PRICING AND HEDGING EXOTIC OPTIONS IN INSURANCE AND FINANCE

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ABSTRACT

This thesis concerns the theoretical pricing and hedging of options, financial instruments that give a payoff at a set date based on the price of one, or several other financial assets, known as the underlying assets. The underlying assets are usually taken to be stocks, but can also be bonds, securities, portfolios, or other financial instruments. In this thesis we study two types of options - life contingent options and barrier Asian options. Because the options examined in this thesis are relatively uncommon, with a novel mechanism of action, they are known as exotic options. The analysis takes place in a stylised mathematical model of a financial market known as the Black-Scholes model. We show for the life contingent option that there exists a minimal super-hedging portfolio and determine the associated initial investment. We also give a characterisation of when replication of the option is possible. Next, we investigate the pricing problem for barrier Asian options with short maturity times. Due to the nature of Asian options, closed form formulae for the fair price of the option are relatively difficult to obtain. Using novel results from the theory of stochastic calculus, we obtain closed form asymptotic formulae for the price of short maturity barrier Asian options. Finally, we demonstrate our results with some explicit examples.

Keywords: Life contingent options, Hedging, Option pricing, Barrier Asian options, Stochastic calculus

HARGA DAN PELINDUNGAN NILAI OPSYEN EKSOTIK DALAM INSURANS DAN KEWANGAN

ABSTRAK

Tesis ini melibatkan teori penentuan harga opsyen dan pelindungan nilai opsyen. Opsyen adalah instrumen kewangan yang memberi pulangan pada tarikh yang ditetapkan berdasarkan harga satu, atau beberapa aset kewangan yang lain, yang dikenali sebagai aset pendasar. Aset pendasar biasanya terdiri daripada saham, tetapi juga boleh terdiri daripada bon, sekuriti, portfolio atau instrumen kewangan lain. Dua jenis opsyen akan dikaji dalam tesis, iaitu opsyen kontingen hayat dan opsyen Asia halangan. Oleh kerana opsyen-opsyen yang dikaji dalam tesis ini agak jarang, dengan mekanisme tindakan yang terbaru, ia juga dikenali sebagai opsyen eksotik. Analisis dilakukan dalam model matematik gaya pasaran kewangan yang dikenali sebagai model Black-Scholes. Kami menunjukkan bahawa untuk opsyen kontinjen hayat terdapat portfolio super-pelindungan nilai yang minimum dan menentukan pelaburan permulaannya. Kami juga memberikan pencirian apabila replikasi opsyen boleh dilakukan. Seterusnya, kami mengaji masalah harga untuk opsyen Asia halangan dengan masa matang yang singkat. Disebabkan oleh sifat opsyen Asia, formula bentuk tertutup untuk harga saksama opsyen agak sukar diperolehi. Dengan menggunakan hasil novel daripada teori kalkulus stokastik, kami memperoleh formula asimptotik bentuk tertutup untuk harga opsyen Asia halangan dengan kematangan pendek. Akhir sekali, kami menunjukkan hasil kami dengan beberapa contoh yang terperinci.

Kata kunci: Opsyen kontingen hayat, Pelindungan nilai, Harga opsyen, Opsyen Asia halangan, Kalkulus stokastik

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LIST OF SYMBOLS AND ABBREVIATIONS

Ω	: Probability space
$\mathbb{P}, \mathbb{Q}$	: Probability measures
$\mathbb{E}$	: Expectation
W_t	: Standard Brownian motion
$\mathcal{F}$	: Sigma-algebra
$\mathcal{F}_t$	: Filtration
$\mathbf{1}$	: Indicator function
$\int$	: Integral
exp	: Exponential function
Φ	: Cumulative distribution function of standard normal random variable
τ	: Stopping time
$\mathcal{A}(\tau)$	: Sigma algebra generated by a stopping time
O	: Landau big-O asymptotic notation
o	: Landau little-o asymptotic notation
sup	: Supremum
inf	: Infimum
SDE	: Stochastic differential equation
FBSDE	: Forward backward stochastic differential equation
GMDB	: Guaranteed minimum death benefits
CEV	: Constant elasticity of variance

CHAPTER 1: INTRODUCTION

1.1 Overview

In the field of finance, an *option* is a financial instrument that gives the holder a payoff that is a function of the price of one or several underlying assets, at a certain future time known as the *exercise time*. The underlying assets are usually stocks, but can also be taken to be bonds, securities, or other financial instruments. The most common types of options are the so called *European* and *American* options.

The *European* call (resp. put) option gives the holder the right, but not the obligation to buy (resp. sell) a unit of the underlying at a fixed price at the exercise time, where both the price and the exercise time are fixed in advance.

The *American* call (resp. put) option on the other hand gives the holder the same right to buy (resp. sell) at a fixed price, but the exercise time is allowed to be chosen by the holder. At any time from the purchase date up to a certain fixed *expiry date*, he may *exercise* the option and buy (resp. sell) a unit of the underlying at the fixed price. If the option is not exercised by the expiry date, the option simply ceases to exist, and can no longer be exercised.

In financial markets, much like stocks, options can be traded and exchanged for money, goods and other financial instruments. Thus key aspects of options trading are the *pricing* and *hedging* of options.

The pricing problem is the question of what the current fair trading price of an option is, given that we know the current and historical prices of the underlying assets, but not the future prices. Since there is inherent uncertainty involved in the future prices of the underlying assets, one must take into account this randomness when determining the theoretical fair price of the option. Thus in the mathematical analysis, the tools of probability theory are often used.

On the other hand, the hedging problem is question of how to mitigate the inherent risk of holding an option via managing a portfolio consisting of other financial instruments such as stocks and bonds. Most commonly, the hedging portfolio consists of a combination of the underlying assets and risk-free bonds. The hedging of options and other financial instruments is of great practical interest to traders. For instance, an employee may be rewarded a large amount of options contingent on the stock of the company he is working at; this is a common bonus reward scheme at large companies. However, as the exercise time is sometime in the future, there is great uncertainty as to what the actual realised gain will be at the exercise time. The employee, being in a so called *long position* in the option, may wish to take a *short position* in a hedging portfolio - that is, the employee sells the portfolio in advance at current prices with the promise to buy it back at a later date, at the future price. The hedging portfolio thus aims to match as well as possible the option payoff. In the case where the option payoff can be exactly matched by a portfolio, we call the portfolio a replicating portfolio. On the other hand, the company itself holds a short position in the option, and thus may themselves wish to hedge their position.

