therefore $AD^t - CB^t = I$

(iii) if $i > n$ and $j > n$

$$0 = \omega(X_i, X_j)$$

$$= \omega\left(\sum_{k=1}^{n} (b_{i-nk}Y_k + d_{i-nk}Y_{k+n}),\ \sum_{s=1}^{n} (b_{j-ns}Y_s + d_{j-ns}Y_{s+n})\right)$$

$$= \sum_{k=1}^{n} (-b_{i-nk}d_{j-nk} + d_{i-nk}b_{j-nk})$$

or $$-BD^t + DB^t = 0$$

$$BD^t = DB^t .$$

All elements of $GL(2n;\mathbb{R})$ that satisfied these conditions form a group (in fact, it is a Lie subgroup of $GL(2n;\mathbb{R})$) since

(a) I belongs to this set.

(b) The inverse of any element belongs to this set since $\{X_1,\ldots,X_{2n}\}$ and $\{Y_1,\ldots,Y_{2n}\}$ play a similar role in deriving these equations.

(c) If $\begin{bmatrix} A_1 & C_1 \\ B_1 & D_1 \end{bmatrix}$ and $\begin{bmatrix} A_2 & C_2 \\ B_2 & D_2 \end{bmatrix}$ belong to this set, then

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} A_1 & C_1 \\ B_1 & D_1 \end{bmatrix}\begin{bmatrix} A_2 & C_2 \\ B_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1A_2 + C_1B_2 & A_1C_2 + C_1D_2 \\ B_1A_2 + D_1B_2 & B_1C_2 + D_1D_2 \end{bmatrix}$$

also belongs to this set. It can be verified as follows:

$$AC^t = (A_1A_2 + C_1B_2)\left(C_2^tA_1^t + D_2^tC_1^t\right)$$

$$= A_1A_2C_2^tA_1^t + A_1A_2D_2^tC_1^t + C_1B_2C_2^tA_1^t + C_1B_2D_1^tC_1^t$$

$$= A_1C_2A_2^tA_1^t + A_1(I + C_2B_2^t)C_1^t + C_1(D_2A_2^t - I)A_1^t + C_1D_1B_2^tC_1^t$$

$$= A_1C_2A_2^tA_1^t + A_1C_2B_2^tC_1^t + C_1D_2A_2^tA_1^t + C_1D_1B_2^tC_1^t$$

$$= (A_1C_2 + C_1D_2)\left(A_2^tA_1^t + B_2^tC_1^t\right) = CA^t .$$

$$AD^t - CB^t = (A_1A_2 + C_1B_2)\left(C_2^tB_1^t + D_2^tD_1^t\right)$$

$$- (A_1C_2 + C_1D_2)\left(A_2^tB_1^t + B_2^tD_1^t\right)$$

$$= A_1A_2C_2^tB_1^t + A_1A_2D_2^tD_1^t + C_1B_2C_2^tB_1^t + C_1B_2D_2^tD_1^t$$

$$- A_1C_2A_2B_1 - A_1C_2B_2D_1 - C_1D_2A_2B_1 - C_1D_2B_2D_1$$

$$= A_1\left(A_2C_2^t - C_2A_2^t\right)B_1^t + A_1\left(A_2D_2^t - C\ B_2^t\right)D_1^t$$

$$+ C_1\left(B_2C_2^t - D\ A_2^t\right)B_1^t + C_1\left(B_2D_2^t - D\ B_2^t\right)D_1^t$$

$$= A_1D_1^t - C_1B_1^t = I$$

since

$$A_2C_2^t - C_2A_2^t = 0, \quad A_2D_2^t - C_2A_2^t = 0, \quad A_2D_2^t - C_2B_2^t = I,$$

$$B_2C_2^t - D_2A_2^t = -I \quad \text{and} \quad B_2D_2^t - D_2B_2^t = 0.$$

Similarly, we have $BD^t = DB^t$. Thus, the structural group of M is reducible to the above mentioned Lie group.

Conversely, if the structural group is reducible to a Lie group consisting all elements of the form $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$ which fulfill the conditions given, then the 2-form ω with components $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ with respect to any adapted frames can be defined. It is easy to show that the 2-form ω is well defined.

Q.E.D.

Clark and Goel [7] proved that if M is an almost cotangent manifold, then there is an open covering of M such that the first n vector fields of the adapated frame for the 2-form ω span a global n-dimensional distribution. In the opposite direction, we can obtain the following result.

Proposition 4.3.

Let ω be a 2-form with components $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ relative to a family of 2n-frames that covered the manifold M. If the first n local vector fields of this set of frames span a global n-dimensional distribution, then M is an almost cotangent manifold.

