COMMUTING ADDITIVE MAPS AND SOME RELATED MAPS ON TRIANGULAR MATRICES

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COMMUTING ADDITIVE MAPS AND SOME RELATED MAPS ON TRIANGULAR MATRICES

ABSTRACT

Let $\mathbb{F}$ be a ring with identity and let $n \geqslant 2$ be an integer. Denote by $T_n(\mathbb{F})$ the ring of $n \times n$ upper triangular matrices over $\mathbb{F}$ with centre $Z(T_n(\mathbb{F}))$ and unity I_n. Let $1 \leqslant i \leqslant j \leqslant n$ be integers and let $E_{ij} \in T_n(\mathbb{F})$ denote the standard matrix unit whose (i,j)th entry is one and zero elsewhere. In this thesis, the following results have been obtained:

Let $1 < k \leqslant n$ be an integer and let $\mathbb{F}$ be a field. We characterise commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank k matrices, i.e., additive maps ψ satisfying $\psi(A)A = A\psi(A)$ for all rank k matrices $A \in T_n(\mathbb{F})$ and show that

- when either $k < n$ or $|\mathbb{F}| \geqslant 3$, there exist $\lambda, \alpha \in \mathbb{F}$ and an additive map $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ such that

  $$\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n}$$

  for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\alpha \neq 0$ only if $k = n$ and $|\mathbb{F}| = 3$,

- when $k = n \geqslant 4$ and $|\mathbb{F}| = 2$, there exist $\lambda, \alpha, \beta_1, \beta_2 \in \mathbb{F}$, $H, K \in T_n(\mathbb{F})$ and $X_1, \ldots, X_n \in T_n(\mathbb{F})$ satisfying $X_1 + \cdots + X_n = 0$ such that

  $$\psi(A) = \lambda A + \mathrm{tr}\,(H^t A)I_n + \mathrm{tr}\,(K^t A)E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A) + \sum_{i=1}^{n} a_{ii}X_i$$

  for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\mathrm{tr}\,(A)$ and A^t are the trace and the transpose of A respectively, and $\Psi_{\alpha,\beta_1,\beta_2} : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is the additive map defined by

  $$\Psi_{\alpha,\beta_1,\beta_2}(A) = (\alpha a_{12} + \beta_1(a_{n-1,n} + a_{nn}))E_{1,n-1} + (\alpha a_{n-1,n} + \beta_2(a_{11} + a_{12}))E_{2n}$$

  for all $A = (a_{ij}) \in T_n(\mathbb{F})$,

- when $k = n = 3$ and $|\mathbb{F}| = 2$, there exist $\lambda, \alpha, \beta, \gamma \in \mathbb{F}$, $H, K \in T_3(\mathbb{F})$ and $X_1, X_2, X_3 \in T_3(\mathbb{F})$ satisfying $X_1 + X_2 + X_3 = 0$ such that

$$\psi(A) = \lambda A + \text{tr}\,(H^t A)I_3 + \text{tr}\,(K^t A)E_{13} + \Psi_{\alpha,\beta}(A) + \Phi_\gamma(A) + \sum_{i=1}^{3} a_{ii}X_i$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$, where $\Psi_{\alpha,\beta} : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ and $\Phi_\gamma : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ are the additive maps defined by

$$\Psi_{\alpha,\beta}(A) = \alpha(a_{23} + a_{33})E_{12} + \beta(a_{11} + a_{12})E_{23},$$

$$\Phi_\gamma(A) = \gamma((a_{12} + a_{22})E_{22} + (a_{11} + a_{12} + a_{23} + a_{33})E_{33} + a_{13}(E_{12} + E_{23}))$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$, and

- when $k = n = 2$ and $|\mathbb{F}| = 2$, there exist $\lambda_1, \lambda_2 \in \mathbb{F}$ and $X_1, X_2 \in T_2(\mathbb{F})$ such that

$$\psi(A) = (a_{11} + a_{12})X_1 + (a_{22} + a_{12})X_2 + \lambda_1 a_{12} I_2 + \lambda_2 a_{12} E_{12}$$

for all $A = (a_{ij}) \in T_2(\mathbb{F})$.

Let $\mathbb{F}$ be a division ring. We classify centralizing additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank one matrices, i.e., additive maps ψ satisfying $\psi(A)A - A\psi(A) \in Z(T_n(\mathbb{F}))$ for all rank one matrices $A \in T_n(\mathbb{F})$. We show that centralizing additive maps on rank one upper triangular matrices are equivalent to commuting additive maps on rank one upper triangular matrices over division rings. The structure of commuting additive maps on rank one upper triangular matrices over noncommutative division rings is relatively simpler than the corresponding result on commuting additive maps on rank one upper triangular matrices over fields. Let $\mathbb{F}_2$ denote the Galois field of two elements. We obtain a complete description of 2-power commuting additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on rank n matrices, i.e., additive maps ψ satisfying $\psi(A)A^2 = A^2\psi(A)$ for all rank n matrices $A \in T_n(\mathbb{F}_2)$.

Keywords: commuting additive maps, upper triangular matrices, ranks, functional identities, linear preserver problems.

PEMETAAN KALIS TUKAR TERTIB BERDAYA TAMBAH DAN BEBERAPA PEMETAAN YANG BERKAITAN PADA MATRIKS SEGITIGA

ABSTRAK

Biar $\mathbb{F}$ suatu gelanggang dengan identiti dan biar $n \geqslant 2$ suatu integer. Tandakan $T_n(\mathbb{F})$ gelanggang bagi matriks segitiga atas $n \times n$ terhadap $\mathbb{F}$ dengan pusat $Z(T_n(\mathbb{F}))$ dan identiti I_n. Biar $1 \leqslant i \leqslant j \leqslant n$ merupakan integer dan biar $E_{ij} \in T_n(\mathbb{F})$ menandakan unit matriks piawai yang masukan (i, j) ialah satu dan sifar bagi yang lain. Dalam tesis ini, hasil berikut telah diperolehi:

Biar $1 < k \leqslant n$ suatu integer dan biar $\mathbb{F}$ suatu medan. Kami mencirikan pemetaan berdaya tambah kalis tukar tertib $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ pada matriks $A \in T_n(\mathbb{F})$ berpangkat k, iaitu, pemetaan berdaya tambah ψ memenuhi $\psi(A)A = A\psi(A)$ bagi semua matriks $A \in T_n(\mathbb{F})$ berpangkat k dan membuktikan bahawa

- apabila $k < n$ atau $|\mathbb{F}| \geqslant 3$, wujudnya $\lambda, \alpha \in \mathbb{F}$ dan suatu pemetaan berdaya tambah $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ supaya

$$\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n}$$

  bagi semua $A = (a_{ij}) \in T_n(\mathbb{F})$, di mana $\alpha \neq 0$ hanya jika $k = n$ dan $|\mathbb{F}| = 3$,

- apabila $k = n \geqslant 4$ dan $|\mathbb{F}| = 2$, wujudnya $\lambda, \alpha, \beta_1, \beta_2 \in \mathbb{F}$, $H, K \in T_n(\mathbb{F})$ dan $X_1, \ldots, X_n \in T_n(\mathbb{F})$ yang memenuhi $X_1 + \cdots + X_n = 0$ supaya

$$\psi(A) = \lambda A + \operatorname{tr}(H^t A)I_n + \operatorname{tr}(K^t A)E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A) + \sum_{i=1}^{n} a_{ii}X_i$$

  bagi semua $A = (a_{ij}) \in T_n(\mathbb{F})$, di mana $\operatorname{tr}(A)$ and A^t ialah surihan dan transposisi bagi A, masing-masing, dan $\Psi_{\alpha,\beta_1,\beta_2} : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ ialah pemetaan berdaya tambah yang ditakrifkan sebagai

$$\Psi_{\alpha,\beta_1,\beta_2}(A) = (\alpha a_{12} + \beta_1(a_{n-1,n} + a_{nn}))E_{1,n-1} + (\alpha a_{n-1,n} + \beta_2(a_{11} + a_{12}))E_{2n}$$

  bagi semua $A = (a_{ij}) \in T_n(\mathbb{F})$,

- apabila $k = n = 3$ dan $|\mathbb{F}| = 2$, wujudnya $\lambda, \alpha, \beta, \gamma \in \mathbb{F}$, $H, K \in T_3(\mathbb{F})$ dan $X_1, X_2, X_3 \in T_3(\mathbb{F})$ yang memenuhi $X_1 + X_2 + X_3 = 0$ supaya

$$\psi(A) = \lambda A + \operatorname{tr}(H^t A) I_3 + \operatorname{tr}(K^t A) E_{13} + \Psi_{\alpha,\beta}(A) + \Phi_\gamma(A) + \sum_{i=1}^{3} a_{ii} X_i$$

bagi semua $A = (a_{ij}) \in T_3(\mathbb{F})$, di mana $\Psi_{\alpha,\beta} : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ dan $\Phi_\gamma : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ adalah pemetaan berdaya tambah yang ditakrifkan sebagai

$$\Psi_{\alpha,\beta}(A) = \alpha(a_{23} + a_{33})E_{12} + \beta(a_{11} + a_{12})E_{23},$$

$$\Phi_\gamma(A) = \gamma((a_{12} + a_{22})E_{22} + (a_{11} + a_{12} + a_{23} + a_{33})E_{33} + a_{13}(E_{12} + E_{23}))$$

bagi semua $A = (a_{ij}) \in T_3(\mathbb{F})$, dan

- apabila $k = n = 2$ dan $|\mathbb{F}| = 2$, wujudnya $\lambda_1, \lambda_2 \in \mathbb{F}$ dan $X_1, X_2 \in T_2(\mathbb{F})$ supaya

$$\psi(A) = (a_{11} + a_{12})X_1 + (a_{22} + a_{12})X_2 + \lambda_1 a_{12} I_2 + \lambda_2 a_{12} E_{12}$$

bagi semua $A = (a_{ij}) \in T_2(\mathbb{F})$.

Biar $\mathbb{F}$ suatu gelanggang pembahagian. Kami mencirikan pemetaan berdaya tambah memusat $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ pada matriks berpangkat satu, iaitu, pemetaan berdaya tambah ψ yang memenuhi $\psi(A)A - A\psi(A) \in Z(T_n(\mathbb{F}))$ bagi semua matriks $A \in T_n(\mathbb{F})$ berpangkat satu. Kami membuktikan bahawa pemetaan berdaya tambah memusat pada matriks segitiga atas berpangkat satu adalah setara dengan pemetaan berdaya tambah kalis tukar tertib pada matriks segitiga atas berpangkat satu terhadap gelanggang pembahagian. Struktur pemetaan berdaya tambah kalis tukar tertib pada matriks segitiga atas berpangkat satu terhadap gelanggang pembahagian tak kalis tukar tertib adalah lebih ringkas berbanding dengan pemetaan berdaya tambah kalis tukar tertib pada matriks segitiga atas berpangkat satu terhadap medan. Biar $\mathbb{F}_2$ menandakan medan dengan dua unsur. Kami memperolehi suatu pemerihalan lengkap bagi pemetaan berdaya tambah kalis tukar tertib 2-kuasa $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ pada matriks berpangkat n, iaitu, pemetaan berdaya tambah ψ memenuhi $\psi(A)A^2 = A^2\psi(A)$ bagi semua matriks $A \in T_n(\mathbb{F}_2)$ berpangkat n.

Kata kunci: pemetaan berdaya tambah kalis tukar tertib, matriks segitiga atas, pangkat, identiti fungsian, masalah pengekal linear.

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LIST OF SYMBOLS

$\lvert\mathbb{F}\rvert$	:	the cardinality of the field $\mathbb{F}$
char $\mathbb{F}$	:	the characteristic of the field $\mathbb{F}$
$\mathbb{F}_2$	:	the Galois field of two elements
$Z(\mathcal{R})$	:	the centre of the ring $\mathcal{R}$
$M_n(\mathbb{F})$	:	the algebra of $n \times n$ matrices over the field $\mathbb{F}$
$T_n(\mathbb{F})$	:	the algebra of $n \times n$ upper triangular matrices over the field $\mathbb{F}$
$Z(T_n(\mathbb{F}))$	:	the centre of the algebra of $n \times n$ upper triangular matrices over a field $\mathbb{F}$
E_{ij}	:	the standard matrix unit whose (i, j)th entry is one and zero elsewhere
I_n	:	the $n \times n$ identity matrix
J_n	:	the $n \times n$ matrix with one on the minor diagonal and zero elsewhere
A^t	:	the transpose of the matrix A
A^+	:	the flip of the upper triangular matrix A, $J_n A^t J_n$
tr (A)	:	the trace of the square matrix A
$[a, b]$	:	the commutator $ab - ba$ of the elements a and b in a ring
δ_{ij}	:	the Kronecker delta function

CHAPTER 1: INTRODUCTION

1.1 Background of the study

Linear preserver problems represent one of the very active research areas in matrix theory concerning the characterisation of linear maps that leave certain properties, relations or subsets invariant. This research subject has a long history in linear algebra which traces back to Frobenius (1897) on determinant preserving linear maps on matrix spaces. Over the past few decades, the study of linear preserver problems is prospering into a fruitful ground of discovery for many researchers and yet there are still many open problems and interactions of linear algebra with other research areas in functional identities, geometry of matrices, operator algebras, etc. Recently, linear preserver problems on triangular matrices have received substantial attention. The results of linear preserver problems on triangular matrices often have much complicated and different structures compared to the corresponding result on other matrices. For a survey of linear preserver problems and its developments, we refer the reader to the special survey issue of Linear and Multilinear Algebra (volume 33, no.1-2 (1992), pp.1-119) in Pierce et al. (1992) and C.-K. Li and Pierce (2001).

Let $\mathcal{R}$ be a ring with centre $Z(\mathcal{R})$ and let $\mathcal{S}$ be a nonempty subset of $\mathcal{R}$. A map $\psi : \mathcal{R} \to \mathcal{R}$ is called *commuting* on $\mathcal{S}$ if $[\psi(x), x] = 0$ for all $x \in \mathcal{S}$, ψ is said to be *centralizing* on $\mathcal{S}$ if $[\psi(x), x] \in Z(\mathcal{R})$ for all $x \in \mathcal{S}$, and ψ is m*-power commuting* on $\mathcal{S}$ if $[\psi(x), x^m] = 0$ for all $x \in \mathcal{S}$, where $[x, y] = xy - yx$ is the commutator of $x, y \in \mathcal{R}$, and $m \geqslant 2$ is an integer. In 1993, Brešar (1993a) obtained the structural result of commuting additive maps $\psi : \mathcal{R} \to \mathcal{R}$ on a prime ring $\mathcal{R}$. His works have actuated the development of the theory of functional identities. Moreover, since such a problem has a wealth of applications, there have been much research activities on commuting maps, centralizing maps, power commuting maps on various rings, algebras and linear spaces. For a survey of the subject and its historical developments, we refer the reader to the book "Functional Identities" by Brešar, Chebotar and Martindale 3rd and the survey paper by Brešar, see Brešar et al. (2007) and Brešar (2004). More recently, inspired by the study of linear preserver problems on sets of matrices that are not closed under addition, Franca (2012, 2013a) initiated the study of commuting additive maps on invertible, singular and rank k square matrices. He deduced the results from the classical result of Brešar (1993a). His work has advanced the study of functional identities to the set of matrices that are not closed under addition, which leads to many open problems in this area.

1.2 Objective of the study

Motivated by the research on linear preserver problems on rank k matrices, Franca (2013a) initiated the study of functional identities on rank k square matrices, which has inspired a new line of research in this research area. Many works have been done, see for example commuting additive maps in Franca (2017); Franca and Louza (2017); Xu and Yi (2014), centralizing additive maps in C.-K. Liu (2014a), power commuting additive maps in Chou and Liu (2021); C.-K. Liu and Yang (2017), m-commuting additive maps in Franca and Louza (2019), strong commutativity preserving maps in C.-K. Liu (2014b); C.-K. Liu et al. (2018), commuting traces maps in Franca (2013b) and additivity preserving maps in Chooi and Kwa (2019); Xu and Liu (2017); Xu et al. (2016).

Let $n \geqslant 2$ and $1 \leqslant k \leqslant n$ be integers. Let $T_n(\mathbb{F})$ denote the ring of $n \times n$ upper triangular matrices over a ring $\mathbb{F}$. The main objective of this thesis is to obtain complete descriptions of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ and some related additive maps on rank k matrices. More precisely, we characterise the following additive maps:

(a) commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank k matrices, where $\mathbb{F}$ is a field with $|\mathbb{F}| \geqslant 3$ and $1 < k \leqslant n$ is a fixed integer,

(b) commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ for all invertible matrices, where $\mathbb{F}$ is the Galois field of two elements,

(c) centralizing additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ for all rank one matrices, where $\mathbb{F}$ is a division ring,

(d) 2-power commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ for all invertible matrices, where $\mathbb{F}$ is the Galois field of two elements.

1.3 Significance of the study

Let $n \geqslant 2$ and $1 \leqslant k \leqslant n$ be integers. This thesis is devoted to the study of functional identities on rank k upper triangular matrices which is motivated by the study of linear preserver problems on sets of matrices that are not closed under addition. This new line of research has advanced the study of linear preserver problems of functional identities on matrix rings (the ring of square matrices and the ring of upper triangular matrices). Moreover, this study has also established an interesting mathematical interaction between linear preserver problems and functional identities on rank k matrices.

1.4 Organisation of the thesis

This thesis is divided into eight chapters. In Chapter 1, we begin with a background of the study and a general introduction of our main themes. We state the research objectives and the significance of the study. This is followed by a brief overview of each chapter in the organisation of the thesis.

In Chapter 2, we start by introducing some preliminary results that are needed in this thesis. Next we give a brief introduction of linear preserver problems on matrices. We then proceed with a literature review of the study of functional identities in commuting maps, centralizing maps, power commuting maps and some related maps. Finally, the research methodology employed in the research will be given.

Chapter 3 is devoted to the study of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank k matrices, where $n \geqslant 2$ and $1 < k \leqslant n$ are integers and $\mathbb{F}$ is a field with $|\mathbb{F}| \geqslant 3$. We assert a few technical lemmas by adapting known techniques from matrix theory and obtain a characterisation of such additive maps.

Chapter 4 is primarily concerning the study of commuting additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on rank n matrices, where $n \geqslant 2$ is an integer and $\mathbb{F}_2$ is the Galois field of two elements. Some irregular nonstandard forms of commuting additive maps on rank n upper triangular matrices will be illustrated. We obtain a complete description of commuting additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on rank n upper triangular matrices for $n \geqslant 2$. As a by-product, we give a classification for commuting additive maps on 2×2 invertible full matrices.

In Chapter 5, we study commuting additive maps on rank one upper triangular matrices over fields. We start with some intriguing irregular forms of commuting additive maps, and then we continue to present a complete characterisation of commuting additive maps on rank one upper triangular matrices over fields.

In Chapter 6, we investigate the structure of centralizing additive maps on rank one upper triangular matrices over division rings. We show that centralizing additive maps and commuting additive maps on rank one upper triangular matrices over division rings are equivalent. We give a characterisation of centralizing additive maps on rank one upper triangular matrices over noncommutative division rings. As a side remark, the structure of centralizing additive maps on rank one upper triangular matrices over fields is much more fertile and complex than the corresponding result on noncommutative division rings.

Chapter 7 is devoted to the study of 2-power commuting additive maps on invertible upper triangular matrices over the Galois field of two elements. We give some interesting examples of the maps which are of nonstandard forms and we prove the main results.

In Chapter 8, we provide a summary of the overall findings in this study and suggest some potential open problems that would be possible for future research work.

CHAPTER 2: LITERATURE REVIEW AND METHODOLOGY

This chapter starts with some preliminary results which will be employed in the forthcoming chapters. We will give a brief introduction of linear preserver problems on matrices. We then proceed with a literature review of the study of functional identities in commuting maps, centralizing maps, power commuting maps and some related maps. We end this chapter with a brief discussion of the methodology used in this research.

2.1 Preliminary results

Let $\mathbb{F}$ be a field and let $n \geqslant 1$ be an integer. Recall that $T_n(\mathbb{F})$ is the ring of all $n \times n$ upper triangular matrices over $\mathbb{F}$ and $E_{ij} \in T_n(\mathbb{F})$ is the standard matrix unit whose (i, j)-th entry is one and zero elsewhere.

We start with the following lemma proved in (Chooi & Lim, 1998, Lemma 4.1).

Lemma 2.1.1. (Chooi & Lim, 1998, Lemma 4.1). *Let $\mathbb{F}$ be a field and let $n \geqslant 1$ and $1 \leqslant k \leqslant n$ be integers. Then $A \in T_n(\mathbb{F})$ is of rank k if and only if there exist invertible matrices $P, Q \in T_n(\mathbb{F})$ such that*

$$A = P\left(\sum_{i=1}^{k} E_{s_i,t_i}\right)Q$$

for some integers $1 \leqslant s_i \leqslant t_i \leqslant n$ for $i = 1, \ldots, k$ with $s_1 < \cdots < s_k$ and $t_i \neq t_j$ whenever $1 \leqslant i \neq j \leqslant k$.

Let $M_n(\mathbb{F})$ be the ring of all $n \times n$ matrices over $\mathbb{F}$. Given $A \in T_n(\mathbb{F})$, we denote by $A^+ = J_n A^t J_n \in T_n(\mathbb{F})$, where A^t is the transpose of A and $J_n \in M_n(\mathbb{F})$ with one on the minor diagonal and zero elsewhere. We now prove a technical lemma that will be used in the study of commuting additive maps on rank k upper triangular matrices.

Lemma 2.1.2. *Let $\mathbb{F}$ be a field with at least three elements and let $n \geqslant 3$ be an integer. Suppose that $1 < k < n$ is a fixed integer. Then each rank one or rank two matrix in $T_n(\mathbb{F})$ can be represented as a sum of three rank k matrices in $T_n(\mathbb{F})$ among which the sum of any two is of rank k.*

Proof. Let $A \in T_n(\mathbb{F})$. First, consider A is of rank one. By Lemma 2.1.1, we assume without loss of generality that $A = E_{ij}$ for some integers $1 \leqslant i \leqslant j \leqslant n$.

Case I: $i = j$. When $1 \leqslant i < n$, we select distinct integers $s_1, \ldots, s_{k-1} \in \{1, \ldots, n-1\}\backslash\{i\}$ and $\alpha \in \mathbb{F}\backslash\{0, 1\}$. Let

$$X_i = E_{ii} + (\alpha - 1)E_{i,i+1} + \sum_{j=1}^{k-1}(\alpha - 1)E_{s_j,s_j+1},$$

$$Y_i = E_{i,i+1} + \sum_{j=1}^{k-1} E_{s_j,s_j+1} \quad \text{and} \quad Z_i = -\alpha Y_i.$$

Then $X_i, Y_i, Z_i \in T_n(\mathbb{F})$ are of rank k such that $E_{ii} = X_i + Y_i + Z_i$ and among which the sum of any two is of rank k. When $i = n$, since $E_{nn} = E_{11}^+$, we have $E_{nn} = X_1^+ + Y_1^+ + Z_1^+$ as required.

Case II: $i < j$. We select distinct integers $s_1, \ldots, s_{k-1} \in \{1, \ldots, n\}\backslash\{i\}$ and $\alpha \in \mathbb{F}\backslash\{0, 1\}$. We set

$$X = E_{ij} + (\alpha - 1)E_{ii} + \sum_{j=1}^{k-1}(\alpha - 1)E_{s_j,s_j},$$

$$Y = E_{ii} + \sum_{j=1}^{k-1} E_{s_j,s_j} \quad \text{and} \quad Z = -\alpha Y.$$

Then $E_{ij} = X + Y + Z$ is the sum of three rank k matrices $X, Y, Z \in T_n(\mathbb{F})$ among which the sum of any two is of rank k.

Consider now A is of rank two. By Lemma 2.1.1, we may assume $A = E_{ij} + E_{pq}$ for some integers $1 \leqslant i \leqslant j \leqslant n$, $1 \leqslant p \leqslant q \leqslant n$, $i < p$ and $j \neq q$. We argue in the following cases.

Case A: $i = j$ and $p = q$. When $i = 1$ and $p = n$, we select distinct integers $s_1, \ldots, s_{k-2} \in \{2, \ldots, n-2\}$ and $\alpha \in \mathbb{F}\backslash\{0, 1\}$. Let

$$X = E_{11} + (\alpha - 1)E_{12} + E_{nn} + (\alpha - 1)E_{n-1,n} + \sum_{j=1}^{k-2}(\alpha - 1)E_{s_j,s_j+1},$$

$$Y = E_{12} + E_{n-1,n} + \sum_{j=1}^{k-2} E_{s_j,s_j+1} \quad \text{and} \quad Z = -\alpha Y.$$

Then $E_{11} + E_{nn} = X + Y + Z$ is the sum of three rank k matrices $X, Y, Z \in T_n(\mathbb{F})$ among which the sum of any two is of rank k.

When $1 \leqslant i < p < n$, we select distinct integers $s_1, \ldots, s_{k-2} \in \{1, \ldots, n-1\} \backslash \{i, p\}$ and $\alpha \in \mathbb{F} \backslash \{0, 1\}$. Set

$$X_{ip} = E_{ii} + (\alpha - 1)E_{i,i+1} + E_{pp} + (\alpha - 1)E_{p,p+1} + \sum_{j=1}^{k-2} (\alpha - 1)E_{s_j, s_j+1},$$

$$Y_{ip} = E_{i,i+1} + E_{p,p+1} + \sum_{j=1}^{k-2} E_{s_j, s_j+1} \quad \text{and} \quad Z_{ip} = -\alpha Y_{ip}.$$

Then $E_{ii} + E_{pp} = X_{ip} + Y_{ip} + Z_{ip}$ is the sum of three rank k matrices $X_{ip}, Y_{ip}, Z_{ip} \in T_n(\mathbb{F})$ among which the sum of any two is of rank k.

When $1 < i < p = n$, since $E_{ii} + E_{nn} = (E_{11} + E_{n+1-i,\, n+1-i})^+$, we have $E_{ii} + E_{nn} = X^+_{1,n+1-i} + Y^+_{1,n+1-i} + Z^+_{1,n+1-i}$ as desired.

Case B: $i = j$ or $p = q$. When $i = j$ and $p < q$, we have $1 \leqslant i < p < q \leqslant n$. We select distinct integers $s_1, \ldots, s_{k-2} \in \{1, \ldots, n\} \backslash \{i, p, q\}$ and $\alpha \in \mathbb{F} \backslash \{0, 1\}$. Set

$$X_{ipq} = E_{ii} + (\alpha - 1)E_{ip} + E_{pq} + (\alpha - 1)E_{qq} + \sum_{j=1}^{k-2} (\alpha - 1)E_{s_j, s_j},$$

$$Y_{ipq} = E_{ip} + E_{qq} + \sum_{j=1}^{k-2} E_{s_j, s_j} \quad \text{and} \quad Z_{ipq} = -\alpha Y_{ipq}.$$

Then $E_{ii} + E_{pq} = X_{ipq} + Y_{ipq} + Z_{ipq}$ is the sum of three rank k matrices $X_{ipq}, Y_{ipq}, Z_{ipq} \in T_n(\mathbb{F})$ among which the sum of any two is of rank k.

When $i < j$ and $p = q$, we have either $1 \leqslant i < j < p \leqslant n$ or $1 \leqslant i < p < j \leqslant n$. Consider the case $1 \leqslant i < j < p \leqslant n$. Note that $E_{ij} + E_{pp} = (E_{hh} + E_{st})^+$, where $h = n + 1 - p$, $s = n + 1 - j$ and $t = n + 1 - i$, with $1 \leqslant h < s < t \leqslant n$. It follows that $E_{ij} + E_{pp} = X^+_{hst} + Y^+_{hst} + Z^+_{hst}$ is the sum of three rank k matrices $X_{hst}, Y_{hst}, Z_{hst} \in T_n(\mathbb{F})$ among which the sum of any two is of rank k as required. Consider now the case $1 \leqslant i < p < j \leqslant n$. We select distinct integers $s_1, \ldots, s_{k-2} \in \{1, \ldots, n\} \backslash \{i, p, j\}$ and $\alpha \in \mathbb{F} \backslash \{0, 1\}$. Let

$$X = E_{pp} + E_{ij} + (\alpha - 1)E_{ip} + (\alpha - 1)E_{jj} + \sum_{j=1}^{k-2} (\alpha - 1)E_{s_j, s_j},$$

$$Y = E_{ip} + E_{jj} + \sum_{j=1}^{k-2} E_{s_j, s_j} \quad \text{and} \quad Z = -\alpha Y.$$

Then $E_{pp}+E_{ij}=X+Y+Z$ is the sum of three rank k matrices $X,Y,Z\in T_n(\mathbb{F})$ among which the sum of any two is of rank k.

Case C: $i<j$ and $p<q$. We select distinct integers $s_1,\ldots,s_{k-2}\in\{1,\ldots,n\}\backslash\{i,p\}$ and $\alpha\in\mathbb{F}\backslash\{0,1\}$. Let

$$X=E_{ij}+(\alpha-1)E_{ii}+E_{pq}+(\alpha-1)E_{pp}+\sum_{j=1}^{k-2}(\alpha-1)E_{s_j,s_j},$$

$$Y=E_{ii}+E_{pp}+\sum_{j=1}^{k-2}E_{s_j,s_j}\quad\text{and}\quad Z=-\alpha Y.$$

Then $E_{ij}+E_{pq}=X+Y+Z$ is the sum of three rank k matrices $X,Y,Z\in T_n(\mathbb{F})$ among which the sum of any two is of rank k. We are done. □

Lemma 2.1.3. *Let $\mathbb{F}$ be a field. If $A\in T_n(\mathbb{F})$ is such that $[A,E_{ij}]=0$ for all $1\leqslant i\leqslant j\leqslant n$, then $A=\lambda I_n$ for some $\lambda\in\mathbb{F}$. In particular, the centre Z of $T_n(\mathbb{F})$ is $\mathbb{F}I_n$.*

Proof. Let $A=(a_{ij})\in T_n(\mathbb{F})$. Note that $AE_{st}=E_{st}A$ leads to

$$E_{kk}(AE_{st})=E_{kk}(E_{st}A)\ \Rightarrow\ a_{ks}E_{kt}=\delta_{ks}E_{kt}A$$

for every integer $1\leqslant k\leqslant s\leqslant t\leqslant n$, where δ_{ks} is the Kronecker symbol. In particular, $a_{ks}E_{st}=0$ when $k<s$. Hence A is diagonal. Thus $E_{kt}A=a_{kk}E_{kt}$, and so $a_{tt}=a_{kk}$ for all $1\leqslant k\leqslant t\leqslant n$. So $A=\lambda I_n$ for some $\lambda\in\mathbb{F}$. Hence $Z(T_n(\mathbb{F}))=\mathbb{F}I_n$. □

Next, we establish two lemmas that will be employed in the study of commuting additive maps on invertible upper triangular matrices. In what follows, let $n\geqslant 2$ be an integer and let $\mathbb{F}_2$ denote the Galois field of two elements. In view of Lemma 2.1.3, we note that the centre of $T_n(\mathbb{F}_2)$ is $Z(T_n(\mathbb{F}_2))=\mathbb{F}_2I_n$. For each integer $0\leqslant\ell\leqslant n-1$, we denote

$$D_\ell=\sum_{i=1}^{n-\ell}E_{i,i+\ell}\in T_n(\mathbb{F}_2). \tag{2.1}$$

In particular, $D_0=I_n$ and $D_{n-1}=E_{1n}$.

Let $A=(a_{ij})\in T_n(\mathbb{F}_2)$ and let $1\leqslant s\leqslant n-\ell$ be an integer. Notice that $AE_{s,s+\ell}=\sum_{i=1}^{s}a_{is}\,E_{i,s+\ell}$ and $E_{s,s+\ell}\,A=\sum_{j=s+\ell}^{n}a_{s+\ell,j}E_{sj}$. It follows from (2.1) that

$$AD_\ell=\sum_{s=1}^{n-\ell}\left(\sum_{t=1}^{s}a_{ts}\,E_{t,\ell+s}\right)=\sum_{s=0}^{n-\ell-1}\ \sum_{i=1}^{n-\ell-s}a_{i,\,i+s}\,E_{i,\,i+\ell+s},$$

$$D_\ell A = \sum_{s=1}^{n-\ell}\left(\sum_{t=\ell+s}^{n} a_{\ell+s,t}E_{st}\right) = \sum_{s=0}^{n-\ell-1}\sum_{i=1}^{n-\ell-s} a_{i+\ell,\,i+\ell+s}\,E_{i,\,i+\ell+s}.$$

In particular, for any integers $0 \leqslant s, t \leqslant n-1$, we have

$$D_sD_t = \begin{cases} \sum_{i=1}^{n-(s+t)} E_{i,\,i+s+t} & \text{when } s+t<n, \\ 0 & \text{when } s+t \geqslant n. \end{cases}$$

Then $D_sD_t = D_{s+t}$ when $s+t<n$. Furthermore, since $D_s = (D_1)^s$, we have

$$D_sD_t = D_{s+t} = (D_1)^{s+t}$$

when $s+t<n$, and $D_sD_t = 0 = (D_1)^{s+t}$ when $s+t \geqslant n$. Consequently, $[D_s, D_t] = 0$ for every integer $0 \leqslant s, t \leqslant n-1$, where $[X, Y]$ is the commutator of $X, Y \in T_n(\mathbb{F}_2)$. This proves the following result.

Let $A = \sum_{i=0}^{n-1}\alpha_i D_i$ and $B = \sum_{i=0}^{n-1}\beta_i D_i$ be upper triangular Toeplitz matrices for some $\alpha_i, \beta_i \in \mathbb{F}_2$. By the bilinearity of $[\,\cdot\,,\,\cdot\,]$, we obtain

$$[A, B] = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\alpha_i\beta_j[D_i, D_j] = 0.$$

We summarise the observation as a lemma.

Lemma 2.1.4. *Let $A = \sum_{i=0}^{n-1}\alpha_i D_i$ and $B = \sum_{i=0}^{n-1}\beta_i D_i$ be Toeplitz matrices in $T_n(\mathbb{F}_2)$. Then $[A, B] = 0$.*

Lemma 2.1.5. *Let $B = D_1 + \alpha D_\ell$ for some integer $1 < \ell < n$ and let $\alpha \in \mathbb{F}_2$. If $A \in T_n(\mathbb{F}_2)$ satisfies $[A, B] = 0$, then $A = \sum_{i=0}^{n-1}\lambda_i D_i$ for some $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{F}_2$.*

Proof. When $n = 2$, the result is clear since $B = E_{12} \in T_2(\mathbb{F}_2)$. We now consider $n \geqslant 3$. Recall that $1 < \ell < n$ and $\alpha \in \mathbb{F}_2$ are fixed. Let $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. Note that

$$[A, D_1] = \sum_{s=0}^{n-2}\sum_{i=1}^{n-1-s}(a_{i,\,i+s} + a_{i+1,\,i+1+s})E_{i,\,i+1+s},$$

$$
\begin{aligned}
[A, D_\ell] &= \sum_{s=0}^{n-\ell-1} \sum_{i=1}^{n-\ell-s} (a_{i,\,i+s} + a_{i+\ell,\,i+\ell+s}) E_{i,\,i+\ell+s} \\
&= \sum_{s=\ell-1}^{n-2} \sum_{i=1}^{n-1-s} (a_{i,\,i-\ell+1+s} + a_{i+\ell,\,i+1+s}) E_{i,\,i+1+s}.
\end{aligned}
$$

Since $[A, B] = 0$, it follows that $[A, D_1] = \alpha[D_\ell, A]$. We thus obtain

$$
\sum_{s=0}^{\ell-2} \sum_{i=1}^{n-1-s} (a_{i,i+s} + a_{i+1,\,i+1+s}) E_{i,i+1+s} + \sum_{s=\ell-1}^{n-2} \sum_{i=1}^{n-1-s} \left((a_{i,\,i+s} + a_{i+1,\,i+1+s}) + \alpha(a_{i,\,i-\ell+1+s} + a_{i+\ell,\,i+1+s})\right) E_{i,\,i+1+s} = 0.
$$

It follows that $\sum_{s=0}^{\ell-2} \sum_{i=1}^{n-1-s} (a_{i+1,\,i+1+s} + a_{i,i+s}) E_{i,i+1+s} = 0$, which leads to

$$
a_{i,i+s} = a_{i+1,\,i+1+s}. \tag{2.2}
$$

It follows that for every integer $0 \leqslant s \leqslant \ell - 2$ and $1 \leqslant i \leqslant n - 1 - s$,

$$
\sum_{s=\ell-1}^{n-2} \sum_{i=1}^{n-1-s} \left((a_{i,\,i+s} + a_{i+1,\,i+1+s}) + \alpha(a_{i,\,i-\ell+1+s} + a_{i+\ell,\,i+1+s})\right) E_{i,\,i+1+s} = 0. \tag{2.3}
$$

Consider $s = \ell - 1$. By the result of (2.2), i.e., $a_{ii} = a_{i+1,i+1}$ for $i = 1, \ldots, n-1$, we get

$$
a_{i,\,i-\ell+1+s} = a_{ii} = a_{i+\ell,\,i+\ell} = a_{i+\ell,\,i+1+s}
$$

for every $i = 1, \ldots, n - 1 - s$. In view of (2.3), for $s = \ell - 1$, we obtain

$$
a_{i,\,i+s} = a_{i+1,\,i+1+s} \tag{2.4}
$$

for every $i = 1, \ldots, n - 1 - s$. Consequently, Equation (2.3) is reduced to

$$
\sum_{s=\ell}^{n-2} \sum_{i=1}^{n-s-1} \left((a_{i,\,i+s} + a_{i+1,\,i+1+s}) + \alpha(a_{i,\,i-\ell+1+s} + a_{i+\ell,\,i+1+s})\right) E_{i,\,i+1+s} = 0. \tag{2.5}
$$

We now consider $s = \ell$. By the result of (2.2) or (2.4), i.e., $a_{i,i+1} = a_{i+1,i+2}$ for $i = 1, \ldots, n-2$, we obtain

$$
a_{i,\,i-\ell+1+s} = a_{i,i+1} = a_{i+\ell,\,i+\ell+1} = a_{i+\ell,\,i+1+s}
$$

for every $i = 1, \ldots, n-1-s$. In view of (2.5), for $s = \ell$, we get

$$a_{i,\, i+s} = a_{i+1,\, i+1+s}$$

for every $i = 1, \ldots, n-1-s$. Hence Equation (2.5) is reduced to

$$\sum_{s=\ell+1}^{n-2} \sum_{i=1}^{n-s-1} \left((a_{i,\, i+s} + a_{i+1,\, i+1+s}) + \alpha (a_{i,\, i-\ell+1+s} + a_{i+\ell,\, i+1+s}) \right) E_{i,\, i+1+s} = 0. \quad (2.6)$$

Continuing this process in (2.6), we thus obtain $a_{i,\, i+s} = a_{i+1,\, i+1+s}$ for all integers $0 \leqslant s \leqslant n-1$ and $1 \leqslant i \leqslant n-1-s$. Therefore $A = \sum_{i=0}^{n-1} \lambda_i D_i$ for some $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{F}_2$. The lemma is proved. □

2.2 Literature review

2.2.1 Linear preserver problems on matrices

Linear preserver problems on matrices represent an active and continuing research subject in matrix theory that deals with the characterisation of linear maps on matrices leaving certain properties or relation of subsets invariant. The formulation of linear preserver problems is natural and simple. The study often gives a deeper understanding of the matrix functions, relations or identities under consideration. Let $\mathcal{M}$ be a linear space of matrices. The study of linear preserver problems usually falls into some of the following typical problems.

(i) **Type I:** Suppose that ϕ is a scalar-valued, vector-valued, or set-valued function on $\mathcal{M}$. The aim of this type of problem is to characterise linear maps $\psi : \mathcal{M} \to \mathcal{M}$ preserving the function ϕ, i.e.,

$$\phi(\psi(A))) = \phi(A)$$

for all $A \in \mathcal{M}$. One of the classical examples in this type of linear preserver problems is the result of Frobenius (1897) which characterised bijective linear operators on complex matrices that preserve the determinant. Let $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an invertible linear map satisfying $\det(\psi(A)) = \det(A)$ for all $A \in M_n(\mathbb{C})$. Frobenius (1897) proved that there exist invertible matrices P and Q in $M_n(\mathbb{C})$ with $\det(PQ) = 1$ such that either

$$\psi(A) = PAQ \tag{2.7}$$

for all $A \in M_n(\mathbb{C})$, or

$$\psi(A) = PA^tQ \tag{2.8}$$

for all $A \in M_n(\mathbb{C})$. For more examples of this type of linear preserver problems, see, for example Botta (1967); Chooi et al. (2017); Huang et al. (2016).

(ii) **Type II:** Let S be a nonempty subset of $\mathcal{M}$. In this type of linear preserver problems, the aim is to determine the structure of linear maps $\psi : \mathcal{M} \to \mathcal{M}$ leaving the subset S invariant, i.e.,

$$\psi(S) \subseteq S.$$

Let $1 \leqslant k \leqslant n$ be an integer and let R_k denote the totality of rank k complex matrices of $M_n(\mathbb{C})$. Marcus and Moyls (1959) characterised linear maps $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ that preserve rank k matrices, i.e., $\psi(R_k) \subseteq R_k$. They showed that there exist invertible matrices P and Q in $M_n(\mathbb{C})$ such that either

$$\psi(A) = PAQ$$

for all $A \in M_n(\mathbb{C})$, or

$$\psi(A) = PA^tQ$$

for all $A \in M_n(\mathbb{C})$. For more examples of Type II linear preserver problems, see, for instance Costara (2020); Marcus and Purves (1959); Omladič and Šemrl (1998); Song et al. (2016).

(iii) **Type III:** Suppose that $\sim$ is a relation on $\mathcal{M}$. The aim of this type of linear preserver problems is to classify linear maps $\psi : \mathcal{M} \to \mathcal{M}$ satisfying the following relation:

$$\psi(A) \sim \psi(B) \quad \text{whenever} \quad A \sim B \text{ in } \mathcal{M},$$

or

$$\psi(A) \sim \psi(B) \quad \text{if and only if} \quad A \sim B \text{ in } \mathcal{M}.$$

Pierce and Watkins (1978) classified invertible linear maps $\psi : M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$ preserving commutativity, i.e., $\psi(A)\psi(B) = \psi(B)\psi(A)$ whenever $AB = BA$ for all $A, B \in M_n(\mathbb{F})$. They proved that there exists an invertible matrix P in $M_n(\mathbb{F})$, $\lambda \in \mathbb{F}$ and a linear functional f on $M_n(\mathbb{F})$ such that either

$$\psi(A) = \lambda P^{-1}AP + f(A)I_n \tag{2.9}$$

for all $A \in M_n(\mathbb{F})$, or

$$\psi(A) = \lambda P^{-1}A^tP + f(A)I_n \tag{2.10}$$

for all $A \in M_n(\mathbb{F})$. More examples of Type III linear preserver problems can be referred to the works, for example Chebotar et al. (2003); Hiai (1987); Petek and Radiċ (2020).

(iv) **Type IV:** Let $G : \mathcal{M} \rightarrow \mathcal{M}$ be a matrix function. The aim of this type of linear preserver problems is to study the structure of linear maps $\psi : \mathcal{M} \rightarrow \mathcal{M}$ that commute with the matrix function G, i.e.,

$$G(\psi(A)) = \psi(G(A))$$

for all $A \in \mathcal{M}$. Let $G(A) = \text{adj}A$, the classical adjoint of $A \in M_n(\mathbb{C})$. Sinkhorn (1982) initiated the study of classical adjoint-commuting linear maps $\psi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ satisfying $\psi(\text{adj}(A)) = \text{adj}(\psi(A))$ for all $A \in M_n(\mathbb{C})$. He showed that there exists an invertible matrix P in $M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$ with $\lambda^{n-2} = 1$ such that either

$$\psi(A) = \lambda PAP^{-1}$$

for all $A \in M_n(\mathbb{C})$, or

$$\psi(A) = \lambda PA^tP^{-1}$$

for all $A \in M_n(\mathbb{C})$. For more examples of this type of linear preserver problems, the reader may referred to see, for instance Chan and Lim (1992); Chooi and Ng (2010); Khachorncharoenkul et al. (2020).

The literature on linear preserver problems is vast and rich. For an extensive expository survey of the subject and its developments, we refer the reader to the papers in Linear and Multilinear Algebra (volume 33, no.1-2 (1992), pp.1-119) in Pierce et al. (1992), or to the expository papers in Guterman et al. (2000); C.-K. Li and Pierce (2001); C.-K. Li and Tsing (1992); Mbekhta (2012), or to the books/book chapter in Molnár (2007); Šemrl (2014); Zhang et al. (2007) and references therein. Many works have been done in linear preserver problems related to the theory of functional identities. For works in this area, see, for example Beidar et al. (2002), (Brešar et al., 2007, Chapter 7) and (Brešar, 2004, Section 5.3).

It is worth mentioning that linear preservers $\psi : \mathcal{M} \to \mathcal{M}$ of the form (2.7), (2.8), (2.9) or (2.10) are said to be of the *standard form* in the study of linear preserver problems. Nevertheless, in some situations, it is interesting to discover linear preservers which are of irregular form or nonstandard form. Chooi and Lim (1998) studied some linear preservers on upper triangular matrices and discovered that the structure of linear preservers on upper triangular matrices is more fertile and complex than the corresponding result on full matrices. A subspace H of $T_n(\mathbb{F})$ is a called a rank one subspace if $A \in H$ implies that either $A = 0$ or A is of rank one. They classified linear maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ preserving rank one upper triangular matrices and showed that either Im ψ is an n-dimensional rank one subspace, or there exist invertible matrices $P, Q \in T_n(\mathbb{F})$ such that

$$\psi(A) = PAQ$$

for all $A \in T_n(\mathbb{F})$, or

$$\psi(A) = PA^+Q$$

for all $A \in T_n(\mathbb{F})$. Here, $A^+ = J_nA^tJ_n$ and $J_n \in M_n(\mathbb{F})$ is the matrix with one on the minor diagonal and zero elsewhere. In the paper, they characterised linear maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ that preserve determinant and showed that there exists a permutation σ

of degree n and nonzero scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ with $\lambda_1 \cdots \lambda_n = 1$ such that

$$[\psi(A)]_{ss} = \lambda_s a_{\sigma(s),\sigma(s)}, \quad s = 1, \ldots, n$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$. Molnár and Šemrl (1998) studied bijective linear maps preserving rank one idempotents, linear maps preserving commutativity in both directions and bijective linear maps preserving commutativity, on upper triangular matrices. Bell and Sourour (2000) characterised surjective additive maps preserving rank one block upper triangular matrices as well as additive maps preserving rank one matrices in both directions on block upper triangular matrices. Since then, there has been considerable interest in studying linear preservers on $T_n(\mathbb{F})$, see, for example rank one nonincreasing linear maps in Chooi and Lim (2001), linear maps preserving numerical range in C.-K. Li et al. (2001), linear maps preserving generalised numerical ranges in Cheung and Li (2001) and coherence invariant maps in Chooi and Lim (2002). The study of linear preservers on $T_n(\mathbb{F})$ is more challenging and the structures are usually more complicated than the corresponding result on $M_n(\mathbb{F})$.

2.2.2 Functional identities

A functional identity on a ring $\mathcal{R}$ is an identical relation holding for all elements in $\mathcal{R}$ which involves some functions on $\mathcal{R}$. The goal in the study of functional identities is to determine the form of functions satisfying certain identities, or, when this is not possible, to describe the structure of the ring admitting the functional identity in question. The theory of functional identities is a relatively new subject whose roots lie in the Ph.D. thesis of Brešar in the year 1990.

Let $\mathcal{R}$ be a ring and let $f, g : \mathcal{R} \to \mathcal{R}$ be maps such that

$$f(x)y + g(y)x = 0 \tag{2.11}$$

for all $x, y \in \mathcal{R}$. This is a very basic example of a functional identity. A trivial solution when (2.11) is fulfilled is when $f = g = 0$. If $\mathcal{R}$ is commutative, then nontrivial solutions when (2.11) holds are f is the identity function and $g = -f$. Let $\mathbb{D}$ be a division ring.

Brešar (1995b) proved the following theorem.

Theorem 2.2.1. (Brešar, 1995b, Lemma 4.5) *If* $f_1, f_2, f_3, f_4 : \mathbb{D} \to \mathbb{D}$ *are additive maps satisfying*

$$f_1(x)y + xf_2(y) + f_3(y)x + yf_4(x) = 0 \tag{2.12}$$

for all $x, y \in \mathbb{D}$*, then there exist additive maps* $\mu, \eta : \mathbb{D} \to Z(\mathbb{D})$ *and* $a, b \in \mathbb{D}$ *such that*

$$\begin{aligned} f_1(x) &= -xa + \mu(x), \\ f_2(x) &= ax - \eta(x), \\ f_3(x) &= -xb + \eta(x), \\ f_4(x) &= bx - \mu(x) \end{aligned} \tag{2.13}$$

for all $x \in \mathbb{D}$.

For more studies on functional identities, see, for example Brešar (1995a, 2016, 2020); Brešar et al. (2015); Brešar and Špenko (2014, 2015); Catalano (2018); Cezayirlioğlu and Demir (2021); Dar and Jing (2022); Han (2017); T.-K. Lee (2019); Wang (2013). Also, the study of functional identities on triangular matrix rings can be found in, for example Beidar et al. (2000); Eremita (2013, 2015, 2016); Wang (2015, 2016b, 2019); Yuan and Chen (2020). For a full account on functional identities and its historical developments, we refer the reader to the book 'Functional Identities' by Brešar, Chebotar and Martindale 3rd in Brešar et al. (2007) and the survey paper in Brešar (2000).

The theory of functional identities is closely related to the study of commuting maps, centralizing maps and power commuting maps on rings. One of the earliest results in the study of commuting maps is Posner's theorem (Posner, 1957, Theorem 2) which states that a prime ring admitting a nonzero commuting derivation must be commutative. In 1993, Brešar first described the structures of commuting additive maps on a prime ring and centralizing additive maps on a prime ring of characteristic not two. He proved the following results.

Theorem 2.2.2. (Brešar, 1993a, Theorem 3.2) *Let* $\mathcal{R}$ *be a prime ring. Suppose that an additive mapping* $\psi : \mathcal{R} \mapsto \mathcal{R}$ *is commuting on* $\mathcal{R}$*. Then there exists an additive map*

$\mu : \mathcal{R} \to \mathcal{C}$ and an element λ in the extended centroid $\mathcal{C}$ of $\mathcal{R}$ such that

$$\psi(A) = \lambda A + \mu(A) \tag{2.14}$$

for all $A \in \mathcal{R}$.

Theorem 2.2.3. (Brešar, 1993a, Theorem A) *Let $\mathcal{R}$ be a prime ring of characteristic not two. If an additive map $\psi : \mathcal{R} \mapsto \mathcal{R}$ is centralizing on $\mathcal{R}$, then there exists an additive map $\mu : \mathcal{R} \to \mathcal{C}$ and an element λ in the extended centroid $\mathcal{C}$ of $\mathcal{R}$ such that*

$$\psi(A) = \lambda A + \mu(A) \tag{2.15}$$

for all $A \in \mathcal{R}$.

The form as described in (2.14) or (2.15) is called the *standard form*. These results have been generalised in several directions, see, for example commuting maps in Beidar (1998); Brešar and Miers (1995); Costara (2021); Lapuangkham and Leerawat (2021); P.-H. Lee and Lee (1997); T.-K. Lee and Lee (1996); Xiao and Wei (2010), and centralizing and related maps in Ara and Mathieu (1993); Brešar et al. (1993); P.-H. Lee and Wang (2009); T.-C. Lee (1998); T.-K. Lee (1997); Y. Li and Wei (2012); Qi (2016); Wang (2016b). For a full account on commuting maps and its historical developments, we refer the reader to the survey paper by Brešar (2004). The study of commuting additive maps on triangular algebras was initiated by Cheung (2001). He showed that the structure of commuting linear maps on triangular algebras is of the standard form (2.14). Beidar et al. (2000) studied commuting linear maps on upper triangular matrices over fields. Eremita (2017) investigated commuting additive maps on upper triangular matrices over unital rings. On the other hand, Brešar's structural results are extremely influential and have stimulated considerable study in some related maps on various algebras, rings and matrix spaces such as: m-power commuting maps in Ahmed (2019); Beidar et al. (1997); Brešar and Hvala (1995); Chacron (2021); Chacron and Lee (2019); Franca and Louza (2021); Inceboz et al. (2016); T.-K. Lee et al. (2004); Qi (2016); Słowik and Ahmed (2021), skew-commuting maps in Brešar (1993c); Fošner (2015); Park and Jung (2002), m-commuting maps in Brešar (1992, 1996); Du and Wang (2012); Y. Li et al. (2019); C.-K. Liu (2020); C.-K. Liu and Pu (2021); Qi and Hou (2015); Xiao and Yang (2021), strong commutativity preserving maps in Brešar and Miers (1994); Chen and Zhao (2021); T.-K. Lee and Wong (2012); Qi and Hou (2012) and commuting traces maps in Benkovič and Eremita (2004); Brešar (1993b); Brešar and Šemrl (2003); Eremita (2017); Franca (2015, 2016); Franca and Louza (2018); P.-H. Lee et al. (1997); Wang (2016a).

In 2012, inspired by the study of linear preserver problems on sets of matrices that are not closed under addition and the structural result in (Brešar, 1993a, Theorem 3.2), Franca (2012) initiated the study of commuting additive maps $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ on invertible (respectively, singular) $n \times n$ matrices over a field $\mathbb{F}$. He showed ψ is of the standard form (2.14). This result has been generalised by Franca (2013a) and Xu and Yi (2014) for commuting additive maps $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ on rank k matrices for some fixed integer $1 < k \leqslant n$. Subsequently, the description of commuting additive maps on rank one matrices over fields and on rank one matrices over noncommutative division rings was obtained in Franca (2017); Franca and Louza (2017), respectively. Extending Franca's results, C.-K. Liu (2014a) studied centralizing additive maps on the set of singular and invertible matrices. Chooi, Mutalib, and Tan (2021) characterised centralizing additive maps on rank k block triangular matrices over fields. C.-K. Liu (2014b); C.-K. Liu et al. (2018) advanced the study of strong commutativity preserving maps to rank k matrices. Chooi and Wong (2021) gave a characterisation of commuting additive maps on tensor products of matrix algebras over fields. More recently, Chooi and Tan (2021) successfully described commuting additive maps on rank k symmetric matrices over a field of characteristic not two. Many interesting results have been obtained, see, for example Chooi and Kwa (2019, 2020); Franca (2013b); Franca and Louza (2019); H. Liu and Xu (2017); Xu and Liu (2017); Xu et al. (2016); Xu and Zhu (2018). Lately, m-power commuting additive maps have been generalised to set of matrices that are not closed under addition in C.-K. Liu and Yang (2017) and Chou and Liu (2021).

2.3 Methodology

Our approaches of research methodology comprises of three components.

The first component is preliminary background review to identify essential research problems. We study and understand specific matrix theoretic techniques and ideas employed in determining the commuting maps on fixed-rank square matrices. We then identify suitable research problems on commuting additive maps on fixed-rank triangular matrices over certain underlying rings for further investigation.

The second component involves mathematical calculations and analysis. We perform a series of careful algebraic calculation via basis constructive approach. This is followed by an analysis of trait and pattern of the computation to derive useful information in planning the sequence of our proof.

The last component is characterisation and refinement of maps. We formulate theorems and prove the structural results with rigorous and valid mathematical arguments. By factorising the additive group generators, we scrutinise and refine the commuting additive map obtained to ensure its form is the simplest and ultimate.

CHAPTER 3: COMMUTING ADDITIVE MAPS ON RANK *k* UPPER TRIANGULAR MATRICES OVER FIELDS

3.1 Introduction

Let $\mathcal{R}$ be a ring with centre $Z(\mathcal{R})$ and let $\mathcal{S}$ be a nonempty subset of $\mathcal{R}$. Let $\psi : \mathcal{R} \to \mathcal{R}$ be a map. Recall that a map ψ is commuting on $\mathcal{S}$ if $[\psi(A), A] = 0$ for all $A \in \mathcal{S}$, where $[X, Y]$ is the commutator of $X, Y \in \mathcal{R}$. The study of commuting additive maps was recently extended to subsets of matrices that are not closed under addition. Let $n \geqslant 2$ be an integer and let $M_n(\mathbb{F})$ denote the ring of all $n \times n$ matrices over a field $\mathbb{F}$ with unity I_n. Franca (2012) proved that if $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is an additive map that satisfies $[\psi(A), A] = 0$ for all invertible (singular) matrices $A \in M_n(\mathbb{F})$, then there exists a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : M_n(\mathbb{F}) \to \mathbb{F}$, such that ψ is of the standard form $\psi(A) = \lambda A + \mu(A)I_n$ for all $A \in M_n(\mathbb{F})$, except when $\mathbb{F} = \mathbb{F}_2$, the Galois field of two elements. Let $n \geqslant 3$ be an integer. Fix an integer $2 \leqslant k \leqslant n - 1$. Under the assumption char $\mathbb{F} = 0$ or char $\mathbb{F} > 3$, Franca (2013a) proved that commuting additive maps $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ on all rank k matrices are of the standard form (2.14) i.e. there exists $\lambda \in \mathbb{F}$ and an additive map $\mu : M_n(\mathbb{F}) \to \mathbb{F}$ such that $\psi(A) = \lambda A + \mu(A)I_n$ for all $A \in M_n(\mathbb{F})$. Improving Franca's result (Franca, 2013a, Theorem 3), Xu and Yi (2014) gave a new proof for commuting additive maps $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ on rank k matrices, for fixed integers $2 \leqslant k \leqslant n$, by getting rid of the assumption char $\mathbb{F} \neq 2, 3$. Recall that $T_n(\mathbb{F})$ is the ring of $n \times n$ upper triangular matrices over $\mathbb{F}$ with centre $Z(T_n(\mathbb{F}))$ and unity I_n. Inspired by the aforementioned results, in this chapter, we successfully address the question of describing the form of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank k matrices with $|\mathbb{F}| \geqslant 3$ and $2 \leqslant k \leqslant n$ a fixed integer. The result highlights that ψ is "almost" of the standard form (2.14) when $|\mathbb{F}| = 3$ and $k = n$.

3.2 Irregular nonstandard examples

We characterise commuting additive maps on rank k, $2 \leqslant k \leqslant n$, upper triangular matrices over fields of at least three elements. Surprisingly, unlike the case of commuting additive maps of $M_n(\mathbb{F})$ on rank k matrices, it turns out that ψ is not necessarily of the standard form (2.14) when $k = n$ and $\mathbb{F}$ is the Galois field of three elements. For instance, consider the additive map $\varphi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is defined to be

$$\varphi(A) = (a_{11} + a_{nn})E_{1n}$$

for every $A = (a_{ij}) \in T_n(\mathbb{F})$. Here, $E_{ij} \in T_n(\mathbb{F})$ denotes the standard matrix unit whose (i, j)-th entry is one and zero elsewhere. Let $A = (a_{ij}) \in T_n(\mathbb{F})$ be invertible. Then $a_{11}, a_{nn} \neq 0$. If $|\mathbb{F}| = 3$, then $a_{11} + a_{nn} = 0$ whenever $a_{11} \neq a_{nn}$. Thus $(a_{11} + a_{nn})a_{nn} = a_{11}(a_{11} + a_{nn})$, and so

$$\begin{aligned}\varphi(A)A &= (a_{11} + a_{nn})E_{1n}A \\ &= (a_{11} + a_{nn})a_{nn}E_{1n} \\ &= a_{11}(a_{11} + a_{nn})E_{1n} \\ &= A(a_{11} + a_{nn})E_{1n} \\ &= A\varphi(A).\end{aligned}$$

Hence φ is a commuting additive map on rank n triangular matrices when $|\mathbb{F}| = 3$.

We now give some examples to show the indispensability of the conditions $k \geqslant 2$ and $|\mathbb{F}| \geqslant 3$ in Theorem 3.3.1.

Example 3.2.1. Let $\mathbb{F}$ be a field and let $n \geqslant 3$ be an integer. Let $\psi_i : T_n(\mathbb{F}) \to T_n(\mathbb{F})$, $i = 1, 2$, be the additive maps defined by

$$\psi_1(A) = a_{pq}E_{1n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where p and q are fixed integers satisfying $1 < p \leqslant q < n$; and

$$\psi_2(A) = a_{22}E_{13} + a_{33}E_{24} + \cdots + a_{n-1,n-1}E_{n-2,n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$. We show that each additive map ψ_i is commuting on rank one matrices. Let $X = (x_{ij}) \in T_n(\mathbb{F})$ be of rank one. As we see in Lemma 2.1.1, there exist invertible matrices $P, Q \in T_n(\mathbb{F})$ such that $X = PE_{st}Q$ for some integers $1 \leqslant s \leqslant t \leqslant n$. Since P and Q are invertible upper triangular matrices, it follows that when $(s, t) = (h, h)$

for some $1 \leqslant h \leqslant n$, we have $x_{hh} \neq 0$, $x_{ij} = 0$ for every $1 \leqslant i \leqslant j < h$ and $x_{ij} = 0$ for every $h < i \leqslant j \leqslant n$.

Consider first the additive map ψ_1. When $(s,t) \in \{(1,1),(n,n)\}$, we have $\psi_1(X) = 0$, and so $[\psi_1(X), X] = 0$. When $(s,t) \notin \{(1,1),(n,n)\}$, we have $x_{11} = x_{nn} = 0$. Thus

$$\psi_1(X)X = (x_{pq}E_{1n})X = 0 = X(x_{pq}E_{1n}) = X\psi_1(X).$$

Next, we consider the additive map ψ_2. When $(s,t) \in \{(1,1),(n,n)\}$ or $1 \leqslant s < t \leqslant n$, we have $\psi_2(X) = 0$, so $[\psi_2(X), X] = 0$. When $(s,t) = (\ell,\ell)$ for some $1 < \ell < n$, we have $\psi_2(X) = x_{\ell\ell}E_{\ell-1,\ell+1}$. Since $E_{\ell-1,\ell+1}PE_{\ell\ell} = 0$ and $E_{\ell\ell}QE_{\ell-1,\ell+1} = 0$, it follows that

$$\psi_2(X)X = x_{\ell\ell}(E_{\ell-1,\ell+1}PE_{\ell\ell})Q = 0 = x_{\ell\ell}P(E_{\ell\ell}QE_{\ell-1,\ell+1}) = X\psi_2(X).$$

Consequently, both ψ_1 and ψ_2 are commuting additive maps on rank one matrices.

Example 3.2.2. Let $|\mathbb{F}| = 2$ and let $n \geqslant 3$ be an integer. Let $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ be the additive map defined by

$$\psi(A) = a_{11}E_{2n} + a_{12}E_{1n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$. Let $X = (x_{ij}) \in T_n(\mathbb{F})$ be of rank one. Then there exist invertible matrices $P, Q \in T_n(\mathbb{F})$ such that $X = PE_{st}Q$ for some integers $1 \leqslant s \leqslant t \leqslant n$. When $(s,t) \notin \{(1,1),(1,2),(2,2)\}$, we have $x_{11} = x_{12} = 0$, and so $[\psi(X), X] = 0$.