1.2 A Brief History of Options

The theoretical analysis of options dates back to the early 1900s - see, for instance, Haug (2009). At the time the only prescribed option was the so called *London option*, which gave the holder the right, but not the obligation to buy a unit of the stock at a future date, at its current price. This is, of course, what we now know today as an *in the money* European call option. Already at the time, quite a sophisticated understanding had been achieved of option pricing. In particular the concepts of *arbitrage free pricing* and *put call parity* were already known to practitioners at the time.

It could be said that the pioneering work that started off the modern theory of option pricing was the doctoral thesis Bachelier (1900) of Louis Bachelier, entitled *The Theory of Speculation*. In the thesis, Bachelier introduced the first mathematical model of Brownian motion and its use in pricing stock options. It was notable for being the first paper to use advanced mathematics in the pricing of options. The theory was further developed by the work of economists such as Paul Samuelson and Robert C. Merton. For a comprehensive overview of their work, we refer to the survey articles Merton (2010) and Schaefer (1998) respectively.

In 1961, Sprenkle introduced the Geometric Brownian motion model, in which stock prices at time $t \geq 0$ are assumed to be given by the *stochastic differential equation* (SDE)

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t), \tag{1.1}$$

where W_t is standard Brownian motion, and μ and σ are constants known as the *drift* and *volatility* respectively. He also discovered what we now know as the *Black-Scholes* formula for the fair price of an European call option, though his methods differed from the methods used by economists Black and Scholes at a later date.

In 1973, Black and Scholes rediscovered the formula, in what is now perhaps the most famous work on option pricing, Black and Scholes (1973). Their work was hailed as a breakthrough in financial mathematics - not because of the formula discovered, which was already known at the time - but due to the technique known as *delta hedging* that they used to derive the result. Roughly, it consisted of making instantaneous adjustments in the holdings of a portfolio in order to counter balance the stochastic part of the evolution of the option's price.

While the technique is considered to be difficult to apply in real life due to the presence of trading delays, transaction fees, and the impossibility of real time continuous trading, it remains an key theoretical device for the analysis of option prices, and an important stepping stone toward more sophisticated models.

Several years later, a Black-Scholes type formula was derived in Jacka (1991) for the American call/put option, based on optimal stopping theory and the no arbitrage principle. For further seminal work on American options, we refer to Bensoussan (1984), Karatzas (1988). More recent work includes Myneni (1992) and Siu and Elliott (2022).

Since then, various different kinds of options such as Asian, Bermudan, and digital options have emerged in the financial markets, and subsequently have been analysed in various recent works, see for instance Bayraktar and Xing (2011) and Boyle and Potapchik (2008) for Asian options and Rogers (2016) for Bermudan options.

1.3 Our Contribution

In this thesis, we will be concerned with the analysis of two types of options - *life contingent options*, and *barrier Asian options*. Since these have a novel mechanism of action, they are not classified as standard type options. Instead, the name *exotic option* is given to them to signify their significant deviation from the contractual structures of standard options. Despite the naming convention, many kinds of exotic options are available and regularly used in the marketplace. Though not nearly an exhaustive list, some examples include ratchet options, rainbow options, basket options and Bermudan options.

As the mechanism of action of exotic options can be rather complex, they are also of great intrinsic interest mathematically. Indeed, the analysis of ratchet options, for instance, has stimulated a great deal of development in the theory of *martingale optimal transport*, while the analysis of Bermudan options naturally involves the computation of multiple iterated conditional expectations, and thus has necessitated the development of efficient numerical algorithms to compute such expressions.

Now we elaborate more on the options we will be looking at in our thesis. Life contingent options, like the European option, give a payoff that depends on the current price of one, or several underlying assets. However, unlike the European and American options, the exercise time is not fixed nor under the option holder's control. Instead, it is taken to be the time of occurrence of an event of interest. This can be a financial event, such as the bankruptcy time of a company, the time a merger between two companies happens, or a non financial event such as the death of an individual. In fact, the first situation in which these options arose was in a insurance context, where the exercise time was taken to be the death of the holder. Thus it resembles a life insurance contract, except the payoff is in the form of an option payoff instead of a fixed payoff.

Because of the inherent uncertainty in when the event of interest will occur, this presents a novel difficulty in the analysis of such options. Indeed, it is not initially clear how to price such an option, since in the analysis of both the standard cases, the European and American options, an important role is played by the fact that the exercise time is either fixed or under the user's control. In both cases the exercise time is *adapted* to the natural filtration of the stock price, and thus is amenable to probabilistic analysis. Further, the hedging problem presents even more difficulties, since the inherent unpredictability of the event of interest means that exact replication is, intuitively not possible in general. Indeed, we shall prove this fact in the course of our work - in fact we give a precise characterisation of when exact replication is possible. We elaborate on our precise results in more detail later in this section.

The next type of option we analyze are short maturity barrier Asian options. This is an exotic option with two significant novelties - the Asian component signifies that the payoff of the option is not contingent merely on the current price of the underlying assets, but in general can depend on the entire history of the asset prices from the purchase time of the option to the exercise time. On the other hand, the barrier component means that the option may only be exercised if the prices of the underlying assets reach a certain level prior to the exercise time. In the most common scenario, the *up and in* barrier option, the option may be exercised if the stock price reaches a level that is currently above its current price. The short maturity aspect is not, properly speaking, part of the definition of the exotic option. It signifies that we analyse these options with their exercise time taken to be very close to the current date.

