# Appendix A

# Dynamics of Kelvin Wave and Mixed Rossby-Gravity Wave

The dynamics of Kelvin wave and mixed Rossby-gravity wave can easily be deduced theoretically by applying the linear perturbation technique in the log-pressure system as given by Holton (1979).

## A.1 The Log-Pressure System

In the log-pressure system, the vertical coordinate is defined as

$$z^* \equiv -H \ln\left(\frac{p}{p_0}\right) \tag{A.1}$$

where $p_0$ is a standard reference usually taken to be 100 kPa and $H$ is a standard scale height, $H \equiv RT_0/g$, with $T_0$ a global average temperature. For an isothermal atmosphere at temperature $T_0$, $z^*$ is exactly equal to height. For an atmosphere with variable temperature, $z^*$ will be only approximately equivalent to the actual height. The vertical velocity is then defined as

$$w^* \equiv \frac{dz^*}{dt} \tag{A.2}$$

The horizontal momentum equation in the log-pressure system is the same as that in the isobaric system:

$$\frac{d\mathbf{V}}{dt} + f\mathbf{K}\times\mathbf{V} = -\nabla\Phi \tag{A.3}$$

However, the operator $d/dt$ is now defined as

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla + w^*\frac{\partial}{\partial z^*}$$

The hydrostatic equation

$$\frac{\partial\Phi}{\partial p} = -\alpha$$

can be transformed to the log-pressure system by eliminating $\alpha$ with the ideal gas law to get

$$\frac{\partial\Phi}{\partial \ln p} = -RT$$

and then manipulated with the aid of equation (A.1) to get

$$\frac{\partial\Phi}{\partial z^*} = \frac{RT}{H} \tag{A.4}$$

The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

can also be conveniently transformed to the log-pressure coordinate form. Substituting equation (A.1) into equation (A.2) gets

$$w^* = -\frac{H}{p}\frac{dp}{dt} = -\frac{H\omega}{p}$$

from which further manipulation and differentiation gives

$$\frac{\partial \omega}{\partial p} = -\frac{\partial}{\partial p}\left(\frac{pw^*}{H}\right) = \frac{\partial w^*}{\partial z^*} - \frac{w^*}{H}$$

Thus, in log-pressure coordinates the continuity equation becomes simply

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w^*}{\partial z^*} - \frac{w^*}{H} = 0 \qquad \text{(A.5)}$$

Finally, the first law of thermodynamics

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{q}$$

can be rewritten in the following log-pressure coordinate form with the aid of equation (A.4):

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{\partial \Phi}{\partial z^*} + w^* N^2 = \frac{\kappa \dot{q}}{H} \qquad \text{(A.6)}$$

where

$$N^2 \equiv \frac{R}{H}\left( \frac{\partial T}{\partial z^*} + \frac{\kappa T}{H} \right)$$

and $\kappa \equiv R/c_p$. In the stratosphere the buoyancy frequency squared, $N^2$, is approximately constant with a value $N^2 \approx 4\times10^{-4}\text{s}^{-2}$.



## A.2 The Kelvin Wave

It is convenient to use the governing equations in log-pressure coordinates (A.3), (A.5) and (A.6) referred to an equatorial $\beta$ plane in which the Coriolis parameter is approximated as

$$f = \frac{2\Omega y}{a} \equiv \beta y$$

where $y$ is the distance from the equator. Thus, $\beta$ is the rate at which $f$ changes with latitude at the equator. Assuming as a basic state an atmosphere at rest with no diabatic heating. If the perturbations are assumed to be zonally propagating waves, then

$$u = u'(y, z^*)e^{i(kx-\nu t)}$$

$$v = v'(y, z^*)e^{i(kx-\nu t)}$$

$$w^* = w^{*'}(y, z^*)e^{i(kx-\nu t)}$$

$$\Phi = \Phi'(y, z^*)e^{i(kx-\nu t)}$$

From equations (A.3), (A.5) and (A.6), the following set of linearized perturbation equations can be obtained:

$$-i\nu u' - \beta y v' = -ik\Phi' \quad \text{(A.7)}$$

$$-i\nu v' + \beta y u' = -\frac{\partial \Phi'}{\partial y} \quad \text{(A.8)}$$

$$iku' + \frac{\partial v'}{\partial y} + \left(\frac{\partial}{\partial z^*} - \frac{1}{H}\right)w^{*'} = 0 \quad \text{(A.9)}$$

$$-i\nu\frac{\partial \Phi'}{\partial z^*} + N^2 w^{*'} = 0 \quad \text{(A.10)}$$