When $(s,t) = (1,1)$, we have $x_{ij} = 0$ for all $2 \leqslant i \leqslant j \leqslant n$. Thus

$$\psi(X)X = (x_{11}E_{2n}+x_{12}E_{1n})X = 0 = 2(x_{11}x_{12})E_{1n} = X(x_{11}E_{2n}+x_{12}E_{1n}) = X\psi(X).$$

When $(s,t) \in \{(1,2),(2,2)\}$, we get $x_{11} = 0 = x_{nn}$. Then

$$\psi(X)X = (x_{12}E_{1n})X = 0 = X(x_{12}E_{1n}) = X\psi(X).$$

Hence ψ is a commuting additive map on rank one matrices.

The following example shows that the commuting additive map $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank k matrices is not of the form given in (3.1) when $|\mathbb{F}| = 2$ and $k = n$.

Example 3.2.3. Let $|\mathbb{F}| = 2$ and let $\psi : T_2(\mathbb{F}) \to T_2(\mathbb{F})$ be the additive map defined by

$$\psi(A) = a_{12}(E_{11} + E_{12} + E_{22})$$

for all $A = (a_{ij}) \in T_2(\mathbb{F})$. Note that $A_1 = E_{11} + E_{22}$ and $A_2 = E_{11} + E_{22} + E_{12}$ are the only rank two matrices in $T_2(\mathbb{F})$, and $\psi(A_1)A_1 = 0 = A_1\psi(A_1)$ and $\psi(A_2)A_2 = A_1 = A_2\psi(A_2)$. Then ψ is a commuting additive map on rank two matrices.

3.3 Main results

We obtain a complete description of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank k matrices for $|\mathbb{F}| \geqslant 3$ and $2 \leqslant k \leqslant n$ which highlights that ψ are "almost" of the standard form as given in (3.1) when $|\mathbb{F}| = 3$ and $k = n$.

Theorem 3.3.1. *Let $2 \leqslant k \leqslant n$ be an integer and let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$. Let $T_n(\mathbb{F})$ be the ring of $n \times n$ upper triangular matrices over $\mathbb{F}$ with centre $Z(T_n(\mathbb{F}))$ and unity I_n. Then $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is an additive map satisfying $[\psi(A), A] = 0$ for all rank k matrices $A \in T_n(\mathbb{F})$ if and only if there exists an additive map $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ and $\lambda, \alpha \in \mathbb{F}$ in which $\alpha = 0$ when $|\mathbb{F}| > 3$ or $k < n$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n} \tag{3.1}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$.

Recall that a map $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is centralizing on rank k matrices if $[\psi(A), A] \in Z(T_n(\mathbb{F}))$ for all rank k matrices $A \in T_n(\mathbb{F})$. The following result can be found in Chooi, Mutalib, and Tan (2021).

Lemma 3.3.2. (Chooi, Mutalib, & Tan, 2021, Theorem 3.8) *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Let $1 < r \leqslant n$ be a fixed integer such that $r \neq n$ when $|\mathbb{F}| = 2$. Then $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is a centralizing additive map on rank r matrices if and only if there* *exist scalars $\lambda, \alpha \in \mathbb{F}$ and an additive map $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n}$$

for every $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\alpha \neq 0$ only if $r = n$ and $|\mathbb{F}| = 3$.

Invoking Theorem 3.3.1 and Lemma 3.3.2, we deduce a structural characterisation of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank r matrices over a field $\mathbb{F}$, where $1 < r \leqslant n$ is an integer such that $r \neq n$ when $|\mathbb{F}| = 2$.

Theorem 3.3.3. *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Let $1 < r \leqslant n$ be a fixed integer. Then $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is a commuting additive map on rank r matrices if and only if when $r < n$ or $|\mathbb{F}| \neq 2$, there exist scalars $\lambda, \alpha \in \mathbb{F}$ and an additive map $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\alpha \neq 0$ only if $r = n$ and $|\mathbb{F}| = 3$.

3.4 Proofs

Throughout this section, unless stated otherwise, let $n \geqslant 2$ be an integer and let $\mathbb{F}$ be a field. Recall that $[A, B]$ is the commutator of $A, B \in M_n(\mathbb{F})$. Our discussion begins with the following lemma by adopting an idea of (Xu & Yi, 2014, Lemma 2.5).

Lemma 3.4.1. *Let $1 \leqslant k \leqslant n$ be an integer and let $\mathscr{F}$ be a subset of $M_n(\mathbb{F})$ closed under addition. Let $\psi : \mathscr{F} \to \mathscr{F}$ be a commuting additive map on rank k matrices. If $A \in \mathscr{F}$ is a sum of three rank k matrices in $\mathscr{F}$ among which the sum of any two is of rank k, then $[\psi(A), A] = 0$.*

Proof. Let $A = X_1 + X_2 + X_3$ for some rank k matrices X_i's in $\mathscr{F}$ such that $X_i + X_j$ is of rank k for each pair of distinct integers $1 \leqslant i, j \leqslant 3$. For each $1 \leqslant i \neq j \leqslant 3$, since $[\psi(X_i + X_j), X_i + X_j] = 0$, $[\psi(X_i), X_i] = 0$ and $[\psi(X_j), X_j] = 0$, we get

$[\psi(X_j), X_i] + [\psi(X_i), X_j] = 0$. Thus

$$\begin{aligned}[\psi(A), A] &= [\psi(X_1) + \psi(X_2) + \psi(X_3), X_1 + X_2 + X_3] \\ &= \sum_{i=1}^{3}[\psi(X_i), X_i] + \sum_{1\leqslant i<j\leqslant 3}([\psi(X_j), X_i] + [\psi(X_i), X_j]) = 0\end{aligned}$$

as desired. $\square$

Lemma 3.4.2. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$ and let $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ be a commuting additive map on rank n matrices. Then there exists a unique additive map $\tau : \mathbb{F} \to \mathbb{F}$ such that*

$$\psi(\lambda I_n) + \tau(\lambda)E_{1n} \in Z(T_n(\mathbb{F}))$$

for every $\lambda \in \mathbb{F}$. Moreover, $\tau = 0$ when $|\mathbb{F}| > 3$.

Proof. Let $\lambda \in \mathbb{F}$. The result clearly holds when $\lambda = 0$. Consider $\lambda \neq 0$. Since $|\mathbb{F}| \geqslant 3$, there exists a nonzero $\alpha \in \mathbb{F}$ such that $\alpha \neq \lambda$. Let $1 \leqslant i < j \leqslant n$ and $B = E_{ij} - \alpha I_n$. Then B and $\lambda I_n + B$ are of rank n. Since $[\psi(\lambda I_n), \lambda I_n] = 0$, $[\psi(B), B] = 0$ and $[\psi(\lambda I_n + B), \lambda I_n + B] = 0$, we get $[\psi(\lambda I_n), B] + [\psi(B), \lambda I_n] = 0$. Since $[\psi(B), \lambda I_n] = 0$, it follows that

$$0 = [\psi(\lambda I_n), B] = \psi(\lambda I_n)(E_{ij} - \alpha I_n) - (E_{ij} - \alpha I_n)\psi(\lambda I_n) = \psi(\lambda I_n)E_{ij} - E_{ij}\psi(\lambda I_n).$$

Then $\psi(\lambda I_n)E_{ij} = E_{ij}\psi(\lambda I_n)$ for all $1 \leqslant i < j \leqslant n$. Note that α and $\alpha - \lambda$ are distinct nonzero scalars. When $|\mathbb{F}| > 3$, there exists a nonzero $\beta \in \mathbb{F}\backslash\{\alpha, \alpha - \lambda\}$. Let $1 \leqslant i \leqslant n$. We take $C = \beta E_{ii} - \alpha I_n \in T_n(\mathbb{F})$. Clearly, C and $\lambda I_n + C = (\lambda - \alpha)I_n + \beta E_{ii}$ are of rank n. Since $[\psi(C), \lambda I_n] = 0$, it follow that $[\psi(\lambda I_n + C), \lambda I_n + C] = 0$ yields $[\psi(\lambda I_n), C] = 0$. Therefore $\psi(\lambda I_n)E_{ii} = E_{ii}\psi(\lambda I_n)$ for all $1 \leqslant i \leqslant n$. By Lemma 2.1.3, we have $\psi(\lambda I_n) \in Z(T_n(\mathbb{F})) = \mathbb{F}I_n$. Consequently, the result follows with τ the zero map on $\mathbb{F}$.

Consider now $|\mathbb{F}| = 3$. Suppose that $\psi(\lambda I_n) = (a_{ij})$. By virtue of $\psi(\lambda I_n)E_{ij} = E_{ij}\psi(\lambda I_n)$ for every $1 \leqslant i < j \leqslant n$, we have $E_{kk}\psi(\lambda I_n)E_{ij} = E_{kk}E_{ij}\psi(\lambda I_n)$ for every

$1 \leqslant k \leqslant i < j \leqslant n$. Hence

$$a_{ki}E_{kj} = \delta_{ki}E_{kj}\psi(\lambda I_n)$$

for all $1 \leqslant k \leqslant i < j \leqslant n$. So $a_{ki} = 0$ for $1 \leqslant k < i < n$. When $k = i$, for each $1 \leqslant i < j \leqslant n$,

$$\begin{aligned} a_{ii}E_{ij} &= E_{ij}\psi(\lambda I_n) \\ &= E_{ij}\left(\sum_{s=1}^{n} a_{ss}E_{ss} + \sum_{s=1}^{n-1} a_{sn}E_{sn}\right) \\ &= \begin{cases} a_{jj}E_{ij} + a_{jn}E_{in} & \text{if } 1 < j < n, \\ a_{nn}E_{in} & \text{if } j = n. \end{cases} \end{aligned}$$

Then $a_{in} = 0$ for all $1 < i < n$, and $a_{ii} = a_{jj}$ for all $1 \leqslant i \neq j \leqslant n$. So $\psi(\lambda I_n) = a_{11}I_n + a_{1n}E_{1n}$. Consequently, there exist maps τ, $\eta : \mathbb{F} \to \mathbb{F}$ such that

$$\psi(\lambda I_n) + \tau(\lambda)E_{1n} = \eta(\lambda)I_n \in Z(T_n(\mathbb{F}))$$

for every $\lambda \in \mathbb{F}$. By the additivity of ψ and the linear independence of I_n and E_{1n}, it can be verified that τ and η are additive maps which are uniquely determined by ψ. □

Lemma 3.4.3. *Let $\mathbb{F}$ be the Galois field of three elements and let $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ be a commuting additive map on rank n matrices. If $\varphi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is the map defined by*

$$\varphi(A) = \psi(A) - \tau(a_{11} + a_{nn})E_{1n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\tau : \mathbb{F} \to \mathbb{F}$ is the additive map uniquely determined by ψ as described in Lemma 3.4.2*, then φ is a commuting additive map on rank n matrices such that $\varphi(Z(T_n(\mathbb{F}))) \subseteq Z(T_n(\mathbb{F}))$.*

Proof. Note that φ is additive by the additivity of ψ and τ. We now claim that φ is commuting on rank n triangular matrices. Let $A = (a_{ij}) \in T_n(\mathbb{F})$ be of rank n. Then $[\psi(A), A] = 0$ and

$$[\varphi(A), A] = A\tau(a_{11} + a_{nn})E_{1n} - \tau(a_{11} + a_{nn})E_{1n}A$$

$$= (a_{11}\tau(a_{11}+a_{nn}) - \tau(a_{11}+a_{nn})a_{nn})E_{1n}$$
$$= (a_{11}-a_{nn})\tau(a_{11}+a_{nn})E_{1n}.$$

Clearly, $[\varphi(A), A] = 0$ when $a_{11} = a_{nn}$. Consider $a_{11} \neq a_{nn}$. Since A is invertible, $a_{11}, a_{nn} \neq 0$. Thus $a_{11} + a_{nn} = 0$ by virtue of $|\mathbb{F}| = 3$. Hence $[\varphi(A), A] = 0$ for all rank n matrices $A \in T_n(\mathbb{F})$.

Let $X = \lambda I_n \in Z(T_n(\mathbb{F}))$ for some $\lambda \in \mathbb{F}$. Then $\varphi(X) = \psi(\lambda I_n) - \tau(\lambda+\lambda)E_{1n} = \psi(\lambda I_n) + \tau(\lambda)E_{1n} - \tau(\lambda+\lambda+\lambda)E_{1n} = \psi(\lambda I_n) + \tau(\lambda)E_{1n}$ since $|\mathbb{F}| = 3$. It follows from Lemma 3.4.2 that $\varphi(X) \in Z(T_n(\mathbb{F}))$. Hence $\varphi(Z(T_n(\mathbb{F}))) \subseteq Z(T_n(\mathbb{F}))$. □

Let $2 \leqslant k \leqslant n$ be fixed integers and let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$. Using some ideas from Franca (2012); Xu and Yi (2014), we now prove Theorem 3.3.1.

Proof of Theorem 3.3.1. The sufficiency is trivial when $|\mathbb{F}| > 3$ or $k < n$. When $|\mathbb{F}| = 3$ and $k = n$, the result follows immediately from Lemma 3.4.3.

Consider the necessity. Let $A = (a_{ij}) \in T_n(\mathbb{F})$. Then

$$[\psi(A), A] = \sum_{1\leqslant i\leqslant j\leqslant n} [\psi(a_{ij}E_{ij}), a_{ij}E_{ij}] + \sum_{(i,j)\neq(s,t)} [\psi(a_{ij}E_{ij}), a_{st}E_{st}]. \tag{3.2}$$

To prove $[\psi(A), A] = 0$, it suffices to claim

$$[\psi(a_{ij}E_{ij}), a_{ij}E_{ij}] = 0 \quad \text{and} \quad [\psi(a_{ij}E_{ij}), a_{st}E_{st}] + [\psi(a_{st}E_{st}), a_{ij}E_{ij}] = 0$$

for all $1 \leqslant i \leqslant j \leqslant n$ and $1 \leqslant s \leqslant t \leqslant n$ with $(i,j) \neq (s,t)$. We argue in the following two cases.

Case I: $1 < k < n$. Then $n \geqslant 3$. Let $A = (a_{ij}) \in T_n(\mathbb{F})$. For any $1 \leqslant i \leqslant j \leqslant n$ and $1 \leqslant s \leqslant t \leqslant n$ with $(i,j) \neq (s,t)$, the rank of $a_{ij}E_{ij} + a_{st}E_{st}$ is at most two. By Lemma 2.1.2, if $a_{ij}E_{ij} + a_{st}E_{st}$ is nonzero, then it can be expressed as a sum of three rank k matrices in $T_n(\mathbb{F})$ among which the sum of any two is of rank k. It follows from Lemma 3.4.1 that $[\psi(a_{ij}E_{ij} + a_{st}E_{st}), a_{ij}E_{ij} + a_{st}E_{st}] = 0$. Likewise, $[\psi(a_{ij}E_{ij}), a_{ij}E_{ij}] = 0 = [\psi(a_{st}E_{st}), a_{st}E_{st}]$. Hence $[\psi(a_{ij}E_{ij}), a_{st}E_{st}] + [\psi(a_{st}E_{st}), a_{ij}E_{ij}] = 0$. By (3.2), $[\psi(A), A] = 0$ for all $A \in T_n(\mathbb{F})$.

Case II: $k = n$. Let $\varphi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ be the map defined by

$$\varphi(A) = \begin{cases} \psi(A) - \tau(a_{11} + a_{nn})E_{1n} & \text{when } |\mathbb{F}| = 3, \\ \psi(A) & \text{when } |\mathbb{F}| > 3 \end{cases}$$

for every $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\tau : \mathbb{F} \to \mathbb{F}$ is the additive map uniquely determined by ψ as described in Lemma 3.4.2. Then φ is a commuting additive map of rank n matrices such that $\varphi(Z(T_n(\mathbb{F}))) \subseteq Z(T_n(\mathbb{F}))$ by Lemmas 3.4.2 and 3.4.3.

Let $A = (a_{ij}) \in T_n(\mathbb{F})$. We first claim that

$$[\varphi(a_{ij}E_{ij}), a_{ij}E_{ij}] = 0 \tag{3.3}$$

for every $1 \leqslant i \leqslant j \leqslant n$. Consider $a_{ij} \neq 0$. Since $|\mathbb{F}| \geqslant 3$, there exists $\alpha \in \mathbb{F}\backslash\{0\}$ such that $a_{ij}E_{ij} + \alpha I_n$ is of rank n. By virtue of $[\varphi(a_{ij}E_{ij} + \alpha I_n), a_{ij}E_{ij} + \alpha I_n] = 0$ and $\varphi(Z(T_n(\mathbb{F}))) \subseteq Z(T_n(\mathbb{F}))$, the claim is proved.

Next, we claim that

$$[\varphi(a_{ij}E_{ij}), a_{st}E_{st}] + [\varphi(a_{st}E_{st}), a_{ij}E_{ij}] = 0 \tag{3.4}$$

for every $1 \leqslant i \leqslant j \leqslant n$ and $1 \leqslant s \leqslant t \leqslant n$ with $(i, j) \neq (s, t)$. Note that if $i < j$ or $s < t$ or $|\mathbb{F}| > 3$, then there exists a $\beta \in \mathbb{F}\backslash\{0\}$ such that $a_{ij}E_{ij} + a_{st}E_{st} + \beta I_n$ is of rank n. By virtue of $[\varphi(a_{ij}E_{ij} + a_{st}E_{st} + \beta I_n), a_{ij}E_{ij} + a_{st}E_{st} + \beta I_n] = 0$, $\varphi(Z(T_n(\mathbb{F}))) \subseteq Z(T_n(\mathbb{F}))$ and (3.3), the claim is proved. We now consider $i = j$, $s = t$ and $|\mathbb{F}| = 3$. Note first that

$$[\varphi(E_{ii}), E_{ss}] + [\varphi(E_{ss}), E_{ii}] = 0 \tag{3.5}$$

due to the fact that $E_{ii} + E_{ss} + \gamma I_n$ is of rank n for some nonzero $\gamma \in \mathbb{F}$. Note that φ is linear when $|\mathbb{F}| = 3$. It follows from (3.5) that

$$\begin{aligned}[\varphi(a_{ii}E_{ii}), a_{ss}E_{ss}] + [\varphi(a_{ss}E_{ss}), a_{ii}E_{ii}] &= a_{ii}a_{ss}\left([\varphi(E_{ii}), E_{ss}] + [\varphi(E_{ss}), E_{ii}]\right) \\ &= 0.\end{aligned}$$

Therefore the claim (3.4) is proved. Together with the results (3.3) and (3.4), we conclude that φ is an additive map satisfying $\varphi(A)A = A\varphi(A)$ for all $A \in T_n(\mathbb{F})$.

It follows from (Eremita, 2015, Corollary 3.1) or (Eremita, 2017, Proposition 3.1) that there exists an additive map $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ and $\lambda \in \mathbb{F}$ such that

$$\psi(A) = \lambda A + \mu(A)I_n + \tau(a_{11} + a_{nn})E_{1n}$$

for all $A \in T_n(\mathbb{F})$, where $\tau = 0$ when $|\mathbb{F}| > 3$ or $1 < k < n$. Furthermore, when $\mathbb{F} = \{0, 1, -1\}$ is the Galois field of three elements and $k = n$, the additivity of τ implies linearity of τ. Then either $\tau = 0$ or τ is bijective. We thus have $\tau = 0$, τ is the identity, or

$$\tau(0) = 0, \ \ \tau(1) = -1 \ \text{ and } \ \tau(-1) = 1.$$

Consequently, there exists a scalar $\alpha \in \mathbb{F}$ such that $\tau(x) = \alpha x$ for every $a \in \mathbb{F}$. This completes the proof. □

Remark: The results in this chapter have been published in Chooi et al. (2020).

CHAPTER 4: COMMUTING ADDITIVE MAPS ON INVERTIBLE UPPER TRIANGULAR MATRICES OVER THE GALOIS FIELD OF TWO ELEMENTS

4.1 Introduction

Let $n \geqslant 2$ be an integer and let $\mathbb{F}_2$ be the Galois field of two elements. Let $T_n(\mathbb{F}_2)$ be the ring of $n \times n$ upper triangular matrices over $\mathbb{F}_2$ with centre $Z(T_n(\mathbb{F}_2))$ and unity I_n. In Franca (2012), an example of a nonstandard commuting additive map on invertible 2×2 matrices over $\mathbb{F}_2$ was illustrated. Motivated by this example and Theorem 3.3.3, in this chapter we give a complete description of commuting additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on invertible matrices, i.e., additive maps ψ satisfying $[\psi(A), A] = 0$ for every invertible matrix $A \in T_n(\mathbb{F}_2)$, in Theorems 4.3.1, 4.3.2 and 4.3.3 for $n \geqslant 4$, $n = 3$ and $n = 2$, respectively, where $[X, Y]$ is the commutator of $X, Y \in T_n(\mathbb{F}_2)$. Surprisingly, unlike the situation in commuting additive maps on invertible square matrices, the structure of commuting additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on invertible matrices is much more complex and fertile. Since the set of commuting additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on invertible matrices is an additive group, it is plausible to start our discussion in the upcoming section by presenting some of the generators or basic maps that are not of the standard form (2.14).

4.2 Irregular nonstandard examples

For the sake of simplicity, we adopt $a - b = a + b$ for $a, b \in \mathbb{F}_2$ throughout our discussion. For each pair of integers $1 \leqslant i, j \leqslant n$, let $E_{ij} \in T_n(\mathbb{F}_2)$ be the standard matrix unit whose (i, j)th entry is one and zero elsewhere.

Example 4.2.1. Let $n \geqslant 2$ be an integer and let $H = (h_{ij}) \in T_n(\mathbb{F}_2)$ be a fixed matrix. Let $\varsigma : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be the map defined by

$$\varsigma(A) = \mathrm{tr}\,(H^t A)E_{1n} = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} h_{ij} E_{1n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$, where $\mathrm{tr}\,(A)$ is the trace of A. Then ς is an additive map on

$T_n(\mathbb{F}_2)$. Let $A = (a_{ij}) \in T_n(\mathbb{F}_2)$ be invertible. Then $a_{ii} = 1$ for $i = 1, \ldots, n$. Thus

$$A\varsigma(A) = \sum_{1 \leqslant i \leqslant j \leqslant n} h_{ij} a_{ij} E_{1n} = \varsigma(A)A.$$

Hence ς is a commuting additive map on invertible matrices. For example,

$$A \mapsto \text{tr}\,(A)E_{1n} \quad \text{and} \quad A \mapsto \Sigma(A)E_{1n}$$

are commuting additive maps on invertible matrices of this type. Here, $\Sigma(A)$ denotes the sum of all entries of A.

Example 4.2.2. Let $n \geqslant 2$ be an integer and let $\alpha, \beta_1, \beta_2 \in \mathbb{F}_2$ be some fixed scalars. Let $\Psi_{\alpha,\beta_1,\beta_2} : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be the map defined by

$$\Psi_{\alpha,\beta_1,\beta_2}(A) = (\alpha a_{12} + \beta_1(a_{n-1,n} + a_{nn}))E_{1,n-1} + (\alpha a_{n-1,n} + \beta_2(a_{11} + a_{12}))E_{2n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. Then $\Psi_{\alpha,\beta_1,\beta_2}$ is an additive map on $T_n(\mathbb{F}_2)$. We now verify that $[A, \Psi_{\alpha,\beta_1,\beta_2}(A)] = 0$ for all invertible matrices $A \in T_n(\mathbb{F}_2)$. Let $A = (a_{ij}) \in T_n(\mathbb{F}_2)$ be invertible. Then $A = I_n + U$ where $U = \sum_{1 \leqslant i < j \leqslant n} a_{ij} E_{ij}$. Since $[I_n, \Psi_{\alpha,\beta_1,\beta_2}(A)] = 0$, it follows that $[A, \Psi_{\alpha,\beta_1,\beta_2}(A)] = [U, \Psi_{\alpha,\beta_1,\beta_2}(A)]$. Therefore

$$U\Psi_{\alpha,\beta_1,\beta_2}(A) = a_{12}E_{12}(\alpha a_{n-1,n} + \beta_2(1 + a_{12}))E_{2n} = \alpha a_{12} a_{n-1,n} E_{1n}$$

because $a_{12}(1 + a_{12}) = 0$, and

$$\Psi_{\alpha,\beta_1,\beta_2}(A)U = (\alpha a_{12} + \beta_1(a_{n-1,n} + 1))E_{1,n-1}(a_{n-1,n}E_{n-1,n}) = \alpha a_{12} a_{n-1,n} E_{1n}$$

because $(a_{n-1,n} + 1)a_{n-1,n} = 0$. Then $\Psi_{\alpha,\beta_1,\beta_2}$ is a commuting additive map on invertible matrices. For instance, the following maps

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & a_{34}+a_{44} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11}+a_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are commuting additive maps on invertible matrices of this type.

Example 4.2.3. Let $n \geqslant 2$ be an integer and let $X_1, \ldots, X_n \in T_n(\mathbb{F}_2)$ be some fixed matrices such that $X_1 + \cdots + X_n = 0$. Let $\phi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be the map defined by

$$\phi(A) = \sum_{i=1}^{n} a_{ii} X_i$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. Notice that ϕ is additive and $\phi(A) = 0$ whenever A is invertible. Then ϕ is a commuting additive map on invertible matrices which vanishes on invertible matrices. For instance, the following

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}+a_{22} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_{22}+a_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11}+a_{22}+a_{33}+a_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}+a_{22} & 0 & \cdots & 0 \\ 0 & a_{22}+a_{33} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn}+a_{11} \end{pmatrix}$$

are commuting additive maps on invertible matrices of this type which vanish on invertible matrices.

4.3 Main results

Recall that $T_n(\mathbb{F}_2)$ is the ring of $n \times n$ upper triangular matrices over $\mathbb{F}_2$ with centre $Z(T_n(\mathbb{F}_2))$ and unity I_n. We obtain the following characterisations of commuting additive maps on invertible upper triangular matrices over the Galois field of two elements.

Theorem 4.3.1. *Let $n \geqslant 4$ be an integer. Then $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is a commuting additive map on invertible matrices if and only if there exist scalars $\lambda, \alpha, \beta_1, \beta_2 \in \mathbb{F}_2$, matrices $H, K \in T_n(\mathbb{F}_2)$ and $X_1, \ldots, X_n \in T_n(\mathbb{F}_2)$ satisfying $X_1 + \cdots + X_n = 0$ such that*

$$\psi(A) = \lambda A + \mathrm{tr}\,(H^t A)I_n + \mathrm{tr}\,(K^t A)E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A) + \sum_{i=1}^{n} a_{ii}X_i$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$ where $\Psi_{\alpha,\beta_1,\beta_2} : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the additive map defined by

$$\Psi_{\alpha,\beta_1,\beta_2}(A) = (\alpha a_{12} + \beta_1(a_{n-1,n} + a_{nn}))E_{1,n-1} + (\alpha a_{n-1,n} + \beta_2(a_{11} + a_{12}))E_{2n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$.

Theorem 4.3.2. *$\psi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ is a commuting additive map on invertible matrices if and only if there exist scalars $\lambda, \alpha, \beta, \gamma \in \mathbb{F}_2$, matrices $H, K \in T_3(\mathbb{F}_2)$ and*

$X_1, X_2, X_3 \in T_3(\mathbb{F}_2)$ *satisfying* $X_1 + X_2 + X_3 = 0$ *such that*

$$\psi(A) = \lambda A + \text{tr}\,(H^t A) I_3 + \text{tr}\,(K^t A) E_{13} + \Psi_{0,\alpha,\beta}(A) + \Phi_\gamma(A) + \sum_{i=1}^{3} a_{ii} X_i$$

for every $A = (a_{ij}) \in T_3(\mathbb{F}_2)$*, where* $\Psi_{0,\alpha,\beta} : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ *is the additive map defined in* (4.4) *and* $\Phi_\gamma : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ *is defined by*

$$\Phi_\gamma(A) = \gamma((a_{12} + a_{22})E_{22} + (a_{11} + a_{12} + a_{23} + a_{33})E_{33} + a_{13}(E_{12} + E_{23}))$$

for every $A = (a_{ij}) \in T_3(\mathbb{F}_2)$.

Theorem 4.3.3. $\psi : T_2(\mathbb{F}_2) \to T_2(\mathbb{F}_2)$ *is a commuting additive map on invertible matrices if and only if there exist some scalars* $\lambda_1, \lambda_2 \in \mathbb{F}_2$ *and matrices* $X_1, X_2 \in T_2(\mathbb{F}_2)$ *such that*

$$\psi(A) = (a_{11} + a_{12})X_1 + (a_{22} + a_{12})X_2 + \lambda_1 a_{12} I_2 + \lambda_2 a_{12} E_{12}$$

for every $A = (a_{ij}) \in T_2(\mathbb{F}_2)$.

We remark that Example 3.2.3 can be derived from Theorem 4.3.3 by setting $X_1 = X_2 = 0$ and $\lambda_1 = \lambda_2 = 1$.

Remark 4.3.4. In view of Theorem 4.3.3, we have the following observation.

(i) Let $\lambda \in \mathbb{F}_2$ be a fixed scalar. If $X_1 = \lambda E_{11}$, $X_2 = \lambda E_{22}$ and $\lambda_1 = \lambda_2 = \lambda$, then

$$\psi(A) = \lambda A \quad \text{for every } A \in T_2(\mathbb{F}_2).$$

(ii) Let $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{F}_2$ be fixed scalars. If $X_1 = \epsilon_1 I_2$, $X_2 = \epsilon_2 I_2$, $\lambda_1 = \epsilon_1 + \epsilon_2 + \epsilon_3$ and $\lambda_2 = 0$, then

$$\psi(A) = (\epsilon_1 a_{11} + \epsilon_2 a_{22} + \epsilon_3 a_{12}) I_2 \quad \text{for every } A = (a_{ij}) \in T_2(\mathbb{F}_2).$$

(iii) Let $\pi_1, \pi_2, \pi_3 \in \mathbb{F}_2$ be fixed scalars. If $X_1 = \pi_1 E_{12}$, $X_2 = \pi_2 E_{12}$, $\lambda_1 = 0$ and $\lambda_2 = \pi_1 + \pi_2 + \pi_3$, then

$$\psi(A) = (\pi_1 a_{11} + \pi_2 a_{22} + \pi_3 a_{12}) E_{12} \quad \text{for every } A = (a_{ij}) \in T_2(\mathbb{F}_2).$$

(iv) Let $X \in T_2(\mathbb{F}_2)$ be a fixed matrix. If $X_1 = X_2 = X$ and $\lambda_1 = \lambda_2 = 0$, then $X_1 + X_2 = 0$ and

$$\psi(A) = a_{11}X_1 + a_{22}X_2 \quad \text{for every } A = (a_{ij}) \in T_2(\mathbb{F}_2).$$

(v) Let $\alpha, \beta_1, \beta_2 \in \mathbb{F}_2$ be fixed scalars. If $X_1 = \beta_2 E_{22}$, $X_2 = \beta_1 E_{11}$, $\lambda_1 = \alpha$ and $\lambda_2 = 0$, then

$$\psi(A) = (\alpha a_{12} + \beta_1(a_{12} + a_{22}))E_{11} + (\alpha a_{12} + \beta_2(a_{11} + a_{12}))E_{22} = \Psi_{\alpha,\beta_1,\beta_2}(A)$$

for every $A = (a_{ij}) \in T_2(\mathbb{F}_2)$.

Together with Theorem 3.3.3 in Chapter 3, Theorems 4.3.1, 4.3.2 and 4.3.3, we obtain a complete structural characterisation of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank r matrices over an arbitrary field $\mathbb{F}$, where $1 < r \leqslant n$ is a fixed integer.

Theorem 4.3.5. *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Let $1 < r \leqslant n$ be a fixed integer. Then $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is a commuting additive map on rank r matrices if and only if*

- *when $r < n$ or $|\mathbb{F}| \neq 2$, there exist scalars $\lambda, \alpha \in \mathbb{F}$ and an additive map $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\alpha \neq 0$ only if $r = n$ and $|\mathbb{F}| = 3$,

- *when $r = n \geqslant 4$ and $|\mathbb{F}| = 2$, there exist scalars $\lambda, \alpha, \beta_1, \beta_2 \in \mathbb{F}$, matrices $H, K \in T_n(\mathbb{F})$ and $X_1, \ldots, X_n \in T_n(\mathbb{F})$ satisfying $X_1 + \cdots + X_n = 0$ such that*

$$\psi(A) = \lambda A + \text{tr}\,(H^tA)I_n + \text{tr}\,(K^tA)E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A) + \sum_{i=1}^{n} a_{ii}X_i$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\Psi_{\alpha,\beta_1,\beta_2} : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is the additive map *defined by*

$$\Psi_{\alpha,\beta_1,\beta_2}(A) = (\alpha a_{12} + \beta_1(a_{n-1,n} + a_{nn}))E_{1,n-1} + (\alpha a_{n-1,n} + \beta_2(a_{11} + a_{12}))E_{2n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$,

- *when* $r = n = 3$ *and* $|\mathbb{F}| = 2$, *there exist scalars* $\lambda, \alpha, \beta, \gamma \in \mathbb{F}$, *matrices* $H, K \in T_3(\mathbb{F})$ *and* $X_1, X_2, X_3 \in T_3(\mathbb{F})$ *satisfying* $X_1 + X_2 + X_3 = 0$ *such that*

$$\psi(A) = \lambda A + \mathrm{tr}\,(H^t A)I_3 + \mathrm{tr}\,(K^t A)E_{13} + \Psi_{\alpha,\beta}(A) + \Phi_\gamma(A) + \sum_{i=1}^{3} a_{ii}X_i$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$, *where* $\Psi_{\alpha,\beta} : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ *and* $\Phi_\gamma : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ *are the additive maps defined by*

$$\Psi_{\alpha,\beta}(A) = (\alpha(a_{23} + a_{33}))E_{12} + (\beta(a_{11} + a_{12}))E_{23},$$

$$\Phi_\gamma(A) = \gamma((a_{12} + a_{22})E_{22} + (a_{11} + a_{12} + a_{23} + a_{33})E_{33} + a_{13}(E_{12} + E_{23}))$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$, *and*

- *when* $r = n = 2$ *and* $|\mathbb{F}| = 2$, *there exist scalars* $\lambda_1, \lambda_2 \in \mathbb{F}$ *and matrices* $X_1, X_2 \in T_2(\mathbb{F})$ *such that*

$$\psi(A) = (a_{11} + a_{12})X_1 + (a_{22} + a_{12})X_2 + \lambda_1 a_{12} I_2 + \lambda_2 a_{12} E_{12}$$

for all $A = (a_{ij}) \in T_2(\mathbb{F})$.

4.4 Proofs

Throughout this section, unless stated otherwise, let $n \geqslant 2$ be an integer and let $\mathbb{F}_2$ be the Galois field of two elements.

Lemma 4.4.1. *Let* $n \geqslant 3$ *be an integer and let* $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ *be a commuting additive map on invertible matrices. Then the following assertions hold.*

(i) *For each integer* $1 < i < n$ *and* $\alpha \in \mathbb{F}_2$, *there exist* $\alpha_{i0}, \ldots, \alpha_{i,n-1} \in \mathbb{F}_2$ *such that* $\psi(I_n + D_1 + \alpha D_i) = \sum_{j=0}^{n-1} \alpha_{ij} D_j$.

(ii) *For each integer* $1 < i < n$*, there exist* $\lambda_{i0}, \ldots, \lambda_{i,n-1} \in \mathbb{F}_2$ *such that* $\psi(D_i) = \sum_{j=0}^{n-1} \lambda_{ij} D_j$.

(iii) $[\psi(I_n), D_1] = [\psi(D_1), D_1]$ *and* $[\psi(I_n), D_i] = [\psi(D_1), D_i] = 0$ *for every* $i = 2, \ldots, n-1$.

(iv) *For each* $i \in \{0, 1\}$*, there exist* $\alpha_{i0}, \ldots, \alpha_{i,n-3}, a_i, b_{i1}, b_{i2}, c_i \in \mathbb{F}_2$ *such that*

$$\psi(D_i) = \left(\sum_{j=0}^{n-3} \alpha_{ij} D_j \right) + (\alpha_{i,n-3} + a_i) E_{2,n-1} + b_{i1} E_{1,n-1} + b_{i2} E_{2n} + c_i E_{1n}$$

with $\alpha_{0,n-3} + \alpha_{1,n-3} = a_0 + a_1$ *and* $b_{01} + b_{11} = b_{02} + b_{12}$.

Proof. (i) Let $\alpha \in \mathbb{F}_2$ and let $1 < i < n$ be an integer. Since $[\psi(I_n + D_1 + \alpha D_i), I_n + D_1 + \alpha D_i] = 0$ and $[\psi(I_n + D_1 + \alpha D_i), I_n] = 0$, we get $[\psi(I_n + D_1 + \alpha D_i), D_1 + \alpha D_i] = 0$. The result readily follows from Lemma 2.1.5.

(ii) Let $1 < i < n$ be an integer. By (i) and the additivity of ψ, we obtain

$$\psi(D_i) = \psi(I_n + D_1 + D_i) + \psi(I_n + D_1) = \sum_{j=0}^{n-1} \lambda_{ij} D_j$$

for some $\lambda_{i0}, \ldots, \lambda_{i,n-1} \in \mathbb{F}_2$.

(iii) Let $1 \leqslant i < n$ be an integer. By (i) and Lemma 2.1.4, $[\psi(I_n + D_1), D_i] = 0$. Then $[\psi(I_n), D_i] = [\psi(D_1), D_i]$. For $1 < i < n$, we note that $[\psi(I_n + D_i), D_i] = 0$. So $[\psi(I_n), D_i] = [\psi(D_i), D_i]$. Moreover, $[\psi(D_i), D_i] = 0$ by (ii) and Lemma 2.1.4. Consequently, $[\psi(D_1), D_i] = [\psi(I_n), D_i] = 0$ for every $i = 2, \ldots, n-1$ as desired.

(iv) Denote $\psi(I_n) = (a_{ij}) \in T_n(\mathbb{F}_2)$. Since $[\psi(I_n), D_2] = 0$, it follows that for each integer $0 \leqslant \ell \leqslant n-3$,

$$a_{i,\, i+\ell} = a_{i+2,\, i+2+\ell} \text{ for every } i = 1, \ldots, n-\ell-2. \tag{4.1}$$

Moreover, for $n \geqslant 4$, $[\psi(I_n), D_3] = 0$ implies that for each $0 \leqslant \ell \leqslant n-4$,

$$a_{i,\, i+\ell} = a_{i+3,\, i+3+\ell} \text{ for every } i = 1, \ldots, n-\ell-3. \tag{4.2}$$

By (4.1) and (4.2), we obtain

$$\psi(I_n) = \left(\sum_{j=0}^{n-3} \alpha_{0j} D_j\right) + (\alpha_{0,n-3} + a_0)E_{2,n-1} + b_{01}E_{1,n-1} + b_{02}E_{2n} + c_0 E_{1n} \tag{4.3}$$

for some scalars $\alpha_{00}, \ldots, \alpha_{0,n-3}, a_0, b_{01}, b_{02}, c_0 \in \mathbb{F}_2$ as required.

Notice that $[\psi(I_n) + \psi(D_1), D_1] = 0$ by (iii). It follows from Lemma 2.1.5 and (4.3) that

$$\psi(D_1) = \left(\sum_{j=0}^{n-3} \alpha_{1j} D_j\right) + (\alpha_{1,n-3} + a_1)E_{2,n-1} + b_{11}E_{1,n-1} + b_{12}E_{2n} + c_1 E_{1n}$$

for some scalars $\alpha_{10}, \ldots, \alpha_{1,n-3}, a_1, b_{11}, b_{12}, c_1 \in \mathbb{F}_2$ such that $\alpha_{0,n-3} + \alpha_{1,n-3} = a_0 + a_1$ and $b_{01} + b_{11} = b_{02} + b_{12}$. □

Lemma 4.4.2. *Let $n \geqslant 3$ be an integer and let $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be a commuting additive map on invertible matrices. Then the following assertions hold.*

(i) $[\psi(I_n), E_{ij}] = [\psi(E_{ij}), E_{ij}]$ *for all integers $1 \leqslant i < j \leqslant n$.*

(ii) $[\psi(D_1), E_{ij}] = [\psi(E_{ij}), D_1]$ *for all integers $1 \leqslant i < j \leqslant n$.*

(iii) $[\psi(E_{ij}), E_{st}] = [\psi(E_{st}), E_{ij}]$ *for all integers $1 \leqslant i < j \leqslant n$ and $1 \leqslant s < t \leqslant n$.*

Proof. (i) Let $1 \leqslant i < j \leqslant n$ be integers. Since $[\psi(I_n + E_{ij}), I_n + E_{ij}] = 0$ and $[\psi(I_n + E_{ij}), I_n] = 0$, it follows that $[\psi(I_n) + \psi(E_{ij}), E_{ij}] = 0$. Thus $[\psi(I_n), E_{ij}] = [\psi(E_{ij}), E_{ij}]$ as desired.

(ii) Let $1 \leqslant i < j \leqslant n$ be integers. Note that $[\psi(I_n + D_1 + E_{ij}), I_n + D_1 + E_{ij}] = 0$ implies that $[\psi(I_n) + \psi(D_1) + \psi(E_{ij}), D_1 + E_{ij}] = 0$. Since $[\psi(I_n), D_1] = [\psi(D_1), D_1]$ and $[\psi(I_n), E_{ij}] = [\psi(E_{ij}), E_{ij}]$, it follows that $[\psi(D_1), E_{ij}] = [\psi(E_{ij}), D_1]$.

(iii) Let $1 \leqslant i < j \leqslant n$ and $1 \leqslant s < t \leqslant n$ be integers. Then $[\psi(I_n + E_{ij} + E_{st}), I_n + E_{ij} + E_{st}] = 0$ yields $[\psi(I_n) + \psi(E_{ij}) + \psi(E_{st}), E_{ij} + E_{st}] = 0$. Since $[\psi(I_n) + \psi(E_{ij}), E_{ij}] = 0$ and $[\psi(I_n) + \psi(E_{st}), E_{st}] = 0$ by (i), we obtain $[\psi(E_{ij}), E_{st}] = [\psi(E_{st}), E_{ij}]$. □

Lemma 4.4.3. *Let $n \geqslant 4$ be an integer and let $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be a commuting additive map on invertible matrices. Then there exist $\lambda, \alpha, \beta_1, \beta_2 \in \mathbb{F}_2$ and additive maps*

$\mu : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ and $\tau : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ such that

$$\psi(A) = \lambda A + \mu(A)I_n + \tau(A)E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A)$$

for every strictly upper triangular matrix $A \in T_n(\mathbb{F}_2)$ and $A = I_n$, where $\Psi_{\alpha,\beta_1,\beta_2} : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the additive map defined by

$$\Psi_{\alpha,\beta_1,\beta_2}(A) = (\alpha a_{12} + \beta_1(a_{n-1,n} + a_{nn}))E_{1,n-1} + (\alpha a_{n-1,n} + \beta_2(a_{11} + a_{12}))E_{2n} \quad (4.4)$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$.

Proof. By Lemma 4.4.1 (i), (ii) and (iv), we have

$$\psi(I_n) = \left(\sum_{i=0}^{n-3} \alpha_i D_i\right) + (\alpha_{n-3} + a)E_{2,n-1} + bE_{1,n-1} + cE_{2n} + dE_{1n}$$

for some $\alpha_0, \ldots, \alpha_{n-3}, a, b, c, d \in \mathbb{F}_2$, and

$$\psi(D_1) = \left(\sum_{i=0}^{n-3} (\alpha_i + \gamma_i) D_i\right) + (\alpha_{n-3} + a)E_{2,n-1} + (b + \gamma_{n-2})E_{1,n-1} + (c + \gamma_{n-2})E_{2n} + (d + \gamma_{n-1})E_{1n}$$

for some $\gamma_0, \ldots, \gamma_{n-1} \in \mathbb{F}_2$. For each integer $1 \leqslant p < q \leqslant n$, we let $\psi(E_{pq}) = (a_{ij}^{(p,q)}) \in T_n(\mathbb{F}_2)$ where $a_{ij}^{(p,q)} \in \mathbb{F}_2$ for all integers $1 \leqslant i \leqslant j \leqslant n$. Note first that

$$[\psi(E_{pq}), D_1] = \sum_{j=0}^{n-2} \sum_{i=1}^{n-1-j} (a_{i,\,i+j}^{(p,q)} + a_{i+1,\,i+1+j}^{(p,q)})E_{i,\,i+1+j},$$

$$[\psi(E_{pq}), E_{pq}] = (a_{pp}^{(p,q)} + a_{qq}^{(p,q)})E_{pq} + \sum_{i=1}^{n-q} a_{q,q+i}^{(p,q)} E_{p,q+i} + \sum_{i=1}^{p-1} a_{p-i,p}^{(p,q)} E_{p-i,q},$$

$$[\psi(I_n), E_{pq}] = \begin{cases} aE_{1,n-1} + cE_{1n} + \sum_{i=1}^{n-4} \alpha_i E_{1,2+i} & \text{if } (p,q) = (1,2), \\ aE_{2n} + bE_{1n} + \sum_{i=1}^{n-4} \alpha_i E_{n-1-i,n} & \text{if } (p,q) = (n-1,n), \\ \sum_{i=1}^{n-q} \alpha_i E_{p,q+i} + \sum_{i=1}^{p-1} \alpha_i E_{p-i,q} & \text{otherwise,} \end{cases}$$

and

$$[\psi(D_1),E_{pq}]=\begin{cases}(a+\gamma_{n-3})E_{1,n-1}+(c+\gamma_{n-2})E_{1n} & \\ \quad +\sum_{i=1}^{n-4}(\alpha_i+\gamma_i)E_{1,2+i} & \text{if } (p,q)=(1,2),\\ (a+\gamma_{n-3})E_{2n}+(b+\gamma_{n-2})E_{1n} & \\ \quad +\sum_{i=1}^{n-4}(\alpha_i+\gamma_i)E_{n-1-i,n} & \text{if } (p,q)=(n-1,n),\\ \sum_{i=1}^{n-q}(\alpha_i+\gamma_i)E_{p,q+i}+\sum_{i=1}^{p-1}(\alpha_i+\gamma_i)E_{p-i,q} & \text{otherwise.}\end{cases}$$

We first consider $\psi(E_{12})=(a_{ij}^{(1,2)})$. By $[\psi(E_{12}),D_1]=[\psi(D_1),E_{12}]$, we have

$$\sum_{j=0}^{n-2}\sum_{i=1}^{n-1-j}(a_{i,\,i+j}^{(1,2)}+\,a_{i+1,\,i+1+j}^{(1,2)})E_{i,\,i+1+j}$$
$$=\;(a+\gamma_{n-3})E_{1,n-1}+(c+\gamma_{n-2})E_{1n}+\sum_{i=1}^{n-4}(\alpha_i+\gamma_i)E_{1,2+i}.$$

Then

$$a_{ii}^{(1,2)}=a_{11}^{(1,2)}\quad\text{for } i=2,\ldots,n, \tag{4.5}$$

$$a_{i,i+n-3}^{(1,2)}=a_{1,n-2}^{(1,2)}+a+\gamma_{n-3}\quad\text{for } i=2,3, \tag{4.6}$$

$$a_{2n}^{(1,2)}=a_{1,n-1}^{(1,2)}+c+\gamma_{n-2}, \tag{4.7}$$

and when $n\geqslant 5$, we get

$$a_{i,i+j}^{(1,2)}=a_{1,j+1}^{(1,2)}+\alpha_j+\gamma_j \tag{4.8}$$

for $j=1,\ldots,n-4$ and $i=2,\ldots,n-j$. By virtue of $[\psi(I_n),E_{12}]=[\psi(E_{12}),E_{12}]$, together with (4.5)–(4.8), we obtain

$$aE_{1,n-1}\;+\;cE_{1n}+\sum_{i=1}^{n-4}\alpha_iE_{1,2+i}$$
$$=(a_{1,n-2}^{(1,2)}+a+\gamma_{n-3})E_{1,n-1}+(a_{1,n-1}^{(1,2)}+c+\gamma_{n-2})E_{1n}$$
$$+\sum_{i=1}^{n-4}(a_{1,i+1}^{(1,2)}+\alpha_i+\gamma_i)E_{1,2+i}.$$

We thus have

$$a_{1,i+1}^{(1,2)}=\gamma_i\quad\text{for } i=1,\ldots,n-2. \tag{4.9}$$

It follows from (4.6)–(4.8) that

$$a_{i,i+n-3}^{(1,2)} = a \quad \text{for } i = 2, 3, \tag{4.10}$$

$$a_{2n}^{(1,2)} = c, \tag{4.11}$$

and when $n \geqslant 5$, we have

$$a_{i,i+j}^{(1,2)} = \alpha_j \tag{4.12}$$

for $j = 1, \ldots, n-4$ and $i = 2, \ldots, n-j$.

Consider now $\psi(E_{23}) = (a_{ij}^{(2,3)})$. By virtue of $[\psi(E_{23}), D_1] = [\psi(D_1), E_{23}]$, we have

$$\sum_{j=0}^{n-2} \sum_{i=1}^{n-1-j} (a_{i,\,i+j}^{(2,3)} + a_{i+1,\,i+1+j}^{(2,3)}) E_{i,\,i+1+j} = (\alpha_1 + \gamma_1) E_{13} + \sum_{i=1}^{n-3} (\alpha_i + \gamma_i) E_{2,3+i}.$$

Then

$$a_{ii}^{(2,3)} = a_{11}^{(2,3)} \quad \text{for } i = 2, \ldots, n, \tag{4.13}$$

$$a_{23}^{(2,3)} = a_{12}^{(2,3)} + \alpha_1 + \gamma_1 \quad \text{and} \quad a_{i,i+1}^{(2,3)} = a_{12}^{(2,3)} \quad \text{for } i = 3, \ldots, n-1, \tag{4.14}$$

$$a_{24}^{(2,3)} = a_{13}^{(2,3)} \quad \text{and} \quad a_{2n}^{(2,3)} = a_{1,n-1}^{(2,3)}, \tag{4.15}$$

and when $n \geqslant 5$, we get

$$a_{2,j+2}^{(2,3)} = a_{1,j+1}^{(2,3)} \quad \text{and} \quad a_{i,i+j}^{(2,3)} = a_{1,j+1}^{(2,3)} + \alpha_j + \gamma_j \tag{4.16}$$

for $j = 2, \ldots, n-3$ and $i = 3, \ldots, n-j$. By $[\psi(I_n), E_{23}] = [\psi(E_{23}), E_{23}]$, together with (4.13), (4.14) and (4.16) yield

$$\alpha_1 E_{13} + \sum_{i=1}^{n-3} \alpha_i E_{2,3+i} = a_{12}^{(2,3)} E_{13} + a_{12}^{(23)} E_{24} + \sum_{i=2}^{n-3} (a_{1,i+1}^{(2,3)} + \alpha_i + \gamma_i) E_{2,3+i}.$$

We thus obtain

$$a_{12}^{(2,3)} = \alpha_1, \tag{4.17}$$

and when $n \geqslant 5$, we have

$$a_{1,i+1}^{(2,3)} = \gamma_i \quad \text{for } i = 2, \ldots, n-3. \tag{4.18}$$

It follows from (4.14) and (4.17) that

$$a_{23}^{(2,3)} = \gamma_1 \quad \text{and} \quad a_{i,i+1}^{(2,3)} = \alpha_1 \quad \text{for } i = 3, \ldots, n-1. \tag{4.19}$$

By (4.16) and (4.18), we get

$$a_{2,j+2}^{(2,3)} = \gamma_j \quad \text{and} \quad a_{i,i+j}^{(2,3)} = \alpha_j \tag{4.20}$$

for $j = 2, \ldots, n-3$ and $i = 3, \ldots, n-j$ when $n \geqslant 5$.

In view of (4.5), (4.9), (4.10) and (4.12), we see that

$$[\psi(E_{12}), E_{23}] = \gamma_1 E_{13} + aE_{2n} + \sum_{i=1}^{n-4} \alpha_i E_{2,3+i}.$$

Next, by (4.13), (4.15), (4.19) and (4.20), we obtain

$$[\psi(E_{23}), E_{12}] = a_{1,n-1}^{(2,3)} E_{1n} + \sum_{i=1}^{n-3} \gamma_i E_{1,2+i}.$$

It follows from $[\psi(E_{12}), E_{23}] = [\psi(E_{23}), E_{12}]$ that

$$a = 0,$$

and when $n \geqslant 5$, we have

$$\alpha_i = 0 \quad \text{and} \quad \gamma_j = 0$$

for $i = 1, \ldots, n-4$ and $j = 2, \ldots, n-3$. We thus obtain

$$\psi(I_n) = \alpha_0 I_n + \alpha_{n-3}(E_{1,n-2} + E_{3n}) + bE_{1,n-1} + cE_{2n} + dE_{1n}, \tag{4.21}$$

$$\psi(D_1) = (\alpha_0 + \gamma_0)I_n + \gamma_1 D_1 + \alpha_{n-3}(E_{1,n-2} + E_{3n}) + X, \tag{4.22}$$

where $X = (b+\gamma_{n-2})E_{1,n-1} + (c+\gamma_{n-2})E_{2n} + (d+\gamma_{n-1})E_{1n}$. Together with (4.5), (4.9), (4.10), (4.11) and (4.12), we obtain

$$\psi(E_{12}) = a_{11}^{(1,2)} I_n + \gamma_1 E_{12} + \gamma_{n-2} E_{1,n-1} + cE_{2n} + a_{1n}^{(1,2)} E_{1n}. \tag{4.23}$$

We next claim that $\alpha_{n-3} = 0$. Consider $\psi(E_{13}) = (a_{ij}^{(1,3)})$. From $[\psi(E_{13}), D_1] = [\psi(D_1), E_{13}]$, we obtain

$$\sum_{j=0}^{n-2}\sum_{i=1}^{n-1-j}(a_{i,\,i+j}^{(1,3)} + a_{i+1,\,i+1+j}^{(1,3)})E_{i,\,i+1+j} = \sum_{i=1}^{n-3}(\alpha_i+\gamma_i)E_{1,3+i} = \gamma_1 E_{14} + \alpha_{n-3}E_{1n}.$$

Then

$$a_{ii}^{(1,3)} = a_{11}^{(1,3)} \quad \text{for } i = 2,\ldots,n, \tag{4.24}$$

$$a_{i,i+1}^{(1,3)} = a_{12}^{(1,3)} \quad \text{for } i = 2,\ldots,n-1; \tag{4.25}$$

when $n = 4$, we obtain

$$a_{24}^{(1,3)} = a_{13}^{(1,3)} + \gamma_1 + \alpha_{n-3}; \tag{4.26}$$

when $n \geqslant 5$, we get

$$a_{i,i+2}^{(1,3)} = a_{13}^{(1,3)} + \gamma_1 \quad \text{for } i = 2,\ldots,n-2, \tag{4.27}$$

$$a_{2n}^{(1,3)} = a_{1,n-1}^{(1,3)} + \alpha_{n-3}, \tag{4.28}$$

and when $n \geqslant 6$,

$$a_{i,i+j}^{(1,3)} = a_{1,j+1}^{(1,3)} \tag{4.29}$$

for $j = 3,\ldots,n-3$ and $i = 2,\ldots,n-j$. By virtue of $[\psi(I_n), E_{13}] = [\psi(E_{13}), E_{13}]$, together with (4.21), (4.24), (4.25), (4.27) and (4.29), we get

$$\alpha_{n-3}E_{1n} = \begin{cases} a_{12}^{(1,3)}E_{14} & \text{if } n = 4, \\ a_{12}^{(1,3)}E_{14} + (a_{13}^{(1,3)} + \gamma_1)E_{15} + \sum_{i=3}^{n-3} a_{1,i+1}^{(1,3)}E_{1,i+3} & \text{if } n \geqslant 5. \end{cases}.$$

It follows that

$$\alpha_{n-3} = \begin{cases} a_{12}^{(1,3)} & \text{if } n = 4, \\ a_{13}^{(1,3)} + \gamma_1 & \text{if } n = 5, \\ a_{1,n-2}^{(1,3)} & \text{if } n \geqslant 6. \end{cases} \tag{4.30}$$

By (4.23), we have $[\psi(E_{12}), E_{13}] = 0$. From $[\psi(E_{12}), E_{13}] = [\psi(E_{13}), E_{12}]$, together with (4.24)–(4.29), we obtain

$$0 = \begin{cases} a_{12}^{(1,3)} E_{13} + (a_{13}^{(1,3)} + \gamma_1 + \alpha_{n-3}) E_{14} & \text{if } n = 4, \\ a_{12}^{(1,3)} E_{13} + (a_{13}^{(1,3)} + \gamma_1) E_{14} + (a_{14}^{(1,3)} + \alpha_{n-3}) E_{15} & \text{if } n = 5, \\ a_{12}^{(1,3)} E_{13} + (a_{13}^{(1,3)} + \gamma_1) E_{14} + (a_{1,n-1}^{(1,3)} + \alpha_{n-3}) E_{1n} & \\ \qquad + \sum_{i=4}^{n-2} a_{1i}^{(1,3)} E_{1,i+1} & \text{if } n \geqslant 6. \end{cases} \tag{4.31}$$

In view of (4.30) and (4.31), we conclude that $\alpha_{n-3} = 0$ as claimed. By (4.21) and (4.22), we obtain

$$\psi(I_n) = \alpha_0 I_n + b E_{1,n-1} + c E_{2n} + d E_{1n}, \tag{4.32}$$

$$\psi(D_1) = (\alpha_0 + \gamma_0) I_n + \gamma_1 D_1 + (b + \gamma_{n-2}) E_{1,n-1} + (c + \gamma_{n-2}) E_{2n} + (d + \gamma_{n-1}) E_{1n}. \tag{4.33}$$

We now consider $\psi(E_{n-1,n}) = (a_{ij}^{(n-1,n)})$. By (4.33), $[\psi(E_{n-1,n}), D_1] = [\psi(D_1), E_{n-1,n}]$ yields

$$\sum_{j=0}^{n-2} \sum_{i=1}^{n-1-j} (a_{i,i+j}^{(n-1,n)} + a_{i+1,i+1+j}^{(n-1,n)}) E_{i,i+1+j} = (b + \gamma_{n-2}) E_{1n} + \gamma_1 E_{n-2,n}.$$

Then

$$a_{ii}^{(n-1,n)} = a_{11}^{(n-1,n)} \quad \text{for } i = 2, \ldots, n,$$

$$a_{n-1,n}^{(n-1,n)} = a_{12}^{(n-1,n)} + \gamma_1 \quad \text{and} \quad a_{i,i+1}^{(n-1,n)} = a_{12}^{(n-1,n)} \quad \text{for } i = 2, \ldots, n-2,$$

$$a_{2n}^{(n-1,n)} = a_{1,n-1}^{(n-1,n)} + b + \gamma_{n-2},$$

and when $n \geqslant 5$, we have

$$a_{i,i+j}^{(n-1,n)} = a_{1,1+j}^{(n-1,n)}$$

for $j = 2, \ldots, n-3$ and $i = 2, \ldots, n-j$. Likewise, by $[\psi(I_n), E_{n-1,n}] = [\psi(E_{n-1,n}), E_{n-1,n}]$ and (4.32), we get

$$bE_{1n} = (a_{n-1,n-1}^{(n-1,n)} + a_{nn}^{(n-1,n)})E_{n-1,n} + \sum_{i=1}^{n-2} a_{n-1-i,n-1}^{(n-1,n)} E_{n-1-i,n}.$$

Then $a_{1,n-1}^{(n-1,n)} = b$ and $a_{i,n-1}^{(n-1,n)} = 0$ for $i = 2, \ldots, n-2$. We thus obtain

$$\psi(E_{n-1,n}) = a_{11}^{(n-1,n)} I_n + \gamma_1 E_{n-1,n} + bE_{1,n-1} + \gamma_{n-2} E_{2n} + a_{1n}^{(n-1,n)} E_{1n}. \tag{4.34}$$

Next, consider $\psi(E_{1n}) = (a_{ij}^{(1,n)})$. By applying (4.33) in $[\psi(E_{1n}), D_1] = [\psi(D_1), E_{1n}]$, we obtain

$$\sum_{j=0}^{n-2} \sum_{i=1}^{n-1-j} (a_{i,\,i+j}^{(1,n)} + a_{i+1,\,i+1+j}^{(1,n)}) E_{i,\,i+1+j} = 0.$$

Then $\psi(E_{1n}) = \sum_{i=0}^{n-1} a_{1,i+1}^{(1,n)} D_i$. By using (4.23) and $[\psi(E_{1n}), E_{12}] = [\psi(E_{12}), E_{1n}]$, we get

$$\sum_{i=2}^{n-1} a_{1i}^{(1,n)} E_{1,i+1} = 0.$$

Then $a_{1i}^{(1,n)} = 0$ for $i = 2, \ldots, n-1$. We thus obtain

$$\psi(E_{1n}) = a_{11}^{(1,n)} I_n + a_{1n}^{(1,n)} E_{1n}. \tag{4.35}$$

Finally, consider $\psi(E_{pq}) = (a_{ij}^{(p,q)})$ for $1 \leqslant p < q \leqslant n$ with $(p,q) \notin \{(1,2), (n-1,n), (1,n)\}$. Since $p \neq n-1, n$ and $q \neq 1, 2$, by applying (4.33) in $[\psi(E_{pq}), D_1] = [\psi(D_1), E_{pq}]$, we obtain

$$\sum_{j=0}^{n-2} \sum_{i=1}^{n-1-j} (a_{i,\,i+j}^{(p,q)} + a_{i+1,\,i+1+j}^{(p,q)}) E_{i,\,i+1+j} = \begin{cases} \gamma_1 E_{1,q+1} & \text{if } p = 1 \text{ and } q < n, \\ \gamma_1 E_{p-1,n} & \text{if } q = n \text{ and } p > 1, \\ \gamma_1 E_{p-1,q} + \gamma_1 E_{p,q+1} & \text{otherwise.} \end{cases} \tag{4.36}$$

When $p = 1$ and $q < n$, we get

$$a^{(1,q)}_{1+i,q+i} = \gamma_1 + a^{(1,q)}_{1q} \quad \text{for } i = 1, \dots, n-q,$$

$$a^{(1,q)}_{1+i,1+i+j} = a^{(1,q)}_{1,1+j}$$

for $j = 0, \dots, q-2, q, \dots, n-2$ and $i = 1, \dots, n-1-j$. When $q = n$ and $p > 1$, we obtain

$$a^{(p,n)}_{pn} = \gamma_1 + a^{(p,n)}_{1,n-p+1} \quad \text{and} \quad a^{(p,n)}_{i,n-p+i} = a^{(p,n)}_{1,n-p+1} \ \text{ for } i = 2, \dots, p-1,$$

$$a^{(p,n)}_{1+i,1+i+j} = a^{(p,n)}_{1,1+j}$$

for $j = 0, \dots, n-p-1, n-p+1, \dots, n-2$ and $i = 1, \dots, n-1-j$. When $p \neq 1$ and $q \neq n$,

$$a^{(p,q)}_{1+i,1+i+j} = a^{(p,q)}_{1,1+j}$$

for $j = 0, \dots, q-p-1, q-p+1, \dots, n-2$ and $i = 1, \dots, n-1-j$,

$$a^{(p,q)}_{1+i,1+q-p+i} = a^{(p,q)}_{1,1+q-p} \ \text{ for } i = 1, \dots, p-2,$$

$$a^{(p,q)}_{pq} = \gamma_1 + a^{(p,q)}_{1,1+q-p} \quad \text{and} \quad a^{(p,q)}_{p+1,q+1} = a^{(p,q)}_{1,1+q-p},$$

$$a^{(p,q)}_{1+i,1+q-p+i} = a^{(p,q)}_{1,1+q-p} \quad \text{for } i = p+1, \dots, n-1-q+p.$$

Consequently, we obtain

$$\psi(E_{pq}) = \begin{cases} \sum_{i=1}^{n-q} \gamma_1 E_{i+1,q+i} + \sum_{i=0}^{n-1} a^{(1,q)}_{1,i+1} D_i & \text{if } p = 1, \\ \gamma_1 E_{pq} + \sum_{i=0}^{n-1} a^{(p,q)}_{1,i+1} D_i & \text{if } p \neq 1. \end{cases} \tag{4.37}$$

We claim that

$$\psi(E_{pq}) = a^{(p,q)}_{11} I_n + \gamma_1 E_{pq} + a^{(p,q)}_{1n} E_{1n}. \tag{4.38}$$

We distinguish our argument between two cases:

Case I: $p = 1$. By (4.23), we have $\psi(E_{12}) = a_{11}^{(1,2)} I_n + \gamma_1 E_{12} + \gamma_{n-2} E_{1,n-1} + cE_{2n} + a_{1n}^{(1,2)} E_{1n}$. Then $[\psi(E_{12}), E_{1q}] = 0$ as $q \neq 1, 2$. By (4.37),

$$[\psi(E_{1q}), E_{12}] = (a_{1q}^{(1,q)} + \gamma_1)E_{1,q+1} + \sum_{i=2, i\neq q}^{n-1} a_{1i}^{(1,q)} E_{1,i+1}.$$

Then $[\psi(E_{1q}), E_{12}] = [\psi(E_{12}), E_{1q}]$ yields $a_{1q}^{(1q)} = \gamma_1$ and $a_{1i}^{(1q)} = 0$ for all $1 < i < n$ with $i \neq q$. Claim (4.38) follows immediately from (4.37).