The combination of both the barrier and Asian component significantly complicates the analysis of these options. Indeed, unlike in the case of European options, where closed form formulae for prices and hedging portfolios are abundant, even the most basic Asian type options in the simplest market models admit no closed form solution for their price. Compounding this difficulty is the presence of the barrier condition, which vaguely speaking creates a large discontinuity in the payoff, since it defaults to a payoff of zero if the barrier condition is not met.

Now we elaborate more on our exact contributions. For the life contingent option, we will be interested in the hedging problem. As mentioned earlier, due to the inherent uncertainty involved in the exercise time, we discover that exact replication is not possible in general. We thus turn to the framework of *superreplication*, or *superhedging* to answer the hedging problem. This is the question of how to create a portfolio that almost surely pays off an amount greater than or equal to the option payoff at the exercise time, whenever the exercise time may be. Further, we wish to do so with minimal initial investment cost. We find that although exact replication is in general not possible, there always exists a minimal superreplicating portfolio, and further this minimal replicating portfolio is unique. We also give an exact characterization of when replication is possible.

Next, we investigate the pricing problem for the short maturity barrier Asian option.

We derive exact asymptotic formulae for the price of barrier Asian options in the short maturity limit - that is, the results become more and more accurate as the maturity time gets shorter. We provide quantitative rates of convergence for the price of the option to the limiting expression, in terms of the time to maturity. The analysis relies on asymptotic results for the geometric Brownian motion, which we term as *large noise limits*. These results, to our best knowledge are novel and have not been explored in the current literature.

1.4 Literature Review

Option pricing and hedging is a vast topic in probability and financial mathematics. To begin the literature review, we give just a few general examples of contemporary work in the field.

American options are studied in papers such as Gapeev (2012), and Goudenège et al. (2023). In Gapeev (2012), the pricing problem for the American option in a market model with partial observation is considered. Investors are able to observe the price of the underlying asset, but not the dividend policy for the asset, which is only observable by the issuing firm. The authors characterise the optimal stopping boundary, and further provide closed form estimates for the rational price of the option and the stopping boundary.

On the other hand, the hedging problem for American options is considered in Goudenège et al. (2023). Here the main novelty is the presence of transaction costs when purchasing and selling portfolio assets. Numerical algorithms for approximate optimal hedging of the option under transaction costs are provided, and compared to existing algorithms. Concerning exotic options, lookback options are studied in G. Zhang and Li (2023) and Chan and Zhu (2014) among many others. The paper G. Zhang and Li (2023) considers the numerical pricing of lookback options in a general Markovian model. The results obtained apply to a wide variety of market models, and their efficiency in such models is demonstrated via numerical simulation. Meanwhile, the paper Chan and Zhu (2014) takes an analytic approach, obtaining exact and explicit formulae for the rational price of European style lookback options in a regime switching model, where the market is allowed to switch randomly between "market states" such as high or low volatility regimes.

General studies on Asian options include works such as Han and Liu (2018) and Pirjol and Zhu (2023). In Han and Liu (2018), the pricing problem for Asian options in a market model with uncertain volatility is considered. The worst-case price of the Asian option is characterised via a solution to a nonlinear PDE, and approximate solutions to the PDE are given. The paper Pirjol and Zhu (2023) is concerned with the sensitivities of the Asian option price to changes in the model parameters, a classic topic in option pricing. The so called "Greeks", various sensitivities are computed for the Asian option using large deviation techniques.

Next, we look at work in the field that is more closely related to our thesis topics. We first discuss relevant work on life contingent options. The authors of Gerber et al. (2012) determine the expected payoff of life contingent options within the framework of the geometric Brownian motion model by using discounted density approach. There, the random exercise time is modelled by a linear combination of exponentially distributed random variables independent of the underlying price processes. Closed form expressions of the expected payoff are obtained for various types of payoffs, including European-type options, which give the holder the right to buy (resp. sell) a unit of underlying at the expiry date; as well as digital, lookback, and barrier options. The results are extended to a jump diffusion model of stock prices in Gerber et al. (2013). An underlying asset price model with jumps is also considered in Z. Zhang et al. (2020). In this paper, valuation formulae for a class of payoff functions are obtained under the assumption that the risky asset price to follow a geometric Levy process, and the pricing method is implemented numerically via spline function methods.

Meanwhile, the valuation problem for life contingent options in a discrete time model is considered by Gerber et al. (2015). They use the technique of geometric stopping of a random walk to derive closed form expressions for the expected payoffs of European, barrier and lookback life contingent options.

On the other hand, the hedging problem for life contingent options, an equally important problem in financial literature, has been far less frequently studied compared to the problem of valuation. Kélani and Quittard-Pinon (2017) considered the hedging problem in incomplete markets with the independence assumption of the mortality risk and market risk. They obtained a concise formula for the optimal hedging ratio under the framework of local risk minimization. In this framework, the portfolio is not required to be self financing, but its value process is a martingale. The objective is to hedge the option while minimizing the variance of the cumulative cost process of the portfolio. W. Wang et al. (2021) considered the hedging problem in a more intricate market model where the risky asset price follows a Hawkes jump-diffusion process, which is a jump process with self-exciting jumps. They obtained explicit expressions of the locally risk minimizing strategies for unit-linked life insurance contracts.

Another hedging framework that is widely used in incomplete markets is that of quantile hedging. In this framework, one attempts to find a self financing portfolio that successfully hedges the option with maximal probability, given constraints on the initial value of the portfolio. This framework is explored in Y. Wang (2009). Under various assumptions, quantile hedges are derived for life contingent options, referred to in their paper as guaranteed minimum death benefits. Meanwhile, Eyraud-Loisel and Royer-Carenzi (2010) considered hedging problems for an insider trader. They studied hedging problem for American-style option and model it with backward stochastic differential equations with random terminal time. However, strong additional conditions were included in the set up.

In particular, the portfolio holder is assumed to have access to additional information not included in the asset filtration, and his portfolio is allowed to consist of an additional asset other than the two included in the standard market model.