Proof:

Again, let $\{X_1, \ldots, X_{2n}\}$ and $\{Y_1, \ldots, Y_{2n}\}$ be two frames of ω related by

$$\begin{bmatrix} X_1 \\ \vdots \\ X_{2n} \end{bmatrix} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_{2n} \end{bmatrix} .$$

The existence of a n-dimensional distribution as stated implies C = 0. Then by Proposition 2,

$$D = A^{-t}$$

and

$$A^{-t}B^{t} = BA^{-1} .$$

Therefore

$$B^{t}A = A^{t}B .$$

Hence M is an almost cotangent manifold.

Q.E.D.

Clark and Goel [7] also proved that an almost cotangent manifold admits an almost sympletic structure. Conversely, for a 2n-dimensional almost sympletic manifold, if the first n vector fields of the adapted frames of the almost sympletic structure span a global n-dimensional distribution, then it admits an almost cotangent structure [7]. This is a consequence of the following argument: A 2n-dimensional almost sympletic manifold has a reduction to a group with elements $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$ where A, B, C and D are $n \times n$ matrices such that

$$B^tA - A^tB = 0$$

$$B^tC - A^tD = -I$$

and
$$D^tC = C^tD \ .$$

If $C = 0$, then $D = A^{-t}$ and $B^tA = A^tB$.

An *almost cotangent metric* is a metric with components $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ relative to a set of adapted frames of the almost cotangent structure that covered M.

In [7], Clark and Goel proved that every Riemannian metric on an almost cotangent manifold determines an almost cotangent metric. Thus the structural group of an almost cotangent manifold is reducible to a group with elements of the form $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ where $A \in O(n)$. It is a subgroup of $GL(n;\mathbb{C})$. Thus, any almost cotangent manifold admits an almost complex structure. In particular, the cotangent bundle is an almost complex manifold.

Conversely, an almost complex manifold with a global n-dimensional distribution that is spanned by the first n vector fields of the adapted frames admits an almost cotangent structure.

The following is an easy consequence of the reduction of the structural group.

Proposition 4.4.

An almost cotangent manifold admits a metric with signature (n,n).

Example 1

Let M be an n-dimensional manifold with an atlas $\{(U_\alpha, \phi_\alpha)\}$. Let M_m^* be the dual vector space of M_m, $TM^* = \bigcup_{m \in M} M_m^*$ be the cotangent bundle of M. Let π^* be the natural projection

$$\pi^* : TM^* \to M .$$

Let $\tilde{\varphi}_\alpha : (\pi^*)^{-1}(U_\alpha) \to \mathbb{R}^{2n}$ such that

$$\tilde{\varphi}_\alpha(\tau) = \left(x_1^\alpha(\pi^*(\tau)),\ x_2^\alpha(\pi^*(\tau)), \dots, x_n^\alpha(\pi^*(\tau)), \tau\left(\frac{\partial}{\partial x_1^\alpha}\right), \dots, \tau\left(\frac{\partial}{\partial x_n^\alpha}\right)\right)$$

where $\phi_\alpha(m) = \left(x_1^\alpha(m), \dots, x_n^\alpha(m)\right)$. On $(\pi^*)^{-1}(U_\alpha) \cap (\pi^*)^{-1}(U_\beta)$, the Jacobian matrix of the change of coordinate is

$$\left[\begin{array}{c|c} \dfrac{\partial x_i^\alpha}{\partial x_j^\beta} & 0 \\ \hline \dfrac{\partial^2 x_k^\beta}{\partial x_i^\alpha\, \partial x_r^\alpha} \dfrac{\partial x_r^\alpha}{\partial x_j^\beta}\, \tau\left(\dfrac{\partial}{\partial x_k^\beta}\right) & \dfrac{\partial x_j^\beta}{\partial x_i^\alpha} \end{array}\right].$$

Let $A = \left[\dfrac{\partial x_i^\alpha}{\partial x_j^\beta} \right]$ and $B = \left[\dfrac{\partial^2 x_k^\beta}{\partial x_i^\alpha \, \partial x_r^\alpha} \dfrac{\partial x_r^\alpha}{\partial x_j^\beta} \tau\left(\dfrac{\partial}{\partial x_k^\beta}\right)\right]$, by the virtue of inverse function theorem, $\left[\dfrac{\partial x_j^\beta}{\partial x_i^\alpha} \right] = A^{-t}$. Hence, the cotangent bundle admits a natural almost cotangent structure.

Proposition 4.5.

Let g be the almost cotangent metric on an almost cotangent manifold M with the 2-form ω. A (1,1) tensor ϕ defined by $\omega(X,\phi Y) = g(X,Y)$ is an almost hermitian structure with respect to the metric g on M.