Case II: $p \neq 1$. Since $p \neq 1, n-1, n$ and $q \neq 1, 2$, it follows from (4.23) that

$$[\psi(E_{12}), E_{pq}] = \begin{cases} \gamma_1 E_{1q} & \text{if } p = 2, \\ 0 & \text{if } p > 2. \end{cases}$$

By (4.37), we have $\psi(E_{pq}) = \gamma_1 E_{pq} + \sum_{i=0}^{n-1} a_{1,i+1}^{(p,q)} D_i$. Since $q \neq 1$, we get $\psi(E_{pq})E_{12} = a_{11}^{(p,q)} E_{12}$. Therefore

$$[\psi(E_{pq}), E_{12}] = \begin{cases} \gamma_1 E_{1q} + \sum_{i=2}^{n-1} a_{1i}^{(2,q)} E_{1,i+1} & \text{if } p = 2, \\ \sum_{i=2}^{n-1} a_{1i}^{(p,q)} E_{1,i+1} & \text{if } p > 2. \end{cases}$$

By virtue of $[\psi(E_{pq}), E_{12}] = [\psi(E_{12}), E_{pq}]$, we obtain $\sum_{i=2}^{n-1} a_{1i}^{(p,q)} E_{1,i+1} = 0$, and thus $a_{1i}^{(p,q)} = 0$ for $i = 2, \ldots, n-1$. It follows from (4.37) that claim (4.38) is proved.

It follows from the results of (4.23), (4.32), (4.34), (4.35) and (4.38) that we let $\mu : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ be any linear map such that $\mu(I_n) = \alpha_0 + \gamma_1$ and

$$\mu(E_{ij}) = a_{11}^{(i,j)}$$

for every integer $1 \leqslant i < j \leqslant n$, and let $\tau : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ be any linear map such that $\tau(I_n) = d$, $\tau(E_{1n}) = \gamma_1 + a_{1n}^{(1,n)}$ and

$$\tau(E_{ij}) = a_{1n}^{(i,j)}$$

for every integer $1 \leqslant i < j \leqslant n$ with $(i, j) \neq (1, n)$. We define the map $\zeta : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ by

$$\zeta(A) = \gamma_1 A + \mu(A)I_n + \tau(A)E_{1n} + \Psi_{\gamma_{n-2},b,c}(A)$$

for every $A \in T_n(\mathbb{F}_2)$ where $\Psi_{\gamma_{n-2},b,c} : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the additive map defined in (4.4). Notice that ζ is a commuting additive map on invertible matrices by Examples 4.2.1 and 4.2.2. Moreover, since $\zeta(E_{ij}) = \psi(E_{ij})$ for all integers $1 \leqslant i < j \leqslant n$, and $\Psi_{\gamma_{n-2},b,c}(I_n) = bE_{1,n-1} + cE_{2n}$, we have

$$\begin{aligned}\zeta(I_n) &= \gamma_1 I_n + \mu(I_n)I_n + \tau(I_n)E_{1n} + \Psi_{\gamma_{n-2},b,c}(I_n) \\ &= \gamma_1 I_n + (\alpha_0 + \gamma_1)I_n + dE_{1n} + bE_{1,n-1} + cE_{2n} \\ &= \alpha_0 I_n + bE_{1,n-1} + cE_{2n} + dE_{1n} \\ &= \psi(I_n).\end{aligned}$$

Since ψ is linear, it follows that $\psi(A) = \zeta(A)$ for all strictly upper triangular matrices $A \in T_n(\mathbb{F}_2)$ and $A = I_n$. The proof is complete. □

We next prove the following particularly interesting results.

Lemma 4.4.4. *Let $n \geqslant 2$ be an integer and let $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be an additive map. Then the following are equivalent.*

- **(i)** *ψ is a commuting map on invertible matrices that vanishes on invertible matrices.*
- **(ii)** *$\psi(I_n) = 0$ and $\psi(E_{ij}) = 0$ for all integers $1 \leqslant i < j \leqslant n$.*
- **(iii)** *There exist matrices $X_1, \ldots, X_n \in T_n(\mathbb{F}_2)$ satisfying $X_1 + \cdots + X_n = 0$ such that*

$$\psi(A) = \sum_{i=1}^{n} a_{ii} X_i$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$.

Proof. (i) $\Longrightarrow$ (ii). Let $1 \leqslant i < j \leqslant n$ be integers. Then $\psi(E_{ij}) = \psi(I_n + E_{ij}) + \psi(I_n) = 0$.

(ii) $\Longrightarrow$ (iii). Let $X_i = \psi(E_{ii}) \in T_n(\mathbb{F}_2)$ for $i = 1, \ldots, n$. Then $X_1 + \cdots + X_n = \psi(I_n) = 0$ and

$$\psi(E_{ij}) = \begin{cases} 0 & \text{when } 1 \leqslant i < j \leqslant n, \\ X_i & \text{when } 1 \leqslant i = j \leqslant n. \end{cases}$$

It follows from the linearity of ψ that $\psi(A) = \sum_{i=1}^{n} a_{ii} X_i$ for every $A = (a_{ij}) \in T_n(\mathbb{F}_2)$.

(iii) $\implies$ (i). Let $A = (a_{ij}) \in T_n(\mathbb{F}_2)$ be invertible. Then $a_{ii} = 1$ for $i = 1, \ldots, n$, and thus $\psi(A) = \sum_{i=1}^n X_i = 0$. Consequently, $\psi(A)A = 0 = A\psi(A)$ as required. □

Lemma 4.4.5. *Let $\mathbb{F}$ be a field and let $\mathcal{V}$ be an n-dimensional linear space over $\mathbb{F}$. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathcal{V}$. Then $\tau : \mathcal{V} \to \mathbb{F}$ is linear if and only if there exist unique scalars $\tau_1, \ldots, \tau_n \in \mathbb{F}$ such that*

$$\tau(u) = \sum_{i=1}^n \tau_i u_i$$

for every $u = \sum_{i=1}^n u_i e_i \in \mathcal{V}$.

Proof. The sufficiency is clear. For the necessity, let $\tau(e_i) = \tau_i \in \mathbb{F}$ for every $1 \leqslant i \leqslant n$. For each $u = \sum_{i=1}^n u_i e_i \in \mathcal{V}$, it follows from the linearity of τ that $\tau(u) = \sum_{i=1}^n u_i \tau(e_i) = \sum_{i=1}^n \tau_i u_i$.

For the uniqueness, suppose there exist scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $\tau(u) = \sum_{i=1}^n \alpha_i u_i$ for every $u = \sum_{i=1}^n u_i e_i$. Then $\sum_{i=1}^n (\alpha_i - \tau_i) u_i = 0$ for every $u = \sum_{i=1}^n u_i e_i$. Choosing $u = e_i$, we thus obtain $\alpha_i = \tau_i$ for all $1 \leqslant i \leqslant n$ as desired. □

Let n be a positive integer and let $\tau : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ be an additive map. Then τ is linear. Considering the standard basis of $T_n(\mathbb{F}_2)$, it follows from Lemma 4.4.5 that there exists a matrix $H = (h_{ij}) \in T_n(\mathbb{F}_2)$ such that

$$\tau(A) = \sum_{1 \leqslant i \leqslant j \leqslant n} h_{ij} a_{ij} = \operatorname{tr}(H^t A) \tag{4.39}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. Notice that $\operatorname{tr}(H^t A)$ is the sum of all entries of $H \circ A = (h_{ij} a_{ij})$, the Hadamard product of H and A.

As a side remark, a result that is similar to (4.39) for symmetric matrices has been obtained in (Orel, 2019, Lemma 3.4).

We are now ready to prove our main results. We start with the proof of Theorem 4.3.1. Let $n \geqslant 4$ be an integer.

Proof of Theorem 4.3.1. By Examples 4.2.1–4.2.3 and Lemma 4.4.4, the sufficiency holds. For the necessity, in view of Lemma 4.4.3, there exist scalars $\lambda, \alpha, \beta_1, \beta_2 \in \mathbb{F}_2$ and additive maps $\mu : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ and $\tau : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ such that

$$\psi(A) = \lambda A + \mu(A)I_n + \tau(A)E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A)$$

for all strictly upper triangular matrices $A \in T_n(\mathbb{F}_2)$ and $A = I_n$ where $\Psi_{\alpha,\beta_1,\beta_2}$ is the additive map defined in (4.4). Let $\varphi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be the map defined by

$$\varphi(A) = \psi(A) + \lambda A + \mu(A)I_n + \tau(A)E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A)$$

for all $A \in T_n(\mathbb{F}_2)$. Then $\varphi(E_{ij}) = 0$ for every integer $1 \leqslant i < j \leqslant n$ and $\varphi(I_n) = 0$. By Lemma 4.4.4, there exist matrices $X_1, \ldots, X_n \in T_n(\mathbb{F}_2)$ satisfying $X_1 + \cdots + X_n = 0$ such that $\varphi(A) = \sum_{i=1}^n a_{ii}X_i$ for every $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. We thus obtain

$$\psi(A) = \lambda A + \mu(A)I_n + \tau(A)E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A) + \sum_{i=1}^{n} a_{ii}X_i$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. It follows that $\mu : T_n(\mathbb{F}_2) \to \mathbb{F}_2$. By (4.39), there exist matrices $H, K \in T_n(\mathbb{F}_2)$ such that

$$\mu(A) = \mathrm{tr}\,(H^tA) \quad \text{and} \quad \tau(A) = \mathrm{tr}\,(K^tA)$$

for every $A \in T_n(\mathbb{F}_2)$. This completes the proof. □

Next we prove Theorem 4.3.2.

Proof of Theorem 4.3.2. We first claim that Φ_γ is a commuting additive map on invertible matrices. Evidently, Φ_γ is additive. Let $A = (a_{ij}) \in T_3(\mathbb{F}_2)$ be invertible. Then $A = I_3 + U$ where $U = a_{12}E_{12} + a_{23}E_{23} + a_{13}E_{13}$, and thus $[A, \Phi_\gamma(A)] = [U, \Phi_\gamma(A)]$. Note that

$$\begin{aligned}
U\Phi_\gamma(A) &= \gamma U\left((a_{12}+1)E_{22} + (a_{12}+a_{23})E_{33} + a_{13}(E_{12}+E_{23})\right)\\
&= \gamma(a_{12}E_{12}(a_{12}+1)E_{22} + (a_{13}E_{13}+a_{23}E_{23})(a_{12}+a_{23})E_{33} + (a_{12}E_{12})a_{13}E_{23})\\
&= \gamma(a_{13}(a_{12}+a_{23})E_{13} + a_{23}(a_{12}+a_{23})E_{23} + a_{12}a_{13}E_{13})\\
&= \gamma(a_{13}a_{23}E_{13} + a_{23}a_{12}E_{23} + a_{23}E_{23})
\end{aligned}$$

since $a_{12}(a_{12}+1)=0$ and $a_{23}^2=a_{23}$. On the other hand, we see that

$$\begin{aligned}\Phi_\gamma(A)U &= \gamma\left((a_{12}+1)E_{22}+(a_{12}+a_{23})E_{33}+a_{13}(E_{12}+E_{23})\right)U \\ &= \gamma((a_{12}+1)E_{22}(a_{23}E_{23})+a_{13}E_{12}(a_{23}E_{23})) \\ &= \gamma(a_{12}a_{23}E_{23}+a_{23}E_{23}+a_{13}a_{23}E_{13}).\end{aligned}$$

Hence Φ_γ is a commuting additive map on invertible matrices. Moreover, by Examples 4.2.1–4.2.3 and Lemma 4.4.4, the sufficiency is proved.

For the necessity, in view of Lemma 4.4.1 (i), (ii) and (iv), we have

$$\psi(I_3)=\alpha_0(E_{11}+E_{33})+aE_{22}+bE_{12}+cE_{23}+dE_{13}, \tag{4.40}$$

$$\psi(D_1)=(\alpha_0+\gamma_0)(E_{11}+E_{33})+(a+\gamma_0)E_{22}+(b+\gamma_1)E_{12}+(c+\gamma_1)E_{23}+(d+\gamma_2)E_{13}$$

for some scalars $a,b,c,d,\alpha_0,\gamma_0,\gamma_1,\gamma_2\in\mathbb{F}_2$. Let $\psi(E_{12})=(p_{ij})\in T_3(\mathbb{F}_2)$. By $[\psi(E_{12}),D_1]=[\psi(D_1),E_{12}]$, we have $p_{33}=p_{22}=p_{11}+a+\alpha_0$ and $p_{23}=p_{12}+c+\gamma_1$. By $[\psi(I_3),E_{12}]=[\psi(E_{12}),E_{12}]$, we obtain $p_{12}=\gamma_1$, and so $p_{23}=c$. Then

$$\psi(E_{12})=p_{11}E_{11}+(p_{11}+a+\alpha_0)(E_{22}+E_{33})+\gamma_1E_{12}+cE_{23}+p_{13}E_{13}. \tag{4.41}$$

Let $\psi(E_{23})=(q_{ij})\in T_3(\mathbb{F}_2)$. Likewise, by $[\psi(E_{23}),D_1]=[\psi(D_1),E_{23}]$, we obtain $q_{22}=q_{11}$, $q_{33}=q_{11}+a+\alpha_0$ and $q_{23}=q_{12}+b+\gamma_1$. It follows from $[\psi(I_3),E_{23}]=[\psi(E_{23}),E_{23}]$ that $q_{12}=b$, and thus $q_{23}=\gamma_1$. Therefore

$$\psi(E_{23})=q_{11}(E_{11}+E_{22})+(q_{11}+a+\alpha_0)E_{33}+bE_{12}+\gamma_1E_{23}+q_{13}E_{13}. \tag{4.42}$$

Let $\psi(E_{13})=(r_{ij})\in T_3(\mathbb{F}_2)$. By $[\psi(E_{13}),D_1]=[\psi(D_1),E_{13}]$, we obtain $r_{11}=r_{22}=r_{33}$ and $r_{23}=r_{12}$. Hence $\psi(E_{13})=r_{11}I_3+r_{12}(E_{12}+E_{23})+r_{13}E_{13}$. Next, by $[\psi(E_{13}),E_{12}]=[\psi(E_{12}),E_{13}]$, we obtain $r_{12}=a+\alpha_0$. Then

$$\psi(E_{13})=r_{11}I_3+(a+\alpha_0)(E_{12}+E_{23})+r_{13}E_{13}. \tag{4.43}$$

We set $\lambda = \gamma_1$, $x = \alpha_0 + \lambda$ and $\gamma = a + \alpha_0$. In view of (4.40)–(4.43), we obtain

$$\psi(I_3) = \lambda I_3 + xI_3 + (bE_{12} + cE_{23}) + \gamma E_{22} + dE_{13}, \tag{4.44}$$

$$\psi(E_{12}) = \lambda E_{12} + p_{11}I_3 + (0E_{12} + cE_{23}) + \gamma(E_{22} + E_{33}) + p_{13}E_{13}, \tag{4.45}$$

$$\psi(E_{23}) = \lambda E_{23} + q_{11}I_3 + (bE_{12} + 0E_{23}) + \gamma E_{33} + q_{13}E_{13}, \tag{4.46}$$

$$\psi(E_{13}) = \lambda E_{13} + r_{11}I_3 + \gamma(E_{12} + E_{23}) + (r_{13} + \lambda)E_{13}. \tag{4.47}$$

By virtue of (4.44)–(4.47), we let $\mu : T_3(\mathbb{F}_2) \to \mathbb{F}_2$ be any linear map such that $\mu(I_3) = x$ and

$$\mu(E_{12}) = p_{11}, \;\; \mu(E_{23}) = q_{11}, \;\; \mu(E_{13}) = r_{11},$$

let $\tau : T_3(\mathbb{F}_2) \to \mathbb{F}_2$ be any linear map such that $\tau(I_3) = d$ and

$$\tau(E_{12}) = p_{13}, \;\; \tau(E_{23}) = q_{13}, \;\; \tau(E_{13}) = r_{13} + \lambda,$$

and let $\Phi_\gamma : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ be the linear map defined by

$$\Phi_\gamma(A) = \gamma((a_{12} + a_{22})E_{22} + (a_{11} + a_{12} + a_{23} + a_{33})E_{33} + a_{13}(E_{12} + E_{23}))$$

for every $A = (a_{ij}) \in T_3(\mathbb{F}_2)$. We next define the map $\xi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ by

$$\xi(A) = \lambda A + \mu(A)I_3 + \tau(A)E_{13} + \Psi_{0,b,c}(A) + \Phi_\gamma(A) \tag{4.48}$$

for every $A \in T_3(\mathbb{F}_2)$ where $\Psi_{0,b,c}$ is the additive map defined in (4.4). Clearly, ξ is a commuting additive map on invertible matrices. By (4.45)–(4.48), we have $\xi(E_{ij}) = \psi(E_{ij})$ for every integer $1 \leqslant i < j \leqslant 3$. In view of (4.44) and (4.48), we obtain $\xi(I_3) = \psi(I_3)$ by virtue of $\Psi_{0,b,c}(I_3) = bE_{12} + cE_{23}$ and $\Phi_\gamma(I_3) = \gamma E_{22}$.

Let $\vartheta : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ be the map defined by

$$\vartheta(A) = \psi(A) + \xi(A)$$

for every $A \in T_3(\mathbb{F}_2)$. Since $\vartheta(I_3) = 0$ and $\vartheta(E_{ij}) = 0$ for every $1 \leqslant i < j \leqslant 3$, it follows from Lemma 4.4.4 that ϑ is a commuting additive map on invertible matrices which vanishes on invertible matrices. Then there exist matrices $X_1, X_2, X_3 \in T_3(\mathbb{F}_2)$ satisfying $X_1 + X_2 + X_3 = 0$ such that $\vartheta(A) = \sum_{i=1}^{3} a_{ii} X_i$ for every $A = (a_{ij}) \in T_3(\mathbb{F}_2)$. By virtue of (4.39) and (4.48), there exist matrices $H, K \in T_3(\mathbb{F}_2)$ such that

$$\psi(A) = \lambda A + \operatorname{tr}(H^t A) I_3 + \operatorname{tr}(K^t A) E_{13} + \Psi_{0,\alpha,\beta}(A) + \Phi_\gamma(A) + \sum_{i=1}^{3} a_{ii} X_i$$

for every $A = (a_{ij}) \in T_3(\mathbb{F}_2)$. This completes the proof. $\square$

We now prove Theorem 4.3.3.

Proof of Theorem 4.3.3. For the sufficiency, we first see that ψ is additive. Let $A \in T_2(\mathbb{F}_2)$ be invertible. Then either $A = I_2$ or $A = I_2 + E_{12}$. Clearly, $\psi(A)A = A\psi(A)$ when $A = I_2$. Consider $A = I_2 + E_{12}$. Then

$$\psi(A)A = (\lambda_1 I_2 + \lambda_2 E_{12})(I_2 + E_{12}) = (I_2 + E_{12})(\lambda_1 I_2 + \lambda_2 E_{12}) = A\psi(A).$$

For the necessity, we notice that $[\psi(I_2 + E_{12}), I_2 + E_{12}] = 0$ and $[\psi(I_2 + E_{12}), I_2] = 0$ yield $[\psi(I_2 + E_{12}), E_{12}] = 0$. By Lemma 2.1.5, we obtain $\psi(I_2 + E_{12}) = \lambda_1 I_2 + \lambda_2 E_{12}$ for some scalars $\lambda_1, \lambda_2 \in \mathbb{F}_2$. Let $X_i = \psi(E_{ii}) \in T_2(\mathbb{F}_2)$ for $i = 1, 2$. Then $\psi(I_2) = X_1 + X_2$ and $\psi(E_{12}) = \psi(I_2) + \psi(I_2 + E_{12}) = X_1 + X_2 + \lambda_1 I_2 + \lambda_2 E_{12}$. Consequently, we have

$$\psi(E_{11}) = X_1, \quad \psi(E_{22}) = X_2 \quad \text{and} \quad \psi(E_{12}) = X_1 + X_2 + \lambda_1 I_2 + \lambda_2 E_{12}.$$

Let $\upsilon : T_2(\mathbb{F}_2) \to T_2(\mathbb{F}_2)$ be the additive map defined by

$$\upsilon(A) = (a_{11} + a_{12}) X_1 + (a_{22} + a_{12}) X_2 + \lambda_1 a_{12} I_2 + \lambda_2 a_{12} E_{12}$$

for every $A = (a_{ij}) \in T_2(\mathbb{F}_2)$. Then $\upsilon(E_{ij}) = \psi(E_{ij})$ for every $1 \leqslant i \leqslant j \leqslant 2$, and so $\psi(A) = \upsilon(A)$ for every $A \in T_2(\mathbb{F}_2)$ as desired. $\square$

Let $M_2(\mathbb{F}_2)$ denote the ring of 2×2 matrices over $\mathbb{F}_2$ and let $E_{ij} \in M_2(\mathbb{F}_2)$ be the standard matrix unit whose (i, j)th entry is one and zero elsewhere. We end the discussion with a characterisation of commuting additive maps $\psi : M_2(\mathbb{F}_2) \to M_2(\mathbb{F}_2)$ on invertible matrices, i.e., additive maps ψ satisfying $[\psi(A), A] = 0$ for all invertible matrices $A \in M_2(\mathbb{F}_2)$.

Theorem 4.4.6. *$\psi : M_2(\mathbb{F}_2) \to M_2(\mathbb{F}_2)$ is a commuting additive map on invertible matrices if and only if there exist scalars $\alpha, \beta, \lambda \in \mathbb{F}_2$ and a matrix $H \in M_2(\mathbb{F}_2)$ such that*

$$\psi(A) = \lambda A + \mathrm{tr}\,(H^t A)I_2 + \Gamma_{\alpha,\beta}(A)$$

for all $A \in M_2(\mathbb{F}_2)$. Here, $\Gamma_{\alpha,\beta} : M_2(\mathbb{F}_2) \to M_2(\mathbb{F}_2)$ is the additive map defined by

$$\Gamma_{\alpha,\beta}(A) = \alpha a_{11} Q + (\alpha a_{22} + \beta(a_{12} + a_{21} + a_{22}))R$$

for all $A = (a_{ij}) \in M_2(\mathbb{F}_2)$ where $Q = E_{11} + E_{12} + E_{21}$ and $R = I_2 + Q$.

Proof. Let $A_1 = I_2$, $A_2 = I_2 + E_{12}$, $A_3 = I_2 + E_{21}$, $A_4 = Q$, $A_5 = R$ and $A_6 = E_{12} + E_{21}$. Notice that $A_1, \ldots, A_6$, are the only invertible matrices in $M_2(\mathbb{F}_2)$, and

$$\Gamma_{\alpha,\beta}(A_2) = \alpha I_2,\ \Gamma_{\alpha,\beta}(A_3) = \alpha I_2,\ \Gamma_{\alpha,\beta}(A_4) = \alpha A_4,\ \Gamma_{\alpha,\beta}(A_5) = (\alpha+\beta)A_5,\ \Gamma_{\alpha,\beta}(A_6) = 0.$$

Therefore $[\Gamma_{\alpha,\beta}(A_i), A_i] = 0$ for $i = 1, \ldots, 6$. Hence ψ is a commuting additive map on invertible matrices of $M_2(\mathbb{F}_2)$ as required.

For the necessity, let $\psi(A_i) = (a_{st}^{(i)}) \in M_2(\mathbb{F}_2)$ for $i = 1, 2, 3, 4$. From $[\psi(A_i), A_i] = 0$ for $i = 2, 3, 4$, we obtain

$$\psi(A_2) = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ 0 & a_{11}^{(2)} \end{pmatrix},\ \psi(A_3) = \begin{pmatrix} a_{11}^{(3)} & 0 \\ a_{21}^{(3)} & a_{11}^{(3)} \end{pmatrix},\ \psi(A_4) = \begin{pmatrix} a_{11}^{(4)} & a_{12}^{(4)} \\ a_{12}^{(4)} & a_{11}^{(4)} + a_{12}^{(4)} \end{pmatrix}.$$

Then

$$\begin{aligned} \psi(E_{11}) &= \psi(A_2) + \psi(A_3) + \psi(A_4) \\ &= \begin{pmatrix} a_{11}^{(2)} + a_{11}^{(3)} + a_{11}^{(4)} & a_{12}^{(2)} + a_{12}^{(4)} \\ a_{21}^{(3)} + a_{12}^{(4)} & a_{11}^{(2)} + a_{11}^{(3)} + a_{11}^{(4)} + a_{12}^{(4)} \end{pmatrix}, \end{aligned} \tag{4.49}$$

$$\psi(E_{12}) = \psi(A_1) + \psi(A_2) = \begin{pmatrix} a_{11}^{(1)} + a_{11}^{(2)} & a_{12}^{(1)} + a_{12}^{(2)} \\ a_{21}^{(1)} & a_{22}^{(1)} + a_{11}^{(2)} \end{pmatrix}, \tag{4.50}$$

$$\psi(E_{21}) = \psi(A_1) + \psi(A_3) = \begin{pmatrix} a_{11}^{(1)} + a_{11}^{(3)} & a_{12}^{(1)} \\ a_{21}^{(1)} + a_{21}^{(3)} & a_{22}^{(1)} + a_{11}^{(3)} \end{pmatrix}, \tag{4.51}$$

$$\begin{aligned} \psi(E_{22}) &= \psi(A_1) + \psi(E_{11}) \\ &= \begin{pmatrix} a_{11}^{(1)} + a_{11}^{(2)} + a_{11}^{(3)} + a_{11}^{(4)} & a_{12}^{(1)} + a_{12}^{(2)} + a_{12}^{(4)} \\ a_{21}^{(1)} + a_{21}^{(3)} + a_{12}^{(4)} & a_{22}^{(1)} + a_{11}^{(2)} + a_{11}^{(3)} + a_{11}^{(4)} + a_{12}^{(4)} \end{pmatrix}. \end{aligned} \tag{4.52}$$

Since $A_5 = E_{12} + E_{21} + E_{22}$, it follows from $0 = [\psi(A_5), A_5] = [\psi(E_{12}) + \psi(E_{21}) + \psi(E_{22}), A_5]$ that

$$a_{21}^{(1)} = a_{12}^{(1)} \quad \text{and} \quad a_{22}^{(1)} = a_{11}^{(1)} + a_{12}^{(1)}. \tag{4.53}$$

Likewise, since $A_6 = E_{12} + E_{21}$, it follows from $[\psi(A_6), A_6] = 0$ that

$$a_{21}^{(3)} = a_{12}^{(2)}. \tag{4.54}$$

Let $\lambda = a_{12}^{(2)}$ and let $h_{11} = a_{11}^{(2)} + a_{11}^{(3)} + a_{11}^{(4)} + a_{12}^{(4)}$, $h_{12} = a_{11}^{(1)} + a_{11}^{(2)}$, $h_{21} = a_{11}^{(1)} + a_{11}^{(3)}$ and $h_{22} = a_{11}^{(1)} + a_{11}^{(2)} + a_{11}^{(3)} + a_{11}^{(4)}$. It follows from (4.49)–(4.54) that

$$\psi(E_{11}) = \lambda E_{11} + h_{11} I_2 + \begin{pmatrix} a_{12}^{(2)} + a_{12}^{(4)} & a_{12}^{(2)} + a_{12}^{(4)} \\ a_{12}^{(2)} + a_{12}^{(4)} & 0 \end{pmatrix}, \tag{4.55}$$

$$\psi(E_{12}) = \lambda E_{12} + h_{12} I_2 + \begin{pmatrix} 0 & a_{12}^{(1)} \\ a_{12}^{(1)} & a_{12}^{(1)} \end{pmatrix}, \tag{4.56}$$

$$\psi(E_{21}) = \lambda E_{21} + h_{21} I_2 + \begin{pmatrix} 0 & a_{12}^{(1)} \\ a_{12}^{(1)} & a_{12}^{(1)} \end{pmatrix}, \tag{4.57}$$

$$\psi(E_{22}) = \lambda E_{22} + h_{22} I_2 + \begin{pmatrix} 0 & a_{12}^{(2)} + a_{12}^{(4)} + a_{12}^{(1)} \\ a_{12}^{(2)} + a_{12}^{(4)} + a_{12}^{(1)} & a_{12}^{(2)} + a_{12}^{(4)} + a_{12}^{(1)} \end{pmatrix}. \tag{4.58}$$

Let $\mu : M_2(\mathbb{F}_2) \to \mathbb{F}_2$ be the linear map defined by

$$\mu(E_{ij}) = h_{ij} \tag{4.59}$$

for every integer $1 \leqslant i, j \leqslant 2$, and let $\xi : M_2(\mathbb{F}_2) \to M_2(\mathbb{F}_2)$ be the map defined by

$$\xi(A) = \lambda A + \mu(A) I_2 \tag{4.60}$$

for every $A \in M_2(\mathbb{F}_2)$. Then ξ and $\psi + \xi$ are commuting additive maps on invertible matrices of $M_2(\mathbb{F}_2)$. Letting $\alpha = a_{12}^{(2)} + a_{12}^{(4)}$ and $\beta = a_{12}^{(1)}$, by (4.55)–(4.60), we get

$$(\psi + \xi)(A) = \alpha a_{11} Q + (\alpha a_{22} + \beta(a_{12} + a_{21} + a_{22}))R$$

for all $A = (a_{ij}) \in M_2(\mathbb{F}_2)$. It follows from Lemma 4.4.5 that there exists a matrix $H \in M_2(\mathbb{F}_2)$ such that

$$\psi(A) = \lambda A + \text{tr}\,(H^t A) I_2 + \Gamma_{\alpha,\beta}(A)$$

for all $A \in M_2(\mathbb{F}_2)$. This completes our proof. □

We remark that Example 1 in Franca (2012) can be derived from Theorem 4.4.6 by setting $\lambda = 0$, $H = E_{12} + E_{21}$ and $\alpha = \beta = 1$.

Remark: The results in this chapter have been published in Chooi et al. (2019).

CHAPTER 5: COMMUTING ADDITIVE MAPS ON RANK ONE UPPER TRIANGULAR MATRICES OVER FIELDS

5.1 Introduction

Motivated by an example of nonstandard commuting map on rank one matrices over fields when $n \geqslant 3$ in Franca (2013a), Franca (2017) studied commuting additive maps $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ on rank one matrices and discovered that the structure is much more complicated and quite different from the standard form (2.14). Let $T_n(\mathbb{F})$ be the ring of all $n \times n$ matrices over the field $\mathbb{F}$ with centre $Z(T_n(\mathbb{F}))$ and unity I_n. In view of Theorem 4.3.5 and with the aim to complete the study for commuting additive maps for all rank k upper triangular matrices, with $1 \leqslant k \leqslant n$ being a fixed integer, in this chapter we obtain a characterisation of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank one matrices. It is worth pointing out that the structure of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank one matrices is much more fertile. Surprisingly we obtain some irregular forms of commuting additive maps on rank one triangular matrices over fields in which their structures are considerably more complex and astonishing.

5.2 Irregular nonstandard examples

We start our discussion with some irregular nonstandard examples of commuting additive maps on rank one upper triangular matrices over fields. Throughout this section, unless stated otherwise, let $n \geqslant 2$ be an integer and let $\mathbb{F}$ be a field. We recall from Lemma 2.1.1 that a matrix $A \in T_n(\mathbb{F})$ is of rank one if and only if there exist a pair of positive integers $1 \leqslant s \leqslant t \leqslant n$ and invertible matrices $P, Q \in T_n(\mathbb{F})$ such that

$$A = PE_{st}Q. \tag{5.1}$$

Here, $E_{ij} \in T_n(\mathbb{F})$ is the standard matrix unit whose (i, j)th entry is one and zero elsewhere.

Example 5.2.1. Let $n \geqslant 3$ be an integer and let $\chi = (\tau_{ij}) \in T_n(\mathbb{F})$ be a strictly upper triangular matrix. Suppose that $\psi_\chi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is the linear map defined by

$$\psi_\chi(A) = \begin{pmatrix} x_1 & -\tau_{12}a_{12} & -\tau_{13}a_{13} & \cdots & -\tau_{1n}a_{1n} \\ 0 & x_2 & -\tau_{23}a_{23} & \cdots & -\tau_{2n}a_{2n} \\ 0 & 0 & x_3 & \cdots & -\tau_{3n}a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where

$$x_h = \begin{cases} \sum_{i=2}^{n} \tau_{1i} a_{ii} & \text{if } h = 1, \\ \sum_{i=1}^{h-1} \tau_{ih}\, a_{ii} + \sum_{i=h+1}^{n} \tau_{hi}\, a_{ii} & \text{if } 2 \leqslant h \leqslant n-1, \\ \sum_{i=1}^{n-1} \tau_{in} a_{ii} & \text{if } h = n. \end{cases}$$

Let $1 \leqslant s \leqslant t \leqslant n$ be integers. It is not difficult to see that

$$\psi_\chi(aE_{st}) = \begin{cases} \sum_{i=2}^{n} \tau_{1i} aE_{ii} & \text{if } s = t = 1, \\ \sum_{i=1}^{s-1} \tau_{is} aE_{ii} + \sum_{i=s+1}^{n} \tau_{si} aE_{ii} & \text{if } 1 < s = t < n, \\ \sum_{i=1}^{n-1} \tau_{in} aE_{ii} & \text{if } s = t = n, \\ -\tau_{st} aE_{st} & \text{if } s < t \end{cases}$$

for all $a \in \mathbb{F}$. For simplicity of notation, we write

$$\psi_\chi(aE_{st}) = \begin{cases} \sum_{i=1}^{s-1} \tau_{is} aE_{ii} + \sum_{i=s+1}^{n} \tau_{si} aE_{ii} & \text{if } s = t, \\ -\tau_{st} aE_{st} & \text{if } s < t \end{cases} \tag{5.2}$$

for all $a \in \mathbb{F}$ and integers $1 \leqslant s \leqslant t \leqslant n$, where it is understood that $\sum_{i=1}^{s-1} \tau_{is} aE_{ii} = 0$ when $s = 1$, and $\sum_{i=s+1}^{n} \tau_{si} aE_{ii} = 0$ when $s = n$.

We show that ψ_χ is commuting on rank one upper triangular matrices. Let $A =$

$(a_{ij}) \in T_n(\mathbb{F})$ be of rank one. By (5.1), there exist integers $1 \leqslant s \leqslant t \leqslant n$ and scalars $\lambda_1, \ldots, \lambda_s \in \mathbb{F}$ such that

$$A = \sum_{i=1}^{s} \lambda_i (a_{st} E_{it} + \cdots + a_{sn} E_{in})$$

with $a_{st} \neq 0$ and $\lambda_s = 1$. We argue in the following two cases:

Case 1: $s < t$. Then $a_{ii} = 0$ for $i = 1, \ldots, n$, and so

$$\psi_\chi(A) = -\sum_{i=1}^{s} (\tau_{it}(\lambda_i a_{st}) E_{it} + \cdots + \tau_{in}(\lambda_i a_{sn}) E_{in}).$$

Since $s < t$, it follows that $A\psi_\chi(A) = 0$ and $\psi_\chi(A)A = 0$.

Case 2: $s = t$. When $s = 1$, we have $A = \sum_{i=1}^{n} \lambda_1 a_{1i} E_{1i}$ and $\psi_\chi(A) = \sum_{i=2}^{n} \tau_{1i}\lambda_1 a_{11} E_{ii} - \sum_{i=2}^{n} \tau_{1i}\lambda_1 a_{1i} E_{1i}$. So $\psi_\chi(A)A = 0$ and $A\psi_\chi(A) = \sum_{i=2}^{n} \lambda_1 a_{1i} \tau_{1i} \lambda_1 a_{11} E_{1i} - \sum_{i=2}^{n} \lambda_1 a_{11} \tau_{1i} \lambda_1 a_{1i} E_{1i} = 0$. When $s = n$, we get $A = \sum_{i=1}^{n} \lambda_i a_{nn} E_{in}$ and $\psi_\chi(A) = \sum_{i=1}^{n-1} \tau_{in}\lambda_n a_{nn} E_{ii} - \sum_{i=1}^{n-1} \tau_{in}\lambda_i a_{nn} E_{in}$. Then $A\psi_\chi(A) = 0$ and $\psi_\chi(A)A = \sum_{i=1}^{n-1} \tau_{in}\lambda_n a_{nn}\lambda_i a_{nn} E_{in} - \sum_{i=1}^{n-1} \tau_{in}\lambda_i a_{nn}\lambda_n a_{nn} E_{in} = 0$. We now consider $1 < s < n$. Then $A = \sum_{i=1}^{s} \sum_{j=s}^{n} \lambda_i a_{sj} E_{ij}$ and

$$\begin{aligned}\psi_\chi(A) = \sum_{i=1}^{s-1} \tau_{is}\lambda_s a_{ss} E_{ii} + \sum_{i=s+1}^{n} \tau_{si}\lambda_s a_{ss} E_{ii} - \sum_{i=1}^{s-1} (\tau_{is}(\lambda_i a_{ss}) E_{is} + \cdots + \tau_{in}(\lambda_i a_{sn}) E_{in}) \\ - (\tau_{s,s+1}\lambda_s a_{s,s+1} E_{s,s+1} + \cdots + \tau_{sn}\lambda_s a_{sn} E_{sn}).\end{aligned}$$

We see that

$$\begin{aligned}A\psi_\chi(A) &= \sum_{j=s+1}^{n} \left(\sum_{i=1}^{s} \lambda_i a_{sj} E_{ij}\right)(\tau_{sj}\lambda_s a_{ss} E_{jj}) - \left(\sum_{i=1}^{s} \lambda_i a_{ss} E_{is}\right)\left(\sum_{j=s+1}^{n} \tau_{sj}\lambda_s a_{sj} E_{sj}\right) \\ &= \sum_{j=s+1}^{n} \sum_{i=1}^{s} \lambda_i a_{sj} \tau_{sj} \lambda_s a_{ss} E_{ij} - \sum_{i=1}^{s} \sum_{j=s+1}^{n} \lambda_i a_{ss} \tau_{sj} \lambda_s a_{sj} E_{ij} = 0,\end{aligned}$$

and

$$\begin{aligned}\psi_\chi(A)A &= \sum_{i=1}^{s-1} (\tau_{is}\lambda_s a_{ss} E_{ii}) \left(\sum_{j=s}^{n} \lambda_i a_{sj} E_{ij}\right) - \left(\sum_{i=1}^{s-1} \tau_{is}\lambda_i a_{ss} E_{is}\right)\left(\sum_{j=s}^{n} \lambda_s a_{sj} E_{sj}\right) \\ &= \sum_{i=1}^{s-1} \sum_{j=s}^{n} \tau_{is}\lambda_s a_{ss} \lambda_i a_{sj} E_{ij} - \sum_{i=1}^{s-1} \sum_{j=s}^{n} \tau_{is}\lambda_i a_{ss} \lambda_s a_{sj} E_{ij} = 0.\end{aligned}$$

Hence $[\psi_\chi(A), A] = 0$ for all rank one matrices $A \in T_n(\mathbb{F})$ as required. As a side remark, (Chooi, Mutalib, & Tan, 2021, Example 3.10) is a commuting linear map on rank one triangular matrices of this type.

Example 5.2.2. (Chooi, Mutalib, & Tan, 2021, Example 3.10) Let $\mathbb{F}$ be a field and let $\psi : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ be the additive map defined by

$$\psi(A) = a_{33}E_{22} + a_{22}E_{33} - a_{23}E_{23}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$. It is not difficult to see that $\psi(A)A = 0 = A\psi(A)$ for all rank one matrices $A \in T_3(\mathbb{F})$.

Example 5.2.3. Let $n \geqslant 3$ be an integer and let

$$\mathscr{F} = \bigcup_{1<s\leqslant t<n} \{\phi_{ij}^{(s,t)} : \mathbb{F} \to \mathbb{F} : 1 \leqslant i < s \text{ and } t < j \leqslant n\}$$

be a set of additive maps on $\mathbb{F}$. We define the additive map $\psi_{\mathscr{F}} : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ by

$$\psi_{\mathscr{F}}(A) = \sum_{1<s\leqslant t<n} \left(\sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a_{st})E_{ij} \right)$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$. Let $1 \leqslant p \leqslant q \leqslant n$ be integers. We see that

$$\psi_{\mathscr{F}}(aE_{pq}) = \begin{cases} 0 & \text{if } p = 1 \text{ or } q = n, \\ \sum_{i=1}^{p-1} \sum_{j=q+1}^{n} \phi_{ij}^{(p,q)}(a)E_{ij} & \text{if } 1 < p \leqslant q < n \end{cases} \tag{5.3}$$

for all $a \in \mathbb{F}$. We now show that $\psi_{\mathscr{F}}(A)A = 0 = A\psi_{\mathscr{F}}(A)$ for all rank one matrices $A \in T_n(\mathbb{F})$. Let $A = (a_{ij}) \in T_n(\mathbb{F})$ be of rank one. By (5.1), there exist integers $1 \leqslant p \leqslant q \leqslant n$ such that

$$A = \sum_{i=1}^{p} \sum_{j=q}^{n} a_{ij}E_{ij},$$

where $a_{pq} \neq 0$ and $(a_{iq}, \ldots, a_{in}), (a_{pq}, \ldots, a_{pn})$ are linearly dependent for $i = 1, \ldots, p-1$. Note that when $p = 1$, we have $A = \sum_{j=q}^{n} a_{1j}E_{1j}$, and thus $\psi_{\mathscr{F}}(A) = 0$ by (5.3). The claim is proved. Likewise, the result holds when $q = n$. We now consider $2 \leqslant p \leqslant q \leqslant$

$n-1$. Notice that

$$\psi_{\mathscr{F}}(A) = \sum_{s=2}^{p} \sum_{t=q}^{n-1} \psi_{st}(a_{st}),$$

where, for each pair of integers $2 \leqslant s \leqslant p$ and $q \leqslant t \leqslant n-1$, $\psi_{st} : \mathbb{F} \to T_n(\mathbb{F})$ is the additive map given by

$$\psi_{st}(a) = \sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a) E_{ij}$$

for all $a \in \mathbb{F}$. We have

$$A\,\psi_{st}(a_{st}) = \left(\sum_{i=1}^{p} \sum_{j=q}^{n} a_{ij} E_{ij} \right) \left(\sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a_{st}) E_{ij} \right) = 0$$

because $s-1 < p \leqslant q$, and

$$\psi_{st}(a_{st})A = \left(\sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a_{st}) E_{ij} \right) \left(\sum_{i=1}^{p} \sum_{j=q}^{n} a_{ij} E_{ij} \right) = 0$$

because $p \leqslant q < t+1$. Hence $A\,\psi_{st}(a_{st}) = 0 = \psi_{st}(a_{st})A$ for all $2 \leqslant s \leqslant p$ and $q \leqslant t \leqslant n-1$. It follows that $A\psi_{\mathscr{F}}(A) = 0 = \psi_{\mathscr{F}}(A)A$ for all rank one matrices $A \in T_n(\mathbb{F})$. We notice that (Chooi, Mutalib, & Tan, 2021, Examples 3.11, 3.12) are commuting additive maps on rank one triangular matrices of this type.

Example 5.2.4. (Chooi, Mutalib, & Tan, 2021, Example 3.11) Let $\mathbb{F}$ be a field and let $f, g : \mathbb{F} \to \mathbb{F}$ be additive maps. Let $\psi : T_4(\mathbb{F}) \to T_4(\mathbb{F})$ be the additive map defined by

$$\psi(A) = f(a_{22})E_{13} + g(a_{33})E_{24}$$

for all $A = (a_{ij}) \in T_4(\mathbb{F})$. It can be checked that $\psi(A)A = 0 = A\psi(A)$ for all rank one matrices $A \in T_4(\mathbb{F})$.

Example 5.2.5. (Chooi, Mutalib, & Tan, 2021, Example 3.12) Let $\mathbb{F}$ be a field and let $n \geqslant 3$ be an integer. Let $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ be the additive map defined by

$$\psi(A) = (\mu(a_{22}) + \eta(a_{n-1,n-1}))E_{1n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\mu, \eta : \mathbb{F} \to \mathbb{F}$ are additive maps. Then $\psi(A)A = 0 = A\psi(A)$ for all rank one matrices $A \in T_n(\mathbb{F})$.

Let $n \geqslant 3$ and let $\nabla_n = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leqslant i \leqslant j \leqslant n\} \backslash \{(1, n-1), (1, n), (2, n-1), (2, n)\}$.

Example 5.2.6. Let $n \geqslant 3$ be an integer and let

$$\Lambda = \bigcup_{(s,t)\in\nabla_n} \left\{ \lambda_{ij}^{(s,t)} \in \mathbb{F} : \ 1 \leqslant i < j < s \ \text{ or } \ t < i < j \leqslant n \right\}$$

be a set of scalars. Let $\psi_\Lambda : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ be the linear map defined by

$$\psi_\Lambda(A) = \sum_{(s,t)\in\nabla_n} \Psi_{st}(A) + \Phi_{st}(A)$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where for each $(s,t) \in \nabla_n$,

$$\Psi_{st}(A) = \begin{cases} 0 & \text{if } 1 \leqslant s \leqslant 2, \\ \Big(\sum_{1\leqslant i<j<s} \lambda_{ij}^{(s,t)} E_{ij}\Big)\big(\sum_{h=1}^{s-1} a_{st}E_{hh} - a_{ht}E_{hs}\big) & \text{if } 3 \leqslant s \leqslant n, \end{cases} \tag{5.4}$$

$$\Phi_{st}(A) = \begin{cases} \big(\sum_{h=t+1}^{n} a_{st}E_{hh} - a_{sh}E_{th}\big)\Big(\sum_{t<i<j\leqslant n} \lambda_{ij}^{(s,t)} E_{ij}\Big) & \text{if } 1 \leqslant t \leqslant n-2, \\ 0 & \text{if } n-1 \leqslant t \leqslant n \end{cases} \tag{5.5}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$. It can be shown that for each integer $1 \leqslant p \leqslant n$,

$$\psi_\Lambda(aE_{pp}) = \sum_{1\leqslant i<j<p} \lambda_{ij}^{(p,p)} aE_{ij} + \sum_{p<i<j\leqslant n} \lambda_{ij}^{(p,p)} aE_{ij} \tag{5.6}$$

for all $a \in \mathbb{F}$, where $\sum_{1\leqslant i<j<p} \lambda_{ij}^{(p,p)} E_{ij} = 0$ when $p = 1, 2$, and $\sum_{p<i<j\leqslant n} \lambda_{ij}^{(p,p)} E_{ij} = 0$ when $p = n-1, n$; and that for each pair of integers $1 \leqslant p < q \leqslant n$,

$$\psi_\Lambda(aE_{pq}) = \sum_{1\leqslant i<j<p} \lambda_{ij}^{(p,q)} aE_{ij} - \sum_{i=1}^{p-1} \sum_{j=p+1}^{q} \lambda_{ip}^{(j,q)} aE_{ij} + \sum_{q<i<j\leqslant n} \lambda_{ij}^{(p,q)} aE_{ij} - \sum_{i=p}^{q-1} \sum_{j=q+1}^{n} \lambda_{qj}^{(p,i)} aE_{ij} \tag{5.7}$$

for all $a \in \mathbb{F}$, where $\sum_{1\leqslant i<j<p} \lambda_{ij}^{(p,q)} E_{ij} = 0$ when $p = 1, 2$, $\sum_{i=1}^{p-1} \sum_{j=p+1}^{q} \lambda_{ip}^{(j,q)} E_{ij} = 0$ when $p = 1$, $\sum_{q<i<j\leqslant n} \lambda_{ij}^{(p,q)} E_{ij} = 0$ when $q = n-1, n$, and $\sum_{i=p}^{q-1} \sum_{j=q+1}^{n} \lambda_{qj}^{(p,i)} E_{ij} = 0$ when $q = n$.

We first claim (5.6). Let $1 \leqslant p \leqslant n$ be an integer and let $a \in \mathbb{F}$. First, note that $\Psi_{st}(aE_{pp}) = 0 = \Phi_{st}(aE_{pp})$ whenever $(s,t) \neq (p,p)$. When $p = 1, 2$, we have $\Psi_{pp}(aE_{pp}) = 0$ by (5.4), and

$$\psi_\Lambda(aE_{pp}) = \Phi_{pp}(aE_{pp}) = \left(\sum_{h=p+1}^{n} aE_{hh}\right)\left(\sum_{p<i<j\leqslant n} \lambda_{ij}^{(p,p)} E_{ij}\right) = \sum_{p<i<j\leqslant n} \lambda_{ij}^{(p,p)} aE_{ij}.$$

When $p = n-1, n$, we get $\Phi_{pp}(aE_{pp}) = 0$ by (5.5), and

$$\psi_\Lambda(aE_{pp}) = \Psi_{pp}(aE_{pp}) = \left(\sum_{1\leqslant i<j<p} \lambda_{ij}^{(p,p)} E_{ij}\right)\left(\sum_{h=1}^{p-1} aE_{hh}\right) = \sum_{1\leqslant i<j<p} \lambda_{ij}^{(p,p)} aE_{ij}.$$

Consider now $2 < p < n-1$. So

$$\psi_\Lambda(aE_{pp}) = \Psi_{pp}(aE_{pp}) + \Phi_{pp}(aE_{pp}) = \sum_{1\leqslant i<j<p} \lambda_{ij}^{(p,p)} aE_{ij} + \sum_{p<i<j\leqslant n} \lambda_{ij}^{(p,p)} aE_{ij}.$$

Hence (5.6) is proved.

We next prove (5.7). Let $1 \leqslant p < q \leqslant n$ be integers and let $a \in \mathbb{F}$. By (5.4) and (5.5), we notice that $\Psi_{st}(aE_{pq}) = 0$ whenever $t \neq q$, $\Psi_{st}(aE_{pq}) = 0$ whenever $p > s$, $\Phi_{st}(aE_{pq}) = 0$ whenever $s \neq p$, and $\Phi_{st}(aE_{pq}) = 0$ whenever $q < t$. We thus obtain

$$\psi_\Lambda(aE_{pq}) = \Psi_{pq}(aE_{pq}) + \sum_{j=p+1}^{q} \Psi_{jq}(aE_{pq}) + \Phi_{pq}(aE_{pq}) + \sum_{i=p}^{q-1} \Phi_{pi}(aE_{pq}). \tag{5.8}$$

In view of (5.4), $\Psi_{pq}(aE_{pq}) = 0$ when $p = 1, 2$. When $p \geqslant 3$, we have

$$\Psi_{pq}(aE_{pq}) = \left(\sum_{1\leqslant i<j<p} \lambda_{ij}^{(p,q)} E_{ij}\right)\left(\sum_{h=1}^{p-1} aE_{hh}\right) = \sum_{1\leqslant i<j<p} \lambda_{ij}^{(p,q)} aE_{ij}. \tag{5.9}$$

Next, in view of (5.4), since $\Psi_{jq}(aE_{pq}) = 0$, $j = p+1, \ldots, q$, when $p = 1$, it follows that

$$\sum_{j=2}^{q} \Psi_{jq}(aE_{1q}) = 0. \tag{5.10}$$

When $p \geqslant 2$, we have

$$\sum_{j=p+1}^{q} \Psi_{jq}(aE_{pq}) = \sum_{j=p+1}^{q} \left(\sum_{1 \leqslant i < \ell < j} \lambda_{i\ell}^{(j,q)} E_{i\ell} \right) (-aE_{pj}) = \sum_{j=p+1}^{q} \left(\sum_{1 \leqslant i < p} -\lambda_{ip}^{(j,q)} aE_{ij} \right)$$

and thus

$$\sum_{j=p+1}^{q} \Psi_{jq}(aE_{pq}) = -\sum_{i=1}^{p-1} \sum_{j=p+1}^{q} \lambda_{ip}^{(j,q)} aE_{ij}. \tag{5.11}$$

Likewise, in view of (5.5), we have $\Phi_{pq}(aE_{pq}) = 0$ when $q = n-1, n$. When $q \leqslant n-2$, we get

$$\Phi_{pq}(aE_{pq}) = \left(\sum_{h=q+1}^{n} aE_{hh} \right) \left(\sum_{q<i<j \leqslant n} \lambda_{ij}^{(p,q)} E_{ij} \right) = \sum_{q<i<j \leqslant n} \lambda_{ij}^{(p,q)} aE_{ij}. \tag{5.12}$$

By (5.5), since $\Phi_{pi}(aE_{pq}) = 0$, $i = p, \ldots, q-1$, when $q = n$, it follows that

$$\sum_{i=p}^{n-1} \Phi_{pi}(aE_{pn}) = 0. \tag{5.13}$$

When $q \leqslant n-1$, we have

$$\sum_{i=p}^{q-1} \Phi_{pi}(aE_{pq}) = \sum_{i=p}^{q-1} (-aE_{iq}) \left(\sum_{i<\ell<j \leqslant n} \lambda_{\ell j}^{(p,i)} E_{\ell j} \right) = \sum_{i=p}^{q-1} \left(\sum_{q<j \leqslant n} -a\lambda_{qj}^{(p,i)} E_{ij} \right)$$

and thus

$$\sum_{i=p}^{q-1} \Phi_{pi}(aE_{pq}) = \sum_{i=p}^{q-1} \sum_{j=q+1}^{n} -\lambda_{qj}^{(p,i)} aE_{ij}. \tag{5.14}$$

Consequently, in view of (5.8)–(5.14), this completes the proof of (5.7).

We next show that ψ_Λ is commuting on rank one upper triangular matrices. Let $A = (a_{ij}) \in T_n(\mathbb{F})$ be of rank one. By (5.1), there exist integers $1 \leqslant p \leqslant q \leqslant n$ such that

$$A = \sum_{i=1}^{p} \sum_{j=q}^{n} a_{ij} E_{ij}, \tag{5.15}$$

where $a_{pq} \neq 0$, $\{(a_{iq}, \ldots, a_{in}), (a_{jq}, \ldots, a_{jn})\}$ is linearly dependent for every $1 \leqslant i, j \leqslant p$, and $\{(a_{1i}, \ldots, a_{pi}), (a_{1j}, \ldots, a_{pj})\}$ is linearly dependent for every $q \leqslant i, j \leqslant n$.

We first claim that

$$\Psi_{st}(A)A = 0 = A\Psi_{st}(A) \tag{5.16}$$

for every $(s,t) \in \nabla_n$. Let $(s,t) \in \nabla_n$. Clearly, (5.16) holds when $s = 1, 2$. Consider now $s \geqslant 3$. By (5.15), since $a_{ij} = 0$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j < q$, it follows that

$$\sum_{h=1}^{s-1} a_{st} E_{hh} A = \sum_{h=1}^{s-1} a_{st}(a_{hq}E_{hq} + \cdots + a_{hn}E_{hn}),$$

$$\sum_{h=1}^{s-1} a_{ht} E_{hs} A = \sum_{h=1}^{s-1} a_{ht}(a_{sq}E_{hq} + \cdots + a_{sn}E_{hn}).$$

If $a_{st} = 0$, then either $(a_{1t}, \ldots, a_{nt}) = 0$ or $(a_{s1}, \ldots, a_{sn}) = 0$ since A is of rank one. Hence $\Psi_{st}(A)A = 0$ by (5.4). If $a_{st} \neq 0$, then $s \leqslant p$ and $t \geqslant q$ by (5.15). Thus, for each $1 \leqslant h < s$, there exists $\alpha_h \in \mathbb{F}$ such that $(a_{hq}, \ldots, a_{hn}) = \alpha_h(a_{sq}, \ldots, a_{sn})$ and $a_{ht} = \alpha_h a_{st}$. Consequently,

$$\Psi_{st}(A)A = \left(\sum_{1 \leqslant i < j < s} \lambda_{ij}^{(s,t)} E_{ij}\right)\left(\sum_{h=1}^{s-1} a_{st} E_{hh} A - \sum_{h=1}^{s-1} a_{ht} E_{hs} A\right) = 0.$$

We now proceed to claim $A\Psi_{st}(A) = 0$. By (5.4), we notice that

$$\Psi_{st}(A) = \sum_{i=1}^{s-2} \sum_{j=i+1}^{s-1} \lambda_{ij}^{(s,t)} a_{st} E_{ij} - \sum_{i=1}^{s-2} \left(\sum_{j=i+1}^{s-1} \lambda_{ij}^{(s,t)} a_{jt}\right) E_{is}$$

because $\sum_{1 \leqslant i < j \leqslant s-1} \lambda_{ij}^{(s,t)} E_{ij} = \sum_{i=1}^{s-2} \sum_{j=i+1}^{s-1} \lambda_{ij}^{(s,t)} E_{ij}$. If $p = 1$, then $a_{st} = 0$ and $a_{jt} = 0$ for $j = 2, \ldots, s-1$, and so $\Psi_{st}(A) = 0$. Therefore Claim (5.16) is proved. Consider now $p \geqslant 2$. Suppose that $a_{st} = 0$. By (5.15), since the j-th column vector of A is zero for $j = 1, \ldots, q-1$, it follows that

$$AE_{ij} = 0 \quad \text{for } i = 1, \ldots, q-1 \text{ and } j = 1, \ldots, n. \tag{5.17}$$

We argue in two cases:

Case I: $p \geqslant s-1$. By (5.17), we obtain

$$A\Psi_{st}(A) = -\sum_{i=1}^{s-2} \left(\sum_{j=i+1}^{s-1} \lambda_{ij}^{(s,t)} a_{jt}\right) AE_{is} = 0.$$

Case II: $p < s - 1$. Then $a_{jt} = 0$ for $j = p+1, \dots, s-1$ by (5.15). We thus have

$$A\Psi_{st}(A) = -\sum_{i=1}^{p-1} \left(\sum_{j=i+1}^{p} \lambda_{ij}^{(s,t)} a_{jt} \right) AE_{is} = 0$$

by (5.17). Consider now $a_{st} \neq 0$. Then $s \leqslant p$ because $a_{ij} = 0$ for all $i > p$ by (5.15). It follows from (5.17) that

$$A\Psi_{st}(A) = \sum_{i=1}^{s-2} \sum_{j=i+1}^{s-1} \lambda_{ij}^{(s,t)} a_{st} AE_{ij} - \sum_{i=1}^{s-2} \left(\sum_{j=i+1}^{s-1} \lambda_{ij}^{(s,t)} a_{jt} \right) AE_{is} = 0$$

as required. Consequently, Claim (5.16) is proved.

We now proceed to show

$$\Phi_{st}(A)A = 0 = A\Phi_{st}(A) \tag{5.18}$$

for all $(s,t) \in \nabla_n$. Let $(s,t) \in \nabla_n$. Claim (5.18) is proved when $t = n-1, n$. Consider $t \leqslant n-2$. By (5.15), since $a_{ij} = 0$ for all $p < i \leqslant n$ and $1 \leqslant j \leqslant n$, it follows that

$$\sum_{h=t+1}^{n} a_{st} AE_{hh} = \sum_{h=t+1}^{n} a_{st}(a_{1h}E_{1h} + \cdots + a_{ph}E_{ph}),$$

$$\sum_{h=t+1}^{n} a_{sh} AE_{th} = \sum_{h=t+1}^{n} a_{sh}(a_{1t}E_{1h} + \cdots + a_{pt}E_{ph}).$$

If $a_{st} = 0$, then either $(a_{1t}, \dots, a_{nt}) = 0$ or $(a_{s1}, \dots, a_{sn}) = 0$. Thus $A\Phi_{st}(A) = 0$ by (5.5). If $a_{st} \neq 0$, then $s \leqslant p$ and $t \geqslant q$. For each $t < h \leqslant n$, there exists $\beta_h \in \mathbb{F}$ such that $(a_{1h}, \dots, a_{ph}) = \beta_h(a_{1t}, \dots, a_{pt})$ and $a_{sh} = \beta_h a_{st}$. Hence $A\Phi_{st}(A) = 0$. We now claim $\Phi_{st}(A)A = 0$. By (5.5), we note that

$$\Phi_{st}(A) = \sum_{j=t+2}^{n} \sum_{i=t+1}^{j-1} a_{st}\lambda_{ij}^{(s,t)} E_{ij} - \sum_{j=t+2}^{n} \left(\sum_{i=t+1}^{j-1} a_{si}\lambda_{ij}^{(s,t)} \right) E_{tj}$$

because $\sum_{t<i<j\leqslant n} \lambda_{ij}^{(s,t)} E_{ij} = \sum_{j=t+2}^{n} \sum_{i=t+1}^{j-1} \lambda_{ij}^{(s,t)} E_{ij}$. If $q = n$, then $a_{st} = 0$ and $a_{si} = 0$ for $i = t+1, \dots, n-1$, and thus $\Phi_{st}(A) = 0$. Hence the claim is proved. Consider now $q \leqslant n-1$. Suppose that $a_{st} = 0$. Since the i-th row vector of A is zero for

$i = p+1, \dots, n$, we have

$$E_{ij}A = 0 \quad \text{for } j = p+1, \dots, n \text{ and } i = 1, \dots, n. \tag{5.19}$$

If $q \leqslant t+1$, then, by virtue of (5.19), we have

$$\Phi_{st}(A)A = -\sum_{j=t+2}^{n} \left(\sum_{i=t+1}^{j-1} a_{si}\lambda_{ij}^{(s,t)} \right) E_{tj}A = 0.$$

If $q > t+1$, then $a_{si} = 0$ for $i = t+1, \dots, q-1$ by (5.15). We thus have

$$\Phi_{st}(A)A = -\sum_{j=q+1}^{n} \left(\sum_{i=q}^{j-1} a_{si}\lambda_{ij}^{(s,t)} \right) E_{tj}A = 0$$

by (5.19). Consider now $a_{st} \neq 0$. Then $t \geqslant q$. It follows from (5.19) that

$$\Phi_{st}(A)A = \sum_{j=t+2}^{n} \sum_{i=t+1}^{j-1} a_{st}\lambda_{ij}^{(s,t)} E_{ij}A - \sum_{j=t+2}^{n} \left(\sum_{i=t+1}^{j-1} a_{si}\lambda_{ij}^{(s,t)} \right) E_{tj}A = 0.$$

Claim (5.18) is proved. Therefore $\psi_\Lambda(A)A = 0 = A\psi_\Lambda(A)$ for all rank one matrices $A \in T_n(\mathbb{F})$. We remark that Example 3.2.2 is a commuting additive map on rank one triangular matrices of this type.