In this thesis, we consider the hedging problem for life contingent options. Since the exercise time for life contingent options is itself random, this presents a novel difficulty in constructing a hedge process, or a replicating portfolio. In Gerber et al. (2012), Gerber et al. (2013), Gerber et al. (2015) and Z. Zhang et al. (2020), the expected payoff of the life contingent option at the exercise time is derived. However, they do not explore the hedging problem for this type of option. In Kélani and Quittard-Pinon (2017) and Y. Wang (2009), the hedging problem is explored. The frameworks explored are, respectively, the local risk minimization framework, and quantile hedging framework. This leaves the problem of super-replication unexplored. In Eyraud-Loisel and Royer-Carenzi (2010), the existence of a super-replication portfolio is obtained, but the authors make crucial additions to the scenario - in particular the existence of an additional asset, and additional datum in the filtration, the so called *insider information*. Thus the question of superhedging for the life contingent option - in the classical setting of a portfolio consisting only of market assets, and adapted to the market filtration - remains unanswered.

As such, we will be interested in the case where the additional devices in Eyraud-Loisel and Royer-Carenzi (2010) are not provided. We explore the super-replication problem for life contingent options. Given that the market is incomplete, exact replication will rarely be possible, thus we examine the possibility of super-replication instead, that is, a portfolio that almost surely pays off more than or equal to the option payoff at the exercise time. First, we derive the minimal price of a super-replicating portfolio for the life contingent option. We then show there exists a minimal hedge for the life contingent option, given only access to the asset price process as information, and consisting of only the two assets in the market. Next, we give a characterisation of when replication of the life contingent option is possible, and finally derive an explicit expression for a super-replicating portfolio in a simple case.

Next, we turn to short maturity Asian options, which have been studied in many papers such as the recent Pirjol and Zhu (2016), Pirjol and Zhu (2019), Shoshi and SenGupta (2023) and Chatterjee et al. (2018). The paper Pirjol and Zhu (2016) investigates pricing of short maturity Asian options in local volatility models, while the paper Pirjol and Zhu (2019) investigate pricing in the Constant Elasticity of Variance (CEV) model, a well known stochastic volatility model for stock prices. Meanwhile, Shoshi and SenGupta (2023) uses large deviations theory to study short maturity Asian options in a jump diffusion model, while in Pirjol and Zhu (2016) the authors use a Markov chain-based approximation method to price short maturity Asian options in the geometric Brownian Motion model.

Small noise limits have been investigated by numerous authors. In Trevisan (2013), the authors determine the small noise limits for a well known family of irregular SDEs via local time techniques, showing convergence to a unique limiting distribution. The recent 2022 paper Fjordholm et al. (2022) by Fjordholm, Musch, and Pilipenko investigates the small noise limit for an SDE with drift coefficient in L^∞. In this case the limiting solution is unique, and corresponds to the so called Osgood solution of the corresponding deterministic equation. Some other references include Bakhtin (2010), which investigates small noise limits from a dynamical systems viewpoint, and Delarue and Maurelli (2019), where the authors investigate a particular multivariable SDE case.

As will be mentioned in later chapters, the mathematical results used here are strongly linked to the so called extreme value theory for Levy processes, in which the behaviour of jump processes conditional on its running maximum taking a large value is considered. In many cases, it happens that the processes converges to a process with only one large random jump, that is otherwise constant. A seminal reference is Hult and Lindskog (2005), in which the authors prove the single jump limiting behaviour of Levy processes with a suitable notion of regular variation. The results are extended significantly by the same authors in Hult and Lindskog (2007). In Bazhba et al. (2017), the authors establish a weak large deviations principle for this scenario, but conclude that a strong large deviations principle in the classical sense does not hold.

CHAPTER 2: LIFE CONTINGENT OPTIONS

2.1 Overview

We now begin our investigation of life contingent options. We work in a modified Black-Scholes market model. In order to model the random exercise time, we include the existence of a stopping time independent of the asset filtration. That it is independent of the asset filtration corresponds to the fact that the option holder cannot predict the occurence of the event of interest on the basis of the asset price process. In fact, life contingent options in general may be contingent on an event that is linked to the asset price. However, in this thesis we restrict ourselves to the independent case, which is itself of considerable interest.

In order to demonstrate existence of a minimal hedge, we will use an approach inspired by dynamic programming. We induct backward starting from the terminal time, giving rise to a series of forward-backward stochastic differential equations (FBSDE) that may be solved sequentially to yield a minimal hedge.

To determine the necessary and sufficient conditions for existence of a replicating portfolio, we will use martingale theory in order to narrow down the conditions under which a replicating portfolio may exist. The key tools here shall be the optional stopping theorem, and martingale inequalities.

2.2 Market Model and Key Definitions

We first introduce the market model for the problem. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. The probability measure $\mathbb{P}$ is referred to as the physical measure. We consider the following two processes

$$X^0, X^1 : [0,T] \times \Omega \to \mathbb{R}_+, \tag{2.1}$$

known as the *bond* and *stock* price respectively satisfying the stochastic differential equation (SDE)

$$dX_t^0 = rX_t^0\,dt, \tag{2.2}$$

$$dX_t^1 = X_t^1\,(\mu\,dt + \sigma\,dW_t)\,, \tag{2.3}$$

with $X_0^0 = 1, X_0^1 = x_0$ almost surely for some $x_0 \in \mathbb{R}_+$, and $W = \{W_t\}_{t\in[0,T]}$ is a standard Brownian motion with T denoting a fixed and finite time horizon. Here $r, \mu, \sigma \in \mathbb{R}$ are positive constants, known as the *risk free interest rate*, *drift* and *volatility* of the stock respectively. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by the Brownian motion W. We assume $\mathbb{F}$ satisfies the usual conditions, that is, it is right continuous and contains all $\mathbb{P}$-null sets.