For Kelvin wave, this set of equations can be considerably simplified. Setting $v' \equiv 0$ and eliminating $w^{*'}$ between equations (A.9) and (A.10), this set of equations become

$$\nu u' = k\Phi' \quad \text{(A.11)}$$

$$\beta y u' = -\frac{\partial \Phi'}{\partial y} \quad \text{(A.12)}$$

$$\left(\frac{\partial}{\partial z^*} - \frac{1}{H}\right)\frac{\partial \Phi'}{\partial z^*} + \frac{k}{\nu}N^2 u' = 0 \quad \text{(A.13)}$$

Using equation (A.11) to eliminate $\Phi'$ in equations (A.12) and (A.13), the field of $u'$ must satisfy the following two independent equations:

$$\beta y u' = -\frac{\nu}{k}\frac{\partial u'}{\partial y} \tag{A.14}$$

$$\left(\frac{\partial}{\partial z^*} - \frac{1}{H}\right)\frac{\partial u'}{\partial z^*} + \frac{k^2}{\nu^2}N^2 u' = 0 \tag{A.15}$$

Equation (A.14) has the solution

$$u' = u_0\left(z^*\right)\exp\left\{-\beta y^2 k/2\nu\right\} \tag{A.16}$$

Assuming that $k > 0$ then $\nu > 0$ corresponds to an eastward-moving wave. In that case $u'$ will have a Gaussian distribution about the equator. For a westward-moving wave ($\nu < 0$) the solution (A.16) increases in amplitude exponentially away from the equator. This solution cannot satisfy reasonable boundary conditions at the poles and must, therefore, be rejected. Hence, there exists only an eastward-propagating atmospheric Kelvin wave.

Solutions for equation (A.15) can be written in the form

$$u_0(z^*) = \exp(z^*/2H)\left[C_1\exp(i\lambda z^*) + C_2\exp(-i\lambda z^*)\right] \tag{A.17}$$

with

$$\lambda^2 \equiv \frac{N^2 k^2}{\nu^2} - \frac{1}{4H^2}$$

Here the constants $C_1$ and $C_2$ are to be determined by appropriate boundary conditions. For $\lambda^2 > 0$ the solution (A.17) is in the form of a vertically propagating wave. For waves in the equatorial stratosphere that are forced by disturbances in the troposphere, the propagation of energy must have an upward component, implying that the phase velocity of the wave must have a downward component. Hence, the constant $C_1 = 0$ in the solution (A.17), and the Kelvin wave has a structure in the $x$, $z$ plane shown in Figure A.1.

## A.3 The Mixed Rossby-Gravity Wave

A similar analysis is possible for the mixed Rossby-gravity wave. In this case, the full perturbation equations (A.7)-(A.10) must be used. After some considerable algebraic manipulation, it can be shown that the mixed Rossby-gravity wave has the solution

$$\begin{Bmatrix} u' \\ v' \\ \Phi' \end{Bmatrix} = \Psi(z^*) \begin{Bmatrix} +i\beta y(1+k\nu/\beta)/\nu \\ 1 \\ +i\nu y \end{Bmatrix} \exp\left[\frac{-(1+k\nu/\beta)\beta^2 y^2}{2\nu^2}\right] \qquad \text{(A.18)}$$

where the vertical structure $\Psi(z^*)$ is given by

$$\Psi(z^*) = \exp(z^*/2H)[C_1 \exp(i\lambda_0 z^*) + C_2 \exp(-i\lambda_0 z^*)] \qquad \text{(A.19)}$$

with

$$\lambda_0^2 \equiv N^2 \frac{k^2}{\nu^2}\left(1+\frac{\beta}{\nu k}\right)^2 - \frac{1}{4H^2}$$

Again, the constants $C_1$ and $C_2$ are to be determined by appropriate boundary conditions.