5.3 Main results

We obtain a characterisation of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank one triangular matrices over an arbitrary field $\mathbb{F}$ in the following two results.

Theorem 5.3.1. *Let $\mathbb{F}$ be a field. Then $\psi : T_2(\mathbb{F}) \to T_2(\mathbb{F})$ is a commuting additive map on rank one matrices if and only if there exists a scalar $\lambda \in \mathbb{F}$ and an additive map $\mu : T_2(\mathbb{F}) \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_2$$

for every $A \in T_2(\mathbb{F})$.

Let $n \geqslant 3$ be an integer and let $\mathbb{F}$ be a field. We now give a complete characterisation of commuting additive maps $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ on rank one triangular matrices. Recall that $\nabla_n = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leqslant i \leqslant j \leqslant n\} \backslash \{(1,n-1),(1,n),(2,n-1),(2,n)\}$.

Theorem 5.3.2. *Let $\mathbb{F}$ be a field and let $n \geqslant 3$ be an integer. Then $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is a commuting additive map on rank one matrices if and only if there exists a scalar $\lambda \in \mathbb{F}$, an additive map $\mu : T_n(\mathbb{F}) \to \mathbb{F}$, a strictly upper triangular matrix $\chi = (\tau_{ij}) \in T_n(\mathbb{F})$, a set of additive maps $\mathscr{F} = \bigcup_{1<s\leqslant t<n} \{\phi_{ij}^{(s,t)} : \mathbb{F} \to \mathbb{F} : 1 \leqslant i < s \text{ and } t < j \leqslant n\}$ and a set of scalars $\Lambda = \bigcup_{(s,t)\in\nabla_n} \left\{\lambda_{ij}^{(s,t)} \in \mathbb{F} : 1 \leqslant i < j < s \text{ or } t < i < j \leqslant n\right\}$ such that*

$$\psi(A) = \lambda A + \mu(A) I_n + \psi_\chi(A) + \psi_{\mathscr{F}}(A) + \psi_\Lambda(A)$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$. Here, $\psi_\chi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is the linear map defined by

$$\psi_\chi(A) = \begin{pmatrix} x_1 & -\tau_{12}a_{12} & -\tau_{13}a_{13} & \cdots & -\tau_{1n}a_{1n} \\ 0 & x_2 & -\tau_{23}a_{23} & \cdots & -\tau_{2n}a_{2n} \\ 0 & 0 & x_3 & \cdots & -\tau_{3n}a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix} \tag{5.20}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where

$$x_h = \begin{cases} \sum_{i=2}^{n} \tau_{1i} a_{ii} & \text{if } h = 1, \\ \sum_{i=1}^{h-1} \tau_{ih}\, a_{ii} + \sum_{i=h+1}^{n} \tau_{hi}\, a_{ii} & \text{if } 2 \leqslant h \leqslant n-1, \\ \sum_{i=1}^{n-1} \tau_{in} a_{ii} & \text{if } h = n, \end{cases}$$

and $\psi_{\mathscr{F}} : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is the additive map defined by

$$\psi_{\mathscr{F}}(A) = \sum_{1<s\leqslant t<n} \left(\sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a_{st}) E_{ij} \right) \tag{5.21}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, and $\psi_\Lambda : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is the linear map defined by

$$\psi_\Lambda(A) = \sum_{(s,t)\in\nabla_n} \Psi_{st}(A) + \Phi_{st}(A) \tag{5.22}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$*, where for each* $(s,t) \in \nabla_n$*,*

$$\Psi_{st}(A) = \begin{cases} 0 & \text{if } 1 \leqslant s \leqslant 2, \\ \Big(\sum_{1\leqslant i<j<s} \lambda_{ij}^{(s,t)} E_{ij}\Big)\big(\sum_{h=1}^{s-1} a_{st}E_{hh} - a_{ht}E_{hs}\big) & \text{if } 3 \leqslant s \leqslant n, \end{cases}$$

$$\Phi_{st}(A) = \begin{cases} \big(\sum_{h=t+1}^{n} a_{st}E_{hh} - a_{sh}E_{th}\big)\Big(\sum_{t<i<j\leqslant n} \lambda_{ij}^{(s,t)} E_{ij}\Big) & \text{if } 1 \leqslant t \leqslant n-2, \\ 0 & \text{if } n-1 \leqslant t \leqslant n \end{cases}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$*.*

5.4 Proofs

Let $n = 2$ and let $\mathbb{F}$ be a field. We first prove Theorem 5.3.1.

Proof of Theorem 5.3.1. The sufficiency is clear. We now consider the necessity. For each pair of integers $1 \leqslant i \leqslant j \leqslant 2$, there exist additive maps $f_{ij}, g_{ij}, h_{ij} : \mathbb{F} \to \mathbb{F}$ such that

$$\psi(aE_{ij}) = \begin{pmatrix} f_{ij}(a) & h_{ij}(a) \\ 0 & g_{ij}(a) \end{pmatrix}$$

for all $a \in \mathbb{F}$. Since $0 = [\psi(aE_{ij}), aE_{ij}] = \psi(aE_{ij})aE_{ij} - aE_{ij}\psi(aE_{ij})$ for all $a \in \mathbb{F}$ and $1 \leqslant i \leqslant j \leqslant 2$, it follows that $h_{11} = h_{22} = 0$ and $g_{12} = f_{12}$. Next, $0 = [\psi(aE_{11} + bE_{12}), aE_{11} + bE_{12}] = \psi(aE_{11} + bE_{12})(aE_{11} + bE_{12}) - (aE_{11} + bE_{12})\psi(aE_{11} + bE_{12})$ for all $a, b \in \mathbb{F}$ implies that

$$ah_{12}(b) + b(g_{11}(a) - f_{11}(a)) = 0 \tag{5.23}$$

for all $a, b \in \mathbb{F}$. Taking $a = 1$ in (5.23), we get $h_{12}(b) = \lambda b$ for all $b \in \mathbb{F}$ where $\lambda = f_{11}(1) - g_{11}(1)$. Setting $b = 1$ in (5.23), we obtain $f_{11}(a) = g_{11}(a) + \lambda a$ for all $a \in \mathbb{F}$. Likewise, considering $[\psi(bE_{12} + aE_{22}), bE_{12} + aE_{22}] = 0$ for all $a, b \in \mathbb{F}$, we obtain $g_{22}(a) = f_{22}(a) + \lambda a$ for all $a \in \mathbb{F}$. Let $\mu : T_2(\mathbb{F}) \to \mathbb{F}$ be the additive map defined by

$$\mu(A) = g_{11}(a_{11}) + g_{12}(a_{12}) + f_{22}(a_{22})$$

for all $A = (a_{ij}) \in T_2(\mathbb{F})$. Then $\psi(A) = \lambda A + \mu(A) I_2$ for all $A \in T_2(\mathbb{F})$ as desired. $\square$

In what follows, let $n \geqslant 3$ be an integer and let $\mathbb{F}$ be a field. Recall that $\nabla_n = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leqslant i \leqslant j \leqslant n\} \backslash \{(1, n-1), (1, n), (2, n-1), (2, n)\}$. We are now ready to prove Theorem 5.3.2.

Proof of Theorem 5.3.2. It is easily seen that $A \mapsto \lambda A + \mu(A) I_n$ is a commuting additive map on rank one matrices $A \in T_n(\mathbb{F})$. Together with Examples 5.2.1, 5.2.3 and 5.2.6, the sufficiency is proved. We now proceed to show the necessity. For each pair of integers $1 \leqslant s \leqslant t \leqslant n$, there are additive maps $\phi_{ij}^{(s,t)} : \mathbb{F} \to \mathbb{F}$, $i, j = 1, \ldots, n$, such that

$$\psi(aE_{st}) = \sum_{1 \leqslant i \leqslant j \leqslant n} \phi_{ij}^{(s,t)}(a) E_{ij} \tag{5.24}$$

for all $a \in \mathbb{F}$. Since $[\psi(aE_{st}), aE_{st}] = 0$ for all $a \in \mathbb{F}$, it follows from (5.24) that

$$\left(\sum_{1 \leqslant i \leqslant j \leqslant n} \phi_{ij}^{(s,t)}(a) E_{ij} \right) aE_{st} - aE_{st} \left(\sum_{1 \leqslant i \leqslant j \leqslant n} a\phi_{ij}^{(s,t)}(a) E_{ij} \right) = 0$$

for all $a \in \mathbb{F}$. Since $E_{ij} E_{st} = \delta_{js} E_{it}$ for any $1 \leqslant i, j, s, t \leqslant n$, it follows that

$$\left(\sum_{1 \leqslant i \leqslant j \leqslant n} \phi_{ij}^{(s,t)}(a) a (\delta_{js} E_{it}) \right) - \left(\sum_{1 \leqslant i \leqslant j \leqslant n} a\phi_{ij}^{(s,t)}(a) (\delta_{ti} E_{sj}) \right) = 0$$

for all $a \in \mathbb{F}$, where δ_{ij} is the Kronecker delta. We thus obtain

$$a(\phi_{ss}^{(s,t)}(a) - \phi_{tt}^{(s,t)}(a)) E_{st} + \sum_{i=1}^{s-1} a\phi_{is}^{(s,t)}(a) E_{it} - \sum_{j=t+1}^{n} a\phi_{tj}^{(s,t)}(a) E_{sj} = 0$$

for all $a \in \mathbb{F}$ and integers $1 \leqslant s \leqslant t \leqslant n$. Then for every $1 \leqslant s \leqslant t \leqslant n$,

$$\phi_{is}^{(s,t)} = 0 \quad \text{for } i = 1, \ldots, s-1, \tag{5.25}$$

$$\phi_{tj}^{(s,t)} = 0 \quad \text{for } j = t+1, \ldots, n, \tag{5.26}$$

$$\phi_{ss}^{(s,t)} = \phi_{tt}^{(s,t)}. \tag{5.27}$$

When $s = t$, it follows from (5.25) and (5.26) that for each $1 \leqslant s \leqslant n$,

$$\phi_{is}^{(s,s)} = 0 \quad \text{for } i = 1, \ldots, s-1, \tag{5.28}$$

$$\phi_{sj}^{(s,s)} = 0 \quad \text{for } j = s+1, \ldots, n. \tag{5.29}$$

We first study the structure of $\psi(aE_{ss})$ for every $1 \leqslant s \leqslant n$ and $a \in \mathbb{F}$. Let $1 \leqslant s < t \leqslant n$ be integers and let $a, b \in \mathbb{F}$. By virtue of $[\psi(X), X] = 0$ for all $X \in \{aE_{ss} + bE_{st}, aE_{ss}, bE_{st}\}$, we obtain $[\psi(aE_{ss}), bE_{st}] + [\psi(bE_{st}), aE_{ss}] = 0$. It follows from (5.24) that

$$\sum_{i=1}^{s} \phi_{is}^{(s,s)}(a)bE_{it} - \sum_{j=t}^{n} \phi_{tj}^{(s,s)}(a)bE_{sj} + \sum_{i=1}^{s} \phi_{is}^{(s,t)}(b)aE_{is} - \sum_{j=s}^{n} \phi_{sj}^{(s,t)}(b)aE_{sj} = 0. \tag{5.30}$$

Since $s < t$, it follows from (5.30) that

$$\begin{aligned}
(\phi_{ss}^{(s,s)}(a)b - \phi_{tt}^{(s,s)}(a)b - \phi_{st}^{(s,t)}(b)a)E_{st} + \sum_{i=1}^{s-1} \phi_{is}^{(s,s)}(a)bE_{it} - \sum_{j=t+1}^{n} \phi_{tj}^{(s,s)}(a)bE_{sj} \\
+ \phi_{ss}^{(s,t)}(b)aE_{ss} + \sum_{i=1}^{s-1} \phi_{is}^{(s,t)}(b)aE_{is} \\
- \phi_{ss}^{(s,t)}(b)aE_{ss} - \sum_{j=s+1}^{t-1} \phi_{sj}^{(s,t)}(b)aE_{sj} - \sum_{j=t+1}^{n} \phi_{sj}^{(s,t)}(b)aE_{sj} = 0
\end{aligned}$$

for all $a, b \in \mathbb{F}$. By (5.25) and (5.28), we see that $\phi_{is}^{(s,s)} = 0 = \phi_{is}^{(s,t)}$ for $i = 1, \ldots, s-1$. Then

$$\begin{aligned}
(\phi_{ss}^{(s,s)}(a)b &- \phi_{tt}^{(s,s)}(a)b - \phi_{st}^{(s,t)}(b)a)E_{st} \\
&- \sum_{j=s+1}^{t-1} \phi_{sj}^{(s,t)}(b)aE_{sj} - \sum_{j=t+1}^{n} (\phi_{sj}^{(s,t)}(b)a + \phi_{tj}^{(s,s)}(a)b)E_{sj} = 0
\end{aligned}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leqslant s < t \leqslant n$. Consequently, for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ss}^{(s,s)}(a)b = \phi_{tt}^{(s,s)}(a)b + \phi_{st}^{(s,t)}(b)a \quad \text{for all } a, b \in \mathbb{F}, \tag{5.31}$$

$$\phi_{sj}^{(s,t)}(b)a = -\phi_{tj}^{(s,s)}(a)b \quad \text{for all } a, b \in \mathbb{F},\ j = t+1, \ldots, n, \tag{5.32}$$

$$\phi_{sj}^{(s,t)} = 0 \quad \text{for } j = s+1, \ldots, t-1. \tag{5.33}$$

By (5.31) and (5.32), we get $\phi_{st}^{(s,t)}$ and $\phi_{sj}^{(s,t)}, \phi_{tj}^{(s,s)}$, $j = t+1, \ldots, n$, are linear maps on $\mathbb{F}$. Therefore, for each pair of integers $1 \leqslant s < t \leqslant n$, there exist scalars $\tau_{st}, \lambda_{tj}^{(s,s)} \in \mathbb{F}$, $j = t+1, \ldots, n$, such that

$$\phi_{st}^{(s,t)}(a) = -\tau_{st} a \quad \text{for all } a \in \mathbb{F}, \tag{5.34}$$

$$\phi_{tt}^{(s,s)}(a) = \phi_{ss}^{(s,s)}(a) + \tau_{st} a \quad \text{for all } a \in \mathbb{F}, \tag{5.35}$$

$$\phi_{tj}^{(s,s)}(a) = \lambda_{tj}^{(s,s)} a = -\phi_{sj}^{(s,t)}(a) \quad \text{for all } a \in \mathbb{F},\ j = t+1, \ldots, n. \tag{5.36}$$

Likewise, let $1 \leqslant r < s \leqslant n$ be integers and let $a, b \in \mathbb{F}$. By $[\psi(X), X] = 0$ for every $X \in \{aE_{ss} + bE_{rs}, aE_{ss}, bE_{rs}\}$, we obtain $[\psi(aE_{ss}), bE_{rs}] + [\psi(bE_{rs}), aE_{ss}] = 0$. By (5.24), we get

$$\sum_{i=1}^{r} \phi_{ir}^{(s,s)}(a) b E_{is} - \sum_{j=s}^{n} \phi_{sj}^{(s,s)}(a) b E_{rj} + \sum_{i=1}^{s} \phi_{is}^{(r,s)}(b) a E_{is} - \sum_{j=s}^{n} \phi_{sj}^{(r,s)}(b) a E_{sj} = 0. \tag{5.37}$$

Since $r < s$, it follows from (5.37) that

$$\begin{aligned}(\phi_{rr}^{(s,s)}(a)b \; - \; & \phi_{ss}^{(s,s)}(a)b + \phi_{rs}^{(r,s)}(b)a)E_{rs} \\ & + \sum_{i=1}^{r-1} \phi_{ir}^{(s,s)}(a) b E_{is} - \sum_{j=s+1}^{n} \phi_{sj}^{(s,s)}(a) b E_{rj} \\ & + \sum_{i=1}^{r-1} \phi_{is}^{(r,s)}(b) a E_{is} + \sum_{i=r+1}^{s-1} \phi_{is}^{(r,s)}(b) a E_{is} - \sum_{j=s+1}^{n} \phi_{sj}^{(r,s)}(b) a E_{sj} = 0\end{aligned}$$

for all $a, b \in \mathbb{F}$. By (5.26) and (5.29), we obtain

$$\begin{aligned}(\phi_{rr}^{(s,s)}(a)b \; - \; & \phi_{ss}^{(s,s)}(a)b + \phi_{rs}^{(r,s)}(b)a)E_{rs} \\ & + \sum_{i=1}^{r-1} (\phi_{ir}^{(s,s)}(a) b + \phi_{is}^{(r,s)}(b) a) E_{is} + \sum_{i=r+1}^{s-1} \phi_{is}^{(r,s)}(b) a E_{is} = 0\end{aligned}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leqslant r < s \leqslant n$. Hence for every $1 \leqslant r < s \leqslant n$,

$$\phi_{ss}^{(s,s)}(a)b = \phi_{rr}^{(s,s)}(a)b + \phi_{rs}^{(r,s)}(b)a \quad \text{for all } a, b \in \mathbb{F}, \tag{5.38}$$

$$\phi_{is}^{(r,s)}(b)a = -\phi_{ir}^{(s,s)}(a)b \quad \text{for all } a, b \in \mathbb{F},\ i = 1, \ldots, r-1, \tag{5.39}$$

$$\phi_{is}^{(r,s)} = 0 \quad \text{for all } i = r+1, \ldots, s-1. \tag{5.40}$$

In view of (5.34), we see that $\phi_{rs}^{(r,s)}(b) = -\tau_{rs} b$ for all $b \in \mathbb{F}$. By (5.39), we have $\phi_{is}^{(r,s)}, \phi_{ir}^{(s,s)}$, $i = 1, \ldots, r-1$, are linear maps. Then for each $r = 1, \ldots, s-1$, there exist scalars $\lambda_{ir}^{(s,s)} \in \mathbb{F}$, $i = 1, \ldots, r-1$, such that

$$\phi_{rr}^{(s,s)}(a) = \phi_{ss}^{(s,s)}(a) + \tau_{rs} a \quad \text{for all } a \in \mathbb{F}, \tag{5.41}$$

$$\phi_{ir}^{(s,s)}(a) = \lambda_{ir}^{(s,s)} a = -\phi_{is}^{(r,s)}(a) \quad \text{for all } a \in \mathbb{F},\ i = 1, \ldots, r-1. \tag{5.42}$$

By (5.35) and (5.41), for each $1 \leqslant s \leqslant n$,

$$\sum_{i=1}^{n} \phi_{ii}^{(s,s)}(a) E_{ii} = \phi_{ss}^{(s,s)}(a) I_n + \sum_{i=1}^{s-1} \tau_{is} a E_{ii} + \sum_{i=s+1}^{n} \tau_{si} a E_{ii} \tag{5.43}$$

for all $a \in \mathbb{F}$, where $\sum_{i=1}^{s-1} \tau_{is} a E_{ii} = 0$ when $s = 1$, and $\sum_{s+1}^{n} \tau_{si} a E_{ii} = 0$ when $s = n$. By virtue of (5.28), (5.29), (5.36) and (5.42), for each $1 \leqslant s \leqslant n$,

$$\sum_{1 \leqslant i < j \leqslant n} \phi_{ij}^{(s,s)}(a) E_{ij} = \sum_{1 \leqslant i < j < s} \lambda_{ij}^{(s,s)} a E_{ij} + \sum_{s < i < j \leqslant n} \lambda_{ij}^{(s,s)} a E_{ij} + \sum_{i=1}^{s-1} \sum_{j=s+1}^{n} \phi_{ij}^{(s,s)}(a) E_{ij} \tag{5.44}$$

for all $a \in \mathbb{F}$. Here, $\sum_{1 \leqslant i < j < s} \lambda_{ij}^{(s,s)} a E_{ij} = 0$ when $s \in \{1, 2\}$, $\sum_{s < i < j \leqslant n} \lambda_{ij}^{(s,s)} a E_{ij} = 0$ when $s \in \{n-1, n\}$ and $\sum_{i=1}^{s-1} \sum_{j=s+1}^{n} \phi_{ij}^{(s,s)}(a) E_{ij} = 0$ when $s \in \{1, n\}$. By (5.24), (5.43) and (5.44), for each $1 \leqslant s \leqslant n$,

$$\begin{aligned} \psi(aE_{ss}) = \phi_{ss}^{(s,s)}(a) I_n &+ \sum_{i=1}^{s-1} \tau_{is} a E_{ii} + \sum_{i=s+1}^{n} \tau_{si} a E_{ii} + \sum_{1 \leqslant i < j < s} \lambda_{ij}^{(s,s)} a E_{ij} \\ &+ \sum_{s < i < j \leqslant n} \lambda_{ij}^{(s,s)} a E_{ij} + \sum_{i=1}^{s-1} \sum_{j=s+1}^{n} \phi_{ij}^{(s,s)}(a) E_{ij} \end{aligned} \tag{5.45}$$

for all $a \in \mathbb{F}$.

We now continue to study the structure of $\psi(aE_{st})$ for every $1 \leqslant s < t \leqslant n$ and $a \in \mathbb{F}$. Let $1 \leqslant s < t \leqslant n$ and $1 \leqslant p < t$ be integers such that $p \neq s$. Let $a, b \in \mathbb{F}$. Since $[\psi(X), X] = 0$ for every $X \in \{aE_{st} + bE_{pt}, aE_{st}, bE_{pt}\}$, it follows that $[\psi(aE_{st}), bE_{pt}] +$

$[\psi(bE_{pt}), aE_{st}] = 0$. By (5.24), we obtain

$$\sum_{i=1}^{p} \phi_{ip}^{(s,t)}(a)bE_{it} - \sum_{j=t}^{n} \phi_{tj}^{(s,t)}(a)bE_{pj} + \sum_{i=1}^{s} \phi_{is}^{(p,t)}(b)aE_{it} - \sum_{j=t}^{n} \phi_{tj}^{(p,t)}(b)aE_{sj} = 0. \quad (5.46)$$

First consider $1 \leqslant p < s$. It follows from (5.46) that

$$\begin{aligned}(\phi_{pp}^{(s,t)}(a)b \,-\, \phi_{tt}^{(s,t)}(a)b + \phi_{ps}^{(p,t)}(b)a)E_{pt} + (\phi_{ss}^{(p,t)}(b)a - \phi_{tt}^{(p,t)}(b)a)E_{st} \\ + \sum_{i=1}^{p-1}(\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a)E_{it} + \sum_{i=p+1}^{s-1} \phi_{is}^{(p,t)}(b)aE_{it} \\ - \sum_{j=t+1}^{n} \phi_{tj}^{(s,t)}(a)bE_{pj} - \sum_{j=t+1}^{n} \phi_{tj}^{(p,t)}(b)aE_{sj} = 0.\end{aligned}$$

Since $\phi_{tj}^{(s,t)} = 0 = \phi_{tj}^{(p,t)}$ for all $t < j \leqslant n$ by (5.26), and $\phi_{pj}^{(p,t)} = 0$ for all $p < j < t$ by (5.33), we get

$$\begin{aligned}(\phi_{pp}^{(s,t)}(a)b - \phi_{tt}^{(s,t)}(a)b)E_{pt} + (\phi_{ss}^{(p,t)}(b)a - \phi_{tt}^{(p,t)}(b)a)E_{st} \\ + \sum_{i=1}^{p-1}(\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a)E_{it} + \sum_{i=p+1}^{s-1} \phi_{is}^{(p,t)}(b)aE_{it} = 0\end{aligned}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leqslant p < s < t \leqslant n$. Thus for every $1 \leqslant p < s < t \leqslant n$,

$$\phi_{pp}^{(s,t)} = \phi_{tt}^{(s,t)}, \quad (5.47)$$

$$\phi_{ip}^{(s,t)}(a)b = -\phi_{is}^{(p,t)}(b)a \quad \text{for all } a, b \in \mathbb{F},\ i = 1, \ldots, p-1. \quad (5.48)$$

By (5.48), we see that $\phi_{ip}^{(s,t)}$, $\phi_{is}^{(p,t)}$, $i = 1, \ldots, p-1$, are linear maps on $\mathbb{F}$. Then for every integer $1 < p < s < t \leqslant n$, there exist scalars $\lambda_{ip}^{(s,t)} \in \mathbb{F}$, $i = 1, \ldots, p-1$, such that

$$\phi_{ip}^{(s,t)}(a) = \lambda_{ip}^{(s,t)} a = -\phi_{is}^{(p,t)}(a) \quad \text{for all } a \in \mathbb{F}. \quad (5.49)$$

In view of (5.27), (5.47) and (5.49), we conclude that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ii}^{(s,t)} = \phi_{tt}^{(s,t)} \quad \text{for } i = 1, \ldots, s, \quad (5.50)$$

$$\phi_{ij}^{(s,t)}(a) = \lambda_{ij}^{(s,t)} a, \ a \in \mathbb{F}, \ \text{ for all integers } 1 \leqslant i < j < s. \tag{5.51}$$

Consider now $s < p < t$. It follows from (6.35) that

$$\begin{aligned}(\phi_{ss}^{(p,t)}(b)a - \phi_{tt}^{(p,t)}(b)a + \phi_{sp}^{(s,t)}(a)b)E_{st} + (\phi_{pp}^{(s,t)}(a)b - \phi_{tt}^{(s,t)}(a)b)E_{pt} \\ + \sum_{i=1}^{s-1}(\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a)E_{it} + \sum_{i=s+1}^{p-1} \phi_{ip}^{(s,t)}(a)bE_{it} \\ - \sum_{j=t+1}^{n} \phi_{tj}^{(s,t)}(a)bE_{pj} - \sum_{j=t+1}^{n} \phi_{tj}^{(p,t)}(b)aE_{sj} = 0\end{aligned}$$

for all $a, b \in \mathbb{F}$. Since $\phi_{tj}^{(s,t)} = 0 = \phi_{tj}^{(p,t)}$ for all $t < j \leqslant n$ by (5.26), and $\phi_{sj}^{(s,t)} = 0$ for all $s < j < t$ by (5.33), it follows that

$$\begin{aligned}(\phi_{ss}^{(p,t)}(b)a - \phi_{tt}^{(p,t)}(b)a)E_{st} + (\phi_{pp}^{(s,t)}(a)b - \phi_{tt}^{(s,t)}(a)b)E_{pt} \\ + \sum_{i=1}^{s-1}(\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a)E_{it} + \sum_{i=s+1}^{p-1} \phi_{ip}^{(s,t)}(a)bE_{it} = 0\end{aligned}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leqslant s < p < t \leqslant n$. Hence for every $1 \leqslant s < p < t \leqslant n$,

$$\phi_{pp}^{(s,t)} = \phi_{tt}^{(s,t)}, \tag{5.52}$$

$$\phi_{ip}^{(s,t)} = 0 \quad \text{for } i = s+1, \ldots, p-1, \tag{5.53}$$

$$\phi_{ip}^{(s,t)}(a)b = -\phi_{is}^{(p,t)}(b)a \quad \text{for all } a, b \in \mathbb{F}, \ i = 1, \ldots, s-1. \tag{5.54}$$

By (5.50) and (5.52), we have

$$\phi_{ii}^{(s,t)} = \phi_{tt}^{(s,t)} \quad \text{for } i = 1, \ldots, t-1. \tag{5.55}$$

We conclude from (5.53) that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ij}^{(s,t)} = 0 \quad \text{for all } s < i < j < t. \tag{5.56}$$

By (5.54), $\phi_{ip}^{(s,t)}$, $\phi_{is}^{(p,t)}$, $i = 1, \ldots, s-1$, are linear maps on $\mathbb{F}$. Moreover, for each $1 \leqslant s < p < t \leqslant n$, it follows from (5.51) that $\phi_{is}^{(p,t)}(a) = \lambda_{is}^{(p,t)} a$ for all $a \in \mathbb{F}$,

$i = 1, \ldots, s-1$. Consequently, for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ij}^{(s,t)}(a) = -\lambda_{is}^{(j,t)} a, \ a \in \mathbb{F}, \ \text{ for all integers } 1 \leqslant i < s \text{ and } s < j < t. \tag{5.57}$$

We proceed to consider integers $1 \leqslant s < t \leqslant n$ and $s < q \leqslant n$ such that $q \neq t$. Let $a, b \in \mathbb{F}$. Since $[\psi(X), X] = 0$ for all $X \in \{aE_{st} + bE_{sq}, aE_{st}, bE_{sq}\}$, it follows that $[\psi(aE_{st}), bE_{sq}] + [\psi(bE_{sq}), aE_{st}] = 0$. By virtue of (5.24), we obtain

$$\sum_{i=1}^{s} \phi_{is}^{(s,t)}(a) b E_{iq} - \sum_{j=q}^{n} \phi_{qj}^{(s,t)}(a) b E_{sj} + \sum_{i=1}^{s} \phi_{is}^{(s,q)}(b) a E_{it} - \sum_{j=t}^{n} \phi_{tj}^{(s,q)}(b) a E_{sj} = 0, \tag{5.58}$$

We first consider $t < q \leqslant n$. It follows from (5.58) that

$$\begin{aligned}(\phi_{ss}^{(s,t)}(a)b \,-\, \phi_{qq}^{(s,t)}(a)b - \phi_{tq}^{(s,q)}(b)a)E_{sq} + (\phi_{ss}^{(s,q)}(b)a - \phi_{tt}^{(s,q)}(b)a)E_{st} \\ - \sum_{j=q+1}^{n} (\phi_{qj}^{(s,t)}(a)b + \phi_{tj}^{(s,q)}(b)a)E_{sj} + \sum_{i=1}^{s-1} \phi_{is}^{(s,q)}(b) a E_{it} \\ + \sum_{i=1}^{s-1} \phi_{is}^{(s,t)}(a) b E_{iq} - \sum_{j=t+1}^{q-1} \phi_{tj}^{(s,q)}(b) a E_{sj} = 0\end{aligned}$$

for all $a, b \in \mathbb{F}$. Since $\phi_{iq}^{(s,q)} = 0$ for all $s < i < q$ by (5.40), $\phi_{is}^{(s,t)} = 0 = \phi_{is}^{(s,q)}$ for all $1 \leqslant i < s$ by (5.25), and $\phi_{tj}^{(s,q)} = 0$ for all $s < t < j < q$ by (5.56), it follows that

$$\begin{aligned}(\phi_{ss}^{(s,t)}(a)b - \phi_{qq}^{(s,t)}(a)b)E_{sq} \,+\, (\phi_{ss}^{(s,q)}(b)a - \phi_{tt}^{(s,q)}(b)a)E_{st} \\ - \sum_{j=q+1}^{n} (\phi_{qj}^{(s,t)}(a)b + \phi_{tj}^{(s,q)}(b)a)E_{sj} = 0\end{aligned}$$

for all $a, b \in \mathbb{F}$ and integers $1 \leqslant s < t < q \leqslant n$. Therefore for every $1 \leqslant s < t < q \leqslant n$,

$$\phi_{qq}^{(s,t)} = \phi_{ss}^{(s,t)}, \tag{5.59}$$

$$\phi_{qj}^{(s,t)}(a)b = -\phi_{tj}^{(s,q)}(b)a \quad \text{for all } a, b \in \mathbb{F}, \ j = q+1, \ldots, n. \tag{5.60}$$

In view of (5.27), (5.55) and (5.59), we conclude that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ii}^{(s,t)} = \phi_{ss}^{(s,t)} \quad \text{for all } i = 1, \ldots, n. \tag{5.61}$$

By (5.60), we see that $\phi_{qj}^{(s,t)}, \phi_{tj}^{(s,q)}, j = q+1, \ldots, n$, are linear maps on $\mathbb{F}$. Therefore for every $1 \leqslant s < t < q \leqslant n$, there exist scalars $\lambda_{qj}^{(s,t)} \in \mathbb{F}, j = q+1, \ldots, n$, such that

$$\phi_{qj}^{(s,t)}(a) = \lambda_{qj}^{(s,t)} a = -\phi_{tj}^{(s,q)}(a) \quad \text{for all } a \in \mathbb{F}. \tag{5.62}$$

We conclude from (5.62) that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ij}^{(s,t)}(a) = \lambda_{ij}^{(s,t)} a, \ a \in \mathbb{F}, \ \text{ for all integers } t < i < j \leqslant n. \tag{5.63}$$

Next, we consider $s < q < t$. It follows from (5.58) that

$$\begin{aligned}(\phi_{ss}^{(s,t)}(a)b \, - \, \phi_{qq}^{(s,t)}(a)b)E_{sq} + (\phi_{ss}^{(s,q)}(b)a - \phi_{tt}^{(s,q)}(b)a - \phi_{qt}^{(s,t)}(a)b)E_{st} \\ + \sum_{i=1}^{s-1} \phi_{is}^{(s,t)}(a)bE_{iq} + \sum_{i=1}^{s-1} \phi_{is}^{(s,q)}(b)aE_{it} \\ - \sum_{j=q+1}^{t-1} \phi_{qj}^{(s,t)}(a)bE_{sj} - \sum_{j=t+1}^{n} (\phi_{qj}^{(s,t)}(a)b + \phi_{tj}^{(s,q)}(b)a)E_{sj} = 0\end{aligned}$$

for all $a, b \in \mathbb{F}$. By virtue of (5.25), (5.40), (5.56) and (5.61), we obtain

$$\sum_{j=t+1}^{n} (\phi_{qj}^{(s,t)}(a)b + \phi_{tj}^{(s,q)}(b)a)E_{sj} = 0$$

for all $a, b \in \mathbb{F}$ and integers $1 \leqslant s < q < t \leqslant n$. Thus for every $1 \leqslant s < q < t \leqslant n$,

$$\phi_{qj}^{(s,t)}(a)b = -\phi_{tj}^{(s,q)}(b)a \quad \text{for all } a, b \in \mathbb{F}, \ j = t+1, \ldots, n. \tag{5.64}$$

By (5.64), $\phi_{qj}^{(s,t)}, \phi_{tj}^{(s,q)}, j = q+1, \ldots, n$, are linear maps on $\mathbb{F}$. Moreover, for every integer $1 \leqslant s < q < t \leqslant n$, it follows from (5.63) that $\phi_{tj}^{(s,q)}(a) = \lambda_{tj}^{(s,q)} a$ for all $a \in \mathbb{F}$, $j = t+1, \ldots, n$. We conclude from (5.64) that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ij}^{(s,t)}(a) = -\lambda_{tj}^{(s,i)} a, \ a \in \mathbb{F}, \ \text{ for all integers } s < i < t \text{ and } t < j \leqslant n. \tag{5.65}$$

We are now ready to classify the structure of $\psi(aE_{st})$ for all integers $1 \leqslant s < t \leqslant n$ and $a \in \mathbb{F}$. To see this, let $1 \leqslant s < t \leqslant n$ be integers. Since $\phi_{is}^{(s,t)} = 0$ for $i = 1, \ldots, s-1$

by (5.25), and $\phi_{it}^{(s,t)} = -\lambda_{is}^{(t,t)}$ for $i = 1, \ldots, s-1$ by (5.42), it follows from (5.51) and (5.57) that

$$\sum_{1\leqslant i<j\leqslant s} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{i=1}^{s-1}\sum_{j=s+1}^{t} \phi_{ij}^{(s,t)}(a)E_{ij} = \sum_{1\leqslant i<j<s} \lambda_{ij}^{(s,t)} aE_{ij} - \sum_{i=1}^{s-1}\sum_{j=s+1}^{t} \lambda_{is}^{(j,t)} aE_{ij} \tag{5.66}$$

for all $a \in \mathbb{F}$, where it is understood that $\sum_{1\leqslant i<j\leqslant s} \phi_{ij}^{(s,t)}(a)E_{ij} = 0 = \sum_{1\leqslant i<j<s} \lambda_{ij}^{(s,t)} aE_{ij}$ when $s \in \{1, 2\}$, and $\sum_{i=1}^{s-1}\sum_{j=s+1}^{t} \phi_{ij}^{(s,t)}(a)E_{ij} = 0 = \sum_{i=1}^{s-1}\sum_{j=s+1}^{t} \lambda_{is}^{(j,t)} aE_{ij}$ when $s = 1$. Likewise, since $\phi_{tj}^{(s,t)} = 0$ for $j = t+1, \ldots, n$ by (5.26), and $\phi_{sj}^{(s,t)} = -\lambda_{tj}^{(s,s)}$ for $j = t+1, \ldots, n$ by (5.36), it follows from (5.63) and (5.65) that

$$\sum_{t\leqslant i<j\leqslant n} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{i=s}^{t-1}\sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a)E_{ij} = \sum_{t<i<j\leqslant n} \lambda_{ij}^{(s,t)} aE_{ij} - \sum_{i=s}^{t-1}\sum_{j=t+1}^{n} \lambda_{tj}^{(s,i)} aE_{ij} \tag{5.67}$$

for all $a \in \mathbb{F}$, where $\sum_{t\leqslant i<j\leqslant n} \phi_{ij}^{(s,t)}(a)E_{ij} = 0 = \sum_{t<i<j\leqslant n} \lambda_{ij}^{(s,t)} aE_{ij} = 0$ when $t \in \{n-1, n\}$, and $\sum_{i=s}^{t-1}\sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a)E_{ij} = 0 = \sum_{i=s}^{t-1}\sum_{j=t+1}^{n} \lambda_{tj}^{(s,i)} aE_{ij}$ when $t = n$. Next, since $\phi_{sj}^{(s,t)} = 0$ for $j = s+1, \ldots, t-1$ by (5.33), $\phi_{it}^{(s,t)} = 0$ for $i = s+1, \ldots, t-1$ by (5.40), and $\phi_{ij}^{(s,t)} = 0$ for all $s < i < j < t$ by (5.56), it follows from (5.34) that

$$\sum_{s\leqslant i<j\leqslant t} \phi_{ij}^{(s,t)}(a)E_{ij} = -\tau_{st} aE_{st} \tag{5.68}$$

for all $a \in \mathbb{F}$. Since

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant n} \phi_{ij}^{(s,t)}(a)E_{ij} = {} & \sum_{1\leqslant i<j\leqslant s} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{i=1}^{s-1}\sum_{j=s+1}^{t} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{i=1}^{s-1}\sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a)E_{ij} \\ & + \sum_{s\leqslant i<j\leqslant t} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{i=s}^{t-1}\sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{t\leqslant i<j\leqslant n} \phi_{ij}^{(s,t)}(a)E_{ij}\end{aligned}$$

for all $a \in \mathbb{F}$, it follows from (5.66)–(5.68) that

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant n} \phi_{ij}^{(s,t)}(a)E_{ij} = {} & \sum_{1\leqslant i<j<s} \lambda_{ij}^{(s,t)} aE_{ij} - \sum_{i=1}^{s-1}\sum_{j=s+1}^{t} \lambda_{is}^{(j,t)} aE_{ij} + \sum_{i=1}^{s-1}\sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a)E_{ij} \\ & - \tau_{st} aE_{st} + \sum_{t<i<j\leqslant n} \lambda_{ij}^{(s,t)} aE_{ij} - \sum_{i=s}^{t-1}\sum_{j=t+1}^{n} \lambda_{tj}^{(s,i)} aE_{ij}\end{aligned} \tag{5.69}$$

for all $a \in \mathbb{F}$, where $\sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a) E_{ij} = 0$ when either $s = 1$ or $t = n$. Moreover, by virtue of (5.61), for each pair of integers $1 \leqslant s < t \leqslant n$,

$$\sum_{i=1}^{n} \phi_{ii}^{(s,t)}(a) E_{ii} = \phi_{ss}^{(s,t)}(a) I_n \tag{5.70}$$

for all $a \in \mathbb{F}$. Consequently, by (5.24), (5.69) and (5.70), for each pair of integers $1 \leqslant s < t \leqslant n$,

$$\begin{aligned} \psi(aE_{st}) = \phi_{ss}^{(s,t)}(a) I_n - \tau_{st} a E_{st} &+ \sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a) E_{ij} \\ + \sum_{1 \leqslant i < j < s} \lambda_{ij}^{(s,t)} a E_{ij} &- \sum_{i=1}^{s-1} \sum_{j=s+1}^{t} \lambda_{is}^{(j,t)} a E_{ij} \\ + \sum_{t < i < j \leqslant n} \lambda_{ij}^{(s,t)} a E_{ij} &- \sum_{i=s}^{t-1} \sum_{j=t+1}^{n} \lambda_{tj}^{(s,i)} a E_{ij} \end{aligned} \tag{5.71}$$

for all $a \in \mathbb{F}$.

Let $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ be the additive map defined by

$$\mu(A) = \sum_{1 \leqslant s \leqslant t \leqslant n} \phi_{ss}^{(s,t)}(a_{st}) \tag{5.72}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$. Let $\psi_\chi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ and $\psi_\Lambda : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ be the linear maps defined in (5.20) and (5.22), respectively, and let $\psi_{\mathscr{F}} : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ be the additive map defined in (5.21), where $\chi = (\tau_{ij}) \in T_n(\mathbb{F})$ is a strictly upper triangular matrix, and

$$\mathscr{F} = \bigcup_{1 < s \leqslant t < n} \{\phi_{ij}^{(s,t)} : \mathbb{F} \to \mathbb{F} : 1 \leqslant i \leqslant s-1 \text{ and } t+1 \leqslant j \leqslant n\},$$

$$\Lambda = \bigcup_{(s,t) \in \nabla_n} \left\{\lambda_{ij}^{(s,t)} \in \mathbb{F} : 1 \leqslant i < j < s \text{ or } t < i < j \leqslant n\right\}.$$

In view of (5.45) and (5.71), together with (5.2), (5.3), (5.6), (5.7) and (5.72), we see that

$$\psi(aE_{st}) = \mu(aE_{st}) I_n + \psi_\chi(aE_{st}) + \psi_\Lambda(aE_{st}) + \psi_{\mathscr{F}}(aE_{st}) \tag{5.73}$$

for all integers $1 \leqslant s \leqslant t \leqslant n$ and $a \in \mathbb{F}$. By (5.73) and the additivity of ψ, μ, ψ_χ, ψ_Λ and $\psi_{\mathscr{F}}$, we obtain

$$\begin{aligned}
\psi(A) &= \sum_{1\leqslant s\leqslant t\leqslant n} \psi(a_{st}E_{st}) \\
&= \sum_{1\leqslant s\leqslant t\leqslant n} \mu(a_{st}E_{st})I_n + \psi_\chi(a_{st}E_{st}) + \psi_\Lambda(a_{st}E_{st}) + \psi_{\mathscr{F}}(a_{st}E_{st}) \\
&= \mu(A)I_n + \psi_\chi(A) + \psi_\Lambda(A) + \psi_{\mathscr{F}}(A) \\
&= \lambda A + \mu(A)I_n + \psi_\chi(A) + \psi_\Lambda(A) + \psi_{\mathscr{F}}(A)
\end{aligned}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\lambda = 0 \in \mathbb{F}$. This completes the proof. □

Remark 5.4.1. Let $\mathbb{F}$ be a field and let $\psi : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ be the commuting additive map defined by

$$\psi(A) = \mu(A)I_3 + \begin{pmatrix} \tau_{12}a_{22} + \tau_{13}a_{33} & -\tau_{12}a_{12} & -\tau_{13}a_{13} \\ 0 & \tau_{12}a_{11} + \tau_{23}a_{33} & -\tau_{23}a_{23} \\ 0 & 0 & \tau_{13}a_{11} + \tau_{23}a_{22} \end{pmatrix} \tag{5.74}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$, where τ_{ij}, $1 \leqslant i < j \leqslant 3$, are scalars in $\mathbb{F}$ and $\mu : T_3(\mathbb{F}) \to \mathbb{F}$ is an additive map. Given any $\lambda \in \mathbb{F}$, it is not difficult to note that the additive map ψ in (5.74) can be reformed as follows:

$$\psi(A) = \lambda A + \eta(A)I_3 + \begin{pmatrix} \varsigma_{12}a_{22} + \varsigma_{13}a_{33} & -\varsigma_{12}a_{12} & -\varsigma_{13}a_{13} \\ 0 & \varsigma_{12}a_{11} + \varsigma_{23}a_{33} & -\varsigma_{23}a_{23} \\ 0 & 0 & \varsigma_{13}a_{11} + \varsigma_{23}a_{22} \end{pmatrix}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$. Here, $\varsigma_{ij} = \tau_{ij} + \lambda \in \mathbb{F}$ for all $1 \leqslant i < j \leqslant 3$, and $\eta : T_3(\mathbb{F}) \to \mathbb{F}$ is the additive map defined by

$$\eta(A) = \mu(A) - \lambda(a_{11} + a_{22} + a_{33})$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$.

Remark: The results in this chapter have been published in Chooi, Mutalib, and **Tan,**

L. Y. (2021).

CHAPTER 6: CENTRALIZING ADDITIVE MAPS ON RANK ONE UPPER TRIANGULAR MATRICES OVER DIVISION RINGS

6.1 Introduction

Let $\mathcal{R}$ be a ring with centre $Z(\mathcal{R})$ and let $\mathcal{S}$ be a nonempty subset of $\mathcal{R}$. Let $\psi : \mathcal{R} \to \mathcal{R}$ be a map. Recall that a map ψ is centralizing on $\mathcal{S}$ if $[\psi(A), A] \in Z(\mathcal{R})$ for all $A \in \mathcal{S}$, where $[X, Y]$ is the commutator of $X, Y \in \mathcal{R}$. Extending Franca's results, C.-K. Liu (2014a) developed the study of centralizing additive maps to subsets of matrices that are not closed under addition. Let $n \geqslant 2$ be an integer and let $\mathbb{D}$ be a division ring. Recall that $M_n(\mathbb{D})$ is the ring of all $n \times n$ matrices over $\mathbb{D}$ with centre $Z(M_n(\mathbb{D}))$ and unity I_n. C.-K. Liu (2014a) showed that if $\psi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ is an additive map satisfying $[\psi(A), A] \in Z(M_n(\mathbb{D}))$ for all invertible $A \in M_n(\mathbb{D})$, then there exists $\lambda \in Z(\mathbb{D})$ and an additive map $\mu : M_n(\mathbb{D}) \to Z(\mathbb{D})$ such that $\psi(A) = \lambda A + \mu(A) I_n$, unless $|\mathbb{D}| = 2$. Franca and Louza (2017) studied commuting additive maps $\psi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ on rank one matrices over a noncommutative division ring $\mathbb{D}$ and it turns out that ψ is of the standard form (2.14), which is unexpectedly simple compared to Franca (2017) when $\mathbb{D}$ is a field. Let $n \geqslant 2$ be an integer. Recall that $T_n(\mathbb{D})$ is the ring of all $n \times n$ upper triangular matrices over a division ring $\mathbb{D}$ with centre $Z(T_n(\mathbb{D}))$. Inspired by the aforementioned result, in this chapter we study and characterise centralizing additive map $\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ (i.e. ψ satisfying $[\psi(A), A] \in Z(T_n(\mathbb{D}))$) for all rank one matrices $A \in T_n(\mathbb{D})$. We then deduce from this result a complete description of commuting additive maps $\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ on rank one matrices over a noncommutative division ring $\mathbb{D}$. We show that the concept of centralizing and commuting are equivalent in rank one upper triangular matrices over division rings. As we see from Theorem 6.2.2, it is worth mentioning that the structure of ψ is relatively simple compared with the corresponding result in Theorem 5.3.2 when $\mathbb{D}$ is a field, but ψ is not of the standard form.

6.2 Main results

Let $n \geqslant 3$ be an integer. Let $\nabla_n = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leqslant i \leqslant j \leqslant n\} \backslash \{(1, n-1), (1, n), (2, n-1), (2, n)\}$. Recall that $E_{ij} \in T_n(\mathbb{D})$ is the standard matrix unit whose (i, j)th entry is one and zero elsewhere.

Theorem 6.2.1. *Let $n \geqslant 2$ be an integer and let $\mathbb{D}$ be a division ring with centre $Z(\mathbb{D})$. Suppose that $\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is a map. Then the following statements are equivalent:*

(i) *ψ is a centralizing additive map on rank one matrices.*

(ii) *ψ is a commuting additive map on rank one matrices.*

(iii) *There exists $\lambda \in Z(\mathbb{D})$, an additive map $\mu : T_n(\mathbb{D}) \to Z(\mathbb{D})$, a strictly upper triangular matrix $\chi = (\tau_{ij}) \in T_n(\mathbb{D})$, a set of elements $\Lambda = \bigcup_{(s,t)\in\nabla_n}\{\lambda_{ij}^{(s,t)} \in \mathbb{D} : 1 \leqslant i < j < s \text{ or } t < i < j \leqslant n\}$ and a set of additive maps $\mathscr{F} = \bigcup_{1<s\leqslant t<n}\{\phi_{ij}^{(s,t)} : \mathbb{D} \to \mathbb{D} : 1 \leqslant i \leqslant s-1 \text{ and } t+1 \leqslant j \leqslant n\}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \psi_\chi(A) + \psi_{\mathscr{F}}(A) + \psi_\Lambda(A)$$

for all $A \in T_n(\mathbb{D})$, where $\psi_\chi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is the linear map defined by

$$\psi_\chi(A) = \begin{pmatrix} x_1 & -\tau_{12}a_{12} & -\tau_{13}a_{13} & \cdots & -\tau_{1n}a_{1n} \\ 0 & x_2 & -\tau_{23}a_{23} & \cdots & -\tau_{2n}a_{2n} \\ 0 & 0 & x_3 & \cdots & -\tau_{3n}a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix}$$

for all $A = (a_{ij}) \in T_n(\mathbb{D})$, where

$$x_h = \begin{cases} \sum_{i=2}^{n} \tau_{1i}a_{ii} & \text{if } h = 1, \\ \sum_{i=1}^{h-1} \tau_{ih}\, a_{ii} + \sum_{i=h+1}^{n} \tau_{hi}\, a_{ii} & \text{if } 2 \leqslant h \leqslant n-1, \\ \sum_{i=1}^{n-1} \tau_{in}a_{ii} & \text{if } h = n, \end{cases}$$

$\psi_{\mathscr{F}} : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is the additive map defined by

$$\psi_{\mathscr{F}}(A) = \sum_{1<s\leqslant t<n} \left(\sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a_{st})E_{ij} \right)$$

for all $A=(a_{ij})\in T_n(\mathbb{D})$, and $\psi_\Lambda : T_n(\mathbb{D})\to T_n(\mathbb{D})$ is the linear map defined by

$$\psi_\Lambda(A)=\sum_{(s,t)\in\nabla_n}\Psi_{st}(A)+\Phi_{st}(A)$$

for all $A\in T_n(\mathbb{D})$, where for each $(s,t)\in\nabla_n$,

$$\Psi_{st}(A)=\begin{cases}0 & \text{if } 1\leqslant s\leqslant 2,\\ \Big(\sum_{1\leqslant i<j<s}\lambda_{ij}^{(s,t)}E_{ij}\Big)\big(\sum_{h=1}^{s-1}a_{st}E_{hh}-a_{ht}E_{hs}\big) & \text{if } 3\leqslant s\leqslant n,\end{cases}$$

$$\Phi_{st}(A)=\begin{cases}\big(\sum_{h=t+1}^{n}a_{st}E_{hh}-a_{sh}E_{th}\big)\Big(\sum_{t<i<j\leqslant n}\lambda_{ij}^{(s,t)}E_{ij}\Big) & \text{if } 1\leqslant t\leqslant n-2,\\ 0 & \text{if } n-1\leqslant t\leqslant n\end{cases}$$

for all $A=(a_{ij})\in T_n(\mathbb{D})$. Here, $\psi_{\mathscr{F}}=0$ when $n=2$, and $\psi_\chi=0$ and $\psi_\Lambda=0$ when either $n=2$ or $\mathbb{D}$ is noncommutative.

As an immediate consequence of Theorem 6.2.1, we deduce the following result.

Theorem 6.2.2. *Let $n\geqslant 2$ be an integer and let $\mathbb{D}$ be a noncommutative division ring with centre $Z(\mathbb{D})$. Then $\psi : T_n(\mathbb{D})\to T_n(\mathbb{D})$ is a commuting additive map on rank one matrices if and only if there exists an element $\lambda\in Z(\mathbb{D})$, an additive map $\mu : T_n(\mathbb{D})\to Z(\mathbb{D})$ and a set of additive maps $\mathscr{F}=\bigcup_{1<s\leqslant t<n}\{\phi_{ij}^{(s,t)} : \mathbb{D}\to\mathbb{D} : 1\leqslant i\leqslant s-1 \text{ and } t+1\leqslant j\leqslant n\}$ such that*

$$\psi(A)=\lambda A+\mu(A)I_n+\psi_{\mathscr{F}}(A)$$

for all $A\in T_n(\mathbb{D})$, where $\psi_{\mathscr{F}} : T_n(\mathbb{D})\to T_n(\mathbb{D})$ is the additive map defined by

$$\psi_{\mathscr{F}}(A)=\sum_{1<s\leqslant t<n}\left(\sum_{i=1}^{s-1}\sum_{j=t+1}^{n}\phi_{ij}^{(s,t)}(a_{st})E_{ij}\right)$$

for all $A=(a_{ij})\in T_n(\mathbb{D})$ and $\psi_{\mathscr{F}}=0$ when $n=2$.

6.3 Proofs

Throughout this section, unless stated otherwise, let $n\geqslant 2$ be an integer and let $\mathbb{D}$ denote a division ring with centre $Z(\mathbb{D})$. Let $A\in M_n(\mathbb{D})$. The row (respectively, column) space of A is the left (respectively, right) vector space over $\mathbb{D}$ generated by the n rows (respectively, n columns) of A. The row rank (respectively, column rank) of A is the dimension of the row (respectively, column) space of A. The rank of A, denoted by rank A, is the common value of the row rank and column rank of A. See, for instance (Wan, 1996, Corollary 1.18).

The following lemma generalises (Chooi & Lim, 1998, Lemma 4.1) over division rings, where the lemma holds true when the division ring is a field.

Lemma 6.3.1. *Let $\mathbb{D}$ be a division ring and let $n \geqslant 1$ and $1 \leqslant k \leqslant n$ be integers. Then $A \in T_n(\mathbb{D})$ is of rank k if and only if there exist invertible matrices $P, Q \in T_n(\mathbb{D})$ such that*

$$A = P\left(\sum_{i=1}^{k} E_{s_i,t_i}\right) Q$$

for some integers $1 \leqslant s_i \leqslant t_i \leqslant n$ for $i = 1, \ldots, k$ with $s_1 < \cdots < s_k$ and $t_i \neq t_j$ whenever $1 \leqslant i \neq j \leqslant k$.

Proof. The sufficiency is clear. We prove the necessity. Let $A = (a_{ij}) \in T_n(\mathbb{D})$ be of rank $k \geqslant 1$. The result is clear when $n = 1$. Consider $n \geqslant 2$. We denote by R_i and C_i the i-th row and the i-th column of A, respectively. Let R_{p_1} be the nonzero row of A in which $R_i = 0$ for $i = p_1 + 1, \ldots, n$, and let a_{p_1,q_1} be the first nonzero entry from the left of R_{p_1}. We left multiply R_{p_1} by a_{p_1,q_1}^{-1} and obtain $a_{p_1,q_1} = 1$. Then, for each $1 \leqslant i < p_1$ and $q_1 < j \leqslant n$, we apply the following elementary row and column operations on A:

$$R_i \to R_i - a_{i,q_1} R_{p_1} \quad \text{and} \quad C_j \to C_j - C_{q_1} a_{p_1,j}. \tag{6.1}$$

Then, by (Hungerford, 1974, Chapter VII, Theorem 2.8) and (6.1), there exist invertible matrices $H_1, K_1 \in T_n(\mathbb{D})$ such that

$$H_1 A K_1 = E_{p_1,q_1} + B, \tag{6.2}$$

where $E_{p_1,q_1}, B \in T_n(\mathbb{D})$. If $B = 0$, then the lemma is proved. Suppose that $B = (b_{ij}) \neq 0$. In view of the elementary operations performed in (6.1), we see that $b_{i,q_1} = 0$ for $i = 1, \ldots, n$, and $b_{ij} = 0$ for all $p_1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$. We repeat a similar process for B. Then there exist integers $1 \leqslant p_2 \leqslant q_2 \leqslant n$, with $p_2 < p_1$ and $q_2 \neq q_1$, and invertible matrices $H_2, K_2 \in T_n(\mathbb{D})$ such that

$$H_2 B K_2 = E_{p_2,q_2} + C \quad \text{and} \quad H_2 E_{p_1,q_1} K_2 = E_{p_1,q_1}$$

for some $C \in T_n(\mathbb{D})$. It follows from (6.2) that $(H_2H_1)A(K_1K_2) = E_{p_1,q_1} + E_{p_2,q_2} + C$. If $C = 0$, then we are done; otherwise, we continue this process, since A is of rank k, and finally reach to the desired result. □

Lemma 6.3.2. (C.-K. Liu et al., 2018, Lemma 2.1) *Let $\mathbb{D}$ be a division ring. Let $a, b \in \mathbb{D}$ be such that $ax = xb$ for all nonzero $x \in \mathbb{D}$. Then $a = b \in Z(\mathbb{D})$.*

Proof. Taking $x = 1 \in \mathbb{D}$, we get $a = b$. It follows that $ax = xa$ for all all nonzero $x \in \mathbb{D}$. Hence $a = b \in Z(\mathbb{D})$. □

Lemma 6.3.3. *Let $n \geqslant 2$ be an integer. Then the centre of $T_n(\mathbb{D})$ is $Z(T_n(\mathbb{D})) = Z(\mathbb{D})I_n$.*

Proof. We first show that $Z(\mathbb{D})I_n \subseteq Z(T_n(\mathbb{D}))$. Let $X \in Z(\mathbb{D})I_n$. Then $X = \lambda I_n$ for some $\lambda \in Z(\mathbb{D})$. Let $A \in T_n(\mathbb{D})$. Since $\lambda \in Z(\mathbb{D})$, we get $\lambda A = A\lambda$. Then $AX = A(\lambda I_n) = (A\lambda)I_n = (\lambda A)I_n = \lambda(AI_n) = \lambda(I_nA) = (\lambda I_n)A = XA$. Therefore $AX = XA$ for all $A \in T_n(\mathbb{D})$, and so $X \in Z(T_n(\mathbb{D}))$. Hence $Z(\mathbb{D})I_n \subseteq Z(T_n(\mathbb{D}))$.

Consider $A = (a_{ij}) \in Z(T_n(\mathbb{D}))$. Let $1 \leqslant s \leqslant t \leqslant n$ be integers and let $d \in \mathbb{D}$ be nonzero. Since $A(dE_{st}) = (dE_{st})A$, it follows that $E_{kk}(A(dE_{st})) = E_{kk}((dE_{st})A)$, and so

$$a_{ks} d E_{kt} = \delta_{ks} d E_{kt} A \tag{6.3}$$

for every integer $1 \leqslant k \leqslant s$, where δ_{ks} is the Kronecker symbol. In particular, $a_{ks}dE_{st} = 0$ when $k < s$. Then A is diagonal. We note that $E_{st}A = a_{tt}E_{st}$. By (6.3), $a_{ss}dE_{st} = da_{tt}E_{st}$ for every nonzero $d \in \mathbb{D}$ and $1 \leqslant s \leqslant t \leqslant n$. Then $a_{ss} = a_{tt} \in Z(\mathbb{D})$ for all $1 \leqslant s \leqslant t \leqslant n$ by Lemma 6.3.2. Consequently, $A = aI_n$ for some $a \in Z(\mathbb{D})$. Hence $Z(T_n(\mathbb{D})) = Z(\mathbb{D})I_n$. □

We remark that Lemma 6.3.3 extends Lemma 2.1.3 over division rings.