To describe the randomness of the exercise time of life contingent options, we let τ be an almost surely finite random variable independent of $\mathcal{F}$ taking finitely many values $0 < t_1 < \cdots < t_n < t_{n+1} = T$, representing the possible exercise and expiry times. We interpret times $t_1 \ldots t_n$ as being the possible occurrence times of the event of interest (eg, death of the holder) at which time the option may be exercised. On the other hand, if $\tau = t_{n+1} = T$, we interpret it as the option having expired before the event of interest occurs. We denote by $\mathcal{G}$ the natural filtration of the process $G : [0, T] \times \Omega \to \mathbb{R}$ given by $G(t, \cdot) = \mathbf{1}_{\{\tau \leq t\}}$. By construction, τ is then a stopping time of $\mathcal{G}$.

We now introduce the key definitions to do with the hedging problem for the life contingent option.

Definition 2.2.1 (T-year life contingent option)**.** *Let T be the fixed expiration date of the life contingent option and $0 \leq \tau \leq T$ be the stopping time of an event of interest. The* *payoff of a life contingent option is an* $\mathcal{F}_T \vee \mathcal{G}_\tau$ *measurable random variable where* $\mathcal{G}_\tau$ *denotes the* σ*-algebra of the stopping time* τ*, and* $\mathcal{F}_T \vee \mathcal{G}_\tau$ *denotes the* σ*-algebra generated by* $\mathcal{F}_T$ *and* $\mathcal{G}_\tau$*. The time-*τ *payoff of the life contingent option is of the form*

$$f_\tau := \mathbf{1}_{\{\tau<T\}} b(X^1_\tau) + \mathbf{1}_{\{\tau=T\}} c(X^1_T), \tag{2.4}$$

where X^1 *is the underlying price process satisfying Eq.* (2.3) *and* τ *is the random exercise time.* $b, c : \mathbb{R}_+ \to \mathbb{R}_+$ *are bounded Borel measurable functions called as death benefit payoff and compensation payoff, respectively.*

Thus the option pays death benefit payoff, $b(X^1_\tau)$ if the holder dies at time τ, but before time $T = t_{n+1}$, and otherwise pays the compensation payoff $c(X^1_T)$ at time T.

Next, we introduce the definition of a portfolio.

Definition 2.2.2 (Self financing portfolio)**.** *A self financing portfolio is a pair of processes* $H := (H^0, H^1)$ *with values in* $\mathbb{R}^2$ *that is adapted to the filtration* $\mathbb{F}$*, and satisfies the following two conditions:*

(C1) The portfolio must be self financing, that is:

$$X^0_t dH^0_t + X^1_t dH^1_t = 0 \ \textit{almost surely} \tag{2.5}$$

(C2) The portfolio must have nonnegative value at all times:

$$X^0_t H^0_t + X^1_t H^1_t \geq 0 \ \textit{almost surely,} \tag{2.6}$$

for all $t \in [0, T]$.

Intuitively, the self financing condition means that no additional money or assets may be externally added or taken away from the portfolio, apart from the gains and losses obtained from holding the assets. Meanwhile, the nonnegativity constraint is a natural one that says that no borrowing from the bank is allowed, and consequently the simultaneous shorting of both assets is not permitted.

We denote by $V_t(H) := X_t^0 H_t^0 + X_t^1 H_t^1$ the *value process* of the portfolio H at time t, and $V_0(H)$ the *initial investment* of the portfolio. The set of self financing portfolios is denoted by SF.

Remark 1. Note that the initial values H_0^0, H_0^1 may be freely chosen, so long as the non negativity condition in Definition 2.2.2 is satisfied.

Definition 2.2.3 (Super-replication portfolio)**.** *A super-replication portfolio for the lifetime contingent option f is a self financing portfolio whose associated value process V satisfies $V_\tau \geq f_\tau$ almost surely. More precisely, a super-replication portfolio is an element of the set $\mathcal{S}(f, \tau)$ defined by*

$$\mathcal{S}(f,\tau) := \{H \mid H \in SF; X_\tau^0 H_\tau^0 + X_\tau^1 H_\tau^1 \geq f_\tau \text{ almost surely}\}. \tag{2.7}$$

Definition 2.2.4 (Minimal super-replication price)**.** *We define the minimal super-replication price $\pi_0(f)$ for the life contingent option f to be the infimal value of a super-replication portfolio for f, that is,*

$$\pi_0(f) := inf\{V_0(H) \mid H = (H^0, H^1) \in \mathcal{S}(f, \tau)\}, \tag{2.8}$$

where $V_0(H)$ is the initial investment for the portfolio H.

Definition 2.2.5 (Minimal super-replication portfolio)**.** *A minimal super-replication portfolio for f, if it exists, is a super-replication portfolio whose initial value equals the minimal hedging price $\pi_0(f)$.*

We recall that the market involving the two assets X^0 and X^1 is complete, that is, there exists a unique *equivalent martingale measure* $\mathbb{Q}$, which is a probability measure equivalent to $\mathbb{P}$ such that the discounted asset prices are martingales.

2.3 Existence of a minimal super-replication portfolio

We are now ready to state the first main theorem of the chapter.

Theorem 1. *Let b, c, f, τ be as in the setup in Section 2.2. There exists a minimal hedge for the life contingent option f, whose associated initial investment π_0 is*

$$\pi_0 = D_0 \dots D_n(U), \tag{2.9}$$

where the random variable U is defined by

$$U := \begin{cases} c(X_T^1) & \text{if } \mathbb{P}(\tau = T) \neq 0, \\ 0 & \text{if } \mathbb{P}(\tau = T) = 0, \end{cases} \tag{2.10}$$

and the operators $D_k : L^1(\Omega) \to \mathbb{R}$ are defined by the following. For any random variable $Y \in L^1(\Omega)$,

$$D_0(Y) := \mathbb{E}^{\mathbb{Q}}[e^{-rt_1} Y], \tag{2.11}$$

and for $1 \leq k \leq n$,

$$D_k(Y) := \max\left(b(X_{t_k}^1), \mathbb{E}^{\mathbb{Q}}\left[e^{-r(t_{k+1}-t_k)} Y \middle| \mathcal{F}_{t_k}\right]\right). \tag{2.12}$$

Here $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation under the probability measure $\mathbb{Q}$, and we recall that $t_1, \ldots, t_{n+1}$ are the possible exercise or expiry times of the option.