Solution (A.18) indicates that $v'$ has a Gaussian distribution about the equator. This solution is valid for westward-propagating waves ($\nu < 0$) provided that

$$1+\frac{k\nu}{\beta} > 0 \qquad \text{(A.20)}$$

For frequencies that do not satisfy equation (A.20), the wave amplitude will not decay away from the equator and so the boundary conditions at the pole will not be satisfied.

Just like the Kelvin waves, the mixed Rossby-gravity waves must have downward phase velocity in order for upward energy propagation. Thus $C_2 = 0$ is required in equation (A.19). The resulting wave structure in the $x, z$ plane at latitude north of the equator is shown in Figure A.2.

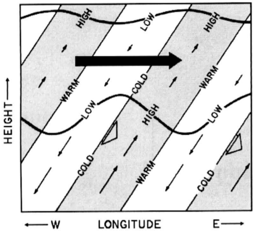


**Figure A.1** Longitude-height section along the equator showing pressure, temperature and wind perturbations for a thermally damped Kelvin wave. Heavy wavy lines indicate material lines; short blunt arrows show phase propagation. Areas of high pressure are shaded. Length of the small thin arrows is proportional to the wave amplitude, which decreases with height due to damping. The large shaded arrow indicates the net mean flow acceleration due to the wave stress divergence. (After Holton, 1979.)

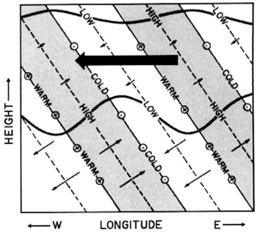


**Figure A.2** Longitude-height section along a latitude circle north of the equator showing pressure, temperature and wind perturbations for a thermally damped mixed Rossby-gravity wave. Areas of high pressure are shaded. Small arrows indicate zonal and vertical wind perturbations with length proportional to the wave amplitude. Arrows pointed into the page (northward) and out of the page (southward) show meridional wind perturbations. The large shaded arrow indicates the net mean flow acceleration due to the wave stress divergence. (After Holton, 1979.)

# Appendix B

# Probability Tables

The following tables are extracted from Hald (1952).

**Table B.1** Probability Points of Gaussian Normal Distribution ($t_g$)

| Degrees of freedom | Probability in per cent | | | |
|---|---|---|---|---|
| | 95* | 97.5⁺ | 99 | 99.5 |
| (Infinite) | 1.645 | 1.960 | 2.326 | 2.576 |

* Probability equivalent to 95 per cent significance point for <u>one-tailed</u> test.

⁺ Probability equivalent to 95 per cent significance point for <u>two-tailed</u> test.

**Table B.2** Probability Points of $\chi^2/\nu$ Distribution

| Degrees of freedom | Probability in per cent | | | | |
|---|---|---|---|---|---|
| | 1 | 5* | 90 | 95+ | 99 |
| 2 | .010 | .052 | 2.305 | 3.000 | 4.605 |
| 3 | .038 | .117 | 2.083 | 2.605 | 3.782 |
| 4 | .074 | .178 | 1.945 | 2.372 | 3.319 |
| 5 | .111 | .229 | 1.848 | 2.214 | 3.017 |
| 6 | .145 | .272 | 1.767 | 2.099 | 2.802 |
| 8 | .206 | .342 | 1.675 | 1.938 | 2.511 |
| 10 | .256 | .394 | 1.600 | 1.831 | 2.321 |
| 15 | .349 | .484 | 1.487 | 1.666 | 2.038 |
| 20 | .413 | .543 | 1.420 | 1.570 | 1.878 |
| 30 | .498 | .616 | 1.343 | 1.459 | 1.696 |
| 40 | .554 | .663 | 1.295 | 1.394 | 1.592 |
| 50 | .594 | .695 | 1.264 | 1.350 | 1.523 |
| 60 | .625 | .720 | - | 1.318 | 1.473 |
| 80 | .669 | .755 | - | 1.274 | 1.404 |
| 100 | .701 | .779 | 1.185 | 1.243 | 1.358 |
| 200 | .782 | .841 | - | 1.170 | 1.247 |
| 400 | .843 | .887 | - | 1.119 | 1.172 |
| 1000 | .899 | .928 | - | 1.075 | 1.107 |

* Probability equivalent to 95 per cent significance point for <u>one-tailed</u> test of spectral gap.

+ Probability equivalent to 95 per cent significance point for <u>one-tailed</u> test of spectral peak.