Lemma 6.3.4. *Let $\mathbb{D}$ be a noncommutative division ring. If $f, g : \mathbb{D} \to \mathbb{D}$ are additive maps satisfying either $xf(y) + yg(x) = 0$ for all $x, y \in \mathbb{D}$, or $f(x)y + g(y)x = 0$ for all $x, y \in \mathbb{D}$, then $f = g = 0$.*

Proof. The case $xf(y) + yg(x) = 0$ has been shown in (C.-K. Liu, 2014a, Lemma 2.7). We consider the case

$$f(x)y + g(y)x = 0. \tag{6.4}$$

Taking $x = 1$ in (6.4), we see that $g(y) = -f(1)y$. Likewise taking $y = 1$ in (6.4), we have $f(x) = -g(1)x$. Letting $x = y = 1$ in (6.4), we get $g(1) = -f(1)$. Hence

$$0 = f(x)y + g(y)x = -g(1)xy - f(1)yx = f(1)xy - f(1)yx = f(1)(xy - yx).$$

It follows from the noncommutativity of $\mathbb{D}$ that $f(1) = 0$. We thus obtain $f = g = 0$. □

Lemma 6.3.5. *Let $\mathbb{D}$ be a noncommutative division ring with centre $Z(\mathbb{D})$ and let $f, g : \mathbb{D} \to \mathbb{D}$ be additive maps. If $f(x)x = xg(x)$ for all $x \in \mathbb{D}$, then there exists $\lambda \in Z(\mathbb{D})$ and an additive map $\mu : \mathbb{D} \to Z(\mathbb{D})$ such that $f(x) = g(x) = \lambda x + \mu(x)$ for all $x \in \mathbb{D}$.*

Proof. Setting $f_1 = f$, $f_2 = f_3 = 0$ and $f_4 = -g$ in Theorem 2.2.1, we get $f(x)x - xg(x) = 0$ for all $x \in \mathbb{D}$. By Theorem 2.2.1, there exists an additive map $\mu : \mathbb{D} \to Z(\mathbb{D})$ and $a, b \in \mathbb{D}$ such that $f(x) = -xa + \mu(x)$, $-g(x) = bx - \mu(x)$ and $ax = xb$ for all $x \in \mathbb{D}$. By Lemma 6.3.2, $a = b \in Z(\mathbb{D})$. Then $f(x) = g(x) = \lambda x + \mu(x)$ for all $x \in \mathbb{D}$, where $\lambda = -a \in Z(\mathbb{D})$. We are done. □

Lemma 6.3.6. *Let $\mathbb{D}$ be a noncommutative division ring with centre $Z(\mathbb{D})$ and let $f, g, h : \mathbb{D} \to \mathbb{D}$ be additive maps.*

(i) *If $f(x)y + xg(y) + yh(x) = 0$ for all $x, y \in \mathbb{D}$ and $f(x) = \lambda x + \mu(x)$ for all $x \in \mathbb{D}$, where $\lambda \in Z(\mathbb{D})$ and $\mu : \mathbb{D} \to Z(\mathbb{D})$ is an additive map, then $g(x) = -\lambda x$ and $h(x) = -\mu(x)$ for all $x \in \mathbb{D}$.*

(ii) *If $f(x)y + g(y)x + yh(x) = 0$ for all $x, y \in \mathbb{D}$ and $h(x) = \lambda x + \mu(x)$ for all $x \in \mathbb{D}$, where $\lambda \in Z(\mathbb{D})$ and $\mu : \mathbb{D} \to Z(\mathbb{D})$ is an additive map, then $f(x) = -\mu(x)$ and $g(x) = -\lambda x$ for all $x \in \mathbb{D}$.*

Proof. (i) Setting $f_1 = f$, $f_2 = g$, $f_3 = 0$ and $f_4 = h$ in Theorem 2.2.1, we get $f(x)y + xg(y) + yh(x) = 0$ for all $x, y \in \mathbb{D}$. Then there exist $a, b \in \mathbb{D}$ and additive maps $\alpha, \beta : \mathbb{D} \to Z(\mathbb{D})$ such that

$$f(x) = -xa + \alpha(x), \;\; g(x) = ax - \beta(x), \;\; 0 = -xb + \beta(x) \;\; \text{and} \;\; h(x) = bx - \alpha(x)$$

for all $x \in \mathbb{D}$. Note that $\lambda x + \mu(x) = -xa + \alpha(x)$ yields $x(\lambda + a) = \alpha(x) - \mu(x)$ for all $x \in \mathbb{D}$. Then $\lambda + a = \alpha(1) - \mu(1) \in Z(\mathbb{D})$. Therefore $yx(\lambda + a) = y(\alpha(x) - \mu(x)) = (\alpha(x) - \mu(x))y = (x(\lambda + a))y = xy(\lambda + a)$ for all $x, y \in \mathbb{D}$. By the noncommutativity of $\mathbb{D}$, we have $a = -\lambda$, and hence $\alpha = \mu$. Moreover, since $\beta(x) = xb$ and $\beta(x) \in Z(\mathbb{D})$ for all $x \in \mathbb{D}$, it follows that $yxb = y\beta(x) = \beta(x)y = xby$ for all $x, y \in \mathbb{D}$. By taking $x = 1$, we get $yb = by$ for all $y \in \mathbb{D}$, so $b \in Z(\mathbb{D})$. Then $yxb = xyb$ for all $x, y \in \mathbb{D}$. Again, by the noncommutativity of $\mathbb{D}$, $b = 0$, and so $\beta = 0$. We thus conclude that $g(x) = -\lambda x$ and $h(x) = -\mu(x)$ for all $x \in \mathbb{D}$.

(ii) Setting $f_1 = f$, $f_2 = 0$, $f_3 = g$ and $f_4 = h$ in Theorem 2.2.1, we get $f(x)y + g(y)x + yh(x) = 0$ for all $x, y \in \mathbb{D}$. Then there exist $a, b \in \mathbb{D}$ and additive maps $\alpha, \beta : \mathbb{D} \to Z(\mathbb{D})$ such that

$$f(x) = -xa + \alpha(x), \;\; 0 = ax - \beta(x), \;\; g(x) = -xb + \beta(x) \;\; \text{and} \;\; h(x) = bx - \alpha(x)$$

for all $x \in \mathbb{D}$. Note that $\lambda x + \mu(x) = bx - \alpha(x)$ yields $(b - \lambda)x = \alpha(x) + \mu(x)$ for all $x \in \mathbb{D}$. Then $b - \lambda = \alpha(1) + \mu(1) \in Z(\mathbb{D})$. Therefore $(b - \lambda)xy = (\alpha(x) + \mu(x))y = y(\alpha(x) + \mu(x)) = y(b - \lambda)x = (b - \lambda)yx$ for all $x, y \in \mathbb{D}$. By the noncommutativity of $\mathbb{D}$, we have $b = \lambda$, and hence $\alpha = -\mu$. Moreover, since $\beta(x) = ax$ and $\beta(x) \in Z(\mathbb{D})$ for all $x \in \mathbb{D}$, it follows that $axy = \beta(x)y = y\beta(x) = yax$ for all $x, y \in \mathbb{D}$. By taking $x = 1$, we get $ay = ya$ for all $y \in \mathbb{D}$, so $a \in Z(\mathbb{D})$. Then $axy = ayx$ for all $x, y \in \mathbb{D}$. Again, by the noncommutativity of $\mathbb{D}$, $a = 0$, and so $\beta = 0$. We thus conclude that $f(x) = -\mu(x)$ and $g(x) = -\lambda x$ for all $x \in \mathbb{D}$. □

Lemma 6.3.7. *Let $n \geqslant 2$ be an integer and let $P, Q \in T_n(\mathbb{D})$. If $A \in T_n(\mathbb{D})$ is of rank one such that $PA - AQ \in Z(T_n(\mathbb{D}))$, then $PA = AQ$.*

Proof. We write $PA = (p_{ij})$ and $AQ = (q_{ij})$ to be matrices in $T_n(\mathbb{D})$. By Lemma 6.3.1, there exist invertible matrices $H, K \in T_n(\mathbb{D})$ such that $A = HE_{st}K$ for some integers $1 \leqslant s \leqslant t \leqslant n$. Then $PA = P'E_{st}K$ and $AQ = HE_{st}Q'$, with $P' = PH$ and $Q' = KQ$ in $T_n(\mathbb{D})$. It follows that $p_{ij} = q_{ij} = 0$ for all $1 \leqslant i \leqslant j < t$, and $p_{ij} = q_{ij} = 0$ for all $s < i \leqslant j \leqslant n$. Consequently, $PA - AQ$ has at most one nonzero diagonal entry. On the other hand, by the hypothesis and Lemma 6.3.3, $PA - AQ = \alpha I_n$ for some $\alpha \in \mathbb{D}$. We infer that $\alpha = 0$, and so $PA = AQ$ as required. □

Let $n \geqslant 2$ be an integer and let $\mathbb{D}$ be a division ring with centre $Z(\mathbb{D})$. Recall that $T_n(\mathbb{D})$ is the ring of all $n \times n$ upper triangular matrices over a division ring $\mathbb{D}$ with centre $Z(T_n(\mathbb{D}))$ and unity I_n We are now ready to prove Theorem 6.2.1.

Proof of Theorem 6.2.1. (i) ⇒ (ii). Let $A \in T_n(\mathbb{D})$ be of rank one. By the hypothesis $\psi(A)A - A\psi(A) \in Z(T_n(\mathbb{D}))$, it follows from Lemma 6.3.7 that $\psi(A)A = A\psi(A)$. Hence ψ is a commuting additive map on rank one matrices.

(iii) ⇒ (i) Trivially, $A \mapsto \lambda A + \mu(A)I_n$ is a centralizing additive map on rank one matrices $A \in T_n(\mathbb{D})$. When $n \geqslant 3$, we infer by a similar argument in the proof of Examples 5.2.1, 5.2.3 and 5.2.6 that ψ_χ, $\psi_{\mathscr{F}}$ and ψ_Λ are centralizing additive maps on rank one matrices. The result follows.

(ii) ⇒ (iii) In view of Theorems 5.3.1 and 5.3.2, we need only consider $\mathbb{D}$ being a noncommutative division ring. First consider $n = 2$. For each pair of integers $1 \leqslant i \leqslant j \leqslant 2$, there exist additive maps $f_{ij}, g_{ij}, h_{ij} : \mathbb{D} \to \mathbb{D}$ such that

$$\psi(aE_{ij}) = \begin{pmatrix} f_{ij}(a) & h_{ij}(a) \\ 0 & g_{ij}(a) \end{pmatrix}$$

for all $a \in \mathbb{F}$. Since $[\psi(aE_{ij}), aE_{ij}] = 0$ for all $a \in \mathbb{D}$ and $1 \leqslant i \leqslant j \leqslant 2$, it follows that $h_{11} = h_{22} = 0$, and $f_{12}(a)a = ag_{12}(a)$, $f_{11}(a)a = af_{11}(a)$ and $g_{22}(a)a = ag_{22}(a)$ for all $a \in \mathbb{D}$. By Lemma 6.3.5, we have

$$f_{12} = g_{12}, \tag{6.5}$$

and there exist $\lambda, \lambda_1 \in Z(\mathbb{D})$ and additive maps $\mu_1, \mu_2 : \mathbb{D} \to Z(\mathbb{D})$ such that

$$f_{11}(a) = \lambda a + \mu_1(a), \tag{6.6}$$

$$g_{22}(a) = \lambda_1 a + \mu_2(a) \tag{6.7}$$

for all $a \in \mathbb{D}$. Since $[\psi(aE_{11} + bE_{12}), aE_{11} + bE_{12}] = 0$ for all $a, b \in \mathbb{D}$, we thus obtain

$$f_{12}(b)a = af_{12}(b) \tag{6.8}$$

for all $a, b \in \mathbb{D}$, and

$$f_{11}(a)b - ah_{12}(b) - bg_{11}(a) = 0 \tag{6.9}$$

for all $a, b \in \mathbb{D}$. By (6.5) and (6.8), we conclude that $f_{12}(a) = g_{12}(a) \in Z(\mathbb{D})$ for all $a \in \mathbb{D}$. Letting $f = f_{11}$, $g = -h_{12}$, $h = -g_{11}$ in Lemma 6.3.6(i), we obtain (6.9). By Lemma 6.3.6(i) and (6.6), we conclude that $h_{12}(a) = \lambda a$ and $g_{11}(a) = \mu_1(a)$ for all $a \in \mathbb{D}$. Next, by virtue of $[\psi(aE_{22} + bE_{12}), aE_{22} + bE_{12}] = 0$ for all $a, b \in \mathbb{D}$, we have

$$f_{22}(a)b + \lambda ba - bg_{22}(a) = 0 \tag{6.10}$$

for all $a, b \in \mathbb{D}$. Taking $f = f_{22}$, $g(b) = \lambda b$ for $b \in \mathbb{D}$ and $h = -g_{22}$ in Lemma 6.3.6(ii), we get (6.10). By Lemma 6.3.6(ii) and (6.7), we thus obtain $\lambda_1 = \lambda$ and $f_{22}(a) = \mu_2(a)$ for all $a \in \mathbb{D}$. Let $\mu : T_2(\mathbb{D}) \to Z(\mathbb{D})$ be the additive map defined by

$$\mu(A) = \mu_1(a_{11}) + f_{12}(a_{12}) + \mu_2(a_{22})$$

for all $A = (a_{ij}) \in T_2(\mathbb{D})$. Then $\psi(A) = \lambda A + \mu(A)I_2$ for all $A \in T_2(\mathbb{D})$ as desired.

We now proceed to $n \geqslant 3$. For each pair of integers $1 \leqslant s \leqslant t \leqslant n$, there exist additive maps $\phi_{ij}^{(s,t)} : \mathbb{D} \to \mathbb{D}$, $i, j = 1, \ldots, n$, such that

$$\psi(aE_{st}) = \sum_{1 \leqslant i \leqslant j \leqslant n} \phi_{ij}^{(s,t)}(a)E_{ij} \tag{6.11}$$

for all $a \in \mathbb{D}$. Let $1 \leqslant i, j \leqslant n$ be integers. Note that $[\psi(aE_{st}), aE_{st}] = 0$ for all $a \in \mathbb{D}$.

By (6.11) and using the fact that $E_{ij}E_{st} = \delta_{js}E_{it}$ for every $1 \leqslant i, j \leqslant n$, we obtain

$$\sum_{1\leqslant i\leqslant j\leqslant n} \phi_{ij}^{(s,t)}(a)a(\delta_{js}E_{it}) - \sum_{1\leqslant i\leqslant j\leqslant n} a\phi_{ij}^{(s,t)}(a)(\delta_{ti}E_{sj}) = 0$$

for all $a \in \mathbb{D}$, where δ_{ij} is the Kronecker delta. Consequently, for every $1 \leqslant s \leqslant t \leqslant n$,

$$(\phi_{ss}^{(s,t)}(a)a - a\phi_{tt}^{(s,t)}(a))E_{st} + \sum_{i=1}^{s-1} \phi_{is}^{(s,t)}(a)aE_{it} - \sum_{j=t+1}^{n} a\phi_{tj}^{(s,t)}(a)E_{sj} = 0$$

for all $a \in \mathbb{D}$. By the linear independence of E_{ij}, $1 \leqslant i \leqslant j \leqslant n$, we infer that for every $1 \leqslant s \leqslant t \leqslant n$,

$$\phi_{is}^{(s,t)} = 0,\ \ i = 1, \ldots, s-1, \tag{6.12}$$

$$\phi_{tj}^{(s,t)} = 0,\ \ j = t+1, \ldots, n, \tag{6.13}$$

$$\phi_{ss}^{(s,t)}(a)a = a\phi_{tt}^{(s,t)}(a) \tag{6.14}$$

for all $a \in \mathbb{D}$. It follows from (6.14) and Lemma 6.3.5 that for each pair of integers $1 \leqslant s \leqslant t \leqslant n$, there exists $\lambda_{st} \in Z(\mathbb{D})$ and an additive map $\mu_{st} : \mathbb{D} \to Z(\mathbb{D})$ such that

$$\phi_{ss}^{(s,t)}(a) = \phi_{tt}^{(s,t)}(a) = \lambda_{st}a + \mu_{st}(a) \tag{6.15}$$

for all $a \in \mathbb{D}$. We first claim that there exists $\lambda \in Z(\mathbb{D})$ such that for each integer $1 \leqslant s \leqslant n$,

$$\psi(aE_{ss}) = \lambda aE_{ss} + \mu_{ss}(a)I_n + \sum_{i=1}^{s-1}\sum_{j=s+1}^{n} \phi_{ij}^{(s,s)}(a)E_{ij} \tag{6.16}$$

for all $a \in \mathbb{D}$, where $\sum_{i=1}^{s-1}\sum_{j=s+1}^{n} \phi_{ij}^{(s,s)}(a)E_{ij} = 0$ when $s = 1, n$. Let $1 \leqslant s < t \leqslant n$ be integers and let $a, b \in \mathbb{D}$. Since $[\psi(A), A] = 0$ for $A \in \{aE_{ss}, bE_{st}, aE_{ss} + bE_{st}\}$, it follows that

$$\psi(aE_{ss})bE_{st} - bE_{st}\psi(aE_{ss}) + \psi(bE_{st})aE_{ss} - aE_{ss}\psi(bE_{st}) = 0.$$

Using (6.11), we obtain

$$\sum_{i=1}^{s} \phi_{is}^{(s,s)}(a)bE_{it} - \sum_{j=t}^{n} b\phi_{tj}^{(s,s)}(a)E_{sj} + \sum_{i=1}^{s} \phi_{is}^{(s,t)}(b)aE_{is} - \sum_{j=s}^{n} a\phi_{sj}^{(s,t)}(b)E_{sj} = 0.$$

Since $\phi_{is}^{(s,s)} = 0 = \phi_{is}^{(s,t)}$ for $i = 1, \ldots, s-1$ by (6.12), it follows that for every $1 \leqslant s < t \leqslant n$,

$$\begin{aligned}(\phi_{ss}^{(s,s)}(a)b - b\phi_{tt}^{(s,s)}(a) - a\phi_{st}^{(s,t)}(b))E_{st} + (\phi_{ss}^{(s,t)}(b)a - a\phi_{ss}^{(s,t)}(b))E_{ss} \\ - \sum_{j=s+1}^{t-1} a\phi_{sj}^{(s,t)}(b)E_{sj} - \sum_{j=t+1}^{n} (a\phi_{sj}^{(s,t)}(b) + b\phi_{tj}^{(s,s)}(a))E_{sj} = 0\end{aligned}$$

for all $a, b \in \mathbb{D}$. Then for each pair of integers $1 \leqslant s < t \leqslant n$,

$$\phi_{sj}^{(s,t)} = 0, \;\; j = s+1, \ldots, t-1, \tag{6.17}$$

$$\phi_{ss}^{(s,t)}(b)a = a\phi_{ss}^{(s,t)}(b) \tag{6.18}$$

for all $a, b \in \mathbb{D}$,

$$a\phi_{sj}^{(s,t)}(b) + b\phi_{tj}^{(s,s)}(a) = 0, \;\; j = t+1, \ldots, n, \tag{6.19}$$

for all $a, b \in \mathbb{D}$, and

$$\phi_{ss}^{(s,s)}(a)b - a\phi_{st}^{(s,t)}(b) - b\phi_{tt}^{(s,s)}(a) = 0 \tag{6.20}$$

for all $a, b \in \mathbb{D}$. By (6.15) and (6.18), for every $1 \leqslant s < t \leqslant n$, $(\lambda_{st}b + \mu_{st}(b))a = a(\lambda_{st}b + \mu_{st}(b))$, and so $\lambda_{st}ab = \lambda_{st}ba$ for all $a, b \in \mathbb{D}$. By the noncommutativity of $\mathbb{D}$, $\lambda_{st} = 0$ for any $1 \leqslant s < t \leqslant n$. Together with (6.15), we thus conclude that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ss}^{(s,t)}(a) = \phi_{tt}^{(s,t)}(a) = \mu_{st}(a) \in Z(\mathbb{D}) \tag{6.21}$$

for all $a \in \mathbb{D}$. In view of (6.19) and Lemma 6.3.4, for every $1 \leqslant s < t \leqslant n$,

$$\phi_{tj}^{(s,s)} = \phi_{sj}^{(s,t)} = 0, \;\; j = t+1, \ldots, n. \tag{6.22}$$

Now, taking $f = \phi_{ss}^{(s,s)}$, $g = -\phi_{st}^{(s,t)}$ and $h = -\phi_{tt}^{(s,s)}$ in Lemma 6.3.6(i), we obtain (6.20).

Together with (6.15), we conclude from Lemma 6.3.6(i) that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{st}^{(s,t)}(a) = \lambda_{ss} a \tag{6.23}$$

for all $a \in \mathbb{D}$, and

$$\phi_{tt}^{(s,s)}(a) = \mu_{ss}(a) \tag{6.24}$$

for all $a \in \mathbb{D}$. Next, let $1 \leqslant r < s \leqslant n$ be integers and let $a, b \in \mathbb{D}$. By virtue of $[\psi(A), A] = 0$ for $A \in \{aE_{ss}, bE_{rs}, aE_{ss} + bE_{rs}\}$, we get $\psi(aE_{ss})bE_{rs} - bE_{rs}\psi(aE_{ss}) + \psi(bE_{rs})aE_{ss} - aE_{ss}\psi(bE_{rs}) = 0$. It follows from (6.11) that

$$\sum_{i=1}^{r} \phi_{ir}^{(s,s)}(a)bE_{is} - \sum_{j=s}^{n} b\phi_{sj}^{(s,s)}(a)E_{rj} + \sum_{i=1}^{s} \phi_{is}^{(r,s)}(b)aE_{is} - \sum_{j=s}^{n} a\phi_{sj}^{(r,s)}(b)E_{sj} = 0.$$

We further obtain

$$\begin{aligned}(\phi_{rr}^{(s,s)}(a)b \,+\, \phi_{rs}^{(r,s)}(b)a - b\phi_{ss}^{(s,s)}(a))E_{rs} + (\phi_{ss}^{(r,s)}(b)a - a\phi_{ss}^{(r,s)}(b))E_{ss} \\ + \sum_{i=1}^{r-1}(\phi_{ir}^{(s,s)}(a)b + \phi_{is}^{(r,s)}(b)a)E_{is} \,-\, \sum_{j=s+1}^{n} b\phi_{sj}^{(s,s)}(a)E_{rj} \\ + \sum_{i=r+1}^{s-1} \phi_{is}^{(r,s)}(b)aE_{is} - \sum_{j=s+1}^{n} a\phi_{sj}^{(r,s)}(b)E_{sj} = 0.\end{aligned}$$

Since $\phi_{sj}^{(r,s)} = 0 = \phi_{sj}^{(s,s)}$ for $j = s+1, \ldots, n$ by (6.13), and $\phi_{ss}^{(r,s)}(a) = \mu_{rs}(a) \in Z(\mathbb{D})$ for all $a \in \mathbb{D}$ by (6.21), it follows that for every $1 \leqslant r < s \leqslant n$,

$$(\phi_{rr}^{(s,s)}(a)b+\phi_{rs}^{(r,s)}(b)a-b\phi_{ss}^{(s,s)}(a))E_{rs}+\sum_{i=1}^{r-1}(\phi_{ir}^{(s,s)}(a)b+\phi_{is}^{(r,s)}(b)a)E_{is}+\sum_{i=r+1}^{s-1}\phi_{is}^{(r,s)}(b)aE_{is} = 0$$

for all $a, b \in \mathbb{D}$. Then for every $1 \leqslant r < s \leqslant n$,

$$\phi_{is}^{(r,s)} = 0, \;\; i = r+1, \ldots, s-1, \tag{6.25}$$

$$\phi_{ir}^{(s,s)}(a)b + \phi_{is}^{(r,s)}(b)a = 0, \;\; i = 1, \ldots, r-1, \tag{6.26}$$

for all $a, b \in \mathbb{D}$, and

$$\phi_{rr}^{(s,s)}(a)b + \phi_{rs}^{(r,s)}(b)a - b\phi_{ss}^{(s,s)}(a) = 0 \tag{6.27}$$

for all $a, b \in \mathbb{D}$. By (6.26) and Lemma 6.3.4, for every $1 \leqslant r < s \leqslant n$,

$$\phi_{ir}^{(s,s)} = \phi_{is}^{(r,s)} = 0, \;\; i = 1, \ldots, r-1. \tag{6.28}$$

Letting $f = \phi_{rr}^{(s,s)}$, $g = \phi_{rs}^{(r,s)}$ and $h = -\phi_{ss}^{(s,s)}$ in Lemma 6.3.6(ii), we obtain (6.27). Together with $\phi_{ss}^{(s,s)}(a) = \lambda_{ss}a + \mu_{ss}(a)$ for all $a \in \mathbb{D}$ in (6.15), and $\phi_{rs}^{(r,s)}(a) = \lambda_{rr}a$ for all $a \in \mathbb{D}$ in (6.23), we conclude from Lemma 6.3.6(ii) that for every $1 \leqslant r < s \leqslant n$,

$$\lambda_{rr} = \lambda_{ss}, \tag{6.29}$$

$$\phi_{rr}^{(s,s)}(a) = \mu_{ss}(a) \tag{6.30}$$

for all $a \in \mathbb{D}$. From (6.29), we get $\lambda_{11} = \cdots = \lambda_{nn}$. We set $\lambda = \lambda_{11}$. By (6.15), (6.24) and (6.30), for each integer $1 \leqslant s \leqslant n$,

$$\sum_{i=1}^{n} \phi_{ii}^{(s,s)}(a)E_{ii} = \lambda a E_{ss} + \mu_{ss}(a)I_n \tag{6.31}$$

for all $a \in \mathbb{D}$. We note that for each $1 \leqslant s \leqslant n$,

$$\sum_{1 \leqslant i < s} \phi_{is}^{(s,s)}(a)E_{is} = 0 \quad \text{and} \quad \sum_{1 \leqslant i < j < s} \phi_{ij}^{(s,s)}(a)E_{ij} = 0 \tag{6.32}$$

for all $a \in \mathbb{D}$ by (6.12) and (6.28), and

$$\sum_{s < j \leqslant n} \phi_{sj}^{(s,s)}(a)E_{sj} = 0 \quad \text{and} \quad \sum_{s < i < j \leqslant n} \phi_{ij}^{(s,s)}(a)E_{ij} = 0 \tag{6.33}$$

for all $a \in \mathbb{D}$ by (6.13) and (6.22). It follows from (6.32) and (6.33) that for each $1 \leqslant s \leqslant n$,

$$\begin{aligned}
\sum_{1 \leqslant i < j \leqslant n} \phi_{ij}^{(s,s)}(a)E_{ij} &= \sum_{1 \leqslant i < j < s} \phi_{ij}^{(s,s)}(a)E_{ij} + \sum_{1 \leqslant i < s} \phi_{is}^{(s,s)}(a)E_{is} + \sum_{i=1}^{s-1}\sum_{j=s+1}^{n} \phi_{ij}^{(s,s)}(a)E_{ij} \\
&\quad + \sum_{s < j \leqslant n} \phi_{sj}^{(s,s)}(a)E_{sj} + \sum_{s < i < j \leqslant n} \phi_{ij}^{(s,s)}(a)E_{ij} \\
&= \sum_{i=1}^{s-1}\sum_{j=s+1}^{n} \phi_{ij}^{(s,s)}(a)E_{ij}
\end{aligned}$$

for all $a \in \mathbb{D}$. Then, together with (6.31), for each integer $1 \leqslant s \leqslant n$,

$$\psi(aE_{ss}) = \sum_{i=1}^{n} \phi_{ii}^{(s,s)}(a)E_{ii} + \sum_{1 \leqslant i < j \leqslant n} \phi_{ij}^{(s,s)}(a)E_{ij} = \lambda aE_{ss} + \mu_{ss}(a)I_n + \sum_{i=1}^{s-1} \sum_{j=s+1}^{n} \phi_{ij}^{(s,s)}(a)E_{ij}$$

for all $a \in \mathbb{D}$. Consequently, Claim (6.16) is proved.

We next claim that for each pair of integers $1 \leqslant s < t \leqslant n$,

$$\psi(aE_{st}) = \lambda aE_{st} + \mu_{st}(a)I_n + \sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a)E_{ij} \tag{6.34}$$

for every $a \in \mathbb{D}$, where $\sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a)E_{ij} = 0$ when $s = 1$ or $t = n$. Let $a, b \in \mathbb{D}$ and let $1 \leqslant s < t \leqslant n$ and $1 \leqslant p \leqslant n$ be integers such that $p \neq s, t$. The proof will be divided into two parts.

Part I: $1 \leqslant p < t$ with $p \neq s$. From $[\psi(A), A] = 0$ for $A \in \{aE_{st}, bE_{pt}, aE_{st} + bE_{pt}\}$, we infer from (6.11) that

$$\sum_{i=1}^{p} \phi_{ip}^{(s,t)}(a)bE_{it} - \sum_{j=t}^{n} b\phi_{tj}^{(s,t)}(a)E_{pj} + \sum_{i=1}^{s} \phi_{is}^{(p,t)}(b)aE_{it} - \sum_{j=t}^{n} a\phi_{tj}^{(p,t)}(b)E_{sj} = 0. \tag{6.35}$$

First consider $1 \leqslant p < s$. By (6.35), we obtain

$$\begin{aligned}(\phi_{pp}^{(s,t)}(a)b - b\phi_{tt}^{(s,t)}(a) + \phi_{ps}^{(p,t)}(b)a)E_{pt} + (\phi_{ss}^{(p,t)}(b)a - a\phi_{tt}^{(p,t)}(b))E_{st} \\ + \sum_{i=1}^{p-1}(\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a)E_{it} + \sum_{i=p+1}^{s-1} \phi_{is}^{(p,t)}(b)aE_{it} \\ - \sum_{j=t+1}^{n} b\phi_{tj}^{(s,t)}(a)E_{pj} - \sum_{j=t+1}^{n} a\phi_{tj}^{(p,t)}(b)E_{sj} = 0.\end{aligned}$$

Note that $\phi_{tj}^{(s,t)} = 0 = \phi_{tj}^{(p,t)}$ for $j = t+1, \ldots, n$ by (6.13), and $\phi_{ps}^{(p,t)} = 0$ by (6.17). Then for every $1 \leqslant p < s < t \leqslant n$,

$$\begin{aligned}(\phi_{pp}^{(s,t)}(a)b - b\phi_{tt}^{(s,t)}(a))E_{pt} + (\phi_{ss}^{(p,t)}(b)a - a\phi_{tt}^{(p,t)}(b))E_{st} \\ + \sum_{i=1}^{p-1}(\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a)E_{it} + \sum_{i=p+1}^{s-1} \phi_{is}^{(p,t)}(b)aE_{it} = 0\end{aligned}$$

for all $a, b \in \mathbb{D}$. Consequently, for every $1 \leqslant p < s < t \leqslant n$,

$$\phi_{pp}^{(s,t)}(a)b - b\phi_{tt}^{(s,t)}(a) = 0 \tag{6.36}$$

for all $a, b \in \mathbb{D}$,

$$\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a = 0, \;\; i = 1, \ldots, p-1, \tag{6.37}$$

for all $a, b \in \mathbb{D}$. Because $\phi_{tt}^{(s,t)}(a) = \mu_{st}(a) \in Z(\mathbb{D})$ for all $a \in \mathbb{D}$ by (6.21), we infer from (6.36) that for every $1 < s < t \leqslant n$,

$$\phi_{ii}^{(s,t)}(a) = \mu_{st}(a), \;\; i = 1, \ldots, s-1, \tag{6.38}$$

for all $a \in \mathbb{D}$. From (6.37), together with Lemma 6.3.4, we see that for every $1 \leqslant p < s < t \leqslant n$, $\phi_{ip}^{(s,t)} = 0$ for $i = 1, \ldots, p-1$. One sees immediately that for every $1 < s < t \leqslant n$,

$$\phi_{ij}^{(s,t)} = 0 \tag{6.39}$$

for all integers $1 \leqslant i < j < s$. We next consider $s < p < t$. It follows from (6.35) that

$$\begin{aligned}(\phi_{ss}^{(p,t)}(b)a \; - \; a\phi_{tt}^{(p,t)}(b) + \phi_{sp}^{(s,t)}(a)b)E_{st} + (\phi_{pp}^{(s,t)}(a)b - b\phi_{tt}^{(s,t)}(a))E_{pt} \\ + \sum_{i=1}^{s-1}(\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a)E_{it} + \sum_{i=s+1}^{p-1}\phi_{ip}^{(s,t)}(a)bE_{it} \\ - \sum_{j=t+1}^{n} b\phi_{tj}^{(s,t)}(a)E_{pj} - \sum_{j=t+1}^{n} a\phi_{tj}^{(p,t)}(b)E_{sj} = 0.\end{aligned}$$

Note that $\phi_{tj}^{(s,t)} = 0 = \phi_{tj}^{(p,t)}$ for $j = t+1, \ldots, n$ by (6.13), and $\phi_{sp}^{(s,t)} = 0$ by (6.17). It follows that for every $1 \leqslant s < p < t \leqslant n$,

$$\begin{aligned}(\phi_{ss}^{(p,t)}(b)a - a\phi_{tt}^{(p,t)}(b))E_{st} + (\phi_{pp}^{(s,t)}(a)b - \; b\phi_{tt}^{(s,t)}(a))E_{pt} \\ + \sum_{i=1}^{s-1}(\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a)E_{it} + \sum_{i=s+1}^{p-1}\phi_{ip}^{(s,t)}(a)bE_{it} = 0\end{aligned}$$

for all $a, b \in \mathbb{D}$. Then for every $1 \leqslant s < p < t \leqslant n$,

$$\phi_{ip}^{(s,t)} = 0, \quad i = s+1, \ldots, p-1, \tag{6.40}$$

$$\phi_{pp}^{(s,t)}(a)b - b\phi_{tt}^{(s,t)}(a) = 0 \tag{6.41}$$

for all $a, b \in \mathbb{D}$, and

$$\phi_{ip}^{(s,t)}(a)b + \phi_{is}^{(p,t)}(b)a = 0, \;\; i = 1, \ldots, s-1, \tag{6.42}$$

for all $a, b \in \mathbb{D}$. From (6.40), we conclude that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ij}^{(s,t)} = 0 \tag{6.43}$$

for all integers $s < i < j < t$. In view of (6.41), since $\phi_{tt}^{(s,t)}(a) = \mu_{st}(a) \in Z(\mathbb{D})$ for $a \in \mathbb{D}$, it follows that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ii}^{(s,t)}(a) = \mu_{st}(a), \;\; i = s+1, \ldots, t-1, \tag{6.44}$$

for all $a \in \mathbb{D}$. By (6.42) and Lemma 6.3.4, we see that for every $1 \leqslant s < p < t \leqslant n$, $\phi_{ip}^{(s,t)} = 0$ for $i = 1, \ldots, s-1$. One sees immediately that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ij}^{(s,t)} = 0 \tag{6.45}$$

for all integers $1 \leqslant i < s$ and $s < j < t$.

Part II: $s < p \leqslant n$ with $p \neq t$. Since $[\psi(A), A] = 0$ for $A \in \{aE_{st}, bE_{sp}, aE_{st} + bE_{sp}\}$, it follows from (6.11) that

$$\sum_{i=1}^{s} \phi_{is}^{(s,t)}(a)bE_{ip} - \sum_{j=p}^{n} b\phi_{pj}^{(s,t)}(a)E_{sj} + \sum_{i=1}^{s} \phi_{is}^{(s,p)}(b)aE_{it} - \sum_{j=t}^{n} a\phi_{tj}^{(s,p)}(b)E_{sj} = 0. \tag{6.46}$$

Consider $t < p \leqslant n$. By (6.46), we see that

$$\begin{aligned}(\phi_{ss}^{(s,t)}(a)b \,-\, b\phi_{pp}^{(s,t)}(a) - a\phi_{tp}^{(s,p)}(b))E_{sp} + (\phi_{ss}^{(s,p)}(b)a - a\phi_{tt}^{(s,p)}(b))E_{st} \\ - \sum_{j=p+1}^{n} (b\phi_{pj}^{(s,t)}(a) + a\phi_{tj}^{(s,p)}(b))E_{sj} + \sum_{i=1}^{s-1} \phi_{is}^{(s,p)}(b)aE_{it} \\ + \sum_{i=1}^{s-1} \phi_{is}^{(s,t)}(a)bE_{ip} - \sum_{j=t+1}^{p-1} a\phi_{tj}^{(s,p)}(b)E_{sj} = 0.\end{aligned}$$

Since $\phi_{tp}^{(s,p)} = 0$ by (6.25), $\phi_{is}^{(s,t)} = 0 = \phi_{is}^{(s,p)}$ for $i = 1, \ldots, s-1$ by (6.12), and $\phi_{tj}^{(s,p)} = 0$ for $j = t+1, \ldots, p-1$ by (6.43), it follows that for every $1 \leqslant s < t < p \leqslant n$,

$$(\phi_{ss}^{(s,t)}(a)b - b\phi_{pp}^{(s,t)}(a))E_{sp} + (\phi_{ss}^{(s,p)}(b)a - a\phi_{tt}^{(s,p)}(b))E_{st} - \sum_{j=p+1}^{n} (b\phi_{pj}^{(s,t)}(a) + a\phi_{tj}^{(s,p)}(b))E_{sj} = 0$$

for all $a, b \in \mathbb{D}$. Then for every $1 \leqslant s < t < p \leqslant n$,

$$\phi_{ss}^{(s,t)}(a)b - b\phi_{pp}^{(s,t)}(a) = 0 \tag{6.47}$$

for all $a, b \in \mathbb{D}$, and

$$b\phi_{pj}^{(s,t)}(a) + a\phi_{tj}^{(s,p)}(b) = 0, \ \ j = p+1, \ldots, n, \tag{6.48}$$

for all $a, b \in \mathbb{D}$. From (6.47), since $\phi_{ss}^{(s,t)}(a) = \mu_{st}(a) \in Z(\mathbb{D})$ for $a \in \mathbb{D}$, we infer that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ii}^{(s,t)}(a) = \mu_{st}(a), \ \ i = t+1, \ldots, n, \tag{6.49}$$

for all $a \in \mathbb{D}$. By (6.48) and Lemma 6.3.4, we see that for every $1 \leqslant s < t < p \leqslant n$, $\phi_{pj}^{(s,t)} = 0$ for $j = p+1, \ldots, n$. From this, we conclude that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ij}^{(s,t)} = 0 \tag{6.50}$$

for all integers $t < i < j \leqslant n$. Next consider $s < p < t$. From (6.46), we infer that

$$(\phi_{ss}^{(s,t)}(a)b - b\phi_{pp}^{(s,t)}(a))E_{sp} + (\phi_{ss}^{(s,p)}(b)a - a\phi_{tt}^{(s,p)}(b) - b\phi_{pt}^{(s,t)}(a))E_{st} + \sum_{i=1}^{s-1} \phi_{is}^{(s,t)}(a)bE_{ip} + \sum_{i=1}^{s-1} \phi_{is}^{(s,p)}(b)aE_{it} - \sum_{j=p+1}^{t-1} b\phi_{pj}^{(s,t)}(a)E_{sj} - \sum_{j=t+1}^{n} (b\phi_{pj}^{(s,t)}(a) + a\phi_{tj}^{(s,p)}(b))E_{sj} = 0.$$

Since $\phi_{ss}^{(s,t)} = \phi_{pp}^{(s,t)}$ and $\phi_{ss}^{(s,p)} = \phi_{tt}^{(s,p)}$ by (6.21), (6.44) and (6.49), $\phi_{pt}^{(s,t)} = 0$ by (6.25), $\phi_{is}^{(s,t)} = 0 = \phi_{is}^{(s,p)}$ for $i = 1, \ldots, s-1$ by (6.12), and $\phi_{pj}^{(s,t)} = 0$ for $j = p+1, \ldots, t-1$ by (6.43), it follows that for every $1 \leqslant s < p < t \leqslant n$,

$$\sum_{j=t+1}^{n} (b\phi_{pj}^{(s,t)}(a) + a\phi_{tj}^{(s,p)}(b))E_{sj} = 0$$

for all $a, b \in \mathbb{D}$. Then for every $1 \leqslant s < p < t \leqslant n$, we have $b\phi_{pj}^{(s,t)}(a) + a\phi_{tj}^{(s,p)}(b) = 0$ for all $a, b \in \mathbb{D}$ and $j = t+1, \ldots, n$. It follows from Lemma 6.3.4 that for every $1 \leqslant s < p < t \leqslant n$, $\phi_{pj}^{(s,t)} = 0$ for $j = t+1, \ldots, n$. One sees immediately that for every $1 \leqslant s < t \leqslant n$,

$$\phi_{ij}^{(s,t)} = 0 \tag{6.51}$$

for all integers $s < i < t$ and $t < j \leqslant n$.

Now we are in the position to prove Claim (6.34). Let $1 \leqslant s < t \leqslant n$ be integers and let $a \in \mathbb{D}$. By (6.21), (6.38), (6.44) and (6.49), we have

$$\sum_{i=1}^{n} \phi_{ii}^{(s,t)}(a)E_{ii} = \mu_{st}(a)I_n. \tag{6.52}$$

Note also that

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant n} \phi_{ij}^{(s,t)}(a)E_{ij} = \sum_{1\leqslant i<j\leqslant s} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{i=1}^{s-1}\sum_{j=s+1}^{t} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{i=1}^{s-1}\sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a)E_{ij} \\ + \sum_{s\leqslant i<j\leqslant t} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{i=s}^{t-1}\sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a)E_{ij} + \sum_{t\leqslant i<j\leqslant n} \phi_{ij}^{(s,t)}(a)E_{ij}.\end{aligned}$$

In view of $\phi_{st}^{(s,t)}(a) = \lambda a$ by (6.23) and (6.29), and $\phi_{ij}^{(s,t)} = 0$ for all $s \leqslant i < j \leqslant t$ except $(i,j) = (s,t)$ by (6.17), (6.25) and (6.43), we obtain

$$\sum_{s\leqslant i<j\leqslant t} \phi_{ij}^{(s,t)}(a)E_{ij} = \lambda a E_{st}.$$

It is straightforward from (6.12) and (6.39) as well as (6.13) and (6.50) that

$$\sum_{1\leqslant i<j\leqslant s} \phi_{ij}^{(s,t)}(a)E_{ij} = 0 \quad \text{and} \quad \sum_{t\leqslant i<j\leqslant n} \phi_{ij}^{(s,t)}(a)E_{ij} = 0.$$

Likewise, one sees immediately from (6.28) and (6.45) as well as (6.22) and (6.51) that

$$\sum_{i=1}^{s-1}\sum_{j=s+1}^{t}\phi_{ij}^{(s,t)}(a)E_{ij}=0 \quad \text{and} \quad \sum_{i=s}^{t-1}\sum_{j=t+1}^{n}\phi_{ij}^{(s,t)}(a)E_{ij}=0.$$

Consequently, we deduce that

$$\sum_{1\leqslant i<j\leqslant n}\phi_{ij}^{(s,t)}(a)E_{ij}=\lambda aE_{st}+\sum_{i=1}^{s-1}\sum_{j=t+1}^{n}\phi_{ij}^{(s,t)}(a)E_{ij}. \tag{6.53}$$

It follows from (6.52) and (6.53) that for every $1\leqslant s<t\leqslant n$,

$$\psi(aE_{st})=\lambda aE_{st}+\mu_{st}(a)I_n+\sum_{i=1}^{s-1}\sum_{j=t+1}^{n}\phi_{ij}^{(s,t)}(a)E_{ij}$$

for all $a\in\mathbb{D}$, which is the desired conclusion.

Let $\mu : T_n(\mathbb{D})\to Z(\mathbb{D})$ be the additive map defined by

$$\mu(A)=\sum_{1\leqslant s\leqslant t\leqslant n}\mu_{st}(a_{st}) \tag{6.54}$$

for all $A=(a_{st})\in T_n(\mathbb{D})$, and let $\psi_{\mathscr{F}} : T_n(\mathbb{D})\to T_n(\mathbb{D})$ be the additive map defined by

$$\psi_{\mathscr{F}}(A)=\sum_{1<s\leqslant t<n}\left(\sum_{i=1}^{s-1}\sum_{j=t+1}^{n}\phi_{ij}^{(s,t)}(a_{st})E_{ij}\right) \tag{6.55}$$

for all $A=(a_{ij})\in T_n(\mathbb{D})$, where $\mathscr{F}=\bigcup_{1<s\leqslant t<n}\{\phi_{ij}^{(s,t)} : \mathbb{D}\to\mathbb{D} : 1\leqslant i\leqslant s-1 \text{ and } t+1\leqslant j\leqslant n\}$. It follows from (6.16), (6.34), (6.54) and (6.55) that

$$\begin{aligned}\psi(A) &= \sum_{1\leqslant s\leqslant t\leqslant n}\psi(a_{st}E_{st})\\ &= \sum_{1\leqslant s\leqslant t\leqslant n}\lambda a_{st}E_{st}+\sum_{1\leqslant s\leqslant t\leqslant n}\mu_{st}(a_{st})I_n+\sum_{1<s\leqslant t<n}\left(\sum_{i=1}^{s-1}\sum_{j=t+1}^{n}\phi_{ij}^{(s,t)}(a_{st})E_{ij}\right)\\ &= \lambda A+\mu(A)I_n+\psi_{\mathscr{F}}(A)\end{aligned}$$

for all $A=(a_{st})\in T_n(\mathbb{D})$. This completes the proof. □

Remark: The results in this chapter have been submitted for publication in Chooi and **Tan, L. Y.** (2022).

CHAPTER 7: 2-POWER COMMUTING ADDITIVE MAPS ON INVERTIBLE UPPER TRIANGULAR MATRICES OVER THE GALOIS FIELD OF TWO ELEMENTS

7.1 Introduction

Let $\mathcal{R}$ be a ring with centre $Z(\mathcal{R})$ and let $m \geqslant 2$ be an integer. Recall that a map $\psi : \mathcal{R} \to \mathcal{R}$ is said to be *m-power commuting* if $[\psi(x), x^m] = 0$ for all $x \in \mathcal{R}$, where $[x, y]$ is the commutator of $x, y \in \mathcal{R}$. Let $\mathcal{R}$ be a prime ring with char$\mathcal{R} \neq 2$. Brešar and Hvala (1995) proved that 2-power commuting additive maps $\psi : \mathcal{R} \to \mathcal{R}$ is of the standard form (2.14). Let $m \geqslant 1$ be an integer. Later, Beidar et al. (1997) extended this result to m-power commuting additive maps and proved that every m-power commuting additive map on a prime ring $\mathcal{R}$ with char $\mathcal{R} = 0$ or char $\mathcal{R} > m$ is of the standard form (2.14). Let $n \geqslant 2$ be an integer. Recall that $M_n(\mathbb{F})$ is the ring of all $n \times n$ matrices over a field $\mathbb{F}$ with centre $Z(M_n(\mathbb{F}))$ and unity I_n. Recently, C.-K. Liu and Yang (2017) generalised the result of Beidar et al. (1997) to subsets of matrices that are not closed under addition and proved that when char $\mathbb{F} = 0$ or char $\mathbb{F} > m$, the m-power commuting additive maps $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ for all invertible matrices $A \in M_n(\mathbb{F})$ are of the standard form, i.e. there exists $\lambda \in \mathbb{F}$ and an additive map $\mu : M_n(\mathbb{F}) \to \mathbb{F}$ such that $\psi(A) = \lambda A + \mu(A)I_n$ for all $A \in M_n(\mathbb{F})$. In the same paper, an analogous result was obtained for m-power commuting additive maps on singular matrices over fields, unless $n = 2$ and char $\mathbb{F} = 2$. Let $\mathbb{D}$ be a division ring and let $n \geqslant 3$ be an integer. Later, Chou and Liu (2021) asserted that additive maps $\psi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ satisfying $[\psi(x), x^{m(x)}] = 0$ for all rank k, $1 < k < n$ matrices $x \in M_n(\mathbb{D})$ are of the standard form (2.14), where $m(x) \geqslant 1$ is an integer depending on x. Let $n \geqslant 2$ be an integer. Recall that $T_n(\mathbb{F}_2)$ is the ring of all $n \times n$ upper triangular matrices over the Galois field of two elements with centre $Z(T_n(\mathbb{F}_2))$ and unity I_n. Inspired by the aforesaid results, in this chapter we study and obtain a complete structural characterisation of additive map $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ that satisfies $[\psi(A), A^2] = 0$ on all invertible matrices $A \in T_n(\mathbb{F}_2)$. Unexpectedly, the structure of ψ is different from the standard form (2.14).

7.2 Irregular nonstandard examples

Throughout this section, unless stated otherwise, let $\mathbb{F}_2$ denote the Galois field of two elements. We begin our discussion with the following irregular examples of 2-power commuting additive maps on invertible upper triangular matrices over $\mathbb{F}_2$.

Example 7.2.1. Let $\lambda_{11}^{(1,1)}, \lambda_{22}^{(1,1)}, \lambda_{12}^{(1,1)}, \lambda_{11}^{(1,2)}, \lambda_{22}^{(1,2)} \in \mathbb{F}_2$. Suppose that $\varsigma : T_2(\mathbb{F}_2) \to T_2(\mathbb{F}_2)$ is the additive map defined by

$$\varsigma(A) = (\lambda_{11}^{(1,2)} + \lambda_{22}^{(1,2)})a_{12}E_{11} + (\lambda_{11}^{(1,1)} + \lambda_{22}^{(1,1)})a_{11}E_{22} + \lambda_{12}^{(1,1)}a_{11}E_{12} \tag{7.1}$$

for all $A = (a_{ij}) \in T_2(\mathbb{F}_2)$. We show that ς is a 2-power commuting additive map on invertible matrices $A \in T_2(\mathbb{F}_2)$. Let $A = (a_{ij}) \in T_2(\mathbb{F}_2)$ be invertible. Then $A \in \{I_2, I_2 + E_{12}\}$ and $A^2 = I_2$. Clearly $[\varsigma(A), A^2] = [\varsigma(A), I_2] = 0$.

Example 7.2.2. Let $1 \leqslant i < j \leqslant 3$ and $1 \leqslant s < t \leqslant 3$ be integers. Let $\lambda, \lambda_{ss}^{(1,1)}, \lambda_{st}^{(1,1)}, \lambda_{ss}^{(i,j)}, \lambda_{st}^{(i,j)} \in \mathbb{F}_2$. Suppose that $\phi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ is the additive map defined by

$$\begin{aligned}\phi(A) = &((\lambda_{11}^{(1,1)} + \lambda_{22}^{(1,1)})a_{11} + (\lambda_{11}^{(1,2)} + \lambda_{22}^{(1,2)})a_{12} + (\lambda_{11}^{(2,3)} + \lambda_{22}^{(2,3)})a_{23}\\ &+ (\lambda_{11}^{(1,3)} + \lambda_{22}^{(1,3)})a_{13})E_{22}\\ &+ (\lambda_{12}^{(1,1)}a_{11} + (\lambda + \lambda)a_{12} + \lambda_{12}^{(1,3)}a_{13} + \lambda_{12}^{(2,3)}a_{23})E_{12}\\ &+ (\lambda_{13}^{(1,1)}a_{11} + \lambda_{13}^{(1,2)}a_{12} + (\lambda_{13}^{(1,3)} + \lambda)a_{13} + \lambda_{13}^{(2,3)}a_{23})E_{13}\\ &+ (\lambda_{23}^{(1,1)}a_{11} + \lambda_{23}^{(1,2)}a_{12} + \lambda_{23}^{(1,3)}a_{13} + (\lambda_{23}^{(2,3)} + \lambda)a_{23})E_{23}.\end{aligned} \tag{7.2}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F}_2)$. We show that ϕ is a 2-power commuting additive map on invertible matrices $A \in T_3(\mathbb{F}_2)$. Let $A \in T_3(\mathbb{F}_2)$ be invertible and let $B = A^2 = (b_{ij}) \in T_3(\mathbb{F}_2)$. Since $A \in T_3(\mathbb{F}_2)$ is invertible, then $A = I_3 + U$, where $U = \sum_{1 \leqslant i < j \leqslant 3} a_{ij}E_{ij}$. Then $B = A^2 = (I_3 + U)(I_3 + U) = I_3 + U^2$. By Lemma 7.4.1, we have $b_{i,i+1} = 0$ for all $i = 1, 2$. Then $U^2 = b_{13}E_{13}$. By $[\phi(A), I_3] = 0$, we see that $[\phi(A), A^2] = [\phi(A), U^2] = 0$, because

$$\begin{aligned}\phi(A)U^2 = \Big(&((\lambda_{11}^{(1,1)} + \lambda_{22}^{(1,1)})a_{11} + (\lambda_{11}^{(1,2)} + \lambda_{22}^{(1,2)})a_{12}\\ &+(\lambda_{11}^{(2,3)} + \lambda_{22}^{(2,3)})a_{23} + (\lambda_{11}^{(1,3)} + \lambda_{22}^{(1,3)})a_{13})E_{22}\end{aligned}$$

$$
\begin{aligned}
&+(\lambda_{12}^{(1,1)}a_{11} + (\lambda+\lambda)a_{12} + \lambda_{12}^{(1,3)}a_{13} + \lambda_{12}^{(2,3)}a_{23})E_{12} \\
&+(\lambda_{13}^{(1,1)}a_{11} + \lambda_{13}^{(1,2)}a_{12} + (\lambda_{13}^{(1,3)}+\lambda)a_{13} + \lambda_{13}^{(2,3)}a_{23})E_{13} \\
&+(\lambda_{23}^{(1,1)}a_{11} + \lambda_{23}^{(1,2)}a_{12} + \lambda_{23}^{(1,3)}a_{13} + (\lambda_{23}^{(2,3)}+\lambda)a_{23})E_{23}\Big)b_{13}E_{13} \\
= {} & 0
\end{aligned}
$$

and

$$
\begin{aligned}
U^2\phi(A) = {} & b_{13}E_{13}\Big(((\lambda_{11}^{(1,1)} + \lambda_{22}^{(1,1)})a_{11} + (\lambda_{11}^{(1,2)} + \lambda_{22}^{(1,2)})a_{12} \\
&+(\lambda_{11}^{(2,3)} + \lambda_{22}^{(2,3)})a_{23} + (\lambda_{11}^{(1,3)} + \lambda_{22}^{(1,3)})a_{13})E_{22} \\
&+(\lambda_{12}^{(1,1)}a_{11} + (\lambda+\lambda)a_{12} + \lambda_{12}^{(1,3)}a_{13} + \lambda_{12}^{(2,3)}a_{23})E_{12} \\
&+(\lambda_{13}^{(1,1)}a_{11} + \lambda_{13}^{(1,2)}a_{12} + (\lambda_{13}^{(1,3)}+\lambda)a_{13} + \lambda_{13}^{(2,3)}a_{23})E_{13} \\
&+(\lambda_{23}^{(1,1)}a_{11} + \lambda_{23}^{(1,2)}a_{12} + \lambda_{23}^{(1,3)}a_{13} + (\lambda_{23}^{(2,3)}+\lambda)a_{23})E_{23}\Big) \\
= {} & 0.
\end{aligned}
$$

For instance, the following maps

$$
\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_{13} & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & a_{12} \\ 0 & a_{23} & a_{13} \\ 0 & 0 & 0 \end{pmatrix}
$$

are 2-power commuting additive maps on invertible matrices in $T_3(\mathbb{F}_2)$ of this type.

Example 7.2.3. Let $\gamma \in \mathbb{F}_2$ and let $\psi_\gamma(A) : T_4(\mathbb{F}_2) \to T_4(\mathbb{F}_2)$ be the additive map defined by

$$
\psi_\gamma(A) = \gamma(a_{23}a_{34}E_{22} + a_{23}E_{44} + a_{13}E_{12} + a_{24}E_{34}) \tag{7.3}
$$

for all $A = (a_{ij}) \in T_4(\mathbb{F}_2)$. We show that ψ_γ is a 2-power commuting additive map on invertible matrices $A \in T_4(\mathbb{F}_2)$. Let $A = (a_{ij}) \in T_4(\mathbb{F}_2)$ be invertible. Then $A = I_4 + a_{12}E_{12} + a_{13}E_{13} + a_{14}E_{14} + a_{23}E_{23} + a_{24}E_{24} + a_{34}E_{34}$ and $A^2 = I_4 + B$, where

$B = a_{12}a_{23}E_{13} + (a_{12}a_{24} + a_{13}a_{34})E_{14} + a_{23}a_{34}E_{24}$. By

$$\begin{aligned}\psi_\gamma(A)B &= \gamma(a_{23}a_{34}E_{22} + a_{23}E_{44} + a_{13}E_{12} + a_{24}E_{34})(a_{12}a_{23}E_{13} \\ &\quad +(a_{12}a_{24} + a_{13}a_{34})E_{14} + a_{23}a_{34}E_{24}) \\ &= \gamma(a_{23}a_{23}a_{34}a_{34}E_{24} + a_{13}a_{23}a_{34}E_{14})\end{aligned}$$

and

$$\begin{aligned}B\psi_\gamma(A) &= \gamma(a_{12}a_{23}E_{13} + (a_{12}a_{24} + a_{13}a_{34})E_{14} + a_{23}a_{34}E_{24})(a_{23}a_{34}E_{22} \\ &\quad +a_{23}E_{44} + a_{13}E_{12} + a_{24}E_{34}) \\ &= \gamma(a_{12}a_{23}a_{24}E_{14} + (a_{12}a_{23}a_{24} + a_{13}a_{23}a_{34})E_{14} + a_{23}a_{23}a_{34})E_{24}) \\ &= \gamma(a_{13}a_{23}a_{34}E_{14} + a_{23}a_{23}a_{34}E_{24}),\end{aligned}$$

we get $\psi_\gamma(A)B - B\psi_\gamma(A) = \gamma a_{23}a_{23}a_{34}(a_{34}+1)E_{24} = 0$ because $a_{34}(a_{34}+1) = 0$. It follows from $[\psi_\gamma, I_4] = 0$ that $[\psi_\gamma(A), A^2] = [\psi_\gamma(A), I_4 + B] = [\psi_\gamma(A), B] = 0$. For instance, the following maps

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{23}a_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{23} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are 2-power commuting additive maps on invertible matrices in $T_4(\mathbb{F}_2)$ of this type.

Example 7.2.4. Let $n \geqslant 4$ be an integer. Denote $\theta_n = \{(1, n-1), (1, n), (2, n-1), (2, n)\}$. Let $\Theta = \bigcup_{(s,t)\in\theta_n}\{\lambda_{st}^{(i,j)} \in \mathbb{F}_2 : 1 \leqslant i < j \leqslant n\}$ be a set of scalars on $\mathbb{F}_2$. Let $\lambda, \lambda_{st}^{(1,1)} \in$

$\mathbb{F}_2$ for $(s,t) \in \theta_n$. Suppose that $\psi_\Theta : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the additive map defined by

$$\begin{aligned}\psi_\Theta(A) = &\left(\lambda_{1,n-1}^{(1,1)} a_{11} + \lambda a_{1,n-1} + \sum_{1\leqslant i<j\leqslant n} \lambda_{1,n-1}^{(i,j)} a_{ij}\right) E_{1,n-1}\\ &+ \left(\lambda_{1n}^{(1,1)} a_{11} + \lambda a_{1n} + \sum_{1\leqslant i<j\leqslant n} \lambda_{1n}^{(i,j)} a_{ij}\right) E_{1n}\\ &+ \left(\lambda_{2,n-1}^{(1,1)} a_{11} + \lambda a_{2,n-1} + \sum_{1\leqslant i<j\leqslant n} \lambda_{2,n-1}^{(i,j)} a_{ij}\right) E_{2,n-1}\\ &+ \left(\lambda_{2n}^{(1,1)} a_{11} + \lambda a_{2n} + \sum_{1\leqslant i<j\leqslant n} \lambda_{2n}^{(i,j)} a_{ij}\right) E_{2n}\end{aligned} \tag{7.4}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. We show that ψ_Θ is a 2-power commuting additive map on invertible matrices $A \in T_n(\mathbb{F}_2)$. Let $A \in T_n(\mathbb{F}_2)$ be invertible and let $B = A^2 = (b_{ij}) \in T_n(\mathbb{F}_2)$. Since $A \in T_n(\mathbb{F}_2)$ is invertible, then $A = I_n + U$, where $U = \sum_{1\leqslant i<j\leqslant n} a_{ij} E_{ij}$. Then $B = A^2 = (I_n + U)(I_n + U) = I_n + U^2$. By Lemma 7.4.1, we have $b_{i,i+1} = 0$ for all $i = 1, \ldots, n-1$. Then

$$\begin{aligned}U^2 &= b_{13}E_{13} + \cdots + b_{1n}E_{1n} + b_{24}E_{24} + \cdots + b_{2n}E_{2n} + \cdots\\ &\quad + b_{n-3,n-1}E_{n-3,n-1} + b_{n-3,n}E_{n-3,n} + b_{n-2,n}E_{n-2,n}.\end{aligned}$$

By $[\psi_\Theta(A), I_n] = 0$, we see that $[\psi_\Theta(A), A^2] = [\psi_\Theta(A), U^2] = 0$, because

$$\begin{aligned}\psi_\Theta(A)U^2 = &\Bigg(\left(\lambda_{1,n-1}^{(1,1)} a_{11} + \lambda a_{1,n-1} + \sum_{1\leqslant i<j\leqslant n} \lambda_{1,n-1}^{(i,j)} a_{ij}\right) E_{1,n-1}\\ &+ \left(\lambda_{1n}^{(1,1)} a_{11} + \lambda a_{1n} + \sum_{1\leqslant i<j\leqslant n} \lambda_{1n}^{(i,j)} a_{ij}\right) E_{1n}\\ &+ \left(\lambda_{2,n-1}^{(1,1)} a_{11} + \lambda a_{2,n-1} + \sum_{1\leqslant i<j\leqslant n} \lambda_{2,n-1}^{(i,j)} a_{ij}\right) E_{2,n-1}\\ &+ \left(\lambda_{2n}^{(1,1)} a_{11} + \lambda a_{2n} + \sum_{1\leqslant i<j\leqslant n} \lambda_{2n}^{(i,j)} a_{ij}\right) E_{2n}\Bigg)\\ &\Big(b_{13}E_{13} + \cdots + b_{1n}E_{1n} + b_{24}E_{24} + \cdots + b_{2n}E_{2n} + \cdots\\ &+ b_{n-3,n-1}E_{n-3,n-1} + b_{n-3,n}E_{n-3,n} + b_{n-2,n}E_{n-2,n}\Big)\\ = &\ 0\end{aligned}$$

and

$$\begin{aligned} U^2\psi_\Theta(A) &= (b_{13}E_{13} + \cdots + b_{1n}E_{1n} + b_{24}E_{24} + \cdots + b_{2n}E_{2n} + \cdots \\ &\quad + b_{n-3,n-1}E_{n-3,n-1} + b_{n-3,n}E_{n-3,n} + b_{n-2,n}E_{n-2,n}) \\ &\quad \Bigg(\bigg(\lambda_{1,n-1}^{(1,1)}a_{11} + \lambda a_{1,n-1} + \sum_{1\leqslant i<j\leqslant n} \lambda_{1,n-1}^{(i,j)}a_{ij}\bigg)E_{1,n-1} \\ &\quad + \bigg(\lambda_{1n}^{(1,1)}a_{11} + \lambda a_{1n} + \sum_{1\leqslant i<j\leqslant n} \lambda_{1n}^{(i,j)}a_{ij}\bigg)E_{1n} \\ &\quad + \bigg(\lambda_{2,n-1}^{(1,1)}a_{11} + \lambda a_{2,n-1} + \sum_{1\leqslant i<j\leqslant n} \lambda_{2,n-1}^{(i,j)}a_{ij}\bigg)E_{2,n-1} \\ &\quad + \bigg(\lambda_{2n}^{(1,1)}a_{11} + \lambda a_{2n} + \sum_{1\leqslant i<j\leqslant n} \lambda_{2n}^{(i,j)}a_{ij}\bigg)E_{2n}\Bigg) \\ &= 0. \end{aligned}$$

For instance, the following maps

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & a_{24} & 0 \\ 0 & 0 & a_{12} & a_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & a_{12} & a_{11} \\ 0 & 0 & a_{24} & a_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are 2-power commuting additive maps on invertible matrices in $T_n(\mathbb{F}_2)$ of this type.

7.3 Main results

Theorem 7.3.1. $\psi : T_2(\mathbb{F}_2) \to T_2(\mathbb{F}_2)$ *is a* 2*-power commuting additive map on invertible matrices if and only if there exists* $\lambda \in \mathbb{F}_2$ *and an additive map* $\mu : T_2(\mathbb{F}_2) \to \mathbb{F}_2$ *such that*

$$\psi(A) = \lambda A + \mu(A)I_2 + \varsigma(A)$$

for all $A \in T_2(\mathbb{F}_2)$*, where* $\varsigma : T_2(\mathbb{F}_2) \to T_2(\mathbb{F}_2)$ *is the additive map defined in* (7.1).