For the proof of Theorem 1, we will need the following lemma on independence of filtrations, as well as a standard result from the theory of option pricing.

Lemma 1. *Let $\mathbb{H} = \{\mathcal{H}_t\}$ and $\mathbb{K} = \{\mathcal{K}_t\}$ be two independent filtrations under the probability measure $\mathbb{P}$. If $\mathbb{Q}$ is another probability measure such that $d\mathbb{Q} = Z\, d\mathbb{P}$ for some $\mathcal{H}_\infty$ measurable random variable Z, then $\mathbb{H}$ and $\mathbb{K}$ remain independent under $\mathbb{Q}$.*

Proof. Let $H \in \mathcal{H}_t$ and $K \in K_r$ for some $t, r \geq 0$. We compute

$$\begin{aligned}\mathbb{Q}(H \cap K) &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_H \mathbf{1}_K] = \mathbb{E}^{\mathbb{P}}[Z \mathbf{1}_H \mathbf{1}_K] \\ &= \mathbb{E}^{\mathbb{P}}[Z\mathbf{1}_H]\mathbb{E}^{\mathbb{P}}[\mathbf{1}_K] = \mathbb{E}^{\mathbb{P}}[Z\mathbf{1}_H]\mathbb{E}^{\mathbb{P}}[\mathbf{1}_K]\mathbb{E}^{\mathbb{P}}[Z] \\ &= \mathbb{E}^{\mathbb{P}}[Z\mathbf{1}_H]\mathbb{E}^{\mathbb{P}}[Z\mathbf{1}_K] = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_H]\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_K] \\ &= \mathbb{Q}(H)\mathbb{Q}(K),\end{aligned} \tag{2.13}$$

where the fourth to last equality is valid because $E^{\mathbb{P}}[Z] = \int_\Omega Z d\mathbb{P} = \int_\Omega d\mathbb{Q} = 1$.

We now recall a key result in the theory of option pricing, which can be found in references such as Elliott and Kopp (2013) (Theorem 7.5.10, page 190) - given a geometric Brownian motion market model with two assets, the minimal initial investment for a hedge for a European option with payoff h_T is $\mathbb{E}^{\mathbb{Q}}[e^{-rT} h_T]$, where $\mathbb{Q}$ denotes an equivalent martingale measure and T is the exercise time.

Stated more precisely, the form of this result we will need is:

Proposition 1. *Let W be a standard Brownian motion on a filtered probability space.*

Suppose X^0 and X^1 are solutions to the SDE

$$dX_t^0 = rX_t^0\, dt, \tag{2.14}$$

$$dX_t^1 = X_t^1\, (\mu\, dt + \sigma\, dW_t)\,, \tag{2.15}$$

for $t \in [q, s]$ with initial condition $X_q^0 = x_0$ and $X_q^1 = x_1$ for some $x_0, x_1 \in \mathbb{R}_+$. Let f_s be a $\mathcal{F}_s$ measurable random variable. Suppose (Z^0, Z^1) is a super-replication portfolio consisting of X^0 and X^1 on $[q, s]$.

Then $Z_q^0 X_q^0 + Z_q^1 X_q^1 \geq \mathbb{E}^{\mathbb{Q}}[e^{r(q-s)} f_s | \mathcal{F}_q]$ almost surely, where $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ under which the discounted process $e^{-r(t-q)} X_t^1$ is a martingale on $[q, s]$ with respect to the filtration $\mathbb{F}$. Further, there exists a self financing portfolio $H = (H^0, H^1)$ with $X_s^0 H_s^0 + X_s^1 H_s^1 = f_s$ almost surely, in which case $X_q^0 H_q^0 + X_q^1 H_q^1 = \mathbb{E}^{\mathbb{Q}}[e^{r(q-s)} f_s | \mathcal{F}_q]$.

As a first step to proving Theorem 1, we obtain a lower bound on the value of any (f, τ) hedge at the times $t_1, \ldots, t_{n+1}$. Write for convenience $t_0 = 0$.

Proposition 2. *Let b, c, f, τ be as in the setup in Section 2.2, and suppose $H \in S(f, \tau)$. Then the associated value process V must satisfy*

$$V_{t_i} \geq D_i \ldots D_n(U), \tag{2.16}$$

for each $0 \leq i \leq n$, almost surely, where the random variables U, and the operators D_i are as defined in Theorem 1.

If $\mathbb{P}(\tau = T) \neq 0$, V must further satisfy

$$V_T \geq c(X_T^1) \tag{2.17}$$

almost surely.

Proof. Let $V(H)$ be the value process of a hedging portfolio $H = (H^0, H^1)$. Assuming first that $\mathbb{P}(\tau = T) \neq 0$, we will show that $V_T(H) \geq c(X_T^1)$ almost surely. Indeed, assume otherwise - then the event $A := \{V_T(H) < c(X_T^1)\}$ has nonzero probability. Since the two variables involved in the defining inequality are $\mathcal{F}_T$ measurable, A is $\mathcal{F}_T$ measurable.

But then since τ is independent of the filtration generated by the Brownian motion, we have that $A \cap \{\tau = T\}$ has nonzero probability as well. Thus $V_\tau < c(X_\tau)$ with nonzero probability, contradicting the definition of a hedge.

By exactly the same reasoning, we conclude that

$$V_{t_i}(H) \geq b(X_{t_i}^1), \text{ almost surely,} \tag{2.18}$$

for each $1 \leq i \leq n$.

It is left to show the inequalities

$$V_{t_i}(H) \geq D_i \dots D_n(U) \text{ almost surely,} \tag{2.19}$$

for each $0 \leq i \leq n$.

We treat the case $1 \leq i \leq n$ and $i = 0$ seperately. For the former, we take a dynamic programming approach and induct backwards on i.