Proof:

Suppose $\phi e_i = \sum_{j=1}^{2n} k_{ij} e_j$ where $\{e_1,\ldots,e_{2n}\}$ is an adapted frame such that ω has components $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ and g has components $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ (g is an almost cotangent metric). Then

$$k_{ij} = \begin{cases} -\delta_{j-ni} & j > n \\ \delta_{j+ni} & j \le n\ . \end{cases}$$

Now, for $r \le n$,

$$\phi^2(e_r) = \phi\left(\sum k_{rj} e_j\right)$$

$$= \phi(-e_{r+n})$$

$$= -e_r$$

and for $r > n$,

$$\phi^2(e_r) = \phi\left(\sum k_{rj}e_j\right)$$

$$= \phi(+e_{r-n})$$

$$= -e_r \ .$$

Thus $\phi^2 = -I$.

Furthermore $\qquad g(\phi X, \phi Y) = -g(\phi^2 X, Y) = -g(-X, Y) = g(X, Y)$

Therefore, ϕ is an almost hermitian structure.

Q.E.D.

4.2. Almost tangent structure

Let M be a 2n-dimensional manifold. An G-structure on M whose group G consists of all matrices of the form $\begin{bmatrix} A & 0 \\ B & A \end{bmatrix}$ where $A \in GL(n;\mathbb{R})$ is called an *almost tangent structure*. A manifold with an almost tangent structure is called an *almost tangent manifold*.

Clark and Goel [6] defined a (1,1) tensor J on an almost cotangent manifold M with components $\begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$ relative to any adapted frame. This is a well defined tensor. Clark and Bruckheimer [5] also proved that the existence of such a (1,1) tensor J determines an almost tangent structure.

The tensor J has rank n and satisfies the condition $J^2 = 0$. Clearly, an almost tangent manifold is orientable. There is an n-dimensional distribution on an almost tangent manifold.

The following proposition gives a sufficient condition for the existence of an almost tangent structure.

Proposition 4.6.

If there is a (1,1) tensor R on M with components $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ relative to a family of 2n-frames that covered M, then the structural group is reducible to $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$. Furthermore, if there is an n-dimensional distribution that is spanned by the first n vector fields of the frames mentioned above, then M admits an almost tangent structure. The existence of the n-dimensional distribution above is equivalent to the existence of a projection tensor ℓ with components $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. The composition $R\ell$ is equal to J.

The proof of Proposition 4.6 is obvious.

The Riemannian metric that has components $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ relative to a set of adapted frames of an almost tangent structure that covered M is called an *almost tangent metric*.

Clark and Goel [6] showed that any Riemannian metric determines an almost tangent metric. Thus, the structural group is reducible to the group G consisting all matrices of the form $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ where $A \in O(n)$. This is a subgroup of U(n). Therefore, an almost tangent manifold admits an almost complex structure. In particular, the tangent bundle has an almost complex structure.

Let M be an almost cotangent manifold with a 2-form ω such that its components are $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ with respect to adapted frames $\{e_1, \ldots, e_{2n}\}$. Consider a (1,1) tensor J defined by

$$J(X) = \sum_{i=1}^{n} \omega(X, e_i) e_i \ .$$

This is a well defined tensor. Therefore

$$J(e_i) = \begin{cases} 0 & i \le n \\ e_{i-n} & i > n . \end{cases}$$

This proves the following:

Proposition 4.7.

An almost cotangent manifold admits an almost tangent structure.

Proposition 4.8.

There exist two (1,1) tensors R and ℓ as defined in Proposition 6 on an almost tangent manifold M. The manifold M also admits a metric with signature (n,n).

Proposition 4.8 is a direct consequence of the reduction of the structural group to a group consisting of elements of the form $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, where $A \in O(n)$.

Let $\phi = J - \ell R$. Proposition 4.8 ensures the existence of R and ℓ and thus ϕ is well defined. The (1,1) tensor ϕ has components $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ which is an almost complex structure. Therefore we have the following proposition:

Proposition 4.9.

An almost tangent manifold admits an almost complex structure.

We can now show that an almost tangent manifold always admits an almost cotangent structure.

Proposition 4.10.

Let M be an almost tangent manifold with a (1,1) tensor J and $\{e_1, \ldots, e_{2n}\}$ be a set of orthonormal basis with respect to an almost tangent metric g. There is a 2-form ω on M defined by $\omega(X,Y) = g(JX,Y) - g(X,JY)$. This 2-form ω and the n-dimensional distribution generated by $\{e_1, \ldots, e_n\}$ define an almost cotangent structure.

Proof:

Let $k_{ij} = \omega(e_i, e_j)$, then

$$k_{ij} = \begin{cases} -\delta_{i\,j-n} & j > n \\ \delta_{i\,j+n} & j \le n \end{cases}$$

and ω is the required 2-form.

Q.E.D.