Theorem 7.3.2. *$\psi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ is a 2-power commuting additive map on invertible matrices if and only if there exists $\lambda \in \mathbb{F}_2$ and an additive map $\mu : T_3(\mathbb{F}_2) \to \mathbb{F}_2$ such that*

$$\psi(A) = \lambda A + \mu(A)I_3 + \phi(A)$$

for all $A \in T_3(\mathbb{F}_2)$, where $\phi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ is the additive map defined in (7.2).

Theorem 7.3.3. *Let $\theta_4 = \{(1,3),(1,4),(2,3),(2,4)\}$ and let $\Theta = \bigcup_{(s,t)\in\theta_4}\{\lambda_{st}^{(i,j)} \in \mathbb{F}_2 : 1 \leqslant i < j \leqslant 4\}$ be a set of scalars on $\mathbb{F}_2$. Then $\psi : T_4(\mathbb{F}_2) \to T_4(\mathbb{F}_2)$ is a 2-power commuting additive map on invertible matrices if and only if there exists $\lambda \in \mathbb{F}_2$ and an additive map $\mu : T_4(\mathbb{F}_2) \to \mathbb{F}_2$ such that*

$$\psi(A) = \lambda A + \mu(A)I_4 + \psi_\gamma(A) + \psi_\Theta(A)$$

for all $A \in T_4(\mathbb{F}_2)$, where $\psi_\gamma : T_4(\mathbb{F}_2) \to T_4(\mathbb{F}_2)$ and $\psi_\Theta : T_4(\mathbb{F}_2) \to T_4(\mathbb{F}_2)$ are additive maps defined in (7.3) *and* (7.4)*, respectively.*

Theorem 7.3.4. *Let $n \geqslant 5$ be an integer. Let $\theta_n = \{(1,n-1),(1,n),(2,n-1),(2,n)\}$ and let $\Theta = \bigcup_{(s,t)\in\theta_n}\{\lambda_{st}^{(i,j)} \in \mathbb{F}_2 : 1 \leqslant i < j \leqslant n\}$ be a set of scalars on $\mathbb{F}_2$. Then $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is a 2-power commuting additive map on invertible matrices if and only if there exists $\lambda \in \mathbb{F}_2$ and an additive map $\mu : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \psi_\Theta(A)$$

for all $A \in T_n(\mathbb{F}_2)$. Here, $\psi_\Theta : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the additive map defined in (7.4).

7.4 Proofs

Lemma 7.4.1. *Let $\mathbb{F}_2$ be the Galois field of two elements and let $n \geqslant 2$ be an integer. Let $A \in T_n(\mathbb{F}_2)$ and $A^2 = (b_{ij})$. If A is invertible, then $b_{i,i+1} = 0$ for $i = 1,\dots,n-1$.*

Proof. We first see that the result holds for $n = 2$ since the only invertible matrices in $T_2(\mathbb{F}_2)$ are I_2 and $I_2 + E_{12}$, where $I_2^2 = (I_2 + E_{12})^2 = I_2$. We now consider $n \geqslant 3$. Let $A \in T_n(\mathbb{F}_2)$ be invertible with $A^2 = (b_{ij}) \in T_n(\mathbb{F}_2)$. Since $A \in T_n(\mathbb{F}_2)$ is invertible, then

$A = I_n + U$, where $U = \sum_{1\leqslant i<j\leqslant n} a_{ij}E_{ij}$. Note that $A^2 = (I_n+U)(I_n+U) = I_n + U^2$, where

$$\begin{aligned}
U^2 &= \Big(\sum_{1\leqslant i<j\leqslant n} a_{ij}E_{ij}\Big)\Big(\sum_{1\leqslant i<j\leqslant n} a_{ij}E_{ij}\Big)\\
&= \Big(\sum_{j=2}^{n} a_{1j}E_{1j} + \sum_{j=3}^{n} a_{2j}E_{2j} + \cdots + a_{n-2,n-1}E_{n-2,n-1} + a_{n-2,n}E_{n-2,n}\\
&\quad + a_{n-1,n}E_{n-1,n}\Big)\\
&\quad \Big(\sum_{j=2}^{n} a_{1j}E_{1j} + \sum_{j=3}^{n} a_{2j}E_{2j} + \cdots + a_{n-2,n-1}E_{n-2,n-1} + a_{n-2,n}E_{n-2,n}\\
&\quad + a_{n-1,n}E_{n-1,n}\Big)\\
&= a_{12}a_{23}E_{13} + \Big(\sum_{j=2}^{3} a_{14}a_{j4}\Big)E_{14} + \cdots + \Big(\sum_{j=2}^{n-1} a_{1j}a_{jn}\Big)E_{1n}\\
&\quad + a_{24}a_{34}E_{24} + \Big(\sum_{j=3}^{4} a_{24}a_{j4}\Big)E_{25} + \cdots + \Big(\sum_{j=3}^{n-1} a_{2j}a_{jn}\Big)E_{2n}\\
&\quad + \qquad\qquad \vdots\\
&\quad + a_{n-3,n-2}a_{n-2,n-1}E_{n-3,n-1} + (a_{n-3,n-2}a_{n-2,n} + a_{n-3,n-1}a_{n-1,n})E_{n-3,n}\\
&\quad + a_{n-2,n-1}a_{n-1,n}E_{n-2,n}. \qquad (7.5)
\end{aligned}$$

By (7.5), we see that $b_{i,i+1} = 0$ for $i = 1, \dots, n-1$. We are done. □

We first prove Theorem 7.3.1.

Proof of Theorem 7.3.1. It is easily seen that $A \mapsto \lambda A + \mu(A)I_2$ is a 2-power commuting additive map on invertible matrices $A \in T_2(\mathbb{F}_2)$. Together with Example 7.2.1, the sufficiency is proved. Consider the necessity. For each pair of integers $1 \leqslant p < q \leqslant 2$, we let

$$\psi(E_{pq}) = (a_{ij}^{(p,q)}) \in T_2(\mathbb{F}_2) \qquad (7.6)$$

and let

$$\psi(I_2) = (a_{ij}^{(1,1)}) \in T_2(\mathbb{F}_2) \qquad (7.7)$$

for all integers $1 \leqslant i \leqslant j \leqslant 2$. Let $A = (a_{ij}) \in T_2(\mathbb{F}_2)$ be invertible. Then $A \in \{I_2, I_2 + E_{12}\}$ and $A^2 = I_2$. By $[\psi(A), A^2] = [\psi(A), I_2] = 0$, we get

$$\psi(I_2) = \begin{pmatrix} a_{11}^{(1,1)} & a_{12}^{(1,1)} \\ 0 & a_{22}^{(1,1)} \end{pmatrix} = a_{12}^{(1,2)} I_2 + (a_{11}^{(1,1)} + a_{12}^{(1,2)}) I_2 + \begin{pmatrix} 0 & a_{12}^{(1,1)} \\ 0 & a_{11}^{(1,1)} + a_{22}^{(1,1)} \end{pmatrix}, \quad (7.8)$$

$$\psi(E_{12}) = \begin{pmatrix} a_{11}^{(1,2)} & a_{12}^{(1,2)} \\ 0 & a_{22}^{(1,2)} \end{pmatrix} = a_{12}^{(1,2)} E_{12} + a_{22}^{(1,2)} I_2 + \begin{pmatrix} a_{11}^{(1,2)} + a_{22}^{(1,2)} & 0 \\ 0 & 0 \end{pmatrix}. \quad (7.9)$$

Let $\lambda = \lambda_{12}^{(1,2)} \in \mathbb{F}_2$. Let $\mu : T_2(\mathbb{F}_2) \to \mathbb{F}_2$ be an additive map such that $\mu(I_2) = a_{11}^{(1,1)} + a_{12}^{(1,2)}$ and $\mu(E_{12}) = a_{22}^{(1,2)}$. Let $\lambda_{11}^{(1,1)} + \lambda_{22}^{(1,1)} = a_{11}^{(1,1)} + a_{22}^{(1,1)}$, $\lambda_{12}^{(1,1)} = a_{12}^{(1,1)}$ and $\lambda_{11}^{(1,2)} + \lambda_{22}^{(1,2)} = a_{11}^{(1,2)} + a_{22}^{(1,2)}$. In view of (7.8) and (7.9), we see that

$$\psi(A) = \lambda A + \mu(A) I_2 + \varsigma(A)$$

for all $A \in T_2(\mathbb{F}_2)$, where $\varsigma : T_2(\mathbb{F}_2) \to T_2(\mathbb{F}_2)$ is the additive map defined in (7.1). We are done. $\square$

We then prove Theorem 7.3.2.

Proof of Theorem 7.3.2. It is easily seen that $A \mapsto \lambda A + \mu(A) I_3$ is a 2-power commuting additive map on invertible matrices $A \in T_3(\mathbb{F}_2)$. Together with Example 7.2.2, the sufficiency is proved. Consider the necessity. For each pair of integers $1 \leqslant p < q \leqslant 3$, we let

$$\psi(E_{pq}) = (a_{ij}^{(p,q)}) \in T_3(\mathbb{F}_2) \quad (7.10)$$

and let

$$\psi(I_3) = (a_{ij}^{(1,1)}) \in T_3(\mathbb{F}_2) \quad (7.11)$$

for all integers $1 \leqslant i \leqslant j \leqslant 3$. Let $A = (a_{ij}) \in T_3(\mathbb{F}_2)$ be invertible. Then $A = A_1 \cup A_2$, where $A_1 = \{I_3 + E_{12} + E_{23}, I_3 + E_{12} + E_{13} + E_{23}\}$ and $A_2 = \{I_3, I_3 + E_{12}, I_3 + E_{13}, I_3 +$

$E_{23}, I_3 + E_{12} + E_{13}, I_3 + E_{13} + E_{23}\}$. Note that

$$A^2 = \begin{cases} I_3 + E_{13} & \text{if } A \in A_1, \\ I_3 & \text{if } A \in A_2. \end{cases} \tag{7.12}$$

Hence we only need to consider $A = I_3 + E_{12} + E_{23}$ and $A = I_3 + E_{12} + E_{13} + E_{23}$. Since $(I_3+E_{12}+E_{23})^2 = I_3+E_{13}$ and $[\psi(I_3+E_{12}+E_{23}), I_3] = 0$, by $[\psi(I_3+E_{12}+E_{23}), E_{13}] = 0$, (7.10) and (7.11), we get

$$a_{33}^{(2,3)} = a_{11}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)}. \tag{7.13}$$

Since $(I_3 + E_{12} + E_{13} + E_{23})^2 = I_3 + E_{13}$ and $[\psi(I_3 + E_{12} + E_{13} + E_{23}), I_3] = 0$, by $[\psi(I_3 + E_{12} + E_{13} + E_{23}), E_{13}] = 0$, (7.10) and (7.11), we obtain

$$a_{33}^{(1,3)} = a_{11}^{(1,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)} + a_{11}^{(2,3)} + a_{33}^{(2,3)}. \tag{7.14}$$

Taking (7.13) into (7.14),

$$a_{11}^{(1,3)} = a_{33}^{(1,3)}. \tag{7.15}$$

We are now ready to classify the structures of $\psi(I_3)$ and $\psi(E_{ij})$ for all $1 \leqslant i < j \leqslant 3$. It follows from (7.11) that

$$\begin{aligned} \psi(I_3) &= \begin{pmatrix} a_{11}^{(1,1)} & a_{12}^{(1,1)} & a_{13}^{(1,1)} \\ 0 & a_{22}^{(1,1)} & a_{23}^{(1,1)} \\ 0 & 0 & a_{23}^{(1,1)} \end{pmatrix} \\ &= a_{12}^{(1,2)} I_3 + (a_{11}^{(1,1)} + a_{12}^{(1,2)}) I_3 \\ &\quad + \begin{pmatrix} 0 & a_{12}^{(1,1)} & a_{13}^{(1,1)} \\ 0 & a_{11}^{(1,1)} + a_{22}^{(1,1)} & a_{23}^{(1,1)} \\ 0 & 0 & a_{11}^{(1,1)} + a_{33}^{(1,1)} \end{pmatrix}. \end{aligned} \tag{7.16}$$

It follows from (7.10) that

$$\psi(E_{12}) = \begin{pmatrix} a_{11}^{(1,2)} & a_{12}^{(1,2)} & a_{13}^{(1,2)} \\ 0 & a_{13}^{(1,2)} & a_{22}^{(1,2)} \\ 0 & 0 & a_{33}^{(1,2)} \end{pmatrix}$$

$$= a_{12}^{(1,2)} E_{12} + a_{11}^{(1,2)} I_3 + \begin{pmatrix} 0 & a_{12}^{(1,2)} + a_{12}^{(1,2)} & a_{13}^{(1,2)} \\ 0 & a_{11}^{(1,2)} + a_{22}^{(1,2)} & a_{23}^{(1,2)} \\ 0 & 0 & a_{11}^{(1,2)} + a_{33}^{(1,2)} \end{pmatrix}. \tag{7.17}$$

By virtue of (7.10) and (7.13),

$$\psi(E_{23}) = \begin{pmatrix} a_{11}^{(2,3)} & a_{12}^{(2,3)} & a_{13}^{(2,3)} \\ 0 & a_{22}^{(2,3)} & a_{23}^{(2,3)} \\ 0 & 0 & a_{11}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)} \end{pmatrix}$$

$$= a_{12}^{(1,2)} E_{23} + a_{11}^{(2,3)} I_3$$

$$+ \begin{pmatrix} 0 & a_{12}^{(2,3)} & a_{13}^{(2,3)} \\ 0 & a_{11}^{(2,3)} + a_{22}^{(2,3)} & a_{23}^{(2,3)} + a_{12}^{(1,2)} \\ 0 & 0 & a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)} \end{pmatrix}. \tag{7.18}$$

By virtue of (7.10) and (7.15),

$$\psi(E_{13}) = \begin{pmatrix} a_{11}^{(1,3)} & a_{12}^{(1,3)} & a_{13}^{(1,3)} \\ 0 & a_{22}^{(1,3)} & a_{23}^{(1,3)} \\ 0 & 0 & a_{11}^{(1,3)} \end{pmatrix}$$

$$= a_{12}^{(1,2)} E_{13} + a_{11}^{(1,3)} I_3 + \begin{pmatrix} 0 & a_{12}^{(1,3)} & a_{13}^{(1,3)} + a_{12}^{(1,2)} \\ 0 & a_{11}^{(1,3)} + a_{22}^{(1,3)} & a_{23}^{(1,3)} \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.19}$$

Let $\lambda = a_{12}^{(1,2)} \in \mathbb{F}_2$. Let $\mu : T_3(\mathbb{F}_2) \to \mathbb{F}_2$ be an additive map such that $\mu(I_3) = a_{11}^{(1,1)}$ and $\mu(E_{ij}) = a_{11}^{(i,j)}$ for every pair of integers $1 \leqslant i < j \leqslant 3$. Let $\lambda_{ss}^{(1,1)} = a_{ss}^{(1,1)}$, $\lambda_{st}^{(1,1)} = a_{st}^{(1,1)}$, $\lambda_{ss}^{(i,j)} = a_{ss}^{(i,j)}$ and $\lambda_{st}^{(i,j)} = a_{st}^{(i,j)}$, where $1 \leqslant s < t \leqslant 3$ are integers. In view of (7.16)–

(7.19), we see that

$$\psi(A) = \lambda A + \mu(A)I_3 + \phi(A)$$

for all $A \in T_3(\mathbb{F}_2)$, where $\phi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ is the additive map defined in (7.2). This completes the proof. $\square$

Next we prove Theorem 7.3.3.

Proof of Theorem 7.3.3. It is easily seen that $A \mapsto \lambda A + \mu(A)I_4$ is a 2-power commuting additive map on invertible matrices $A \in T_4(\mathbb{F}_2)$. Together with Examples 7.2.3 and 7.2.4, the sufficiency is proved. Consider the necessity. For each pair of integers $1 \leqslant p < q \leqslant 4$, we let

$$\psi(E_{pq}) = (a_{ij}^{(p,q)}) \in T_4(\mathbb{F}_2) \tag{7.20}$$

and let

$$\psi(I_4) = (a_{ij}^{(1,1)}) \in T_4(\mathbb{F}_2) \tag{7.21}$$

for all integers $1 \leqslant i \leqslant j \leqslant 4$. Let $A = (a_{ij}) \in T_4(\mathbb{F}_2)$ be invertible. Note that if $A = I_4 + E_{ij}$ for some $1 \leqslant i < j \leqslant 4$, then $A^2 = (I_4 + E_{ij})(I_4 + E_{ij}) = I_4 + E_{ij} + E_{ij} + E_{ij}E_{ij} = I_4$, for all $1 \leqslant i < j \leqslant 4$. Thus $[\psi(A), A^2] = [\psi(I_4 + E_{ij}), I_4] = \psi(A) - \psi(A) = 0$. Hence we first consider $A^2 = I_4 + E_{sq}$ for $1 \leqslant s < t < q \leqslant 4$. Let $A = I_4 + E_{st} + E_{tq}$ for integers $1 \leqslant s < t < q \leqslant 4$. Then $A^2 = (I_4 + E_{st} + E_{tq})(I_4 + E_{st} + E_{tq}) = I_4 + E_{st} + E_{tq} + E_{st} + E_{st}E_{tq} + E_{tq} = I_4 + E_{sq}$. Since $[\psi(A), I_4] = 0$, hence $0 = [\psi(A), A^2] = [\psi(A), I_4 + E_{sq}] = [\psi(A), E_{sq}] = [\psi(I_4 + E_{st} + E_{tq}), E_{sq}]$. By $[\psi(I_4 + E_{12} + E_{23}), E_{13}] = 0$, (7.20) and (7.21), we have

$$a_{33}^{(2,3)} = a_{11}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)}, \tag{7.22}$$

$$a_{34}^{(2,3)} = a_{34}^{(1,1)} + a_{34}^{(1,2)}. \tag{7.23}$$

Since $(I_4 + E_{12} + E_{23} + E_{13})^2 = (I_4 + E_{12} + E_{23} + E_{14})^2 = I_4 + E_{13}$, by $[\psi(I_4 + E_{12} + E_{23} + E_{uv}), E_{13}] = 0$, for every $(u, v) \in \{(1, 3), (1, 4)\}$, we get

$$a_{11}^{(u,v)} = a_{33}^{(u,v)} + a_{11}^{(2,3)} + a_{33}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)}, \tag{7.24}$$

$$a_{34}^{(u,v)} = a_{34}^{(1,1)} + a_{34}^{(1,2)} + a_{34}^{(2,3)}. \tag{7.25}$$

Taking (7.22) into (7.24) and (7.23) into (7.25) with $(u, v) \in \{(1, 3), (1, 4)\}$ respectively, we obtain

$$a_{11}^{(1,3)} = a_{33}^{(1,3)}, \tag{7.26}$$

$$a_{11}^{(1,4)} = a_{33}^{(1,4)}, \tag{7.27}$$

$$a_{34}^{(1,3)} = 0, \tag{7.28}$$

$$a_{34}^{(1,4)} = 0. \tag{7.29}$$

By $[\psi(I_4 + E_{23} + E_{34}), E_{24}] = 0$, we have

$$a_{22}^{(3,4)} = a_{44}^{(3,4)} + a_{22}^{(1,1)} + a_{44}^{(1,1)} + a_{22}^{(2,3)} + a_{44}^{(2,3)}, \tag{7.30}$$

$$a_{12}^{(3,4)} = a_{12}^{(1,1)} + a_{12}^{(2,3)}. \tag{7.31}$$

Since $(I_4 + E_{23} + E_{34} + E_{14})^2 = (I_4 + E_{23} + E_{34} + E_{24})^2 = I_4 + E_{24}$, by $[\psi(I_4 + E_{23} + E_{34} + E_{uv}), E_{24}] = 0$ for every $(u, v) \in \{(1, 4), (2, 4)\}$, we get

$$a_{22}^{(u,v)} = a_{44}^{(u,v)} + a_{22}^{(2,3)} + a_{44}^{(2,3)} + a_{22}^{(1,1)} + a_{44}^{(1,1)} + a_{22}^{(3,4)} + a_{44}^{(3,4)}, \tag{7.32}$$

$$a_{12}^{(u,v)} = a_{12}^{(1,1)} + a_{12}^{(2,3)} + a_{12}^{(3,4)}, \tag{7.33}$$

for every $(u, v) \in \{(1, 4), (2, 4)\}$. Taking (7.30) into (7.32) and (7.31) into (7.33) with $(u, v) \in \{(1, 4), (2, 4)\}$ respectively,

$$a_{22}^{(2,4)} = a_{44}^{(2,4)}, \tag{7.34}$$

$$a_{22}^{(1,4)} = a_{44}^{(1,4)}, \tag{7.35}$$

$$a_{12}^{(1,4)} = 0, \tag{7.36}$$

$$a_{12}^{(2,4)} = 0. \tag{7.37}$$

By $[\psi(I_4 + E_{12} + E_{24}), E_{14}] = 0$, we have

$$a_{11}^{(1,1)} + a_{44}^{(1,1)} + a_{11}^{(1,2)} + a_{44}^{(1,2)} + a_{11}^{(2,4)} + a_{44}^{(2,4)} = 0. \tag{7.38}$$

Since $(I_4 + E_{12} + E_{24} + E_{13})^2 = (I_4 + E_{12} + E_{24} + E_{14})^2 = (I_4 + E_{12} + E_{24} + E_{34})^2 = I_4 + E_{14}$, by $[\psi(I_4 + E_{12} + E_{24} + E_{uv}), E_{14}] = 0$ for every $(u, v) \in \{(1, 3), (1, 4), (3, 4)\}$, we get

$$a_{11}^{(u,v)} = a_{44}^{(u,v)} + a_{11}^{(1,1)} + a_{44}^{(1,1)} + a_{11}^{(1,2)} + a_{44}^{(1,2)} + a_{11}^{(2,4)} + a_{44}^{(2,4)} \tag{7.39}$$

for every $(u, v) \in \{(1, 3), (1, 4), (3, 4)\}$. Taking (7.38) into (7.39) for every $(u, v) \in \{(1, 3), (1, 4), (3, 4)\}$,

$$a_{11}^{(1,3)} = a_{44}^{(1,3)}, \tag{7.40}$$

$$a_{11}^{(1,4)} = a_{44}^{(1,4)}, \tag{7.41}$$

$$a_{11}^{(3,4)} = a_{44}^{(3,4)}. \tag{7.42}$$

It follows from (7.27), (7.35) and (7.41) that

$$a_{11}^{(1,4)} = a_{22}^{(1,4)} = a_{33}^{(1,4)} = a_{44}^{(1,4)}. \tag{7.43}$$

By $[\psi(I_4 + E_{13} + E_{34}), E_{14}] = 0$, we have

$$a_{11}^{(1,1)} + a_{44}^{(1,1)} + a_{11}^{(1,3)} + a_{44}^{(1,3)} + a_{11}^{(3,4)} + a_{44}^{(3,4)} = 0. \tag{7.44}$$

Taking (7.40) and (7.42) into (7.44),

$$a_{11}^{(1,1)} = a_{44}^{(1,1)}. \tag{7.45}$$

Taking (7.34) and (7.45) into (7.38),

$$a_{22}^{(2,4)} = a_{11}^{(2,4)} + a_{11}^{(1,2)} + a_{44}^{(1,2)}. \tag{7.46}$$

Taking (7.42) and (7.45) into (7.30),

$$a_{22}^{(3,4)} = a_{11}^{(3,4)} + a_{11}^{(1,1)} + a_{22}^{(1,1)} + a_{22}^{(2,3)} + a_{44}^{(2,3)}. \tag{7.47}$$

Since $(I_4 + E_{13} + E_{34} + E_{12})^2 = I_4 + E_{14}$, by $[\psi(I_4 + E_{13} + E_{34} + E_{12}), E_{14}] = 0$, we have

$$a_{11}^{(1,2)} = a_{44}^{(1,2)} + a_{11}^{(1,1)} + a_{44}^{(1,1)} + a_{11}^{(1,3)} + a_{44}^{(1,3)} + a_{11}^{(3,4)} + a_{44}^{(3,4)}. \tag{7.48}$$

Taking (7.44) into (7.48),

$$a_{11}^{(1,2)} = a_{44}^{(1,2)}. \tag{7.49}$$

Taking (7.49) into (7.46),

$$a_{11}^{(2,4)} = a_{22}^{(2,4)}. \tag{7.50}$$

We now consider $A^2 = I_4 + E_{st} + E_{sq}$ for $1 \leqslant s < t < q \leqslant 4$. Since $(I_4 + E_{12} + E_{23} + E_{24})^2 = I_4 + E_{13} + E_{14}$, by $[\psi(I_4 + E_{12} + E_{23} + E_{24}), E_{13} + E_{14}] = 0$, we have

$$a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(2,4)} = a_{33}^{(1,1)} + a_{33}^{(1,2)} + a_{33}^{(2,3)} + a_{33}^{(2,4)}, \tag{7.51}$$

$$\begin{aligned} &a_{34}^{(1,1)} + a_{34}^{(1,2)} + a_{34}^{(2,3)} + a_{34}^{(2,4)} \\ &\quad = a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(2,4)} + a_{44}^{(1,1)} + a_{44}^{(1,2)} + a_{44}^{(2,3)} + a_{44}^{(2,4)}. \end{aligned} \tag{7.52}$$

Taking (7.23) and (7.38) into (7.52),

$$a_{34}^{(2,4)} = a_{11}^{(2,3)} + a_{44}^{(2,3)}. \tag{7.53}$$

Taking (7.22) into (7.51),

$$a_{11}^{(2,4)} = a_{33}^{(2,4)}. \tag{7.54}$$

It follows from (7.34), (7.50) and (7.54) that

$$a_{11}^{(2,4)} = a_{22}^{(2,4)} = a_{33}^{(2,4)} = a_{44}^{(2,4)}. \tag{7.55}$$

We next consider $A^2 = I_4 + E_{st} + E_{pt}$ for $1 \leqslant s < p < t \leqslant 4$. Since $(I_4 + E_{13} + E_{23} +$

$E_{34})^2 = I_4 + E_{14} + E_{24}$, by $[\psi(I_4 + E_{13} + E_{23} + E_{34}), E_{14} + E_{24}] = 0$, we have

$$\begin{aligned} a_{12}^{(1,1)} + a_{12}^{(1,3)} + a_{12}^{(2,3)} + a_{12}^{(3,4)} &= a_{11}^{(1,1)} + a_{11}^{(1,3)} + a_{11}^{(2,3)} + a_{11}^{(3,4)} \\ &\quad + a_{44}^{(1,1)} + a_{44}^{(1,3)} + a_{44}^{(2,3)} + a_{44}^{(3,4)}, \end{aligned} \tag{7.56}$$

$$a_{22}^{(1,1)} + a_{22}^{(1,3)} + a_{22}^{(2,3)} + a_{22}^{(3,4)} = a_{44}^{(1,1)} + a_{44}^{(1,3)} + a_{44}^{(2,3)} + a_{44}^{(3,4)}. \tag{7.57}$$

Taking (7.31) and (7.44) into (7.56),

$$a_{12}^{(1,3)} = a_{11}^{(2,3)} + a_{44}^{(2,3)}. \tag{7.58}$$

Taking (7.30) and (7.40) into (7.57),

$$a_{11}^{(1,3)} = a_{22}^{(1,3)}. \tag{7.59}$$

It follows from (7.26), (7.40) and (7.59) that

$$a_{11}^{(1,3)} = a_{22}^{(1,3)} = a_{33}^{(1,3)} = a_{44}^{(1,3)}. \tag{7.60}$$

We now consider $A^2 = I_4 + E_{st} + E_{pq}$ for $1 \leqslant s < p < t < q \leqslant 4$. Since $(I_4 + E_{12} + E_{23} + E_{34})^2 = I_4 + E_{13} + E_{24}$, by $[\psi(I_4 + E_{12} + E_{23} + E_{34}), E_{13} + E_{24}] = 0$, we have

$$a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(3,4)} = a_{33}^{(1,1)} + a_{33}^{(1,2)} + a_{33}^{(2,3)} + a_{33}^{(3,4)}, \tag{7.61}$$

$$a_{12}^{(1,1)} + a_{12}^{(1,2)} + a_{12}^{(2,3)} + a_{12}^{(3,4)} = a_{34}^{(1,1)} + a_{34}^{(1,2)} + a_{34}^{(2,3)} + a_{34}^{(3,4)}, \tag{7.62}$$

$$a_{22}^{(1,1)} + a_{22}^{(1,2)} + a_{22}^{(2,3)} + a_{22}^{(3,4)} = a_{44}^{(1,1)} + a_{44}^{(1,2)} + a_{44}^{(2,3)} + a_{44}^{(3,4)}. \tag{7.63}$$

Taking (7.22) into (7.61),

$$a_{11}^{(3,4)} = a_{33}^{(3,4)}. \tag{7.64}$$

It follows from (7.42) and (7.64) that

$$a_{11}^{(3,4)} = a_{33}^{(3,4)} = a_{44}^{(3,4)}. \tag{7.65}$$

Taking (7.23) and (7.31) into (7.62),

$$a_{34}^{(3,4)} = a_{12}^{(1,2)}. \tag{7.66}$$

Taking (7.30) and (7.49) into (7.63),

$$a_{11}^{(1,2)} = a_{22}^{(1,2)}. \tag{7.67}$$

It follows from (7.49) and (7.67) that

$$a_{11}^{(1,2)} = a_{22}^{(1,2)} = a_{44}^{(1,2)}. \tag{7.68}$$

We are now ready to classify the structures of $\psi(I_4)$ and $\psi(E_{ij})$ for all $1 \leqslant i < j \leqslant 4$. We first classify $\psi(I_4)$. Since $a_{11}^{(1,1)} = a_{44}^{(1,1)}$ by (7.45), it follows from (7.21) that

$$\begin{aligned}\psi(I_4) &= \begin{pmatrix} a_{11}^{(1,1)} & a_{12}^{(1,1)} & a_{13}^{(1,1)} & a_{14}^{(1,1)} \\ 0 & a_{22}^{(1,1)} & a_{23}^{(1,1)} & a_{24}^{(1,1)} \\ 0 & 0 & a_{33}^{(1,1)} & a_{34}^{(1,1)} \\ 0 & 0 & 0 & a_{11}^{(1,1)} \end{pmatrix} \\ &= a_{12}^{(1,2)} I_4 + (a_{11}^{(1,1)} + a_{12}^{(1,2)}) I_4 \\ &\quad + \begin{pmatrix} 0 & a_{12}^{(1,1)} & a_{13}^{(1,1)} & a_{14}^{(1,1)} \\ 0 & a_{11}^{(1,1)} + a_{22}^{(1,1)} & a_{23}^{(1,1)} & a_{24}^{(1,1)} \\ 0 & 0 & a_{11}^{(1,1)} + a_{33}^{(1,1)} & a_{34}^{(1,1)} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.69}$$

Secondly, we classify $\psi(E_{12})$. Since $a_{11}^{(1,2)} = a_{22}^{(1,2)} = a_{44}^{(1,2)}$ by (7.68), it follows from (7.20) that

$$\psi(E_{12}) = \begin{pmatrix} a_{11}^{(1,2)} & a_{12}^{(1,2)} & a_{13}^{(1,2)} & a_{14}^{(1,2)} \\ 0 & a_{11}^{(1,2)} & a_{23}^{(1,2)} & a_{24}^{(1,2)} \\ 0 & 0 & a_{33}^{(1,2)} & a_{34}^{(1,2)} \\ 0 & 0 & 0 & a_{11}^{(1,2)} \end{pmatrix}$$

$$= a_{12}^{(1,2)} E_{12} + a_{11}^{(1,2)} I_4 + \begin{pmatrix} 0 & 0 & a_{13}^{(1,2)} & a_{14}^{(1,2)} \\ 0 & 0 & a_{23}^{(1,2)} & a_{24}^{(1,2)} \\ 0 & 0 & a_{11}^{(1,2)} + a_{33}^{(1,2)} & a_{34}^{(1,2)} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.70}$$

Thirdly, we classify $\psi(E_{23})$. Since $a_{33}^{(2,3)} = a_{11}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)}$ by (7.22) and $a_{34}^{(2,3)} = a_{34}^{(1,1)} + a_{34}^{(1,2)}$ by (7.23), it follows from (7.20) that

$$\begin{aligned} \psi(E_{23}) &= \begin{pmatrix} a_{11}^{(2,3)} & a_{12}^{(2,3)} & a_{13}^{(2,3)} & a_{14}^{(2,3)} \\ 0 & a_{22}^{(2,3)} & a_{23}^{(2,3)} & a_{24}^{(2,3)} \\ 0 & 0 & a_{11}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)} & a_{34}^{(1,1)} + a_{34}^{(1,2)} \\ 0 & 0 & 0 & a_{44}^{(2,3)} \end{pmatrix} \\ &= a_{12}^{(1,2)} E_{23} + a_{11}^{(2,3)} I_4 \\ &\quad + \begin{pmatrix} 0 & a_{12}^{(2,3)} & a_{13}^{(2,3)} & a_{14}^{(2,3)} \\ 0 & a_{11}^{(2,3)} + a_{22}^{(2,3)} & a_{23}^{(2,3)} + a_{12}^{(1,2)} & a_{24}^{(2,3)} \\ 0 & 0 & a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)} & a_{34}^{(1,1)} + a_{34}^{(1,2)} \\ 0 & 0 & 0 & a_{11}^{(2,3)} + a_{44}^{(2,3)} \end{pmatrix} \end{aligned} \tag{7.71}$$

Subsequently, we classify $\psi(E_{34})$. Since $a_{12}^{(3,4)} = a_{12}^{(1,1)} + a_{12}^{(2,3)}$ by (7.31), $a_{22}^{(3,4)} = a_{11}^{(3,4)} + a_{11}^{(1,1)} + a_{22}^{(1,1)} + a_{22}^{(2,3)} + a_{44}^{(2,3)}$ by (7.47), $a_{11}^{(3,4)} = a_{33}^{(3,4)} = a_{44}^{(3,4)}$ by (7.65) and $a_{34}^{(3,4)} = a_{12}^{(1,2)}$ by (7.66), it follows from (7.20) that

$$\begin{aligned} \psi(E_{34}) &= \begin{pmatrix} a_{11}^{(3,4)} & a_{12}^{(1,1)} + a_{12}^{(2,3)} & a_{13}^{(3,4)} & a_{14}^{(3,4)} \\ 0 & a_{11}^{(3,4)} + a_{11}^{(1,1)} + a_{22}^{(1,1)} + a_{22}^{(2,3)} + a_{44}^{(2,3)} & a_{23}^{(3,4)} & a_{24}^{(3,4)} \\ 0 & 0 & a_{11}^{(3,4)} & a_{12}^{(1,2)} \\ 0 & 0 & 0 & a_{11}^{(3,4)} \end{pmatrix} \\ &= a_{12}^{(1,2)} E_{34} + a_{11}^{(3,4)} I_4 \\ &\quad + \begin{pmatrix} 0 & a_{12}^{(1,1)} + a_{12}^{(2,3)} & a_{13}^{(3,4)} & a_{14}^{(3,4)} \\ 0 & a_{11}^{(1,1)} + a_{22}^{(1,1)} + a_{22}^{(2,3)} + a_{44}^{(2,3)} & a_{23}^{(3,4)} & a_{24}^{(3,4)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.72}$$

Next, we classify $\psi(E_{13})$. Since $a_{34}^{(1,3)} = 0$ by (7.28), $a_{12}^{(1,3)} = a_{11}^{(2,3)} + a_{44}^{(2,3)}$ by (7.58) and $a_{11}^{(1,3)} = a_{22}^{(1,3)} = a_{33}^{(1,3)} = a_{44}^{(1,3)}$ by (7.60), it follows from (7.20) that

$$\begin{aligned}\psi(E_{13}) &= \begin{pmatrix} a_{11}^{(1,3)} & a_{11}^{(2,3)} + a_{44}^{(2,3)} & a_{13}^{(1,3)} & a_{14}^{(1,3)} \\ 0 & a_{11}^{(1,3)} & a_{23}^{(1,3)} & a_{24}^{(1,3)} \\ 0 & 0 & a_{11}^{(1,3)} & 0 \\ 0 & 0 & 0 & a_{11}^{(1,3)} \end{pmatrix} \\ &= a_{12}^{(1,2)} E_{13} + a_{11}^{(1,3)} I_4 \\ &\quad + \begin{pmatrix} 0 & a_{11}^{(2,3)} + a_{44}^{(2,3)} & a_{13}^{(1,3)} + a_{12}^{(1,2)} & a_{14}^{(1,3)} \\ 0 & 0 & a_{23}^{(1,3)} & a_{24}^{(1,3)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.73}$$

Then, we classify $\psi(E_{24})$. Since $a_{12}^{(2,4)} = 0$ by (7.37), $a_{34}^{(2,4)} = a_{11}^{(2,3)} + a_{44}^{(2,3)}$ by (7.53) and $a_{11}^{(2,4)} = a_{22}^{(2,4)} = a_{33}^{(2,4)} = a_{44}^{(2,4)}$ by (7.55), it follows from (7.20) that

$$\begin{aligned}\psi(E_{24}) &= \begin{pmatrix} a_{11}^{(2,4)} & 0 & a_{13}^{(2,4)} & a_{14}^{(2,4)} \\ 0 & a_{11}^{(2,4)} & a_{23}^{(2,4)} & a_{24}^{(2,4)} \\ 0 & 0 & a_{11}^{(2,4)} & a_{11}^{(2,3)} + a_{44}^{(2,3)} \\ 0 & 0 & 0 & a_{11}^{(2,4)} \end{pmatrix} \\ &= a_{12}^{(1,2)} E_{24} + a_{11}^{(2,4)} I_4 + \begin{pmatrix} 0 & 0 & a_{13}^{(2,4)} & a_{14}^{(2,4)} \\ 0 & 0 & a_{23}^{(2,4)} & a_{24}^{(2,4)} + a_{12}^{(1,2)} \\ 0 & 0 & 0 & a_{11}^{(2,3)} + a_{44}^{(2,3)} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.74}$$

Finally, we classify $\psi(E_{14})$. Since $a_{34}^{(1,4)} = 0$ by (7.29), $a_{12}^{(1,4)} = 0$ by (7.36) and $a_{11}^{(1,4)} =$

$a_{22}^{(1,4)} = a_{33}^{(1,4)} = a_{44}^{(1,4)}$ by (7.43), it follows from (7.20) that

$$\begin{aligned}\psi(E_{14}) &= \begin{pmatrix} a_{11}^{(1,4)} & 0 & a_{13}^{(1,4)} & a_{14}^{(1,4)} \\ 0 & a_{11}^{(1,4)} & a_{23}^{(1,4)} & a_{24}^{(1,4)} \\ 0 & 0 & a_{11}^{(1,4)} & 0 \\ 0 & 0 & 0 & a_{11}^{(1,4)} \end{pmatrix} \\ &= a_{12}^{(1,2)} E_{14} + a_{11}^{(1,4)} I_4 + \begin{pmatrix} 0 & 0 & a_{13}^{(1,4)} & a_{14}^{(1,4)} + a_{12}^{(1,2)} \\ 0 & 0 & a_{23}^{(1,4)} & a_{24}^{(1,4)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.75}$$

Let $\lambda = a_{12}^{(1,2)} \in \mathbb{F}_2$ and $\gamma = a_{11}^{(2,3)} + a_{44}^{(2,3)} \in \mathbb{F}_2$. Let $\mu : T_4(\mathbb{F}_2) \to \mathbb{F}_2$ be an additive map such that $\mu(I_4) = a_{11}^{(1,1)} + a_{12}^{(1,2)}$ and $\mu(E_{ij}) = a_{11}^{(i,j)}$ for each $1 \leqslant i < j \leqslant 4$. Let $\theta_4 = \{(1,3),(1,4),(2,3),(2,4)\}$. Let $\lambda_{st}^{(i,j)} = a_{st}^{(i,j)}$ and $\lambda_{st}^{(1,1)} = a_{st}^{(1,1)}$, for each $1 \leqslant i < j \leqslant 4$ and $(s,t) \in \theta_4$. In view of (7.69)–(7.75), we see that

$$\psi(A) = \lambda A + \mu(A) I_4 + \psi_\gamma(A) + \psi_\Theta(A)$$

for all $A \in T_4(\mathbb{F}_2)$, where $\lambda \in \mathbb{F}_2$ and ψ_γ and ψ_Θ are the additive maps defined in (7.3) and (7.4) respectively. This completes the proof. □

Finally we prove Theorem 7.3.4.

Proof of Theorem 7.3.4. Throughout the proof, unless otherwise specified, let $n \geqslant 5$ be an integer. It is easily seen that $A \mapsto \lambda A + \mu(A) I_n$ is a 2-power commuting additive map on invertible matrices $A \in T_n(\mathbb{F}_2)$. Together with Example 7.4, the sufficiency is proved. We now prove the necessity. We divide our proof into the following two cases:

Case I: $n = 5$. For each pair of integers $1 \leqslant p < q \leqslant 5$, we let

$$\psi(E_{pq}) = (a_{ij}^{(p,q)}) \in T_5(\mathbb{F}_2) \tag{7.76}$$

and let

$$\psi(I_5) = (a_{ij}^{(1,1)}) \in T_5(\mathbb{F}_2) \tag{7.77}$$

for all integers $1 \leqslant i \leqslant j \leqslant 5$. In what follows, it is understood that (7.76) and (7.77) are used in $[\psi(I_5 + E_{st} + E_{tq}), E_{sq}] = 0$ for $1 \leqslant s < t < q \leqslant 5$. Let $A = (a_{ij}) \in T_5(\mathbb{F}_2)$ be invertible. If $A = I_5 + E_{ij}$ for some $1 \leqslant i < j \leqslant 5$, then $A^2 = I_5 + E_{ij} + E_{ij} = I_5$. Thus $[\psi(A), A^2] = [\psi(I_5 + E_{ij}), I_5] = 0$. Hence we first consider $A^2 = I_5 + E_{sq}$ for integers $1 \leqslant s < t < q \leqslant 5$. Note that $A^2 \neq I_5 + E_{s,s+1}$ for all $1 \leqslant s \leqslant 4$ since $q \neq s + 1$. Let $A = I_5 + E_{st} + E_{tq}$ for integers $1 \leqslant s < t < q \leqslant 5$. Then $A^2 = I_5 + E_{sq}$, where $(s, q) \neq (i, i + 1)$ for all integers $i = 1, \ldots, 4$, by Lemma 7.4.1. By $[\psi(I_5 + E_{st} + E_{tq}), E_{sq}] = 0$, (7.76) and (7.77), for every $1 \leqslant s < t < q \leqslant 5$,

$$\left(\sum_{1\leqslant i\leqslant j\leqslant 5} (a_{ij}^{(1,1)} + a_{ij}^{(s,t)} + a_{ij}^{(t,q)}) E_{ij} \right) E_{sq} - E_{sq} \left(\sum_{1\leqslant i\leqslant j\leqslant 5} (a_{ij}^{(1,1)} + a_{ij}^{(s,t)} + a_{ij}^{(t,q)}) E_{ij} \right) = 0.$$

Then

$$\sum_{1\leqslant i\leqslant s} (a_{is}^{(1,1)} + a_{is}^{(s,t)} + a_{is}^{(t,q)}) E_{iq} - \sum_{q\leqslant j\leqslant 5} (a_{qj}^{(1,1)} + a_{qj}^{(s,t)} + a_{qj}^{(t,q)}) E_{sj} = 0.$$

Thus

$$\begin{aligned} &(a_{ss}^{(1,1)} + a_{ss}^{(s,t)} + a_{ss}^{(t,q)} + a_{qq}^{(1,1)} + a_{qq}^{(s,t)} + a_{qq}^{(t,q)}) E_{sq} \\ &+ \sum_{1\leqslant i\leqslant s-1} (a_{is}^{(1,1)} + a_{is}^{(s,t)} + a_{is}^{(t,q)}) E_{iq} - \sum_{q+1\leqslant j\leqslant 5} (a_{qj}^{(1,1)} + a_{qj}^{(s,t)} + a_{qj}^{(t,q)}) E_{sj} = 0. \end{aligned}$$

Hence for every $1 \leqslant s < t < q \leqslant 5$, we obtain

$$a_{ss}^{(1,1)} + a_{qq}^{(1,1)} + a_{ss}^{(s,t)} + a_{qq}^{(s,t)} + a_{ss}^{(t,q)} + a_{qq}^{(t,q)} = 0, \tag{7.78}$$

$$a_{is}^{(1,1)} + a_{is}^{(s,t)} + a_{is}^{(t,q)} = 0 \quad \text{for } i = 1, \ldots, s - 1, \tag{7.79}$$

$$a_{qj}^{(1,1)} + a_{qj}^{(s,t)} + a_{qj}^{(t,q)} = 0 \quad \text{for } j = q + 1, \ldots, 5. \tag{7.80}$$

Setting $(s,t,q)=(3,4,5)$ and $i\in\{1,2\}$ in (7.79) respectively,

$$a_{13}^{(4,5)}=a_{13}^{(1,1)}+a_{13}^{(3,4)}, \tag{7.81}$$

$$a_{23}^{(4,5)}=a_{23}^{(1,1)}+a_{23}^{(3,4)}. \tag{7.82}$$

Setting $(s,t,q)=(1,2,3)$ and $j\in\{4,5\}$ in (7.80) respectively,

$$a_{34}^{(2,3)}=a_{34}^{(1,1)}+a_{34}^{(1,2)}, \tag{7.83}$$

$$a_{35}^{(2,3)}=a_{35}^{(1,1)}+a_{35}^{(1,2)}. \tag{7.84}$$

Let $1\leqslant s<t<q\leqslant 5$ be integers. For each pair of integers $1\leqslant u<v\leqslant 5$ such that $u\neq t,q$ and $v\neq s,t$, we note that $(I_5+E_{st}+E_{tq}+E_{uv})^2=I_5+E_{sq}$. Since $[\psi(I_5+E_{st}+E_{tq}+E_{uv}),I_5]=0$, by $[\psi(I_5+E_{st}+E_{tq}+E_{uv}),E_{sq}]=0$, for every $1\leqslant u<v\leqslant 5$ such that $u\neq t,q$ and $v\neq s,t$,

$$a_{qq}^{(u,v)}=a_{ss}^{(u,v)}+a_{ss}^{(1,1)}+a_{qq}^{(1,1)}+a_{ss}^{(s,t)}+a_{qq}^{(s,t)}+a_{ss}^{(t,q)}+a_{qq}^{(t,q)}, \tag{7.85}$$

$$a_{is}^{(u,v)}=a_{is}^{(1,1)}+a_{is}^{(s,t)}+a_{is}^{(t,q)} \quad \text{for } i=1,\ldots,s-1, \tag{7.86}$$

$$a_{qj}^{(u,v)}=a_{qj}^{(1,1)}+a_{qj}^{(s,t)}+a_{qj}^{(t,q)} \quad \text{for } j=q+1,\ldots,5. \tag{7.87}$$

For every $1\leqslant s<t<q\leqslant 5$, taking (7.78) into (7.85), (7.79) into (7.86), and (7.80) into (7.87) respectively, for $1\leqslant u<v\leqslant 5$, $u\neq t,\ q$ and $v\neq s,\ t$,

$$a_{qq}^{(u,v)}=a_{ss}^{(u,v)}, \tag{7.88}$$

$$a_{is}^{(u,v)}=0 \quad \text{for } i=1,\ldots,s-1, \tag{7.89}$$

$$a_{qj}^{(u,v)}=0 \quad \text{for } j=q+1,\ldots,5. \tag{7.90}$$

Note that equations (7.88) and (7.90) give

(i) the 3rd row of $\psi(E_{uv})$ for each pair of integers $1\leqslant u<v\leqslant 5$ satisfying $u\neq 2,3$

and $(u,v) \neq (1,2)$.

(ii) the 4th row of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant 5$ satisfying $u \neq 4$ and $(u,v) \neq (2,3)$.

(iii) the 5th row of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant 5$ satisfying $u \neq 5$.

In particular, when $s = 1$, equations (7.88) and (7.90) respectively become

$$a_{11}^{(u,v)} = a_{qq}^{(u,v)} \quad \text{for} \;\; q = 3, 4, 5, \tag{7.91}$$

$$a_{qj}^{(u,v)} = 0 \quad \text{for} \;\; j = q+1, \ldots, 5. \tag{7.92}$$

- When $q = 3$, then $t = 2$. By (7.91) and (7.92), we obtain the 3rd row of $\psi(E_{uv})$ for every pair of integers $1 \leqslant u < v \leqslant 5$ where $u \neq 2, 3$ and $(u,v) \neq (1,2)$. This is because we get $u = 1, 4$ and $v = 3, 4, 5$, where $u \neq 2, 3$ and $v \neq 1, 2$, satisfying $1 \leqslant u < v \leqslant 5$. Hence

$$a_{11}^{(u,v)} = a_{33}^{(u,v)}, \tag{7.93}$$

$$a_{34}^{(u,v)} = a_{35}^{(u,v)} = 0 \tag{7.94}$$

  for all $(u,v) \in \{(1,3), (1,4), (1,5), (4,5)\}$.

- When $q = 4$, then $t \in \{2, 3\}$. By (7.91) and (7.92), we obtain the 4th row of $\psi(E_{uv})$ where $u \neq 4$ and $(u,v) \neq (2,3)$. This is because when $(t,q) = (2,4)$, we get $u = 1, 3$ and $v = 3, 4, 5$, where $u \neq 2, 4$ and $v \neq 1, 2$, satisfying $1 \leqslant u < v \leqslant 5$ and; when $(t,q) = (3,4)$, we get $u = 1, 2$ and $v = 2, 4, 5$, where $u \neq 3, 4$ and $v \neq 1, 3$, satisfying $1 \leqslant u < v \leqslant 5$. Hence

$$a_{11}^{(u,v)} = a_{44}^{(u,v)}, \tag{7.95}$$

$$a_{45}^{(u,v)} = 0 \tag{7.96}$$

  for all $(u,v) \in \{(1,2), (1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$.

- When $q = 5$ and $t \in \{2, 3, 4\}$, by (7.91) and (7.92), we get the 5th row of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant 5$ where $u \neq 5$. This is because when $t = 2$, we get $u = 1, 3, 4$ and $v = 3, 4, 5$, where $u \neq 2, 5$ and $v \neq 1, 2$, satisfying

$1 \leqslant u < v \leqslant 5$; when $t = 3$, we get $u = 1, 2$ and $v = 2, 4, 5$, where $u \neq 3, 5$ and $v \neq 1, 3$, satisfying $1 \leqslant u < v \leqslant 5$ and; when $t = 4$, we get $(u, v) = (2, 3)$. Hence

$$a_{11}^{(u,v)} = a_{55}^{(u,v)} \tag{7.97}$$

for $(u, v) = (i, j)$ for each pair of integers $1 \leqslant i < j \leqslant 5$.

Note that equations (7.88) and (7.89) give

(i) the 2nd column of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant 5$ satisfying $v \neq 2$ and $(u, v) \neq (3, 4)$.

(ii) the 3rd column of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant 5$ satisfying $v \neq 3, 4$ and $(u, v) \neq (4, 5)$.

For every $2 \leqslant s \leqslant 3$, equations (7.88) and (7.89) respectively become

$$a_{ss}^{(u,v)} = a_{qq}^{(u,v)} \quad \text{for } q = s + 2, \ldots, 5, \tag{7.98}$$

$$a_{is}^{(u,v)} = 0 \quad \text{for } i = 1, \ldots, s - 1. \tag{7.99}$$

- For $s = 2$ with $t \in \{3, 4\}$ and $q \in \{4, 5\}$, by (7.98) and (7.99), we obtain the 2nd column of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant 5$ where $v \neq 2$, and $(u, v) \neq (3, 4)$. This is because:
  - when $(t, q) = (3, 4)$, we get $u = 1, 2$ and $v = 4, 5$, where $u \neq 3, 4$ and $v \neq 2, 3$, satisfying $1 \leqslant u < v \leqslant 5$. Hence

$$a_{22}^{(u,v)} = a_{44}^{(u,v)}, \tag{7.100}$$

$$a_{12}^{(u,v)} = 0 \tag{7.101}$$

    for all $(u, v) \in \{(1, 4), (1, 5), (2, 4), (2, 5)\}$;
  - when $(t, q) = (3, 5)$, we get $(u, v) = (4, 5)$, where $u \neq 3, 5$ and $v \neq 2, 3$, satisfying $1 \leqslant u < v \leqslant 5$. Hence

$$a_{22}^{(4,5)} = a_{55}^{(4,5)}, \tag{7.102}$$

$$a_{12}^{(4,5)} = 0; \tag{7.103}$$

– when $(t, q) = (4, 5)$, we get $u = 1, 2, 3$ and $v = 3, 5$, where $u \neq 4, 5$ and $v \neq 2, 4$, satisfying $1 \leqslant u < v \leqslant 5$. Hence

$$a_{22}^{(u,v)} = a_{55}^{(u,v)}, \tag{7.104}$$

$$a_{12}^{(u,v)} = 0 \tag{7.105}$$

for all $(u, v) \in \{(1, 3), (2, 3), (3, 5)\}$.

- Finally, for $s = 3$, by (7.98) and (7.99), we obtain the 3rd column of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant 5$ where $v \neq 3, 4$, and $(u, v) \neq (4, 5)$, because when $(t, q) = (4, 5)$, we get $u = 1, 2, 3$ and $v = 2, 5$, where $u \neq 4, 5$ and $v \neq 3, 4$, satisfying $1 \leqslant u < v \leqslant 5$. Hence

$$a_{33}^{(u,v)} = a_{55}^{(u,v)}, \tag{7.106}$$

$$a_{13}^{(u,v)} = a_{23}^{(u,v)} = 0 \tag{7.107}$$

for all $(u, v) \in \{(1, 2), (1, 5), (2, 5), (3, 5)\}$.

It follows from (7.95), (7.97) and (7.100) that

$$a_{11}^{(2,4)} = a_{22}^{(2,4)} = a_{44}^{(2,4)} = a_{55}^{(2,4)}. \tag{7.108}$$

It follows from (7.97) and (7.104) that

$$a_{11}^{(2,3)} = a_{22}^{(2,3)} = a_{55}^{(2,3)}. \tag{7.109}$$

It follows from (7.95) and (7.97) that

$$a_{11}^{(3,4)} = a_{44}^{(3,4)} = a_{55}^{(3,4)}. \tag{7.110}$$

It follows from (7.95), (7.97) and (7.106) that

$$a_{11}^{(1,2)} = a_{33}^{(1,2)} = a_{44}^{(1,2)} = a_{55}^{(1,2)}. \tag{7.111}$$

It follows from (7.93), (7.97) and (7.102) that

$$a_{11}^{(4,5)} = a_{22}^{(4,5)} = a_{33}^{(4,5)} = a_{55}^{(4,5)}. \tag{7.112}$$

It follows from (7.93), (7.95), (7.97) and (7.104) that

$$a_{11}^{(1,3)} = a_{ii}^{(1,3)} \quad \text{for } i = 2, \dots, 5. \tag{7.113}$$

It follows from (7.93), (7.95), (7.97) and (7.100) that

$$a_{11}^{(1,4)} = a_{ii}^{(1,4)} \quad \text{for } i = 2, \dots, 5, \tag{7.114}$$

$$a_{11}^{(1,5)} = a_{ii}^{(1,5)} \quad \text{for } i = 2, \dots, 5. \tag{7.115}$$

It follows from (7.95), (7.97), (7.100) and (7.106) that

$$a_{11}^{(2,5)} = a_{ii}^{(2,5)} \quad \text{for } i = 2, \dots, 5. \tag{7.116}$$

It follows from (7.95), (7.97), (7.104) and (7.106) that

$$a_{11}^{(3,5)} = a_{ii}^{(3,5)} \quad \text{for } i = 2, \dots, 5. \tag{7.117}$$

Next, by $[\psi(I_5 + E_{12} + E_{2q}), E_{1q}] = 0$ for every $3 \leqslant q \leqslant 5$, we get

$$a_{qq}^{(2,q)} = a_{11}^{(2,q)} + a_{11}^{(1,1)} + a_{qq}^{(1,1)} + a_{11}^{(1,2)} + a_{qq}^{(1,2)} \tag{7.118}$$

for every $3 \leqslant q \leqslant 5$. Taking (7.108), (7.111) and (7.116) into (7.118), for every $4 \leqslant q \leqslant 5$,

$$a_{11}^{(1,1)} = a_{qq}^{(1,1)}. \tag{7.119}$$

Setting $q = 3$ in (7.118),

$$a_{33}^{(2,3)} = a_{11}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)}. \tag{7.120}$$

Taking (7.111) into (7.120),

$$a_{33}^{(2,3)} = a_{11}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)}. \tag{7.121}$$

By $[\psi(I_5 + E_{23} + E_{35}), E_{25}] = 0$, we have

$$a_{12}^{(3,5)} = a_{12}^{(1,1)} + a_{12}^{(2,3)}, \tag{7.122}$$

$$a_{22}^{(1,1)} = a_{55}^{(1,1)} + a_{22}^{(2,3)} + a_{55}^{(2,3)} + a_{22}^{(3,5)} + a_{55}^{(3,5)}. \tag{7.123}$$

It follows from (7.105) that

$$a_{12}^{(2,3)} = a_{12}^{(3,5)} = 0. \tag{7.124}$$

Taking (7.124) into (7.122),

$$a_{12}^{(1,1)} = 0. \tag{7.125}$$

Taking (7.109) and (7.117) into (7.123),

$$a_{22}^{(1,1)} = a_{55}^{(1,1)}. \tag{7.126}$$

It follows from (7.119) and (7.126) that

$$a_{11}^{(1,1)} = a_{22}^{(1,1)} = a_{44}^{(1,1)} = a_{55}^{(1,1)}. \tag{7.127}$$

By $[\psi(I_5 + E_{12} + E_{2s}), E_{1s}] = 0$ for every $3 \leqslant s < t \leqslant 5$, we get

$$a_{sj}^{(1,1)} + a_{sj}^{(1,2)} + a_{sj}^{(2,s)} = 0 \quad \text{for} \quad j = s+1, \ldots, t-1, t+1, \ldots, 5, \tag{7.128}$$

$$a_{st}^{(1,1)} + a_{st}^{(1,2)} + a_{st}^{(2,s)} = 0, \tag{7.129}$$

for every $3 \leqslant s < t \leqslant 5$. Setting $(s,t) = (4,5)$ in (7.129),

$$a_{45}^{(1,1)} + a_{45}^{(1,2)} + a_{45}^{(2,4)} = 0. \tag{7.130}$$

It follows from (7.96) that

$$a_{45}^{(1,2)} = a_{45}^{(2,4)} = 0. \tag{7.131}$$

Taking (7.131) into (7.130),

$$a_{45}^{(1,1)} = 0. \tag{7.132}$$

On the other hand, by $[\psi(I_5 + E_{23} + E_{34}), E_{24}] = 0$, we get

$$a_{12}^{(3,4)} = a_{12}^{(1,1)} + a_{12}^{(2,3)}, \tag{7.133}$$

$$a_{45}^{(2,3)} = a_{45}^{(1,1)} + a_{45}^{(3,4)}. \tag{7.134}$$

It follows from (7.105) that

$$a_{12}^{(2,3)} = 0. \tag{7.135}$$

Taking (7.125) and (7.135) into (7.133),

$$a_{12}^{(3,4)} = 0. \tag{7.136}$$

It follows from (7.96) that

$$a_{45}^{(3,4)} = 0. \tag{7.137}$$

Taking (7.132) and (7.137) into (7.134),

$$a_{45}^{(2,3)} = 0. \tag{7.138}$$

Let $1 \leqslant s < t \leqslant 5$ be integers. Set

$$X_{st} = \begin{cases} a_{14}^{(s,t)} E_{14} + a_{15}^{(s,t)} E_{15} + a_{24}^{(s,t)} E_{24} + a_{25}^{(s,t)} E_{25} & \text{if } 1 \leqslant s < t \leqslant 5, \\ a_{14}^{(1,1)} E_{14} + a_{15}^{(1,1)} E_{15} + a_{24}^{(1,1)} E_{24} + a_{25}^{(1,1)} E_{25} & \text{if } s = t = 1. \end{cases}$$

Up to this point, we obtain the partially completed maps as the following.

In view of (7.77), (7.81)–(7.84), (7.125), (7.127) and (7.132),

$$\begin{aligned}\psi(I_5) = a_{11}^{(1,1)}(E_{11} + E_{22} + E_{44} + E_{55}) + a_{33}^{(1,1)}E_{33} + a_{13}^{(1,1)}E_{13} + a_{23}^{(1,1)}E_{23} \\ + a_{34}^{(1,1)}E_{34} + a_{35}^{(1,1)}E_{35} + X_{11}.\end{aligned} \tag{7.139}$$

In view of (7.76), (7.83), (7.84), (7.109), (7.121), (7.124) and (7.138),

$$\begin{aligned}\psi(E_{23}) = a_{11}^{(2,3)}(E_{11} + E_{22} + E_{55}) + (a_{11}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)})E_{33} + a_{44}^{(2,3)}E_{44} \\ + a_{13}^{(2,3)}E_{13} + a_{23}^{(2,3)}E_{23} + (a_{34}^{(1,1)} + a_{34}^{(1,2)})E_{34} \\ + (a_{35}^{(1,1)} + a_{35}^{(1,2)})E_{35} + X_{23}.\end{aligned} \tag{7.140}$$

In view of (7.76), (7.81), (7.82), (7.110), (7.136) and (7.137),

$$\begin{aligned}\psi(E_{34}) = a_{11}^{(3,4)}(E_{11} + E_{44} + E_{55}) + a_{22}^{(3,4)}E_{22} + a_{33}^{(3,4)}E_{33} + a_{34}^{(3,4)}E_{34} \\ + a_{35}^{(3,4)}E_{35} + a_{13}^{(3,4)}E_{13} + a_{23}^{(3,4)}E_{23} + X_{34}.\end{aligned} \tag{7.141}$$

In view of (7.76), (7.101), (7.108) and (7.131),

$$\begin{aligned}\psi(E_{24}) = a_{11}^{(2,4)}(E_{11} + E_{22} + E_{44} + E_{55}) + a_{33}^{(2,4)}E_{33} + a_{13}^{(2,4)}E_{13} + a_{23}^{(2,4)}E_{23} \\ + a_{34}^{(2,4)}E_{34} + a_{35}^{(2,4)}E_{35} + X_{24}.\end{aligned} \tag{7.142}$$

In view of (7.76), (7.83), (7.84), (7.107), (7.111) and (7.131),

$$\begin{aligned}\psi(E_{12}) = a_{11}^{(1,2)}(E_{11} + E_{33} + E_{44} + E_{55}) + a_{22}^{(1,2)}E_{22} + a_{12}^{(1,2)}E_{12} + a_{34}^{(1,2)}E_{34} \\ + a_{35}^{(1,2)}E_{35} + X_{12}.\end{aligned} \tag{7.143}$$

In view of (7.76), (7.94), (7.96), (7.101) and (7.114),

$$\psi(E_{14}) = a_{11}^{(1,4)}I_n + a_{13}^{(1,4)}E_{13} + a_{23}^{(1,4)}E_{23} + X_{14}. \tag{7.144}$$

In view of (7.76), (7.96), (7.101), (7.107) and (7.116),

$$\psi(E_{25}) = a_{11}^{(2,5)}I_5 + a_{34}^{(2,5)}E_{34} + a_{35}^{(2,5)}E_{35} + X_{25}. \tag{7.145}$$

In view of (7.76), (7.81), (7.82), (7.94), (7.103) and (7.112),

$$\begin{aligned}\psi(E_{45}) = a_{11}^{(4,5)}&(E_{11}+E_{22}+E_{33}+E_{55}) + a_{44}^{(4,5)}E_{44} + a_{45}^{(4,5)}E_{45} \\ &+ (a_{13}^{(1,1)} + a_{13}^{(3,4)})E_{13} + (a_{23}^{(1,1)} + a_{23}^{(3,4)})E_{23} + X_{45}.\end{aligned} \tag{7.146}$$

In view of (7.76), (7.96), (7.105), (7.107) and (7.117),

$$\psi(E_{35}) = a_{11}^{(3,5)}I_5 + a_{34}^{(3,5)}E_{34} + a_{35}^{(3,5)}E_{35} + X_{35}. \tag{7.147}$$

In view of (7.76), (7.94), (7.96), (7.105) and (7.113),

$$\psi(E_{13}) = a_{11}^{(1,3)}I_5 + a_{13}^{(1,3)}E_{13} + a_{23}^{(1,3)}E_{23} + X_{13}. \tag{7.148}$$

Finally, in view of (7.76), (7.94), (7.96), (7.101), (7.107) and (7.115),

$$\psi(E_{15}) = a_{11}^{(1,5)}I_5 + X_{15}. \tag{7.149}$$

Remark that the map $\psi(E_{15})$ in (7.149) is already ultimate.