For the base case, we must show that

$$V_{t_n}(H) \geq D_n(U), \text{ almost surely.} \tag{2.20}$$

We first note that by the Markov property of Ito stochastic differential equations, conditional on $\mathcal{F}_{t_n}$, we find that X^0, X^1, and the portfolio H^0, H^1 restricted to $[t_n, T]$ satisfy the hypotheses of Proposition 1.

Indeed, X^0 and X^1 satisfy the given SDE (2.14), (2.15) on the interval $[t_n, T]$, and by the second and third paragraph above, $V_T(H) = X_T^0 H_T^0 + X_T^1 H_T^1 \geq c(X_T^1)$ almost surely. Thus

$$V_{t_n}(H) \geq \mathbb{E}^{\mathbb{Q}}\left[e^{r(t_n - T)} c(X_T^1) \middle| \mathcal{F}_{t_n}\right]. \tag{2.21}$$

Combining this with the fact that $V_{t_n}(H) \geq b(X_{t_n})$ from (2.18), we conclude the inequality (2.20) as desired.

For the induction step, let $2 \leq k \leq n$, and assume

$$V_{t_k}(H) \geq D_k \ldots D_n (U), \text{ almost surely.} \tag{2.22}$$

We must show that

$$V_{t_{k-1}}(H) \geq D_{k-1} \ldots D_n (U), \text{ almost surely.} \tag{2.23}$$

However, conditional on $\mathcal{F}_{t_{k-1}}$, we again find that X^0, X^1, and the portfolio H^0, H^1 restricted to $[t_{k-1}, t_k]$ satisfy the hypotheses of Proposition 1, Indeed, we again check that X^0 and X^1 satisfy the given SDE (2.14), (2.15) on the interval $[t_{k-1}, t_k]$ and by the induction hypothesis,

$$V_{t_k}(H) = X_{t_k}^0 H_{t_k}^0 + X_{t_k}^1 H_{t_k}^1 \geq D_k \ldots D_n (U), \text{ almost surely.} \tag{2.24}$$

Together with (2.18), we deduce

$$\begin{aligned} V_{t_{k-1}}(H) &\geq \max\left(b(X_{t_{k-1}}^1), \mathbb{E}^{\mathbb{Q}}\left[e^{r(t_{k-1} - t_k)} D_k \ldots D_n (U) | \mathcal{F}_{t_{k-1}}\right]\right) \\ &= D_{k-1} \ldots D_n (U), \text{ almost surely,} \end{aligned} \tag{2.25}$$

which proves the case $1 \le i \le n$ in (2.19) as required.

Finally, one more application of Proposition 2 proves the case $i = 0$ in (2.19).

In the case where $\mathbb{P}(\tau = T) = 0$, we note that by similar considerations as earlier, we still have the inequality

$$V_{t_n}(H) \ge b(X^1_{t_n}), \text{ almost surely,} \tag{2.26}$$

whence the rest of the proof proceeds verbatim.

Now we set out to construct a minimal hedge. Before we do so, we will need the following generalities on regular conditional probabilities. The below is largely based on Jr et al. (2004).

Definition 2.3.1 (Transition probability)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ a measurable space. A transition probability from E to Ω is a function $\nu : E \times \mathcal{F} \to [0, 1]$ which satisfies the following two conditions:*

a) $\nu(x, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$, for all $x \in E$;

b) $\nu(\cdot, A)$ is a $\mathcal{E}$-measurable function on E, for all $A \in \mathcal{F}$.

Definition 2.3.2 (Regular conditional probability)**.** *Let $T : \Omega \to E$ be a measurable function. A regular conditional probability with respect to T is a transition probability $\nu : E \times \mathcal{F} \to [0, 1]$ from $(E, \mathcal{E})$ to $(\Omega, \mathcal{F})$ such that*

$$\mathbb{P}[A \cap T^{-1}(B)] = \int_B \nu(x, A)\, T_*\mathbb{P}(dx), \tag{2.27}$$

for all $x \in E$, $A \in \mathcal{F}$ and $B \in \mathcal{E}$, where $T_*\mathbb{P}$ denotes the image measure of $\mathbb{P}$ under T.

Definition 2.3.3 (Sub-σ-algebra regular conditional probability)**.** *Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$. A sub-σ-algebra regular conditional probability (with respect to $\mathcal{H}$) is a regular conditional probability with respect to the identity map $I : (\Omega, \mathcal{H}) \to (\Omega, \mathcal{F})$.*

The following is Proposition 1.9 in Yong and Zhou (1999), and gives sufficient conditions for regular conditional probabilities to exist. After, we state a proposition which ensures that regular conditional probabilities exist in our setting.

Proposition 3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Radon probability space, and $\mathcal{H}$ a sub σ-algebra of $\mathcal{F}$. Then there exists a regular conditional probability with respect to $\mathcal{H}$.*

We shall need the following technical lemma, whose proof we relegate to the section on technical proofs at the end of the chapter.

Proposition 4. *The market model $(\Omega, \mathcal{F}, \mathbb{P})$ in the setup in Section 2 can be taken to be a Radon probability space.*

We are now ready to construct our minimal hedge, and in doing so, prove Theorem 1.

Proof. Denote by $0 < t_1 < \cdots < t_n < t_{n+1} = T$, the values of τ that occur with nonzero probability, with the exception of $\mathbb{P}(\tau = t_{n+1})$ which is permitted to possibly be 0.

In the notation introduced in the statement of the theorem, write for convenience

$$J_i := D_i \dots D_n(U), \tag{2.28}$$

for $1 \le i \le n$.

We define our hedging process H on $[0, t_1]$ by $H^0 = Y^0$, $H^1 = Y^1$, where Y^0, Y^1 are defined as follows. By Proposition 2, given X^0, X^1 as in the setup, there exists a $\mathbb{F}$-adapted solution (Y^0, Y^1, V) to the following system (FBSDE 1) of forward backward stochastic differential equations

$$dX_t^0 = rX_t^0\, dt, \tag{2.29}$$

$$dX_t^1 = X_t^1\, (\mu\, dt + \sigma\, dW_t)\,, \tag{2.30}$$

$$dV_t = Y_t^0 dX_t^0 + Y_t^1 dX_t^1, \tag{2.31}$$

$$X_t^0 dY_t^0 + X_t^1 dY_t^1 = 0, \tag{2.32}$$

for $t \in [0, t_1]$ under $\mathbb{P}$ with initial condition $X_0^0 = 1$, $X_0^1 = x$, and terminal condition $V_{t_1} = J_1$ almost surely.