We now consider $A^2 = I_5 + E_{pq} + E_{st}$ for integers $1 \leqslant p < q \leqslant 5$ and $1 \leqslant s < t \leqslant 5$ and $(p,q) \neq (s,t)$. By $[\psi(I_5 + E_{12} + E_{23} + E_{24}), E_{13} + E_{14}] = 0$, we have

$$a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(2,4)} = a_{33}^{(1,1)} + a_{33}^{(1,2)} + a_{33}^{(2,3)} + a_{33}^{(2,4)}, \tag{7.150}$$

$$\begin{aligned}a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(2,4)} + a_{44}^{(1,1)} + a_{44}^{(1,2)} + a_{44}^{(2,3)} + a_{44}^{(2,4)}& \\ = a_{34}^{(1,1)} + a_{34}^{(1,2)} + a_{34}^{(2,3)} + a_{34}^{(2,4)}&,\end{aligned} \tag{7.151}$$

$$a_{35}^{(1,1)} + a_{35}^{(1,2)} + a_{35}^{(2,3)} + a_{35}^{(2,4)} = a_{45}^{(1,1)} + a_{45}^{(1,2)} + a_{45}^{(2,3)} + a_{45}^{(2,4)}. \tag{7.152}$$

Taking (7.111) and (7.121) into (7.150),

$$a_{11}^{(2,4)} = a_{33}^{(2,4)}. \tag{7.153}$$

We conclude from (7.108) and (7.153) that

$$a_{11}^{(2,4)} = a_{ii}^{(2,4)} \text{ for } i = 2, \ldots, 5. \tag{7.154}$$

By $[\psi(I_5 + E_{12} + E_{23} + E_{25}), E_{13} + E_{15}] = 0$, we have

$$a_{34}^{(1,1)} + a_{34}^{(1,2)} + a_{34}^{(2,3)} + a_{34}^{(2,5)} = 0, \tag{7.155}$$

$$\begin{aligned} a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(2,5)} + a_{55}^{(1,1)} + a_{55}^{(1,2)} + a_{55}^{(2,3)} + a_{55}^{(2,5)} \\ = a_{35}^{(1,1)} + a_{35}^{(1,2)} + a_{35}^{(2,3)} + a_{35}^{(2,5)}. \end{aligned} \tag{7.156}$$

By $[\psi(I_5 + E_{12} + E_{23}), E_{13}] = 0$ for every $4 \leqslant j \leqslant 5$, we get

$$a_{3j}^{(1,1)} + a_{3j}^{(1,2)} + a_{3j}^{(2,3)} = 0 \tag{7.157}$$

for every $4 \leqslant j \leqslant 5$. Taking (7.138), (7.157) with $j = 5$ and (7.130) into (7.152),

$$a_{35}^{(2,4)} = 0. \tag{7.158}$$

Taking (7.157) with $j = 4$ into (7.155),

$$a_{34}^{(2,5)} = 0. \tag{7.159}$$

Taking (7.108), (7.111), (7.127) and (7.157) with $j = 4$ into (7.151),

$$a_{34}^{(2,4)} = a_{11}^{(2,3)} + a_{44}^{(2,3)}. \tag{7.160}$$

Taking (7.109), (7.111), (7.116), (7.127) and (7.157) with $j = 5$ into (7.156),

$$a_{35}^{(2,5)} = 0. \tag{7.161}$$

Secondly, by $[\psi(I_5 + E_{12} + E_{2s} + E_{st}), E_{1s} + E_{2t}] = 0$ for every $3 \leqslant s < t \leqslant 5$, we have

$$a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,s)} + a_{11}^{(s,t)} = a_{ss}^{(1,1)} + a_{ss}^{(1,2)} + a_{ss}^{(2,s)} + a_{ss}^{(s,t)}, \tag{7.162}$$

$$a_{12}^{(1,1)} + a_{12}^{(1,2)} + a_{12}^{(2,s)} + a_{12}^{(s,t)} = a_{st}^{(1,1)} + a_{st}^{(1,2)} + a_{st}^{(2,s)} + a_{st}^{(s,t)}, \tag{7.163}$$

$$a_{22}^{(1,1)} + a_{22}^{(1,2)} + a_{22}^{(2,s)} + a_{22}^{(s,t)} = a_{tt}^{(1,1)} + a_{tt}^{(1,2)} + a_{tt}^{(2,s)} + a_{tt}^{(s,t)}, \tag{7.164}$$

$$a_{sj}^{(1,1)} + a_{sj}^{(1,2)} + a_{sj}^{(2,s)} + a_{sj}^{(s,t)} = 0 \quad \text{for } j = s+1, \ldots, t-1, t+1, \ldots, 5, \tag{7.165}$$

for every $3 \leqslant s < t \leqslant 5$. Taking (7.128) into (7.165), for every $3 \leqslant s < t \leqslant 5$,

$$a_{sj}^{(s,t)} = 0 \quad \text{for } j = s+1, \ldots, t-1, t+1, \ldots, 5. \tag{7.166}$$

By $[\psi(I_5 + E_{2s} + E_{st}), E_{2t}] = 0$ for every $3 \leqslant s < t \leqslant 5$, we get

$$a_{12}^{(1,1)} + a_{12}^{(2,s)} + a_{12}^{(s,t)} = 0 \tag{7.167}$$

for every $3 \leqslant s < t \leqslant 5$. Taking (7.129) and (7.167) into (7.163), for all $3 \leqslant s < t \leqslant 5$,

$$a_{12}^{(1,2)} = a_{st}^{(s,t)}. \tag{7.168}$$

Setting $(s,t) = (4,5)$ in (7.162),

$$a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,4)} + a_{11}^{(4,5)} = a_{44}^{(1,1)} + a_{44}^{(1,2)} + a_{44}^{(2,4)} + a_{44}^{(4,5)}. \tag{7.169}$$

Taking (7.108), (7.111), (7.127) into (7.169),

$$a_{11}^{(4,5)} = a_{44}^{(4,5)}. \tag{7.170}$$

We conclude from (7.112) and (7.170) that

$$a_{11}^{(4,5)} = a_{ii}^{(4,5)} \quad \text{for } i = 2, \ldots, 5. \tag{7.171}$$

Setting $(s, t) = (3, 4)$ in (7.162),

$$a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(3,4)} = a_{33}^{(1,1)} + a_{33}^{(1,2)} + a_{33}^{(2,3)} + a_{33}^{(3,4)}. \tag{7.172}$$

Taking (7.111) and (7.121) into (7.172),

$$a_{11}^{(3,4)} = a_{33}^{(3,4)}. \tag{7.173}$$

It follows from (7.110) and (7.173) that

$$a_{11}^{(3,4)} = a_{33}^{(3,4)} = a_{44}^{(3,4)} = a_{55}^{(3,4)}. \tag{7.174}$$

Setting $(s, t) = (3, 4)$ in (7.164),

$$a_{22}^{(1,1)} + a_{22}^{(1,2)} + a_{22}^{(2,3)} + a_{22}^{(3,4)} = a_{44}^{(1,1)} + a_{44}^{(1,2)} + a_{44}^{(2,3)} + a_{44}^{(3,4)}. \tag{7.175}$$

Taking (7.109), (7.111), (7.127) and (7.174) into (7.175),

$$a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(3,4)} = a_{22}^{(1,2)} + a_{44}^{(2,3)} + a_{22}^{(3,4)}. \tag{7.176}$$

Setting $(s, t) = (3, 5)$ in (7.164),

$$a_{22}^{(1,1)} + a_{22}^{(1,2)} + a_{22}^{(2,3)} + a_{22}^{(3,5)} = a_{55}^{(1,1)} + a_{55}^{(1,2)} + a_{55}^{(2,3)} + a_{55}^{(3,5)}. \tag{7.177}$$

Taking (7.109), (7.111), (7.117) and (7.127) into (7.177),

$$a_{11}^{(1,2)} = a_{22}^{(1,2)}. \tag{7.178}$$

It follows from (7.111) and (7.178) that

$$a_{11}^{(1,2)} = a_{ii}^{(1,2)} \quad \text{for } i = 2, \ldots, 5. \tag{7.179}$$

Taking (7.179) into (7.176),

$$a_{11}^{(2,3)} + a_{11}^{(3,4)} = a_{44}^{(2,3)} + a_{22}^{(3,4)}. \tag{7.180}$$

Thirdly, by $[\psi(I_5 + E_{13} + E_{34} + E_{45}), E_{14} + E_{35}] = 0$, we have

$$a_{13}^{(1,1)} + a_{13}^{(1,3)} + a_{13}^{(3,4)} + a_{13}^{(4,5)} = a_{45}^{(1,1)} + a_{45}^{(1,3)} + a_{45}^{(3,4)} + a_{45}^{(4,5)}, \tag{7.181}$$

$$a_{23}^{(1,1)} + a_{23}^{(1,3)} + a_{23}^{(3,4)} + a_{23}^{(4,5)} = 0. \tag{7.182}$$

By $[\psi(I_5 + E_{34} + E_{45}), E_{35}] = 0$, we obtain

$$a_{13}^{(1,1)} + a_{13}^{(3,4)} + a_{13}^{(4,5)} = 0, \tag{7.183}$$

$$a_{23}^{(1,1)} + a_{23}^{(3,4)} + a_{23}^{(4,5)} = 0, \tag{7.184}$$

$$a_{33}^{(1,1)} + a_{33}^{(3,4)} + a_{33}^{(4,5)} = a_{55}^{(1,1)} + a_{55}^{(3,4)} + a_{55}^{(4,5)}. \tag{7.185}$$

Taking (7.184) into (7.182),

$$a_{23}^{(1,3)} = 0. \tag{7.186}$$

Taking (7.171) and (7.174) into (7.185),

$$a_{33}^{(1,1)} = a_{55}^{(1,1)}. \tag{7.187}$$

We conclude from (7.127) and (7.187) that

$$a_{11}^{(1,1)} = a_{ii}^{(1,1)} \quad \text{for } \ i = 2, \ldots, 5. \tag{7.188}$$

Taking (7.188) into (7.121),

$$a_{11}^{(2,3)} = a_{33}^{(2,3)}. \tag{7.189}$$

By $[\psi(I_5 + E_{13} + E_{34}), E_{14}] = 0$, we get

$$a_{45}^{(1,1)} + a_{45}^{(1,3)} + a_{45}^{(3,4)} = 0. \tag{7.190}$$

Taking (7.183) and (7.190) into (7.181),

$$a_{13}^{(1,3)} = a_{45}^{(4,5)}. \tag{7.191}$$

Next, by $[\psi(I_5 + E_{23} + E_{34} + E_{45}), E_{24} + E_{35}] = 0$, we have

$$a_{13}^{(1,1)} + a_{13}^{(2,3)} + a_{13}^{(3,4)} + a_{13}^{(4,5)} = 0, \tag{7.192}$$

$$a_{23}^{(1,1)} + a_{23}^{(2,3)} + a_{23}^{(3,4)} + a_{23}^{(4,5)} = a_{45}^{(1,1)} + a_{45}^{(2,3)} + a_{45}^{(3,4)} + a_{45}^{(4,5)}. \tag{7.193}$$

Taking (7.183) into (7.192),

$$a_{13}^{(2,3)} = 0. \tag{7.194}$$

By $[\psi(I_5 + E_{23} + E_{34}), E_{24}] = 0$, we get

$$a_{45}^{(1,1)} + a_{45}^{(2,3)} + a_{45}^{(3,4)} = 0. \tag{7.195}$$

Taking (7.184) and (7.195) into (7.193),

$$a_{23}^{(2,3)} = a_{45}^{(4,5)}. \tag{7.196}$$

We conclude from (7.191) and (7.196) that for every $1 \leqslant r \leqslant 2$,

$$a_{r3}^{(r,3)} = a_{45}^{(4,5)}. \tag{7.197}$$

Setting $(s,t) = (4,5)$ in (7.168),

$$a_{12}^{(1,2)} = a_{45}^{(4,5)}. \tag{7.198}$$

We conclude from (7.197) and (7.198) that for every $1 \leqslant r \leqslant 2$,

$$a_{r3}^{(r,3)} = a_{12}^{(1,2)}. \tag{7.199}$$

We conclude from (7.168) and (7.199) that for every $1 \leqslant r \leqslant 2 < s < t \leqslant 5$,

$$a_{12}^{(1,2)} = a_{r3}^{(r,3)} = a_{st}^{(s,t)}. \tag{7.200}$$

By $[\psi(I_5 + E_{23} + E_{34} + E_{35}), E_{24} + E_{25}] = 0$, we have

$$\begin{aligned} a_{22}^{(1,1)} + a_{22}^{(2,3)} + a_{22}^{(3,4)} + a_{22}^{(3,5)} + a_{55}^{(1,1)} + a_{55}^{(2,3)} + a_{55}^{(3,4)} + a_{55}^{(3,5)} \\ = a_{45}^{(1,1)} + a_{45}^{(2,3)} + a_{45}^{(3,4)} + a_{45}^{(3,5)}. \end{aligned} \tag{7.201}$$

It follows from (7.147) that

$$a_{45}^{(3,5)} = 0. \tag{7.202}$$

Taking (7.123), (7.195) and (7.202) into (7.201),

$$a_{22}^{(3,4)} = a_{55}^{(3,4)}. \tag{7.203}$$

We conclude from (7.174) and (7.203) that

$$a_{11}^{(3,4)} = a_{ii}^{(3,4)} \quad \text{for } \ i = 2, \ldots, 5. \tag{7.204}$$

Taking (7.204) into (7.180),

$$a_{11}^{(2,3)} = a_{44}^{(2,3)}. \tag{7.205}$$

We conclude from (7.109), (7.189) and (7.205) that

$$a_{11}^{(2,3)} = a_{ii}^{(2,3)} \quad \text{for } \ i = 2, \ldots, 5. \tag{7.206}$$

Taking (7.206) into (7.160),

$$a_{34}^{(2,4)} = 0. \tag{7.207}$$

Finally, by $[\psi(I_5 + E_{14} + E_{34} + E_{45}), E_{15} + E_{35}] = 0$, we have

$$\begin{aligned} a_{11}^{(1,1)} + a_{11}^{(1,4)} + a_{11}^{(3,4)} + a_{11}^{(4,5)} + a_{55}^{(1,1)} + a_{55}^{(1,4)} + a_{55}^{(3,4)} + a_{55}^{(4,5)} \\ = a_{13}^{(1,1)} + a_{13}^{(1,4)} + a_{13}^{(3,4)} + a_{13}^{(4,5)}, \end{aligned} \tag{7.208}$$

$$a_{23}^{(1,1)} + a_{23}^{(1,4)} + a_{23}^{(3,4)} + a_{23}^{(4,5)} = 0. \tag{7.209}$$

Taking (7.184) into (7.209),

$$a_{23}^{(1,4)} = 0. \tag{7.210}$$

By $[\psi(I_5 + E_{14} + E_{45}), E_{15}] = 0$, we get

$$a_{11}^{(1,1)} + a_{11}^{(1,4)} + a_{11}^{(4,5)} + a_{55}^{(1,1)} + a_{55}^{(1,4)} + a_{55}^{(4,5)} = 0. \tag{7.211}$$

Taking (7.183), (7.204) and (7.211) into (7.208),

$$a_{13}^{(1,4)} = 0. \tag{7.212}$$

By $[\psi(I_5 + E_{24} + E_{34} + E_{45}), E_{25} + E_{35}] = 0$, we obtain

$$a_{12}^{(1,1)} + a_{12}^{(2,4)} + a_{12}^{(3,4)} + a_{12}^{(4,5)} = a_{13}^{(1,1)} + a_{13}^{(2,4)} + a_{13}^{(3,4)} + a_{13}^{(4,5)}, \tag{7.213}$$

$$\begin{aligned} a_{22}^{(1,1)} + a_{22}^{(2,4)} + a_{22}^{(3,4)} + a_{22}^{(4,5)} + a_{55}^{(1,1)} + a_{55}^{(2,4)} + a_{55}^{(3,4)} + a_{55}^{(4,5)} \\ = a_{23}^{(1,1)} + a_{23}^{(2,4)} + a_{23}^{(3,4)} + a_{23}^{(4,5)}. \end{aligned} \tag{7.214}$$

By $[\psi(I_5 + E_{24} + E_{45}), E_{25}] = 0$, we have

$$a_{22}^{(1,1)} + a_{22}^{(2,4)} + a_{22}^{(4,5)} + a_{55}^{(1,1)} + a_{55}^{(2,4)} + a_{55}^{(4,5)} = 0. \tag{7.215}$$

Taking (7.184), (7.204) and (7.215) into (7.214),

$$a_{23}^{(2,4)} = 0. \tag{7.216}$$

It follows from (7.141) that

$$a_{12}^{(3,4)} = 0. \tag{7.217}$$

Setting $(s, t) = (4, 5)$ in (7.167),

$$a_{12}^{(1,1)} + a_{12}^{(2,4)} + a_{12}^{(4,5)} = 0. \tag{7.218}$$

Taking (7.183), (7.217) and (7.218) into (7.213),

$$a_{13}^{(2,4)} = 0. \tag{7.219}$$

Consequently, using (7.139)–(7.149), we are ready to classify $\psi(I_5)$ and $\psi(E_{ij})$ for each pair of integers $1 \leqslant i < j \leqslant 5$. By virtue of (7.139) and (7.188),

$$\psi(I_5) = a_{11}^{(1,1)} I_5 + a_{13}^{(1,1)} E_{13} + a_{23}^{(1,1)} E_{23} + a_{34}^{(1,1)} E_{34} + a_{35}^{(1,1)} E_{35} + X_{11}. \tag{7.220}$$

By virtue of (7.143) and (7.179),

$$\psi(E_{12}) = a_{11}^{(1,2)} I_5 + a_{12}^{(1,2)} E_{12} + a_{34}^{(1,2)} E_{34} + a_{35}^{(1,2)} E_{35} + X_{12}. \tag{7.221}$$

By virtue of (7.146), (7.171) and (7.200),

$$\psi(E_{45}) = a_{11}^{(4,5)} I_5 + a_{12}^{(1,2)} E_{45} + (a_{13}^{(1,1)} + a_{13}^{(3,4)}) E_{13} + (a_{23}^{(1,1)} + a_{23}^{(3,4)}) E_{23} + X_{45}. \tag{7.222}$$

By virtue of (7.140), (7.188), (7.194), (7.200) and (7.206),

$$\psi(E_{23}) = a_{11}^{(2,3)} I_5 + a_{12}^{(1,2)} E_{23} + (a_{34}^{(1,1)} + a_{34}^{(1,2)}) E_{34} + (a_{35}^{(1,1)} + a_{35}^{(1,2)}) E_{35} + X_{23}. \tag{7.223}$$

By virtue of (7.141), (7.166), (7.200) and (7.204),

$$\psi(E_{34}) = a_{11}^{(3,4)} I_5 + a_{12}^{(1,2)} E_{34} + a_{13}^{(3,4)} E_{13} + a_{23}^{(3,4)} E_{23} + X_{34}. \tag{7.224}$$

It follows from (7.144), (7.210) and (7.212) that

$$\begin{aligned} \psi(E_{14}) = {} & a_{11}^{(1,4)} I_5 + a_{12}^{(1,2)} E_{14} + (a_{12}^{(1,2)} + a_{14}^{(1,4)}) E_{14} + a_{15}^{(1,4)} E_{15} \\ & + a_{2,4}^{(1,4)} E_{24} + a_{25}^{(1,4)} E_{25}. \end{aligned} \tag{7.225}$$

Since the map $\psi(E_{15})$ in (7.149) is already ultimate, it follows from (7.149) that

$$\begin{aligned}\psi(E_{15}) = a_{11}^{(1,5)} I_n + a_{12}^{(1,2)} E_{15} + a_{14}^{(1,5)} E_{14} + (a_{12}^{(1,2)} + a_{15}^{(1,5)}) E_{15} \\ + a_{24}^{(1,5)} E_{24} + a_{25}^{(1,5)} E_{25}.\end{aligned} \tag{7.226}$$

By virtue of (7.142), (7.154), (7.158), (7.207), (7.216) and (7.219),

$$\begin{aligned}\psi(E_{24}) = a_{11}^{(2,4)} I_5 + a_{12}^{(1,2)} E_{24} + a_{14}^{(2,4)} E_{14} + a_{15}^{(2,4)} E_{15} \\ + (a_{12}^{(1,2)} + a_{24}^{(2,4)}) E_{24} + a_{25}^{(2,4)} E_{25}.\end{aligned} \tag{7.227}$$

It follows from (7.145), (7.159) and (7.161) that

$$\begin{aligned}\psi(E_{25}) = a_{11}^{(2,5)} I_5 + a_{12}^{(1,2)} E_{25} + a_{14}^{(2,5)} E_{14} + a_{15}^{(2,5)} E_{15} + a_{24}^{(2,5)} E_{24} \\ + (a_{12}^{(1,2)} + a_{25}^{(2,5)}) E_{25}.\end{aligned} \tag{7.228}$$

By virtue of (7.147), (7.166) and (7.200),

$$\psi(E_{35}) = a_{11}^{(3,5)} I_5 + a_{12}^{(1,2)} E_{35} + X_{35}. \tag{7.229}$$

By virtue of (7.148), (7.186) and (7.200),

$$\psi(E_{13}) = a_{11}^{(1,3)} I_5 + a_{12}^{(1,2)} E_{13} + X_{13}. \tag{7.230}$$

Let $\lambda = a_{12}^{(1,2)} \in \mathbb{F}_2$. Let $\mu : T_5(\mathbb{F}_2) \to \mathbb{F}_2$ be the additive map defined by

$$\mu(A) = a_{11}^{(1,1)} + \sum_{1 \leqslant i < j \leqslant 5} a_{11}^{(i,j)} \tag{7.231}$$

for all $A = (a_{ij}) \in T_5(\mathbb{F}_2)$. Let $\theta_5 = \{(1,4), (1,5), (2,4), (2,5)\}$. Let $\lambda_{st}^{(1,1)} = a_{st}^{(1,1)}$ and $\lambda_{st}^{(i,j)} = a_{st}^{(i,j)}$, for each pair of integers $1 \leqslant i < j \leqslant 5$ and $(s,t) \in \theta_5$. Let $\psi_\Theta : T_5(\mathbb{F}_2) \to T_5(\mathbb{F}_2)$ be the additive map defined in (7.4). In view of (7.220)–(7.231), together with the additivity of ψ, μ and ψ_Θ, we obtain

$$\psi(A) = \sum_{1 \leqslant i \leqslant j \leqslant 5} \psi(E_{ij})$$

$$
\begin{aligned}
&= \psi(I_5) + \sum_{1 \leqslant i < j \leqslant 5} \psi(E_{ij}) \\
&= \lambda A + \mu(A) I_5 + \psi_\Theta(A)
\end{aligned}
$$

for all $A \in T_5(\mathbb{F}_2)$, where $\lambda \in \mathbb{F}_2$. This completes the proof for $n = 5$.

Case I: $n \geqslant 6$. For each pair of integers $1 \leqslant p < q \leqslant n$, we let

$$\psi(E_{pq}) = (a_{ij}^{(p,q)}) \in T_n(\mathbb{F}_2) \tag{7.232}$$

and let

$$\psi(I_n) = (a_{ij}^{(1,1)}) \in T_n(\mathbb{F}_2) \tag{7.233}$$

for all integers $1 \leqslant i \leqslant j \leqslant n$. In what follows, it is understood that (7.232) and (7.233) are used in $[\psi(I_n + E_{st} + E_{tq}), E_{sq}] = 0$ for $1 \leqslant s < t < q \leqslant n$. Let $A = (a_{ij}) \in T_n(\mathbb{F}_2)$ be invertible. If $A = I_n + E_{ij}$, then $A^2 = I_n + E_{ij} + E_{ij} = I_n$. Thus $[\psi(A), A^2] = [\psi(I_n + E_{ij}), I_n] = 0$. Hence we first consider $A^2 = I_n + E_{sq}$ for integers $1 \leqslant s < t < q \leqslant n$. Note that $A^2 \neq I_n + E_{s,s+1}$ for all $1 \leqslant s \leqslant n-1$ since $q \neq s+1$. Let $A = I_n + E_{st} + E_{tq}$ for integers $1 \leqslant s < t < q \leqslant n$. Then $A^2 = I_n + E_{sq}$, where $(s,q) \neq (i, i+1)$ for all $i = 1, \ldots, n-1$, by Lemma 7.4.1. Since $[\psi(I_n + E_{st} + E_{tq}), I_n] = 0$, by $[\psi(I_n + E_{st} + E_{tq}), E_{sq}] = 0$, (7.232) and (7.233), for every $1 \leqslant s < t < q \leqslant n$,

$$\left(\sum_{1 \leqslant i \leqslant j \leqslant n} (a_{ij}^{(1,1)} + a_{ij}^{(s,t)} + a_{ij}^{(t,q)}) E_{ij} \right) E_{sq} - E_{sq} \left(\sum_{1 \leqslant i \leqslant j \leqslant n} (a_{ij}^{(1,1)} + a_{ij}^{(s,t)} + a_{ij}^{(t,q)}) E_{ij} \right) = 0.$$

Then

$$\sum_{1 \leqslant i \leqslant s} (a_{is}^{(1,1)} + a_{is}^{(s,t)} + a_{is}^{(t,q)}) E_{iq} - \sum_{q \leqslant j \leqslant n} (a_{qj}^{(1,1)} + a_{qj}^{(s,t)} + a_{qj}^{(t,q)}) E_{sj} = 0.$$

Thus

$$
\begin{aligned}
&(a_{ss}^{(1,1)} + a_{ss}^{(s,t)} + a_{ss}^{(t,q)} - a_{qq}^{(1,1)} - a_{qq}^{(s,t)} - a_{qq}^{(t,q)}) E_{sq} \\
&+ \sum_{1 \leqslant i \leqslant s-1} (a_{is}^{(1,1)} + a_{is}^{(s,t)} + a_{is}^{(t,q)}) E_{iq} - \sum_{q+1 \leqslant j \leqslant n} (a_{qj}^{(1,1)} + a_{qj}^{(s,t)} + a_{qj}^{(t,q)}) E_{sj} = 0.
\end{aligned}
$$

Hence for every $1 \leqslant s < t < q \leqslant n$, we obtain

$$a_{ss}^{(1,1)} + a_{qq}^{(1,1)} + a_{ss}^{(s,t)} + a_{qq}^{(s,t)} + a_{ss}^{(t,q)} + a_{qq}^{(t,q)} = 0, \tag{7.234}$$

$$a_{is}^{(1,1)} + a_{is}^{(s,t)} + a_{is}^{(t,q)} = 0 \quad \text{for } i = 1, \ldots, s-1, \tag{7.235}$$

$$a_{qj}^{(1,1)} + a_{qj}^{(s,t)} + a_{qj}^{(t,q)} = 0 \quad \text{for } j = q+1, \ldots, n. \tag{7.236}$$

Setting $(s,t,q) = (n-2, n-1, n)$ and $i \in \{1, 2\}$ in (7.235) respectively,

$$a_{1,n-2}^{(n-1,n)} = a_{1,n-2}^{(1,1)} + a_{1,n-2}^{(n-2,n-1)}, \tag{7.237}$$

$$a_{2,n-2}^{(n-1,n)} = a_{2,n-2}^{(1,1)} + a_{2,n-2}^{(n-2,n-1)}. \tag{7.238}$$

Setting $(s,t,q) = (1,2,3)$ and $j \in \{n-1, n\}$ in (7.236) respectively,

$$a_{3,n-1}^{(2,3)} = a_{3,n-1}^{(1,1)} + a_{3,n-1}^{(1,2)}, \tag{7.239}$$

$$a_{3n}^{(2,3)} = a_{3n}^{(1,1)} + a_{3n}^{(1,2)}. \tag{7.240}$$

Let $1 \leqslant s < t < q \leqslant n$ be integers. For each pair of integers $1 \leqslant u < v \leqslant n$ such that $u \neq t, q$ and $v \neq s, t$, we note that $(I_n + E_{st} + E_{tq} + E_{uv})^2 = I_n + E_{sq}$. Since $[\psi(I_n + E_{st} + E_{tq} + E_{uv}), I_n] = 0$, by $[\psi(I_n + E_{st} + E_{tq} + E_{uv}), E_{sq}] = 0$, for every $1 \leqslant u < v \leqslant n$ such that $u \neq t, q$ and $v \neq s, t$,

$$a_{qq}^{(u,v)} = a_{ss}^{(u,v)} + a_{ss}^{(1,1)} + a_{qq}^{(1,1)} + a_{ss}^{(s,t)} + a_{qq}^{(s,t)} + a_{ss}^{(t,q)} + a_{qq}^{(t,q)}, \tag{7.241}$$

$$a_{is}^{(u,v)} = a_{is}^{(1,1)} + a_{is}^{(s,t)} + a_{is}^{(t,q)} \quad \text{for } i = 1, \ldots, s-1, \tag{7.242}$$

$$a_{qj}^{(u,v)} = a_{qj}^{(1,1)} + a_{qj}^{(s,t)} + a_{qj}^{(t,q)} \quad \text{for } j = q+1, \ldots, n. \tag{7.243}$$

For every $1 \leqslant s < t < q \leqslant n$, taking (7.234) into (7.241), (7.235) into (7.242), and (7.236) into (7.243) respectively, for $1 \leqslant u < v \leqslant n$, $u \neq t,\ q$ and $v \neq s,\ t$,

$$a_{qq}^{(u,v)} = a_{ss}^{(u,v)}, \tag{7.244}$$

$$a_{is}^{(u,v)} = 0 \quad \text{for} \;\; i = 1, \ldots, s-1, \tag{7.245}$$

$$a_{qj}^{(u,v)} = 0 \quad \text{for} \;\; j = q+1, \ldots, n. \tag{7.246}$$

Note that equations (7.244) and (7.246) give

(i) the 3rd row of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant n$ satisfying $u \neq 2, 3$ and $(u, v) \neq (1, 2)$.

(ii) the 4th row of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant n$ satisfying $u \neq 4$ and $(u, v) \neq (2, 3)$.

(iii) the qth row of $\psi(E_{uv})$ for every integer $5 \leqslant q \leqslant n$ of each pair of integers $1 \leqslant u < v \leqslant n$ satisfying $u \neq q$.

In particular, when $s = 1$, equations (7.244) and (7.246) respectively become

$$a_{11}^{(u,v)} = a_{qq}^{(u,v)} \quad \text{for} \;\; q = 3, \ldots, n, \tag{7.247}$$

$$a_{qj}^{(u,v)} = 0 \quad \text{for} \;\; j = q+1, \ldots, n. \tag{7.248}$$

- When $q = 3$, then $t = 2$. By (7.247) and (7.248), we obtain the 3rd row of $\psi(E_{uv})$ for every pair of integers $1 \leqslant u < v \leqslant n$ where $u \neq 2, 3$ and $(u, v) \neq (1, 2)$. This is because we get $u = 1, 4, \ldots, n-1$ and $v = 3, \ldots, n$, where $u \neq 2, 3$ and $v \neq 1, 2$, satisfying $1 \leqslant u < v \leqslant n$.
- When $q = 4$, then $t \in \{2, 3\}$ since $2 \leqslant t \leqslant q-1$. By (7.247) and (7.248), we obtain the 4th row of $\psi(E_{uv})$ where $u \neq 4$ and $(u, v) \neq (2, 3)$. This is because when $(t, q) = (2, 4)$, we get $u = 1, 3, 5, 6, \ldots, n-1$ and $v = 3, 4, \ldots, n$, where $u \neq 2, 4$ and $v \neq 1, 2$, satisfying $1 \leqslant u < v \leqslant n$ and; when $(t, q) = (3, 4)$, we get $u = 1, 2$ and $v = 2, 4, 5, 6, \ldots, n$, where $u \neq 3, 4$ and $v \neq 1, 3$, satisfying $1 \leqslant u < v \leqslant n$.
- Continue in this way, for every $5 \leqslant q \leqslant n$ and $t \in \{2, 3, 4\}$, by (7.247) and (7.248), we obtain the qth row of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant n$ where $u \neq q$. This is because when $t = 2$, we get $u = 1, 3, 4, \ldots, n-1$ and $v = 3, 4, \ldots, n$, where $u \neq 2, q$ and $v \neq 1, 2$, satisfying $1 \leqslant u < v \leqslant n$; when $t = 3$, we get $u = 1, 2$ and $v = 2, 4, 5, 6, \ldots, n$, where $u \neq 3, q$ and $v \neq 1, 3$,

satisfying $1 \leqslant u < v \leqslant n$ and; when $t = 4$, we get $u = 2, v = 3$.

Note that equations (7.244) and (7.245) give

(i) the sth column of $\psi(E_{uv})$ for every $2 \leqslant s \leqslant n-3$ of each pair of integers $1 \leqslant u < v \leqslant n$ satisfying $v \neq s$ and $(u, v) \neq (s+1, s+2)$ if $n = s+3$.

(ii) the $(n-2)$th column of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant n$ satisfying $v \neq n-2, n-1$ and $(u, v) \neq (n-1, n)$.

For every $2 \leqslant s \leqslant n-2$, equations (7.244) and (7.245) respectively become

$$a_{ss}^{(u,v)} = a_{qq}^{(u,v)} \quad \text{for} \;\; q = s+2, \ldots, n, \tag{7.249}$$

$$a_{is}^{(u,v)} = 0 \quad \text{for} \;\; i = 1, \ldots, s-1. \tag{7.250}$$

- For every $2 \leqslant s \leqslant n-4$, with $t \in \{s+1, s+2, s+3\}$ and $q \in \{s+2, s+3, s+4\}$, by (7.249) and (7.250), we obtain the sth column of $\psi(E_{uv})$, for every $s = 2, \ldots, n-4$ of each pair of integers $1 \leqslant u < v \leqslant n$, where $v \neq s$. This is because when $(t, q) = (s+1, s+2)$, we get $u = 1, 2, \ldots, n-1$ and $v = 2, 3, \ldots, n$, where $u \neq s+1, s+2$ and $v \neq s, s+1$, satisfying $1 \leqslant u < v \leqslant n$; when $(t, q) = (s+1, s+3)$, we get $u = s+2$ and $v = s+3, \ldots, n$, where $u \neq s+1, s+3$ and $v \neq s, s+1$, satisfying $1 \leqslant u < v \leqslant n$; when $(t, q) = (s+2, s+3)$, we get $u = 1, \ldots, s+1$ and $v = s+1, \ldots, n$, where $u \neq s+2, s+3$ and $v \neq s, s+2$, satisfying $1 \leqslant u < v \leqslant n$; when $(t, q) = (s+3, s+4)$ if $n \geqslant s+4$, we get $(u, v) = (s+1, s+2)$, where $u \neq s+3, s+4$ and $v \neq s, s+3$, satisfying $1 \leqslant u < v \leqslant n$.
- Continue in this way, when $s = n-3$ with $t \in \{n-2, n-1\}$ and $q \in \{n-1, n\}$, by (7.249) and (7.250), we obtain the $(n-3)$th column of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant n$ where $v \neq n-3$ and $(u, v) \neq (n-2, n-1)$. This is because when $(t, q) = (n-2, n-1)$, we get $u = 1, \ldots, n-3$ and $v = 2, \ldots, n-4, n-1, n$, where $u \neq n-2, n-1$ and $v \neq n-3, n-2$, satisfying $1 \leqslant u < v \leqslant n$; when $(t, q) = (n-2, n)$, we get $(u, v) = (n-1, n)$, where $u \neq n-2, n$ and $v \neq n-3, n-2$, satisfying $1 \leqslant u < v \leqslant n$; when $(t, q) = (n-1, n)$, we get $u = 1, \ldots, n-2$ and $v = n-2, n$, where $u \neq n-1, n$ and $v \neq n-3, n-1$, satisfying $1 \leqslant u < v \leqslant n$.

- Finally, when $s = n-2$, by (7.249) and (7.250), we obtain the $(n-2)$th column of $\psi(E_{uv})$ for each pair of integers $1 \leqslant u < v \leqslant n$ where $v \neq n-2, n-1$, and $(u,v) \neq (n-1,n)$, because when $(t,q) = (n-1,n)$, we get $u = 1,\ldots,n-2$ and $v = 2,\ldots,n-3,n$, where $u \neq n-1,n$ and $v \neq n-2,n-1$, satisfying $1 \leqslant u < v \leqslant n$.

It follows from (7.247) and (7.249) that for every $1 \leqslant p < q \leqslant n$,

$$a_{11}^{(1,2)} = a_{ii}^{(1,2)} \quad \text{for } i = 3,\ldots,n, \tag{7.251}$$

$$a_{nn}^{(n-1,n)} = a_{ii}^{(n-1,n)} \quad \text{for } i = 1,\ldots,n-2, \tag{7.252}$$

$$a_{11}^{(2,3)} = a_{22}^{(2,3)} = a_{ii}^{(2,3)} \quad \text{for } i = 4,\ldots,n, \tag{7.253}$$

$$a_{nn}^{(n-2,n-1)} = a_{n-1,n-1}^{(n-2,n-1)} = a_{ii}^{(n-2,n-1)} \quad \text{for } i = 1,\ldots,n-3, \tag{7.254}$$

$$a_{11}^{(p,q)} = a_{ii}^{(p,q)} \quad \text{for } i = 2,\ldots,n, \tag{7.255}$$

for every $(p,q) \notin \{(1,2),(2,3),(n-1,n),(n-2,n-1)\}$.

By $[\psi(I_n + E_{23} + E_{35}), E_{25}] = 0$, we have,

$$a_{55}^{(3,5)} = a_{22}^{(3,5)} + a_{22}^{(1,1)} + a_{55}^{(1,1)} + a_{22}^{(2,3)} + a_{55}^{(2,3)}. \tag{7.256}$$

Taking (7.253) and (7.255) with $(p,q) = (3,5)$ into (7.256),

$$a_{22}^{(1,1)} = a_{55}^{(1,1)}. \tag{7.257}$$

By $[\psi(I_n + E_{34} + E_{45}), E_{35}] = 0$, we get

$$a_{55}^{(4,5)} = a_{33}^{(4,5)} + a_{33}^{(1,1)} + a_{55}^{(1,1)} + a_{33}^{(3,4)} + a_{55}^{(3,4)}. \tag{7.258}$$

It follows from (7.255) with $(p,q) \in \{(3,5),(4,5)\}$ that

$$a_{33}^{(3,4)} = a_{55}^{(3,4)}, \tag{7.259}$$

$$a_{33}^{(4,5)} = a_{55}^{(4,5)}. \tag{7.260}$$

Taking (7.259) and (7.260) into (7.258),

$$a_{33}^{(1,1)} = a_{55}^{(1,1)}. \tag{7.261}$$

By $[\psi(I_n + E_{12} + E_{2q}), E_{1q}] = 0$, for every $4 \leqslant q \leqslant n$, we get

$$a_{qq}^{(2,q)} = a_{11}^{(2,q)} + a_{11}^{(1,1)} + a_{qq}^{(1,1)} + a_{11}^{(1,2)} + a_{qq}^{(1,2)}. \tag{7.262}$$

$$a_{qj}^{(2,p)} = a_{qj}^{(1,1)} + a_{qj}^{(1,2)} \quad \text{for } \; j = q+1, \ldots, n. \tag{7.263}$$

Taking (7.251) and (7.255) with $p = 2$ into (7.262), for every $4 \leqslant q \leqslant n$,

$$a_{11}^{(1,1)} = a_{qq}^{(1,1)}. \tag{7.264}$$

We conclude from (7.257), (7.261) and (7.264) that

$$a_{11}^{(1,1)} = a_{ii}^{(1,1)} \quad \text{for } i = 2, \ldots, n. \tag{7.265}$$

By $[\psi(I_n + E_{12} + E_{23}), E_{13}] = 0$, we obtain

$$a_{33}^{(2,3)} = a_{11}^{(2,3)} + a_{11}^{(1,1)} + a_{33}^{(1,1)} + a_{11}^{(1,2)} + a_{33}^{(1,2)}, \tag{7.266}$$

$$a_{3,n-2}^{(2,3)} = a_{3,n-2}^{(1,1)} + a_{3,n-2}^{(1,2)}. \tag{7.267}$$

Taking (7.251) and (7.265) into (7.266),

$$a_{11}^{(2,3)} = a_{33}^{(2,3)}. \tag{7.268}$$

It follows from (7.253) and (7.268) that

$$a_{11}^{(2,3)} = a_{ii}^{(2,3)} \quad \text{for } i = 2, \ldots, n. \tag{7.269}$$

By $[\psi(I_n + E_{n-2,n-1} + E_{n-1,n}), E_{n-2,n}] = 0$, we have

$$a_{nn}^{(n-1,n)} = a_{n-2,n-2}^{(n-1,n)} + a_{n-2,n-2}^{(1,1)} + a_{nn}^{(1,1)} + a_{n-2,n-2}^{(n-2,n-1)} + a_{nn}^{(n-2,n-1)}. \tag{7.270}$$

Taking (7.252) and (7.265) into (7.270),

$$a_{n-2,n-2}^{(n-2,n-1)} = a_{nn}^{(n-2,n-1)}. \tag{7.271}$$

It follows from (7.254) and (7.271) that

$$a_{11}^{(n-2,n-1)} = a_{ii}^{(n-2,n-1)} \quad \text{for } i = 2, \ldots, n. \tag{7.272}$$

By $[\psi(I_n+E_{n-2,n-1}+E_{n-1,n}), E_{n-2,n}] = 0$ and $[\psi(I_n+E_{n-2,n-1}+E_{n-1,n}+E_{u,v}), E_{n-2,n}] = 0$, for $(u, v) \in \{(1, 2), (2, 3)\}$, we obtain

$$a_{3,n-2}^{(1,2)} = 0, \tag{7.273}$$

$$a_{3,n-2}^{(2,3)} = 0. \tag{7.274}$$

Taking (7.273) and (7.274) into (7.267),

$$a_{3,n-2}^{(1,1)} = 0. \tag{7.275}$$

By $[\psi(I_n + E_{13} + E_{3q}), E_{1q}] = 0$ and $[\psi(I_n + E_{13} + E_{3q} + E_{uv}), E_{1q}] = 0$ for each $(u, v) \in \{(2, q), (1, 2)\}$, where $4 \leqslant q \leqslant n$, we obtain

$$a_{qj}^{(2,q)} = a_{qj}^{(1,2)} = 0 \quad \text{for } j = q + 1, \ldots, n. \tag{7.276}$$

Taking (7.276) into (7.263), for every $4 \leqslant q \leqslant n$,

$$a_{qj}^{(1,1)} = 0 \quad \text{for } j = q + 1, \ldots, n. \tag{7.277}$$

Setting $q = 4$ in (7.277), we get

$$a_{4j}^{(1,1)} = 0 \quad \text{for } j = 5, \ldots, n. \tag{7.278}$$

By $[\psi(I_n + E_{12} + E_{24}), E_{14}] = 0$ and $[\psi(I_n + E_{12} + E_{24} + E_{34}), E_{14}] = 0$, we get

$$a_{4j}^{(3,4)} = 0 \quad \text{for } j = 5, \ldots, n. \tag{7.279}$$

By $[\psi(I_n + E_{23} + E_{34}), E_{24}] = 0$, we have

$$a_{4j}^{(2,3)} = a_{4j}^{(1,1)} + a_{4j}^{(3,4)} \quad \text{for } j = 5, \ldots, n. \tag{7.280}$$

Taking (7.278) and (7.279) into (7.280),

$$a_{4j}^{(2,3)} = 0 \quad \text{for } j = 5, \ldots, n. \tag{7.281}$$

Next, by $[\psi(I_n + E_{q,q+1} + E_{q+1,q+3}), E_{q,q+3}] = 0$ for every $2 \leqslant q \leqslant n - 3$, we have

$$a_{iq}^{(q+1,q+3)} = a_{iq}^{(1,1)} + a_{iq}^{(q,q+1)} \quad \text{for } i = 1, \ldots, q - 1, \tag{7.282}$$

for every $2 \leqslant q \leqslant n-3$. By $[\psi(I_n + E_{q,q+2} + E_{q+2,q+3}), E_{q,q+3}] = 0$ and $[\psi(I_n + E_{q,q+2} + E_{q+2,q+3} + E_{uv}), E_{q,q+3}] = 0$ for each $(u, v) \in \{(q, q+1), (q+1, q+3)\}$, we get

$$a_{iq}^{(q,q+1)} = a_{iq}^{(q+1,q+3)} = 0 \quad \text{for } i = 1, \ldots, q - 1. \tag{7.283}$$

Taking (7.283) into (7.282), for every $2 \leqslant q \leqslant n - 3$,

$$a_{iq}^{(1,1)} = 0 \quad \text{for } i = 1, \ldots, q - 1. \tag{7.284}$$

On the other hand, by $[\psi(I_n + E_{n-3,n-2} + E_{n-2,n-1}), E_{n-3,n-1}] = 0$, we have

$$a_{i,n-3}^{(n-2,n-1)} = a_{i,n-3}^{(1,1)} + a_{i,n-3}^{(n-3,n-2)} \quad \text{for } i = 1, \ldots, n - 4. \tag{7.285}$$

Setting $q = n-3$ in (7.284),

$$a_{i,n-3}^{(1,1)} = 0 \quad \text{for } i = 1\ldots,n-4. \tag{7.286}$$

By $[\psi(I_n + E_{n-3,n-1} + E_{n-1,n}), E_{n-3,n}] = 0$ and
$[\psi(I_n + E_{n-3,n-1} + E_{n-1,n} + E_{n-3,n-2}), E_{n-3,n}] = 0$, we obtain

$$a_{i,n-3}^{(n-3,n-2)} = 0 \quad \text{for } i = 1,\ldots,n-4. \tag{7.287}$$

Taking (7.286) and (7.287) into (7.285),

$$a_{i,n-3}^{(n-2,n-1)} = 0 \quad \text{for } i = 1\ldots,n-4. \tag{7.288}$$

Let $1 \leqslant s < t \leqslant n$ be integers. Set

$$X_{st} = \begin{cases} a_{1,n-1}^{(s,t)}E_{1,n-1} + a_{1n}^{(s,t)}E_{1n} + a_{2,n-1}^{(s,t)}E_{2,n-1} + a_{2n}^{(s,t)}E_{2n} & \text{if } 1 \leqslant s < t \leqslant n, \\ a_{1,n-1}^{(1,1)}E_{1,n-1} + a_{1n}^{(1,1)}E_{1n} + a_{2,n-1}^{(1,1)}E_{2,n-1} + a_{2n}^{(1,1)}E_{2n} & \text{if } s = t = 1. \end{cases}$$

Up to this point, we obtain the partially completed maps as the following. In view of (7.233), (7.237)–(7.240), (7.265), (7.275), (7.277) and (7.284),

$$\begin{aligned} \psi(I_n) = a_{11}^{(1,1)}I_n + a_{1,n-2}^{(1,1)}E_{1,n-2} + a_{2,n-2}^{(1,1)}E_{2,n-2} + a_{3,n-1}^{(1,1)}E_{3,n-1} \\ + a_{3n}^{(1,1)}E_{3n} + X_{11}. \end{aligned} \tag{7.289}$$

Remark that the map $\psi(I_n)$ in (7.289) is already ultimate. In view of (7.232), (7.239), (7.240), (7.248), (7.250), (7.269) and (7.281),

$$\begin{aligned} \psi(E_{23}) = a_{11}^{(2,3)}I_n + a_{13}^{(2,3)}E_{13} + a_{23}^{(2,3)}E_{23} + (a_{3,n-1}^{(1,1)} + a_{3,n-1}^{(1,2)})E_{3,n-1} \\ + (a_{3n}^{(1,1)} + a_{3n}^{(1,2)})E_{3n} + X_{23}. \end{aligned} \tag{7.290}$$

In view of (7.232), (7.248), (7.250) and (7.255) with $(p,q) = (2, n-1)$,

$$\begin{aligned}\psi(E_{2,n-1}) = a_{11}^{(2,n-1)} I_n + a_{1,n-2}^{(2,n-1)} E_{1,n-2} + a_{2,n-2}^{(2,n-1)} E_{2,n-2} + a_{3,n-2}^{(2,n-1)} E_{3,n-2} \\ + a_{3,n-1}^{(2,n-1)} E_{3,n-1} + a_{3n}^{(2,n-1)} E_{3n} + X_{2,n-1}.\end{aligned} \tag{7.291}$$

In view of (7.232), (7.237), (7.238), (7.248), (7.250), (7.272) and (7.288),

$$\begin{aligned}\psi(E_{n-2,n-1}) = a_{11}^{(n-2,n-1)} I_n + a_{n-2,n-1}^{(n-2,n-1)} E_{n-2,n-1} + a_{n-2,n}^{(n-2,n-1)} E_{n-2,n} \\ + a_{1,n-2}^{(n-2,n-1)} E_{1,n-2} + a_{2,n-2}^{(n-2,n-1)} E_{2,n-2} + X_{n-2,n-1}.\end{aligned} \tag{7.292}$$

In view of (7.232), (7.248), (7.250) and (7.255) with $p = 2$ and $4 \leqslant q \leqslant n-2$, for every $4 \leqslant t \leqslant n-2$,

$$\begin{aligned}\psi(E_{2t}) = a_{11}^{(2,t)} I_n + a_{1t}^{(2,t)} E_{1t} + a_{2t}^{(2,t)} E_{2t} + a_{3t}^{(2,t)} E_{3t} + a_{3,n-1}^{(2,t)} E_{3,n-1} \\ + a_{3n}^{(2,t)} E_{3n} + X_{2t}.\end{aligned} \tag{7.293}$$

In view of (7.232), (7.248), (7.250) and (7.255) with $3 \leqslant p \leqslant n-3$ and $q = n-1$, for every $3 \leqslant s \leqslant n-3$,

$$\begin{aligned}\psi(E_{s,n-1}) = a_{11}^{(s,n-1)} I_n + a_{1,n-2}^{(s,n-1)} E_{1,n-2} + a_{2,n-2}^{(s,n-1)} E_{2,n-2} + a_{s,n-2}^{(s,n-1)} E_{s,n-2} \\ + a_{s,n-1}^{(s,n-1)} E_{s,n-1} + a_{sn}^{(s,n-1)} E_{sn} + X_{s,n-1}.\end{aligned} \tag{7.294}$$

In view of (7.232), (7.248), (7.250) and (7.255) with $3 \leqslant p < q \leqslant n-2$, for every $3 \leqslant p < q \leqslant n-2$,

$$\begin{aligned}\psi(E_{pq}) = a_{11}^{(p,q)} I_n + a_{1q}^{(p,q)} E_{1q} + a_{2q}^{(p,q)} E_{2q} + a_{pq}^{(p,q)} E_{pq} + a_{p,n-1}^{(p,q)} E_{p,n-1} \\ + a_{pn}^{(p,q)} E_{pn} + X_{pq}.\end{aligned} \tag{7.295}$$

In view of (7.232), (7.239), (7.240), (7.248), (7.250) and (7.251),

$$\begin{aligned}\psi(E_{12}) = a_{11}^{(1,2)} \left(E_{11} + \sum_{i=3}^{n} E_{ii} \right) + a_{22}^{(1,2)} E_{22} + a_{12}^{(1,2)} E_{12} + a_{3,n-1}^{(1,2)} E_{3,n-1} \\ + a_{3n}^{(1,2)} E_{3n} + X_{12}.\end{aligned} \tag{7.296}$$

In view of (7.232), (7.248), (7.250) and (7.255) with $(p, q) = (1, n-1)$,

$$\psi(E_{1,n-1}) = a_{11}^{(1,n-1)} I_n + a_{1,n-2}^{(1,n-1)} E_{1,n-2} + a_{2,n-2}^{(1,n-1)} E_{2,n-2} + X_{1,n-1}. \tag{7.297}$$

In view of (7.232), (7.248), (7.250) and (7.255) with $(p, q) = (2, n)$,

$$\psi(E_{2n}) = a_{11}^{(2,n)} I_n + a_{3,n-1}^{(2,n)} E_{3,n-1} + a_{3n}^{(2,n)} E_{3n} + X_{2n}. \tag{7.298}$$

In view of (7.232), (7.237), (7.238), (7.248), (7.250) and (7.252),

$$\begin{aligned} \psi(E_{n-1,n}) = a_{11}^{(n-1,n)} \left(E_{nn} + \sum_{i=1}^{n-2} E_{ii} \right) + a_{n-1,n-1}^{(n-1,n)} E_{n-1,n-1} \\ + a_{n-1,n}^{(n-1,n)} E_{n-1,n} + (a_{1,n-2}^{(1,1)} + a_{1,n-2}^{(n-2,n-1)}) E_{1,n-2} \\ + (a_{2,n-2}^{(1,1)} + a_{2,n-2}^{(n-2,n-1)}) E_{2,n-2} + X_{n-1,n}. \end{aligned} \tag{7.299}$$

In view of (7.232), (7.248), (7.250) and (7.255) with $3 \leqslant p \leqslant n-2$ and $q = n$, for every $3 \leqslant s \leqslant n-2$,

$$\psi(E_{sn}) = a_{11}^{(s,n)} I_n + a_{s,n-1}^{(s,n)} E_{s,n-1} + a_{sn}^{(s,n)} E_{sn} + X_{sn}. \tag{7.300}$$

In view of (7.232), (7.248), (7.250) and (7.255) with $p = 1$ and $3 \leqslant q \leqslant n-2$, for every $3 \leqslant t \leqslant n-2$,

$$\psi(E_{1t}) = a_{11}^{(1,t)} I_n + a_{1t}^{(1,t)} E_{1t} + a_{2t}^{(1,t)} E_{2t} + X_{1t}. \tag{7.301}$$

Finally, in view of (7.232), (7.248), (7.250) and (7.255) with $(p, q) = (1, n)$, we get

$$\psi(E_{1n}) = a_{11}^{(1,n)} I_n + X_{1n}. \tag{7.302}$$

Remark that the map $\psi(E_{1n})$ in (7.302) is already ultimate.

We now consider $A^2 = I_n + E_{pq} + E_{st}$ for integers $1 \leqslant p < q \leqslant n$ and $1 \leqslant s < t \leqslant n$ and $(p, q) \neq (s, t)$. By $[\psi(I_n + E_{12} + E_{23} + E_{2t}), E_{13} + E_{1t}] = 0$, for every $4 \leqslant t \leqslant n$,

$$\left(\sum_{1 \leqslant i \leqslant j \leqslant n} (a_{ij}^{(1,1)} + a_{ij}^{(1,2)} + a_{ij}^{(2,3)} + a_{ij}^{(2,t)}) E_{ij} \right) (E_{13} + E_{1t})$$

$$- (E_{13} + E_{1t})\left(\sum_{1\leqslant i\leqslant j\leqslant n}(a_{ij}^{(1,1)} + a_{ij}^{(1,2)} + a_{ij}^{(2,3)} + a_{ij}^{(2,t)})E_{ij}\right) = 0.$$

Then

$$(a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(2,t)})(E_{13} + E_{1t}) - \sum_{3\leqslant j\leqslant n}(a_{3j}^{(1,1)} + a_{3j}^{(1,2)} + a_{3j}^{(2,3)} + a_{3j}^{(2,t)})E_{1j}$$
$$- \sum_{t\leqslant j\leqslant n}(a_{tj}^{(1,1)} + a_{tj}^{(1,2)} + a_{tj}^{(2,3)} + a_{tj}^{(2,t)})E_{1j} = 0.$$

Thus

$$\begin{aligned}
&(a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(2,t)} + a_{33}^{(1,1)} + a_{33}^{(1,2)} + a_{33}^{(2,3)} + a_{33}^{(2,t)})E_{13} \\
&\quad + (a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(2,t)} + a_{tt}^{(1,1)} + a_{tt}^{(1,2)} + a_{tt}^{(2,3)} + a_{tt}^{(2,t)} \\
&\quad + a_{3t}^{(1,1)} + a_{3t}^{(1,2)} + a_{3t}^{(2,3)} + a_{3t}^{(2,t)})E_{1t} \\
&\quad - \sum_{\substack{4\leqslant j\leqslant t-1\\ t+1\leqslant j\leqslant n}}(a_{3j}^{(1,1)} + a_{3j}^{(1,2)} + a_{3j}^{(2,3)} + a_{3j}^{(2,t)})E_{1j} \\
&\quad - \sum_{t+1\leqslant j\leqslant n}(a_{tj}^{(1,1)} + a_{tj}^{(1,2)} + a_{tj}^{(2,3)} + a_{tj}^{(2,t)})E_{1j} = 0.
\end{aligned} \tag{7.303}$$

Hence for every $4 \leqslant t \leqslant n$, we have

$$\begin{aligned}
&a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,3)} + a_{11}^{(2,t)} + a_{tt}^{(1,1)} + a_{tt}^{(1,2)} + a_{tt}^{(2,3)} + a_{tt}^{(2,t)} \\
&\qquad = a_{3t}^{(1,1)} + a_{3t}^{(1,2)} + a_{3t}^{(2,3)} + a_{3t}^{(2,t)},
\end{aligned} \tag{7.304}$$

$$a_{3j}^{(1,1)} + a_{3j}^{(1,2)} + a_{3j}^{(2,3)} + a_{3j}^{(2,t)} = 0 \quad \text{for } j = 4, \ldots, t-1, \tag{7.305}$$

$$a_{3j}^{(1,1)} + a_{3j}^{(1,2)} + a_{3j}^{(2,3)} + a_{3j}^{(2,t)} = a_{tj}^{(1,1)} + a_{tj}^{(1,2)} + a_{tj}^{(2,3)} + a_{tj}^{(2,t)} \quad \text{for } j = t+1, \ldots, n. \tag{7.306}$$

Taking (7.251), (7.255) with $(p, q) = (2, t)$, (7.265) and (7.269) into (7.304), for every $4 \leqslant t \leqslant n$,

$$a_{3t}^{(2,t)} = a_{3t}^{(1,1)} + a_{3t}^{(1,2)} + a_{3t}^{(2,3)}. \tag{7.307}$$

By $[\psi(I_n + E_{12} + E_{2s}), E_{1s}] = 0$, for every $3 \leqslant s < t \leqslant n$, we have

$$a_{sj}^{(1,1)} + a_{sj}^{(1,2)} + a_{sj}^{(2,s)} = 0 \quad \text{for } j = s+1, \ldots, t-1, t+1, \ldots, n, \tag{7.308}$$

$$a_{st}^{(1,1)} + a_{st}^{(1,2)} + a_{st}^{(2,s)} = 0. \tag{7.309}$$

Taking (7.308) with $s = 3$ into (7.305), for every $4 \leqslant t \leqslant n$,

$$a_{3j}^{(2,t)} = 0 \quad \text{for} \quad j = 4, \ldots, t-1. \tag{7.310}$$

Taking (7.309) with $s = 3$ into (7.307), for every $4 \leqslant t \leqslant n$,

$$a_{3t}^{(2,t)} = 0. \tag{7.311}$$

By $[\psi(I_n + E_{12} + E_{2t}), E_{1t}] = 0$ for every $4 \leqslant t \leqslant n$, we have

$$a_{tj}^{(1,1)} + a_{tj}^{(1,2)} + a_{tj}^{(2,t)} = 0 \quad \text{for} \quad j = t+1, \ldots, n, \tag{7.312}$$

for every $4 \leqslant t \leqslant n$. It follows from (7.290) that for every $4 \leqslant t \leqslant n$,

$$a_{tj}^{(2,3)} = 0 \quad \text{for} \quad j = t+1, \ldots, n. \tag{7.313}$$

Taking (7.308) with $s = 3$, (7.312) and (7.313) into (7.306), for every $4 \leqslant t \leqslant n$,

$$a_{3j}^{(2,t)} = 0 \quad \text{for} \quad j = t+1, \ldots, n. \tag{7.314}$$

We conclude from (7.310) and (7.314) that for every $4 \leqslant t \leqslant n$,

$$a_{3j}^{(2,t)} = 0 \;\; \text{for}\; j = 4, \ldots, t-1, t+1, \ldots, n. \tag{7.315}$$

Secondly, by $[\psi(I_n + E_{12} + E_{2s} + E_{st}), E_{1s} + E_{2t}] = 0$ for every $3 \leqslant s < t \leqslant n$, we have

$$a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,s)} + a_{11}^{(s,t)} = a_{ss}^{(1,1)} + a_{ss}^{(1,2)} + a_{ss}^{(2,s)} + a_{ss}^{(s,t)}, \tag{7.316}$$

$$a_{12}^{(1,1)} + a_{12}^{(1,2)} + a_{12}^{(2,s)} + a_{12}^{(s,t)} = a_{st}^{(1,1)} + a_{st}^{(1,2)} + a_{st}^{(2,s)} + a_{st}^{(s,t)}, \tag{7.317}$$

$$a_{22}^{(1,1)} + a_{22}^{(1,2)} + a_{22}^{(2,s)} + a_{22}^{(s,t)} = a_{tt}^{(1,1)} + a_{tt}^{(1,2)} + a_{tt}^{(2,s)} + a_{tt}^{(s,t)}, \tag{7.318}$$

$$a_{sj}^{(1,1)} + a_{sj}^{(1,2)} + a_{sj}^{(2,s)} + a_{sj}^{(s,t)} = 0 \quad \text{for } j = s+1, \ldots, t-1, t+1, \ldots, n, \tag{7.319}$$

for every $3 \leqslant s < t \leqslant n$. Taking (7.308) into (7.319), for every $3 \leqslant s < t \leqslant n$,

$$a_{sj}^{(s,t)} = 0 \quad \text{for } j = s+1, \ldots, t-1, t+1, \ldots, n. \tag{7.320}$$

By $[\psi(I_n + E_{2s} + E_{st}), E_{2t}] = 0$ for every $3 \leqslant s < t \leqslant n$, we have

$$a_{12}^{(1,1)} + a_{12}^{(2,s)} + a_{12}^{(s,t)} = 0 \tag{7.321}$$

for every $3 \leqslant s < t \leqslant n$. Taking (7.309) and (7.321) into (7.317), for every $3 \leqslant s < t \leqslant n$,

$$a_{12}^{(1,2)} = a_{st}^{(s,t)}. \tag{7.322}$$

Setting $(s,t) = (3,4)$ in (7.318),

$$a_{22}^{(1,1)} + a_{22}^{(1,2)} + a_{22}^{(2,3)} + a_{22}^{(3,4)} = a_{44}^{(1,1)} + a_{44}^{(1,2)} + a_{44}^{(2,3)} + a_{44}^{(3,4)}. \tag{7.323}$$

Taking (7.251), (7.255) with $(p,q) = (3,4)$, (7.265) and (7.269) into (7.323),

$$a_{11}^{(1,2)} = a_{22}^{(1,2)}. \tag{7.324}$$

We conclude from (7.251) and (7.324) that

$$a_{11}^{(1,2)} = a_{ii}^{(1,2)} \quad \text{for } i = 2, \ldots, n. \tag{7.325}$$

Setting $(s,t) = (n-1,n)$ in (7.316),

$$a_{11}^{(1,1)} + a_{11}^{(1,2)} + a_{11}^{(2,n-1)} + a_{11}^{(n-1,n)} = a_{n-1,n-1}^{(1,1)} + a_{n-1,n-1}^{(1,2)} + a_{n-1,n-1}^{(2,n-1)} + a_{n-1,n-1}^{(n-1,n)}. \tag{7.326}$$

Taking (7.255) with $(p,q) = (2,n-1)$, (7.265) and (7.325) into (7.326),

$$a_{11}^{(n-1,n)} = a_{n-1,n-1}^{(n-1,n)}. \tag{7.327}$$

We conclude from (7.252) and (7.327) that

$$a_{11}^{(n-1,n)} = a_{ii}^{(n-1,n)} \quad \text{for } i = 2, \ldots, n. \tag{7.328}$$

Thirdly, by $[\psi(I_n + E_{st} + E_{t,t+1} + E_{t+1,t+2}), E_{s,t+1} + E_{t,t+2}] = 0$ for every $1 \leqslant s < t \leqslant n-2$, we have

$$\Bigg(\sum_{1\leqslant i\leqslant j\leqslant n}(a_{ij}^{(1,1)} + a_{ij}^{(s,t)} + a_{ij}^{(t,t+1)} + a_{ij}^{(t+1,t+2)})E_{ij}\Bigg)(E_{s,t+1} + E_{t,t+2})$$
$$-(E_{s,t+1} + E_{t,t+2})\Bigg(\sum_{1\leqslant i\leqslant j\leqslant n}(a_{ij}^{(1,1)} + a_{ij}^{(s,t)} + a_{ij}^{(t,t+1)} + a_{ij}^{(t+1,t+2)})E_{ij}\Bigg) = 0.$$

Then

$$\sum_{1\leqslant i\leqslant s}(a_{is}^{(1,1)} + a_{is}^{(s,t)} + a_{is}^{(t,t+1)} + a_{is}^{(t+1,t+2)})E_{i,t+1}$$
$$+ \sum_{1\leqslant i\leqslant t}(a_{it}^{(1,1)} + a_{it}^{(s,t)} + a_{it}^{(t,t+1)} + a_{it}^{(t+1,t+2)})E_{i,t+2}$$
$$- \sum_{t+1\leqslant j\leqslant n}(a_{t+1,j}^{(1,1)} + a_{t+1,j}^{(s,t)} + a_{t+1,j}^{(t,t+1)} + a_{t+1,j}^{(t+1,t+2)})E_{sj}$$
$$- \sum_{t+2\leqslant j\leqslant n}(a_{t+2,j}^{(1,1)} + a_{t+2,j}^{(s,t)} + a_{t+2,j}^{(t,t+1)} + a_{t+2,j}^{(t+1,t+2)})E_{tj} = 0.$$

Since $1 \leqslant s < t \leqslant n-2$,

$$
\begin{aligned}
(a_{ss}^{(1,1)} &+ a_{ss}^{(s,t)} + a_{ss}^{(t,t+1)} + a_{ss}^{(t+1,t+2)} \\
&\quad + a_{t+1,t+1}^{(1,1)} + a_{t+1,t+1}^{(s,t)} + a_{t+1,t+1}^{(t,t+1)} + a_{t+1,t+1}^{(t+1,t+2)})E_{s,t+1} \\
&+ \sum_{1 \leqslant i \leqslant s-1} (a_{is}^{(1,1)} + a_{is}^{(s,t)} + a_{is}^{(t,t+1)} + a_{is}^{(t+1,t+2)})E_{i,t+1} \\
&+ (a_{st}^{(1,1)} + a_{st}^{(s,t)} + a_{st}^{(t,t+1)} + a_{st}^{(t+1,t+2)} \\
&\quad + a_{t+1,t+2}^{(1,1)} + a_{t+1,t+2}^{(s,t)} + a_{t+1,t+2}^{(t,t+1)} + a_{t+1,t+2}^{(t+1,t+2)})E_{s,t+2} \\
&+ (a_{tt}^{(1,1)} + a_{tt}^{(s,t)} + a_{tt}^{(t,t+1)} + a_{tt}^{(t+1,t+2)} \qquad (7.329) \\
&\quad + a_{t+2,t+2}^{(1,1)} + a_{t+2,t+2}^{(s,t)} + a_{t+2,t+2}^{(t,t+1)} + a_{t+2,t+2}^{(t+1,t+2)})E_{t,t+2} \\
&+ \sum_{\substack{1 \leqslant i \leqslant s-1 \\ s+1 \leqslant i \leqslant t-1}} (a_{it}^{(1,1)} + a_{it}^{(s,t)} + a_{it}^{(t,t+1)} + a_{it}^{(t+1,t+2)})E_{i,t+2} \\
&- \sum_{t+3 \leqslant j \leqslant n} (a_{t+1,j}^{(1,1)} + a_{t+1,j}^{(s,t)} + a_{t+1,j}^{(t,t+1)} + a_{t+1,j}^{(t+1,t+2)})E_{sj} \\
&- \sum_{t+3 \leqslant j \leqslant n} (a_{t+2,j}^{(1,1)} + a_{t+2,j}^{(s,t)} + a_{t+2,j}^{(t,t+1)} + a_{t+2,j}^{(t+1,t+2)})E_{tj} = 0.
\end{aligned}
$$

Hence for every $1 \leqslant s < t \leqslant n-2$, we obtain

$$a_{st}^{(1,1)} + a_{st}^{(s,t)} + a_{st}^{(t,t+1)} + a_{st}^{(t+1,t+2)} = a_{t+1,t+2}^{(1,1)} + a_{t+1,t+2}^{(s,t)} + a_{t+1,t+2}^{(t,t+1)} + a_{t+1,t+2}^{(t+1,t+2)}, \quad (7.330)$$

$$a_{it}^{(1,1)} + a_{it}^{(s,t)} + a_{it}^{(t,t+1)} + a_{it}^{(t+1,t+2)} = 0 \quad \text{for } i = 1, \ldots, s-1, s+1, \ldots, t-1. \quad (7.331)$$

By $[\psi(I_n + E_{t,t+1} + E_{t+1,t+2}), E_{t,t+2}] = 0$ for every $1 \leqslant s < t \leqslant n-2$, we have

$$a_{st}^{(1,1)} + a_{st}^{(t,t+1)} + a_{st}^{(t+1,t+2)} = 0, \quad (7.332)$$

$$a_{it}^{(1,1)} + a_{it}^{(t,t+1)} + a_{it}^{(t+1,t+2)} = 0 \quad \text{for } i = 1, \ldots, s-1, s+1, \ldots, t-1, \quad (7.333)$$

for every $1 \leqslant s < t \leqslant n-2$. Taking (7.333) into (7.331), for every $1 \leqslant s < t \leqslant n-2$,

$$a_{it}^{(s,t)} = 0 \quad \text{for } i = 1, \ldots, s-1, s+1, \ldots, t-1. \quad (7.334)$$

By $[\psi(I_n + E_{st} + E_{t,t+1}), E_{s,t+1}] = 0$ for every $1 \leqslant s < t \leqslant n-2$, we get

$$a^{(1,1)}_{t+1,t+2} + a^{(s,t)}_{t+1,t+2} + a^{(t,t+1)}_{t+1,t+2} = 0, \tag{7.335}$$

for every $1 \leqslant s < t \leqslant n-2$. Taking (7.335) and (7.332) into (7.330), for every $1 \leqslant s < t \leqslant n-2$,

$$a^{(s,t)}_{st} = a^{(t+1,t+2)}_{t+1,t+2}. \tag{7.336}$$

It follows from (7.322) that for every $3 \leqslant t \leqslant n-2$,

$$a^{(t+1,t+2)}_{t+1,t+2} = a^{(1,2)}_{1,2}. \tag{7.337}$$

We conclude from (7.336) and (7.337) that for every $1 \leqslant s < t \leqslant n-2$,

$$a^{(s,t)}_{st} = a^{(1,2)}_{1,2}. \tag{7.338}$$

We conclude from (7.322) and (7.338) that for every $1 \leqslant s < t \leqslant n-2$ and every $3 \leqslant s < t \leqslant n$,

$$a^{(s,t)}_{st} = a^{(1,2)}_{1,2}. \tag{7.339}$$

Finally, by $[\psi(I_n + E_{s,n-1} + E_{n-2,n-1} + E_{n-1,n}), E_{sn} + E_{n-2,n}] = 0$ for every $1 \leqslant s \leqslant n-3$, we have

$$\left(\sum_{1\leqslant i\leqslant j\leqslant n} (a^{(1,1)}_{ij} + a^{(s,n-1)}_{ij} + a^{(n-2,n-1)}_{ij} + a^{(n-1,n)}_{ij})E_{ij}\right)(E_{sn} + E_{n-2,n})$$
$$-(E_{sn} + E_{n-2,n}) \left(\sum_{1\leqslant i\leqslant j\leqslant n} (a^{(1,1)}_{ij} + a^{(s,n-1)}_{ij} + a^{(n-2,n-1)}_{ij} + a^{(n-1,n)}_{ij})E_{ij}\right) = 0.$$

Then

$$\sum_{1\leqslant i\leqslant s} (a^{(1,1)}_{is} + a^{(s,n-1)}_{is} + a^{(n-2,n-1)}_{is} + a^{(n-1,n)}_{is})E_{in}$$
$$+ \sum_{1\leqslant i\leqslant n-2} (a^{(1,1)}_{i,n-2} + a^{(s,n-1)}_{i,n-2} + a^{(n-2,n-1)}_{i,n-2} + a^{(n-1,n)}_{i,n-2})E_{in}$$
$$- (a^{(1,1)}_{nn} + a^{(s,n-1)}_{nn} + a^{(n-2,n-1)}_{nn} + a^{(n-1,n)}_{nn})E_{sn}$$
$$- (a^{(1,1)}_{nn} + a^{(s,n-1)}_{nn} + a^{(n-2,n-1)}_{nn} + a^{(n-1,n)}_{nn})E_{n-2,n} = 0.$$

Thus

$$
\begin{aligned}
(a_{ss}^{(1,1)} &+ a_{ss}^{(s,n-1)} + a_{ss}^{(n-2,n-1)} + a_{ss}^{(n-1,n)} \\
&\quad + a_{nn}^{(1,1)} + a_{nn}^{(s,n-1)} + a_{nn}^{(n-2,n-1)} + a_{nn}^{(n-1,n)})E_{sn} \\
&+ (a_{n-2,n-2}^{(1,1)} + a_{n-2,n-2}^{(s,n-1)} + a_{n-2,n-2}^{(n-2,n-1)} + a_{n-2,n-2}^{(n-1,n)} \\
&\quad + a_{nn}^{(1,1)} + a_{nn}^{(s,n-1)} + a_{nn}^{(n-2,n-1)} + a_{nn}^{(n-1,n)})E_{n-2,n} \\
&+ \sum_{1\leqslant i\leqslant s-1} (a_{is}^{(1,1)} + a_{is}^{(s,n-1)} + a_{is}^{(n-2,n-1)} + a_{is}^{(n-1,n)})E_{in} \\
&+ \sum_{1\leqslant i\leqslant n-3} (a_{i,n-2}^{(1,1)} + a_{i,n-2}^{(s,n-1)} + a_{i,n-2}^{(n-2,n-1)} + a_{i,n-2}^{(n-1,n)})E_{in} = 0.
\end{aligned}
\tag{7.340}
$$

Hence for every $1 \leqslant i \leqslant s-1$ with $1 \leqslant s \leqslant n-3$ or, for $i = 1, \ldots, s, \ldots, n-4$,

$$
\begin{aligned}
a_{is}^{(1,1)} &+ a_{is}^{(s,n-1)} + a_{is}^{(n-2,n-1)} + a_{is}^{(n-1,n)} \\
&+ a_{i,n-2}^{(1,1)} + a_{i,n-2}^{(s,n-1)} + a_{i,n-2}^{(n-2,n-1)} + a_{i,n-2}^{(n-1,n)} = 0.
\end{aligned}
\tag{7.341}
$$

By $[\psi(I_n + E_{s,n-1} + E_{n-1,n}), E_{sn}] = 0$ for every $1 \leqslant s \leqslant n-3$, we have

$$a_{is}^{(1,1)} + a_{is}^{(s,n-1)} + a_{is}^{(n-1,n)} = 0 \quad \text{for } i = 1, \ldots, s-1, \tag{7.342}$$

for every $1 \leqslant s \leqslant n-3$. It follows from (7.292) that for $1 \leqslant s \leqslant n-3$,

$$a_{is}^{(n-2,n-1)} = 0 \quad \text{for } i = 1, \ldots, s-1. \tag{7.343}$$

By $[\psi(I_n + E_{n-2,n-1} + E_{n-1,n}), E_{n-2,n}] = 0$, we have

$$a_{i,n-2}^{(1,1)} + a_{i,n-2}^{(n-2,n-1)} + a_{i,n-2}^{(n-1,n)} = 0 \quad \text{for } i = 1, \ldots, n-3. \tag{7.344}$$

Taking (7.342), (7.343) and (7.344) into (7.341), for every $1 \leqslant s \leqslant n-3$,

$$a_{i,n-2}^{(s,n-1)} = 0 \quad \text{for } i = 1, \ldots, s, \ldots, n-4. \tag{7.345}$$

Consequently, using (7.289)–(7.302), we are ready to classify $\psi(I_n)$ and $\psi(E_{ij})$ for each pair of integers $1 \leqslant i < j \leqslant n$. Since $\psi(I_n)$ in (7.289) is already ultimate, it follows from (7.289) that

$$\psi(I_n) = a_{11}^{(1,1)} I_n + a_{1,n-2}^{(1,1)} E_{1,n-2} + a_{2,n-2}^{(1,1)} E_{2,n-2} + a_{3,n-1}^{(1,1)} E_{3,n-1} + a_{3n}^{(1,1)} E_{3n} + X_{11}. \quad (7.346)$$

It follows from (7.296) and (7.325) that

$$\psi(E_{12}) = a_{11}^{(1,2)} I_n + a_{12}^{(1,2)} E_{12} + a_{3,n-1}^{(1,2)} E_{3,n-1} + a_{3n}^{(1,2)} E_{3n} + X_{12}. \quad (7.347)$$

By virtue of (7.299), (7.328) and (7.339),

$$\begin{aligned} \psi(E_{n-1,n}) = a_{11}^{(n-1,n)} I_n + a_{12}^{(1,2)} E_{n-1,n} + (a_{1,n-2}^{(1,1)} + a_{1,n-2}^{(n-2,n-1)}) E_{1,n-2} \\ + (a_{2,n-2}^{(1,1)} + a_{2,n-2}^{(n-2,n-1)}) E_{2,n-2} + X_{n-1,n}. \end{aligned} \quad (7.348)$$

By virtue of (7.290), (7.334) and (7.339),

$$\psi(E_{23}) = a_{11}^{(2,3)} I_n + a_{12}^{(1,2)} E_{23} + (a_{3,n-1}^{(1,1)} + a_{3,n-1}^{(1,2)}) E_{3,n-1} + (a_{3n}^{(1,1)} + a_{3n}^{(1,2)}) E_{3n} + X_{23}. \quad (7.349)$$

By virtue of (7.292), (7.320) and (7.339),

$$\begin{aligned} \psi(E_{n-2,n-1}) = a_{11}^{(n-2,n-1)} I_n + a_{12}^{(1,2)} E_{n-2,n-1} + a_{1,n-2}^{(n-2,n-1)} E_{1,n-2} \\ + a_{2,n-2}^{(n-2,n-1)} E_{2,n-2} + X_{n-2,n-1}. \end{aligned} \quad (7.350)$$

It follows from (7.297) and (7.345) that

$$\begin{aligned} \psi(E_{1,n-1}) = a_{11}^{(1,n-1)} I_n + a_{12}^{(1,2)} E_{1,n-1} + (a_{12}^{(1,2)} + a_{1,n-1}^{(1,n-1)}) E_{1,n-1} \\ + a_{1n}^{(1,n-1)} E_{1n} + a_{2,n-1}^{(1,n-1)} E_{2,n-1} + a_{2n}^{(1,n-1)} E_{2n}. \end{aligned} \quad (7.351)$$

Since the map $\psi(E_{1n})$ in (7.302) is already ultimate, it follows from (7.302) that

$$\begin{aligned} \psi(E_{1n}) = a_{11}^{(1,n)} I_n + a_{12}^{(1,2)} E_{1n} + a_{1,n-1}^{(1,n)} E_{1,n-1} + (a_{12}^{(1,2)} + a_{1n}^{(1,n)}) E_{1n} \\ + a_{2,n-1}^{(1,n)} E_{2,n-1} + a_{2n}^{(1,n)} E_{2n}. \end{aligned} \quad (7.352)$$

By virtue of (7.291), (7.311), (7.315) and (7.345),

$$\begin{aligned}\psi(E_{2,n-1}) = a_{11}^{(2,n-1)} I_n + a_{12}^{(1,2)} E_{2,n-1} + a_{1,n-1}^{(2,n-1)} E_{1,n-1} + a_{1n}^{(2,n-1)} E_{1n} \\ + (a_{12}^{(1,2)} + a_{2,n-1}^{(2,n-1)}) E_{2,n-1} + a_{2n}^{(2,n-1)} E_{2n}.\end{aligned} \tag{7.353}$$

It follows from (7.298), (7.311) and (7.315) that

$$\begin{aligned}\psi(E_{2n}) = a_{11}^{(2,n)} I_n + a_{12}^{(1,2)} E_{2n} + a_{1,n-1}^{(2,n)} E_{1,n-1} + a_{1n}^{(2,n)} E_{1n} + a_{2,n-1}^{(2,n)} E_{2,n-1} \\ + (a_{12}^{(1,2)} + a_{2n}^{(2,n)}) E_{2n}.\end{aligned} \tag{7.354}$$

By virtue of (7.294), (7.320), (7.339) and (7.345), for every $3 \leqslant s \leqslant n-3$,

$$\psi(E_{s,n-1}) = a_{11}^{(s,n-1)} I_n + a_{12}^{(1,2)} E_{s,n-1} + X_{s,n-1}. \tag{7.355}$$

By virtue of (7.293), (7.311), (7.315), (7.334) and (7.339), for every $4 \leqslant t \leqslant n-2$,

$$\psi(E_{2t}) = a_{11}^{(2,t)} I_n + a_{12}^{(1,2)} E_{2t} + X_{2t}. \tag{7.356}$$

By virtue of (7.300), (7.320) and (7.339), for every $3 \leqslant s \leqslant n-2$,

$$\psi(E_{sn}) = a_{11}^{(s,n)} I_n + a_{12}^{(1,2)} E_{sn} + X_{sn}. \tag{7.357}$$

By virtue of (7.301), (7.334) and (7.339), for every $3 \leqslant t \leqslant n-2$,

$$\psi(E_{1t}) = a_{11}^{(1,t)} I_n + a_{12}^{(1,2)} E_{1t} + X_{1t}. \tag{7.358}$$

By virtue of (7.295), (7.320), (7.334) and (7.339), for every $3 \leqslant p < q \leqslant n-2$,

$$\psi(E_{pq}) = a_{11}^{(p,q)} I_n + a_{12}^{(1,2)} E_{pq} + X_{pq}. \tag{7.359}$$

In view of (7.355)–(7.359), we conclude that

$$\psi(E_{pq}) = a_{11}^{(p,q)} I_n + a_{12}^{(1,2)} E_{pq} + X_{pq} \tag{7.360}$$

for every $(p,q) \notin \theta_n \cup \{(1,2),(2,3),(n-1,n),(n-2,n-1)\}$, where $1 \leqslant p < q \leqslant n$ and $\theta_n = \{(1,n-1),(1,n),(2,n-1),(2,n)\}$.

Let $\lambda = a_{12}^{(1,2)} \in \mathbb{F}_2$. Let $\mu : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ be the additive map defined by

$$\mu(A) = a_{11}^{(1,1)} + \sum_{1 \leqslant i < j \leqslant n} a_{11}^{(i,j)} \tag{7.361}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. Let $\theta_n = \{(1,n-1),(1,n),(2,n-1),(2,n)\}$. Let $\lambda_{st}^{(1,1)} = a_{st}^{(1,1)}$ and $\lambda_{st}^{(i,j)} = a_{st}^{(i,j)}$, for each pair of integers $1 \leqslant i < j \leqslant n$ and $(s,t) \in \theta_n$. Let $\psi_\Theta : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be the additive map defined in (7.4). In view of (7.346)–(7.354), (7.360) and (7.361), together with the additivity of ψ, μ and ψ_Θ, we obtain

$$\begin{aligned}
\psi(A) &= \sum_{1 \leqslant i \leqslant j \leqslant n} \psi(E_{ij}) \\
&= \psi(I_n) + \sum_{1 \leqslant i < j \leqslant n} \psi(E_{ij}) \\
&= \lambda A + \mu(A) I_n + \psi_\Theta(A)
\end{aligned}$$

for all $A \in T_n(\mathbb{F}_2)$, where $\lambda \in \mathbb{F}_2$. This completes the proof. □

CHAPTER 8: CONCLUSIONS AND DISCUSSIONS

In this chapter, we summarise the main results in Chapters 3–7 for convenience. We also propose some potential open problems related to the study in this thesis.

8.1 Main results in Chapters 3 and 4

Theorem 8.1.1. *Let $\mathbb{F}$ be a field and let $n \geqslant 2$ be an integer. Let $1 < k \leqslant n$ be a fixed integer. Then $\psi : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is a commuting additive map on rank k matrices if and only if*

- *when $k < n$ or $|\mathbb{F}| \geqslant 3$, there exist scalars $\lambda, \alpha \in \mathbb{F}$ and an additive map $\mu : T_n(\mathbb{F}) \to \mathbb{F}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \alpha(a_{11} + a_{nn})E_{1n}$$

  for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\alpha \neq 0$ only if $k = n$ and $|\mathbb{F}| = 3$,

- *when $k = n \geqslant 4$ and $|\mathbb{F}| = 2$, there exist scalars $\lambda, \alpha, \beta_1, \beta_2 \in \mathbb{F}$ and matrices $H, K \in T_n(\mathbb{F})$ and $X_1, \ldots, X_n \in T_n(\mathbb{F})$ satisfying $X_1 + \cdots + X_n = 0$ such that*

$$\psi(A) = \lambda A + \mathrm{tr}\,(H^tA)I_n + \mathrm{tr}\,(K^tA)E_{1n} + \Psi_{\alpha,\beta_1,\beta_2}(A) + \sum_{i=1}^{n} a_{ii}X_i$$

  for all $A = (a_{ij}) \in T_n(\mathbb{F})$, where $\mathrm{tr}\,(A)$ and A^t are the trace and the transpose of A respectively, and $\Psi_{\alpha,\beta_1,\beta_2} : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is the additive map defined by

$$\Psi_{\alpha,\beta_1,\beta_2}(A) = (\alpha a_{12} + \beta_1(a_{n-1,n} + a_{nn}))E_{1,n-1} + (\alpha a_{n-1,n} + \beta_2(a_{11} + a_{12}))E_{2n}$$

  for all $A = (a_{ij}) \in T_n(\mathbb{F})$,

- *when $k = n = 3$ and $|\mathbb{F}| = 2$, there exist scalars $\lambda, \alpha, \beta, \gamma \in \mathbb{F}$ and matrices $H, K \in T_3(\mathbb{F})$ and $X_1, X_2, X_3 \in T_3(\mathbb{F})$ satisfying $X_1 + X_2 + X_3 = 0$ such that*

$$\psi(A) = \lambda A + \mathrm{tr}\,(H^tA)I_3 + \mathrm{tr}\,(K^tA)E_{13} + \Psi_{\alpha,\beta}(A) + \Phi_\gamma(A) + \sum_{i=1}^{3} a_{ii}X_i$$

  for all $A = (a_{ij}) \in T_3(\mathbb{F})$, where $\Psi_{\alpha,\beta} : T_3(\mathbb{F}) \to T_3(\mathbb{F})$ and $\Phi_\gamma : T_3(\mathbb{F}) \to T_3(\mathbb{F})$

are the additive maps defined by

$$\Psi_{\alpha,\beta}(A) = (\alpha(a_{23} + a_{33}))E_{12} + (\beta(a_{11} + a_{12}))E_{23},$$

$$\Phi_\gamma(A) = \gamma((a_{12} + a_{22})E_{22} + (a_{11} + a_{12} + a_{23} + a_{33})E_{33} + a_{13}(E_{12} + E_{23}))$$

for all $A = (a_{ij}) \in T_3(\mathbb{F})$*, and*

- *when* $k = n = 2$ *and* $|\mathbb{F}| = 2$*, there exist scalars* $\lambda_1, \lambda_2 \in \mathbb{F}$ *and matrices* $X_1, X_2 \in T_2(\mathbb{F})$ *such that*

$$\psi(A) = (a_{11} + a_{12})X_1 + (a_{22} + a_{12})X_2 + \lambda_1 a_{12} I_2 + \lambda_2 a_{12} E_{12}$$

for all $A = (a_{ij}) \in T_2(\mathbb{F})$.

Theorem 8.1.2. $\psi : M_2(\mathbb{F}_2) \to M_2(\mathbb{F}_2)$ *is a commuting additive map on invertible matrices if and only if there exist scalars* $\alpha, \beta, \lambda \in \mathbb{F}_2$ *and a matrix* $H \in M_2(\mathbb{F}_2)$ *such that*

$$\psi(A) = \lambda A + \mathrm{tr}\,(H^t A)I_2 + \Gamma_{\alpha,\beta}(A)$$

for all $A \in M_2(\mathbb{F}_2)$*. Here,* $\Gamma_{\alpha,\beta} : M_2(\mathbb{F}_2) \to M_2(\mathbb{F}_2)$ *is the additive map defined by*

$$\Gamma_{\alpha,\beta}(A) = \alpha a_{11} Q + (\alpha a_{22} + \beta(a_{12} + a_{21} + a_{22}))R$$

for all $A = (a_{ij}) \in M_2(\mathbb{F}_2)$*, where* $Q = E_{11} + E_{12} + E_{21}$ *and* $R = I_2 + Q$*.*

8.2 Main results in Chapters 5 and 6

Theorem 8.2.1. *Let* $n \geqslant 2$ *be an integer and let* $\mathbb{D}$ *be a division ring with centre* $Z(\mathbb{D})$*. Suppose that* $\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ *is a map. Then the following statements are equivalent:*

(i) ψ *is a centralizing additive map on rank one matrices.*

(ii) ψ *is a commuting additive map on rank one matrices.*

(iii) *There exists* $\lambda \in Z(\mathbb{D})$*, an additive map* $\mu : T_n(\mathbb{D}) \to Z(\mathbb{D})$*, a strictly upper triangular matrix* $\chi = (\tau_{ij}) \in T_n(\mathbb{D})$*, a set of elements* $\Lambda = \bigcup_{(s,t)\in\nabla_n}\{\lambda_{ij}^{(s,t)} \in \mathbb{D} : 1 \leqslant i < j < s \text{ or } t < i < j \leqslant n\}$ *and a set of additive maps* $\mathscr{F} = \bigcup_{1<s\leqslant t<n}\{\phi_{ij}^{(s,t)} :$

$\mathbb{D} \to \mathbb{D} : 1 \leqslant i \leqslant s-1$ and $t+1 \leqslant j \leqslant n\}$ *such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \psi_\chi(A) + \psi_{\mathscr{F}}(A) + \psi_\Lambda(A)$$

for all $A \in T_n(\mathbb{D})$*, where* $\psi_\chi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ *is the linear map defined by*

$$\psi_\chi(A) = \begin{pmatrix} x_1 & -\tau_{12}a_{12} & -\tau_{13}a_{13} & \cdots & -\tau_{1n}a_{1n} \\ 0 & x_2 & -\tau_{23}a_{23} & \cdots & -\tau_{2n}a_{2n} \\ 0 & 0 & x_3 & \cdots & -\tau_{3n}a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix}$$

for all $A = (a_{ij}) \in T_n(\mathbb{D})$*, where*

$$x_h = \begin{cases} \sum_{i=2}^{n} \tau_{1i}a_{ii} & \text{if } h = 1, \\ \sum_{i=1}^{h-1} \tau_{ih}\, a_{ii} + \sum_{i=h+1}^{n} \tau_{hi}\, a_{ii} & \text{if } 2 \leqslant h \leqslant n-1, \\ \sum_{i=1}^{n-1} \tau_{in}a_{ii} & \text{if } h = n, \end{cases}$$

$\psi_{\mathscr{F}} : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ *is the additive map defined by*

$$\psi_{\mathscr{F}}(A) = \sum_{1<s\leqslant t<n} \left(\sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a_{st})E_{ij} \right)$$

for all $A = (a_{ij}) \in T_n(\mathbb{D})$*, and* $\psi_\Lambda : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ *is the linear map defined by*

$$\psi_\Lambda(A) = \sum_{(s,t)\in\nabla_n} \Psi_{st}(A) + \Phi_{st}(A)$$

for all $A \in T_n(\mathbb{D})$*, where for each* $(s,t) \in \nabla_n$*,*

$$\Psi_{st}(A) = \begin{cases} 0 & \text{if } 1 \leqslant s \leqslant 2, \\ \left(\sum_{1\leqslant i<j<s} \lambda_{ij}^{(s,t)} E_{ij}\right)\left(\sum_{h=1}^{s-1} a_{st}E_{hh} - a_{ht}E_{hs}\right) & \text{if } 3 \leqslant s \leqslant n, \end{cases}$$

$$\Phi_{st}(A) = \begin{cases} \left(\sum_{h=t+1}^{n} a_{st}E_{hh} - a_{sh}E_{th}\right)\left(\sum_{t<i<j\leqslant n} \lambda_{ij}^{(s,t)}E_{ij}\right) & \text{if } 1 \leqslant t \leqslant n-2, \\ 0 & \text{if } n-1 \leqslant t \leqslant n \end{cases}$$

for all $A = (a_{ij}) \in T_n(\mathbb{D})$. Here, $\psi_\chi = 0$ and $\psi_\Lambda = 0$ when either $n = 2$ or $\mathbb{D}$ is noncommutative, and $\psi_{\mathscr{F}} = 0$ when $n = 2$.

Theorem 8.2.2. *Let $n \geqslant 2$ be an integer and let $\mathbb{D}$ be a noncommutative division ring with centre $Z(\mathbb{D})$. Then $\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is a commuting additive map on rank one matrices if and only if there exists an element $\lambda \in Z(\mathbb{D})$, an additive map $\mu : T_n(\mathbb{D}) \to Z(\mathbb{D})$ and a set of additive maps $\mathscr{F} = \bigcup_{1<s\leqslant t<n}\{\phi_{ij}^{(s,t)} : \mathbb{D} \to \mathbb{D} : 1 \leqslant i \leqslant s-1 \text{ and } t+1 \leqslant j \leqslant n\}$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \psi_{\mathscr{F}}(A)$$

for all $A \in T_n(\mathbb{D})$, where $\psi_{\mathscr{F}} : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is the additive map defined by

$$\psi_{\mathscr{F}}(A) = \sum_{1<s\leqslant t<n} \left(\sum_{i=1}^{s-1} \sum_{j=t+1}^{n} \phi_{ij}^{(s,t)}(a_{st})E_{ij} \right)$$

for all $A = (a_{ij}) \in T_n(\mathbb{D})$ and $\psi_{\mathscr{F}} = 0$ when $n = 2$.

8.3 Main results in Chapter 7

In view of Theorems 7.3.1–7.3.4, we obtain a complete characterisation of 2-power commuting additive maps on invertible upper triangular matrices over the Galois field of two elements.

Theorem 8.3.1. *Let $\mathbb{F}_2$ be the Galois field of two elements and let $n \geqslant 2$ be an integer. Then $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is a 2-power commuting additive map on invertible matrices $A \in T_n(\mathbb{F}_2)$ if and only if*

- *when $n \geqslant 5$, let $\theta_n = \{(1, n-1), (1, n), (2, n-1), (2, n)\}$, there exist $\lambda, \lambda_{st}^{(1,1)} \in \mathbb{F}_2$ with $(s, t) \in \theta_n$, a set of scalars $\Theta = \bigcup_{(s,t)\in\theta_n}\{\lambda_{st}^{(i,j)} \in \mathbb{F}_2 : 1 \leqslant i < j \leqslant n\}$ and an additive map $\mu : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ such that*

$$\psi(A) = \lambda A + \mu(A)I_n + \psi_\Theta(A)$$

for all $A \in T_n(\mathbb{F}_2)$. Here, $\psi_\Theta : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the additive map defined by

$$\begin{aligned}\psi_\Theta(A) = &\Bigg(\lambda_{1,n-1}^{(1,1)} a_{11} + \lambda a_{1,n-1} + \sum_{1\leqslant i<j\leqslant n} \lambda_{1,n-1}^{(i,j)} a_{ij}\Bigg) E_{1,n-1} \\ &+ \Bigg(\lambda_{1n}^{(1,1)} a_{11} + \lambda a_{1n} + \sum_{1\leqslant i<j\leqslant n} \lambda_{1n}^{(i,j)} a_{ij}\Bigg) E_{1n} \\ &+ \Bigg(\lambda_{2,n-1}^{(1,1)} a_{11} + \lambda a_{2,n-1} + \sum_{1\leqslant i<j\leqslant n} \lambda_{2,n-1}^{(i,j)} a_{ij}\Bigg) E_{2,n-1} \\ &+ \Bigg(\lambda_{2n}^{(1,1)} a_{11} + \lambda a_{2n} + \sum_{1\leqslant i<j\leqslant n} \lambda_{2n}^{(i,j)} a_{ij}\Bigg) E_{2n}\end{aligned}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$,

- *when $n = 4$, let $\theta_4 = \{(1,3),(1,4),(2,3),(2,4)\}$, there exist $\lambda, \lambda_{st}^{(1,1)} \in \mathbb{F}_2$ with $(s,t) \in \theta_4$, a set of scalars $\Theta = \bigcup_{(s,t)\in\theta_4} \{\lambda_{st}^{(i,j)} \in \mathbb{F}_2 : 1 \leqslant i < j \leqslant 4\}$ and an additive map $\mu : T_4(\mathbb{F}_2) \to \mathbb{F}_2$ such that*

  $$\psi(A) = \lambda A + \mu(A) I_4 + \psi_\gamma(A) + \psi_\Theta(A)$$

  for all $A \in T_4(\mathbb{F}_2)$, where $\psi_\gamma : T_4(\mathbb{F}_2) \to T_4(\mathbb{F}_2)$ is the additive map defined by

  $$\psi_\gamma(A) = \gamma(a_{23}a_{34}E_{22} + a_{23}E_{44} + a_{13}E_{12} + a_{24}E_{34})$$

  for all $A = (a_{ij}) \in T_4(\mathbb{F}_2)$; and $\psi_\Theta : T_4(\mathbb{F}_2) \to T_4(\mathbb{F}_2)$ is the additive map defined by

  $$\begin{aligned}\psi_\Theta(A) = &\Bigg(\lambda_{13}^{(1,1)} a_{11} + \lambda a_{13} + \sum_{1\leqslant i<j\leqslant 4} \lambda_{13}^{(i,j)} a_{ij}\Bigg) E_{13} \\ &+ \Bigg(\lambda_{14}^{(1,1)} a_{11} + \lambda a_{14} + \sum_{1\leqslant i<j\leqslant 4} \lambda_{14}^{(i,j)} a_{ij}\Bigg) E_{14} \\ &+ \Bigg(\lambda_{23}^{(1,1)} a_{11} + \lambda a_{23} + \sum_{1\leqslant i<j\leqslant 4} \lambda_{23}^{(i,j)} a_{ij}\Bigg) E_{23} \\ &+ \Bigg(\lambda_{24}^{(1,1)} a_{11} + \lambda a_{24} + \sum_{1\leqslant i<j\leqslant 4} \lambda_{24}^{(i,j)} a_{ij}\Bigg) E_{24}\end{aligned}$$

  for all $A = (a_{ij}) \in T_4(\mathbb{F}_2)$,

- *when $n = 3$, let $1 \leqslant i < j \leqslant 3$ and $1 \leqslant s < t \leqslant 3$ be integers, there exist*

$\lambda, \lambda_{ss}^{(1,1)}, \lambda_{st}^{(1,1)}, \lambda_{ss}^{(i,j)}, \lambda_{st}^{(i,j)} \in \mathbb{F}_2$ and an additive map $\mu : T_3(\mathbb{F}_2) \to \mathbb{F}_2$ such that

$$\psi(A) = \lambda A + \mu(A)I_3 + \phi(A)$$

for all $A \in T_3(\mathbb{F}_2)$, where $\phi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ is the additive map defined by

$$\begin{aligned}\phi(A) = &((\lambda_{11}^{(1,1)} + \lambda_{22}^{(1,1)})a_{11} + (\lambda_{11}^{(1,2)} + \lambda_{22}^{(1,2)})a_{12} + (\lambda_{11}^{(2,3)} + \lambda_{22}^{(2,3)})a_{23} \\ &+ (\lambda_{11}^{(1,3)} + \lambda_{22}^{(1,3)})a_{13})E_{22} \\ &+ (\lambda_{12}^{(1,1)}a_{11} + (\lambda + \lambda)a_{12} + \lambda_{12}^{(1,3)}a_{13} + \lambda_{12}^{(2,3)}a_{23})E_{12} \\ &+ (\lambda_{13}^{(1,1)}a_{11} + \lambda_{13}^{(1,2)}a_{12} + (\lambda_{13}^{(1,3)} + \lambda)a_{13} + \lambda_{13}^{(2,3)}a_{23})E_{13} \\ &+ (\lambda_{23}^{(1,1)}a_{11} + \lambda_{23}^{(1,2)}a_{12} + \lambda_{23}^{(1,3)}a_{13} + (\lambda_{23}^{(2,3)} + \lambda)a_{23})E_{23}.\end{aligned}$$

for all $A = (a_{ij}) \in T_3(\mathbb{F}_2)$, and

- *when $n = 2$, there exist $\lambda, \lambda_{11}^{(1,1)}, \lambda_{22}^{(1,1)}, \lambda_{12}^{(1,1)}, \lambda_{11}^{(1,2)}, \lambda_{22}^{(1,2)} \in \mathbb{F}_2$ and an additive map $\mu : T_2(\mathbb{F}_2) \to \mathbb{F}_2$ such that*

$$\psi(A) = \lambda A + \mu(A)I_2 + \varsigma(A)$$

for all $A \in T_2(\mathbb{F}_2)$, where $\varsigma : T_2(\mathbb{F}_2) \to T_2(\mathbb{F}_2)$ is the additive map defined by

$$\varsigma(A) = (\lambda_{11}^{(1,2)} + \lambda_{22}^{(1,2)})a_{12}E_{11} + (\lambda_{11}^{(1,1)} + \lambda_{22}^{(1,1)})a_{11}E_{22} + \lambda_{12}^{(1,1)}a_{11}E_{12}$$

for all $A = (a_{ij}) \in T_2(\mathbb{F}_2)$.

8.4 Some open problems

Let $\mathcal{S}$ be an additive subgroup of a ring $\mathcal{R}$. We say that a map $\psi : \mathcal{S} \to \mathcal{R}$ is 2-commuting on S if $[[\psi(x), x], x] = 0$ for all $x \in \mathcal{S}$. Let $\mathcal{R}$ be a prime ring with char$\mathcal{R} \neq 2$. Brešar (1992) proved that if an additive mapping $\psi : \mathcal{R} \to \mathcal{R}$ satisfies $[[\psi(x), x], x] = 0$ for all $x \in \mathcal{R}$, then $[\psi(x), x] = 0$ for all $x \in \mathcal{R}$. Let $n \geqslant 4$ be an integer and let $\mathbb{F}$ be a field with char $\mathbb{F} = 0$ or char $\mathbb{F} > 2$. Generalising Brešar's result of 2-commuting additive maps to subsets of matrices that are not closed under addition, Franca and Louza (2019) proved that if $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is an additive map such that $[[\psi(A), A], A] = 0$ for all singular $A \in M_n(\mathbb{F})$, then there exists an element $\lambda \in \mathbb{F}$ and a central map μ such that $\psi(A) = \lambda A + \mu(A)$ for all $A \in M_n(\mathbb{F})$.

We end this chapter with some open problems related to the work in this thesis.

1. Determine the structure of 2-power commuting additive maps on invertible $n \times n$ upper triangular matrices over fields with at least three elements.
2. Determine the structure of 2-power commuting additive maps on rank k, $n \times n$ upper triangular matrices over fields, where $2 \leqslant k \leqslant n-1$ is a fixed integer.
3. Determine the structure of 2-power commuting additive maps on rank one upper triangular matrices over fields.
4. Determine the structure of m-power commuting additive maps on rank k upper triangular matrices over fields, where $1 \leqslant k \leqslant n$ and $m \geqslant 3$ are integers.
5. Determine the structure of 2-commuting additive maps on invertible upper triangular matrices over fields.

REFERENCES

Ahmed, D. A. H. (2019). m-commuting additive maps on upper triangular matrices rings. *Earthline Journal of Mathematical Sciences*, *2*, 505–514.

Ara, P., & Mathieu, M. (1993). An application of local multipliers to centralizing mappings of C^*-algebras. *The Quarterly Journal of Mathematics*, *44*, 129–138.

Beidar, K. I. (1998). On functional identities and commuting additive mappings. *Communications in Algebra*, *26*, 1819–1850.

Beidar, K. I., Brešar, M., Chebotar, M. A., & Fong, Y. (2002). Applying functional identities to some linear preserver problems. *Pacific Journal of Mathematics*, *204*, 257–271.

Beidar, K. I., Brešar, M., & Chebotar, M. A. (2000). Functional identities on upper triangular matrix algebras. *Journal of Mathematical Sciences*, *102*, 4557–4565.

Beidar, K. I., Fong, Y., Lee, P.-H., & Wong, T.-L. (1997). On additive maps of prime rings satisfying the engel condition. *Communications in Algebra*, *25*, 3889–3902.

Bell, J., & Sourour, A. R. (2000). Additive rank-one preserving mappings on triangular matrix algebras. *Linear Algebra and its Applications*, *312*, 13–33.

Benkovič, D., & Eremita, D. (2004). Commuting traces and commutativity preserving maps on triangular algebras. *Journal of Algebra*, *280*, 797–824.

Botta, P. (1967). Linear transformations that preserve the permanent. *Proceedings of the American Mathematical Society*, *18*, 566–569.

Brešar, M. (1992). On a generalization of the notion of centralizing mappings. *Proceedings of the American Mathematical Society*, *114*, 641–649.

Brešar, M. (1993a). Centralizing mappings and derivations in prime rings. *Journal of Algebra*, *156*, 385–394.

Brešar, M. (1993b). Commuting traces of biadditive mappings, commutativity-preserving mappings and lie mappings. *Transactions of the American Mathematical Society*, *335*, 525–546.

Brešar, M. (1993c). On skew-commuting mappings of rings. *Bulletin of the Australian Mathematical Society*, *47*, 291–296.

Brešar, M. (1995a). Functional identities of degree two. *Journal of Algebra*, *172*, 690–720.

Brešar, M. (1995b). On generalized biderivations and related maps. *Journal of Algebra*, *172*, 764–786.

Brešar, M. (1996). Applying the theorem on functional identities. *Nova Journal of Mathematics, Game Theory and Algebra*, *4*, 43–54.

Brešar, M. (2000). Functional identities: a survey. *Contemporary Mathematics*, *259*, 93–110.

Brešar, M. (2004). Commuting maps: a survey. *Taiwanese Journal of Mathematics*, *8*, 361–397.

Brešar, M. (2016). Functional identities on tensor products of algebras. *Journal of Algebra*, *455*, 108–136.

Brešar, M. (2020). Functional identities and zero Lie product determined Banach algebras. *The Quarterly Journal of Mathematics*, *71*, 649–665.

Brešar, M., Chebotar, M. A., & Martindale 3rd, W. S. (2007). *Functional Identities. Frontiers in Mathematics*. Basel: Birkhäuser.

Brešar, M., & Hvala, B. (1995). On additive maps of prime rings. *Bulletin of the Australian Mathematical Society*, *51*, 377–381.

Brešar, M., Martindale 3rd, W. S., & Miers, C. R. (1993). Centralizing maps in prime rings with involution. *Journal of Algebra*, *162*, 342–357.

Brešar, M., & Miers, C. R. (1994). Strong commutativity preserving maps of semiprime rings. *Canadian Mathematical Bulletin*, *37*, 457–460.

Brešar, M., & Miers, C. R. (1995). Commuting maps on lie ideals. *Communications in Algebra*, *23*, 5539–5553.

Brešar, M., Procesi, C., & Špenko, Špela. (2015). Quasi-identities on matrices and the Cayley–Hamilton polynomial. *Advances in Mathematics*, *280*, 439–471.

Brešar, M., & Šemrl, P. (2003). Commuting traces of biadditive maps revisited. *Communications in Algebra*, *31*, 381–388.

Brešar, M., & Špenko, Špela. (2014). Functional identities in one variable. *Journal of Algebra*, *401*, 234–244.

Brešar, M., & Špenko, Špela. (2015). Functional identities on matrix algebras. *Algebras and Representation Theory*, *18*, 1337–1356.

Catalano, L. (2018). On a certain functional identity involving inverses. *Communications in Algebra*, *46*, 3430–3835.

Cezayirlioğlu, N., & Demir, Çağri. (2021). Functional identities of degree 2 vanishing on zero products of xy and yx. *Linear and Multilinear Algebra*.

Chacron, M. (2021). Generalized power commuting antiautomorphisms. *Communications in Algebra*, *49*, 5017–5026.

Chacron, M., & Lee, T.-K. (2019). Division rings with power commuting semi-linear additive maps. *Publicationes Mathematicae-Debrecen*, *95*, 187–203.

Chan, G.-H., & Lim, M.-H. (1992). Linear preservers on powers of matrices. *Linear Algebra and its Applications*, *162*, 615–626.

Chebotar, M. A., Ke, W.-F., Lee, P.-H., & Wong, N.-C. (2003). Mappings preserving zero products. *Studia Mathematica*, *155*, 77–94.

Chen, C. Z., & Zhao, Y. (2021). Strong commutativity preserving maps of upper triangular matrix lie algebras over a commutative ring. *Bulletin of the Korean Mathematical*

Society, *58*, 973–981.

Cheung, W.-S. (2001). Commuting maps of triangular algebras. *Journal of the London Mathematical Society*, *63*, 117–127.

Cheung, W.-S., & Li, C.-K. (2001). Linear operators preserving generalized numerical ranges and radii on certain triangular algebras of matrices. *Canadian Mathematical Bulletin*, *44*, 270–281.

Chooi, W. L., & Kwa, K. H. (2019). Additive maps of rank r tensors and symmetric tensors. *Linear and Multilinear Algebra*, *67*, 1269–1293.

Chooi, W. L., & Kwa, K. H. (2020). Additive maps of rank k bivectors. *The Electronic Journal of Linear Algebra*, *36*, 847–856.

Chooi, W. L., Kwa, K. H., & Lim, M. H. (2017). Coherence invariant maps on tensor products. *Linear Algebra and its Applications*, *516*, 24–46.

Chooi, W. L., Kwa, K. H., & **Tan, L. Y.** (2019). Commuting maps on invertible triangular matrices over $\mathbb{F}_2$. *Linear Algebra and its Applications*, *583*, 77–101.

Chooi, W. L., Kwa, K. H., & **Tan, L. Y.** (2020). Commuting maps on rank k triangular matrices. *Linear and Multilinear Algebra*, *68*, 1021–1030.

Chooi, W. L., & Lim, M. H. (1998). Linear preservers on triangular matrices. *Linear Algebra and its Applications*, *269*, 241–255.

Chooi, W. L., & Lim, M. H. (2001). Rank-one nonincreasing mappings on triangular matrices and some related preserver problems. *Linear and Multilinear Algebra*, *49*, 305–336.

Chooi, W. L., & Lim, M. H. (2002). Coherence invariant mappings on block triangular matrix spaces. *Linear Algebra and its Applications*, *346*, 199–238.

Chooi, W. L., Mutalib, M. H. A., & Tan, L. Y. (2021). Centalizing additive maps on rank r block triangular matrices. *Acta Scientiarum Mathematicarum*, *87*, 63–94.

Chooi, W. L., Mutalib, M. H. A., & **Tan, L. Y.** (2021). Commuting additive maps on rank one triangular matrices. *Linear Algebra and its Applications*, *626*, 34–55.

Chooi, W. L., & Ng, W. S. (2010). On classical adjoint-commuting mappings between matrix algebras. *Linear Algebra and its Applications*, *432*, 2589–2599.

Chooi, W. L., & Tan, Y. N. (2021). A note on commuting additive maps on rank k symmetric matrices. *The Electronic Journal of Linear Algebra*, *37*, 734–746.

Chooi, W. L., & **Tan, L. Y.** (2022). A note on centralizing additive maps on rank one triangular matrices over division rings. *preprint*.

Chooi, W. L., & Wong, J. Y. (2021). Commuting additive maps on tensor products of matrices. *Linear and Multilinear Algebra*.

Chou, P.-H., & Liu, C.-K. (2021). Power commuting additive maps on rank-k linear transformations. *Linear and Multilinear Algebra*, *69*, 403–427.

Costara, C. (2020). Nonlinear invertibility preserving maps on matrix algebras. *Linear Algebra and its Applications*, *602*, 216–222.

Costara, C. (2021). Nonlinear commuting maps on $\mathcal{L}(x)$. *Linear and Multilinear Algebra*, *69*, 551–556.

Dar, N. A., & Jing, W. (2022). On a functional identity involving inverses on matrix rings. *Quaestiones Mathematicae*.

Du, Y., & Wang, Y. (2012). k-commuting maps on triangular algebras. *Linear Algebra and its Applications*, *436*, 1367–1375.

Eremita, D. (2013). Functional identities of degree 2 in triangular rings. *Linear Algebra and its Applications*, *438*, 584–597.

Eremita, D. (2015). Functional identities of degree 2 in triangular rings revisited. *Linear and Multilinear Algebra*, *63*, 534–553.

Eremita, D. (2016). Functional identities in upper triangular matrix rings. *Linear Algebra* *and its Applications*, *493*, 580–605.

Eremita, D. (2017). Commuting traces of upper triangular matrix rings. *Aequationes Mathematicae*, *91*, 563–578.

Fošner, M. (2015). A result concerning additive mappings in semiprime rings. *Mathematica Slovac*, *5*, 1271–1276.

Franca, W. (2012). Commuting maps on some subsets of matrices that are not closed under addition. *Linear Algebra and its Applications*, *437*, 388–391.

Franca, W. (2013a). Commuting maps on rank-k matrices. *Linear Algebra and its Applications*, *438*, 2813–2815.

Franca, W. (2013b). Commuting traces of multiadditive maps on invertible and singular matrices. *Linear and Multilinear Algebra*, *61*, 1528–1535.

Franca, W. (2015). Commuting traces on invertible and singular operators. *Operators and Matrices*, *9*, 305–310.

Franca, W. (2016). Commuting traces of biadditive maps on invertible elements. *Communications in Algebra*, *44*, 2621–2634.

Franca, W. (2017). Weakly commuting maps on the set of rank-1 matrices. *Linear and Multilinear Algebra*, *65*, 475–495.

Franca, W., & Louza, N. (2017). Commuting maps on rank-1 matrices over noncommutative division rings. *Communications in Algebra*, *45*, 4696-4706.

Franca, W., & Louza, N. (2018). Commuting traces of multilinear maps on invertible elements. *Communications in Algebra*, *46*, 2890-2898.

Franca, W., & Louza, N. (2019). Generalized commuting maps on the set of singular matrices. *The Electronic Journal of Linear Algebra*, *35*, 533-542.

Franca, W., & Louza, N. (2021). Power commuting traces of bilinear maps on invertible elements. *Journal of Algebra and Its Applications*, *20*, 2150023.

Frobenius, G. (1897). Über die darstellung der endlichen gruppen durch lineare substitutionen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, 994–1015.

Guterman, A., Li, C.-K., & Šemrl, P. (2000). Some general techniques on linear preserver problems. *Linear Algebra and its Applications*, *315*, 61–81.

Han, D. (2017). Functional identities of degree 2 in CSL algebras. *Bulletin of the Iranian Mathematical Society*, *43*, 1601–1619.

Hiai, F. (1987). Similarity preserving linear maps on matrices. *Linear Algebra and its Applications*, *97*, 127–139.

Huang, H., Liu, C.-N., Szokol, P., Tsai, M.-C., & Zhang, J. (2016). Trace and determinant preserving maps of matrices. *Linear Algebra and its Applications*, *507*, 373–388.

Hungerford, T. W. (1974). *Algebra (Graduate Texts in Mathematics, 73)*. New York: Springer Verlag.

Inceboz, H., Koşan, M. T., & Lee, T.-K. (2016). m-power commuting maps on semiprime rings. *Communications in Algebra*, *42*, 1095–1110.

Khachorncharoenkul, P., Pianskool, S., & Siraworakun, A. (2020). Additive adjugate-commuting preservers between matrix spaces. *Asian-European Journal of Mathematics*, *13*, Article#2050114.

Lapuangkham, S., & Leerawat, U. (2021). On commuting additive mappings on semiprime rings. *Asian-European Journal of Mathematics*, *14*, Article#2150079.

Lee, P.-H., & Lee, T.-K. (1997). Linear identities and commuting maps in rings with involution. *Communications in Algebra*, *25*, 2881-2895.

Lee, P.-H., & Wang, Y. (2009). Supercentralizing maps in prime superalgebras. *Communications in Algebra*, *37*, 840–854.

Lee, P.-H., Wong, T.-L., Lin, J.-S., & Wang, R.-J. (1997). Commuting traces of multiadditive mappings. *Journal of Algebra*, *193*, 709–723.

Lee, T.-C. (1998). Derivations and centralizing maps on skew elements. *Soochow Journal of Mathematics*, *24*, 273–290.

Lee, T.-K. (1997). Derivations and centralizing mappings in prime rings. *Taiwanese Journal of Mathematics*, *1*, 333–342.

Lee, T.-K. (2019). Certain basic functional identities of semiprime rings. *Communications in Algebra*, *47*, 17–29.

Lee, T.-K., & Lee, T.-C. (1996). Commuting additive mappings in semiprime rings. *Bulletin-Institute of Mathematics Academia Sinica*, *24*, 259–268.

Lee, T.-K., Liu, K.-S., & Shiue, W.-K. (2004). n-commuting maps on prime rings. *Publications Mathematicae Debrecen*, *63*, 463–472.

Lee, T.-K., & Wong, T.-L. (2012). Nonadditive strong commutativity preserving maps. *Communications in Algebra*, *40*, 2213–2218.

Li, C.-K., & Pierce, S. (2001). Linear preserver problems. *The American Mathematical Monthly*, *108*, 591–605.

Li, C.-K., & Tsing, N.-K. (1992). Linear preserver problems: A brief introduction and some special techniques. *Linear Algebra and its Applications*, *162*, 217–235.

Li, C.-K., Šemrl, P., & Soares, G. (2001). Linear operators preserving the numerical range (radius) on triangular matrices. *Linear and Multilinear Algebra*, *48*, 281–292.

Li, Y., & Wei, F. (2012). Semi-centralizing maps of generalized matrix algebras. *Linear Algebra and its Applications*, *436*, 1122–1153.

Li, Y., Wei, F., & Fošner, A. (2019). k-commuting mappings of generalized matrix algebras. *Periodica Mathematica Hungarica*, *79*, 50–77.

Liu, C.-K. (2014a). Centralizing maps on invertible or singular matrices over division rings. *Linear Algebra and its Applications*, *440*, 318–324.

Liu, C.-K. (2014b). Strong commutativity preserving maps on subsets of matrices that are not closed under addition. *Linear Algebra and its Applications*, *458*, 280–290.

Liu, C.-K. (2020). Additive n-commuting maps on semiprime rings. *Proceedings of the Edinburgh Mathematical Society*, *63*, 193–216.

Liu, C.-K., Liau, P.-K., & Tsai, Y.-T. (2018). Nonadditive strong commutativity preserving maps on rank-k matrices over division rings. *Operators and Matrices*, *12*, 563–578.

Liu, C.-K., & Pu, Y.-F. (2021). The structure of n-commuting additive maps on lie ideals of prime rings. *Linear Algebra and its Applications*, *631*, 328–361.

Liu, C.-K., & Yang, J.-J. (2017). Power commuting additive maps on invertible or singular matrices. *Linear Algebra and its Applications*, *530*, 127–149.

Liu, H., & Xu, X. (2017). Additive maps on invertible upper triangular matrices. *Journal of Jilin University Science Edition*, *55*, 79–81.

Marcus, M., & Moyls, B. N. (1959). Linear transformations on algebras of matrices. *Canadian Journal of Mathematics*, *11*, 61–66.

Marcus, M., & Purves, R. (1959). Linear transformations on algebras of matrices: The invariance of the elementary symmetric functions. *Canadian Journal of Mathematics*, *11*, 383–396.

Mbekhta, M. (2012). A survey on linear (additive) preserver problems. In F. J. Pérez-Fernández (Ed.), *Advanced Courses of Mathematical Analysis IV* (pp. 174–195). World Scientific.

Molnár, L. (2007). *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces. Lecture Notes in Mathematics. Volume 1895*. Springer-Verlag Berlin Heidelberg.

Molnár, L., & Šemrl, P. (1998). Some linear preserver problems on upper triangular matrices. *Linear and Multilinear Algebra*, *45*, 189–206.

Omladič, M., & Šemrl, P. (1998). Preserving diagonalisability. *Linear Algebra and its Applications*, *285*, 165–179.

Orel, M. (2019). Nonstandard rank-one nonincreasing maps on symmetric matrices. *Linear and Multilinear Algebra*, *67*, 391–432.

Park, K.-H., & Jung, Y.-S. (2002). Skew-commuting and commuting mappings in rings. *Aequationes Mathematicae*, *64*, 136–144.

Petek, T., & Radič, G. (2020). A note on equivalence preserving maps. *Linear and Multilinear Algebra*, *68*, 2289–2297.

Pierce, S., Lim, M. H., Loewy, R., Li, C.-K., Tsing, N.-K., McDonald, B. R., & Beasley, L. (1992). A survey of linear preserver problems. *Linear and Multilinear Algebra*, *33*, 1–129.

Pierce, S., & Watkins, W. (1978). Invariants of linear maps on matrix algebras. *Linear and Multilinear Algebra*, *6*, 185–200.

Posner, E. C. (1957). Derivations in prime rings. *Proceedings of the American Mathematical Society*, *8*, 1093–1100.

Qi, X. (2016). k-power centralizing and k-power skew-centralizing maps on triangular rings. *Bulletin of the Iranian Mathematical Society*, *42*, 539–554.

Qi, X., & Hou, J. (2012). Strong commutativity preserving maps on triangular rings. *Operators and Matrices*, *6*, 147–158.

Qi, X., & Hou, J. (2015). Characterization of k-commuting additive maps on rings. *Linear Algebra and its Applications*, *468*, 48–62.

Sinkhorn, R. (1982). Linear adjugate preservers on the complex matrices. *Linear and Multilinear Algebra*, *12*, 215–222.

Słowik, R., & Ahmed, D. A. H. (2021). m-commuting maps on triangular and strictly triangular infinite matrices. *The Electronic Journal of Linear Algebra*, *37*, 247–255.

Song, S.-Z., Beasley, L. B., Mohindru, P., & Pereira, R. (2016). Preservers of completely positive matrix rank. *Linear and Multilinear Algebra*, *64*, 1258–1265.

Šemrl, P. (2014). Linear preserver problems. In L. Hogben (Ed.), *Handbook of Linear Algebra 2nd Edition. Discrete Mathematics and its Applications* (pp. 30-1–30-9). Boca Raton: Chapman & Hall/CRC.

Wan, Z. (1996). *Geometry of matrices: in memory of Professor LK Hua (1910–1985)*. World Scientific Singapore Berlin Heidelberg.

Wang, Y. (2013). Functional identities in superalgebras. *Journal of Algebra*, *382*, 144-176.

Wang, Y. (2015). Functional identities of degree 2 in arbitrary triangular rings. *Linear Algebra and its Applications*, *479*, 171-184.

Wang, Y. (2016a). Commuting (centralizing) traces and Lie (triple) isomorphisms on triangular algebras revisited. *Linear Algebra and its Applications*, *488*, 45-70.

Wang, Y. (2016b). On functional identities of degree 2 and centralizing maps in triangular rings. *Operator and Matrices*, *10*, 485–499.

Wang, Y. (2019). Functional identities in upper triangular matrix rings revisited. *Linear and Multilinear Algebra*, *67*, 348–359.

Xiao, Z.-K., & Wei, F. (2010). Commuting mappings of generalized matrix algebras. *Linear Algebra and its Applications*, *433*, 2178–2197.

Xiao, Z.-K., & Yang, L.-Q. (2021). Linear n-commuting maps on incidence algebras. *Acta Mathematica Hungarica*, *164*, 470–483.

Xu, X., & Liu, H. (2017). Additive maps on rank-s matrices. *Linear and Multilinear Algebra*, *65*, 806–812.

Xu, X., Pei, Y., & Yi, X. (2016). Additive maps on invertible matrices. *Linear and Multilinear Algebra*, *64*, 1283–1294.

Xu, X., & Yi, X. (2014). Commuting maps on rank-k matrices. *The Electronic Journal of Linear Algebra*, *27*, 735–741.

Xu, X., & Zhu, J. (2018). Central traces of multiadditive maps on invertible matrices. *Linear and Multilinear Algebra*, *66*, 1442–1448.

Yuan, H., & Chen, L. (2020). Functional identities on upper triangular matrix rings. *Open Mathematics*, *18*, 182–193.

Zhang, X., Tang, X., Cao, C., Mo, D., & Chen, Y. (2007). *Preserver Problems on Spaces of Matrices*. Science Press.