Suppose now inductively our processes $(H^0, H^1, V(H))$ have already been defined on $[0, t_i]$ for some $1 \leq i \leq n$, satisfies Eqs. (2.29), (2.30), (2.31), (2.32) and further that $V_{t_k}(H) \geq J_k$ for each $1 \leq k \leq i$.

Consider the regular conditional probability ξ of $\mathbb{P}$ given $\mathcal{F}_{t_i}$.

Now fix $x \in \mathbb{R}_+$, and $\omega \in \Omega$ such that $X_{t_i}^1(\omega) = x$, and consider the system (FBSDE 2)

$$dX_t^0 = rX_t^0\, dt, \tag{2.33}$$

$$dX_t^1 = X_t^1\, (\mu\, dt + \sigma\, dW_t)\,, \tag{2.34}$$

$$dR_t = Z_t^0 dX_t^0 + Z_t^1 dX_t^1, \tag{2.35}$$

$$X_t^0 dZ_t^0 + X_t^1 dZ_t^1 = 0, \tag{2.36}$$

for $t \in [t_i, t_{i+1}]$ under the probability measure $\xi(\omega, \cdot)$ with initial condition $X_{t_i}^0 = e^{rt_i}$, $X_{t_i}^1 = x$, and terminal condition $R_{t_{i+1}} = J_{i+1}$, $\xi(\omega, \cdot)$-almost surely. Note that $\xi(\omega, \cdot)$ is supported on the event $\{X^1_{t_i} = x\}$, since $X^1_{t_i}$ is $\mathcal{F}_{t_i}$ measurable, and $X^1_{t_i}(\omega) = x$.

By Proposition 1, for each such $\omega \in \Omega$, there exists a $\xi(\omega, \cdot)$-almost surely well defined solution $(Z^0, Z^1, R) =: (Z^{0,x}, Z^{1,x}, R^x)$ to (FBSDE 2). We define our process for $t \in (t_i, t_{i+1}]$ by

$$H^0_t = Z^{0,x}_t + e^{-rt_i}(V_{t_i}(H) - Z^{0,x}_{t_i} X^0_{t_i} - Z^{1,x}_{t_i} X^1_{t_i}) \tag{2.37}$$

$$H^1_t = Z^{1,x}_t \tag{2.38}$$

$$\begin{aligned} V_t(H) =& e^{-rt_i}(V_{t_i}(H) - Z^{0,x}_{t_i} X^0_{t_i} - Z^{1,x}_{t_i} X^1_{t_i}) X^0_t + Z^{0,x}_t X^0_t + Z^{1,x}_t X^1_t & (2.39) \\ =& H^0_t X^0_t + H^1_t X^1_t & (2.40) \end{aligned}$$

on the event $\{X^1_{t_i} = x\}$. This defines (H^0, H^1, V), $\mathbb{P}$ almost surely up to $[0, t_{i+1}]$.

Indeed, denoting by E the set on which (H^0, H^1, V) is well defined up to $[0, t_{i+1})$, we have

$$\mathbb{P}(E) = \int_\Omega \xi(\omega, E)\, \mathbb{P}(d\omega) = \int_\Omega 1\, \mathbb{P}(d\omega) = 1. \tag{2.41}$$

We now show that $(H^0, H^1, V(H))$ satisfies the system (FBSDE 1) on $[0, t_{n+1}]$ under $\mathbb{P}$ with initial condition $X^0_0 = 1$, $X^1_0 = x$, and terminal condition $V_{t_{i+1}}(H) \geq J_{i+1}$, almost surely.

That (2.29) and (2.30) are satisfied has already been established *a priori* in the construction of the market model.

For $t \in [t_i, t_{i+1}]$, Eq. (2.31) is satisfied. Indeed, denoting $E_{t_i} := e^{-rt_i}(V_{t_i}(H) - Z^{0,x}_{t_i} X^0_{t_i} -$

$Z_{t_i}^{1,x} X_{t_i}^1$), we compute

$$
\begin{aligned}
dV_t(H) &= E_{t_i} dX_t^0 + dZ_t^{0,x} X_t^0 + dZ_t^{1,x} X_t^1 + Z_t^{0,x} dX_t^0 + Z_t^{1,x} dX_t^1 \\
&= E_{t_i} dX_t^0 + Z_t^{0,x} dX_t^0 + Z_t^{1,x} dX_t^1 \qquad (2.42) \\
&= H_t^0 dX_t^0 + H_t^1 dX_t^1.
\end{aligned}
$$

But we may also compute, for $s \in [0, t_i]$ and $t \in [t_i, t_{i+1}]$,

$$
\begin{aligned}
V_t(H) - V_s(H) &= (V_t(H) - V_{t_i}(H)) + (V_{t_i}(H) - V_s(H)) \\
&= \int_{t_i}^{t} H_r^0 dX_r^0 + \int_{t_i}^{t} H_r^1 dX_r^1 + \int_{s}^{t_i} H_r^0 dX_r^0 + \int_{s}^{t_i} H_r^1 dX_r^1 \qquad (2.43) \\
&= \int_{s}^{t} H_r^0 dX_r^0 + \int_{s}^{t} H_r^1 dX_r^1,
\end{aligned}
$$

which shows that (2.31) holds on $[0, t_{i+1}]$. Finally, we check the self financing condition (2.32). Recall that we have assumed inductively that (2.32) holds on $[0, t_i]$. Thus we need only check that (2.32) is satisfied for times s, t in the interval $[0, t_{i+1}]$, with $s < t$, and $t \in (t_i, t_{i+1}]$.

Assume first $s \leq t_i$. Then we have

$$
\int_{s}^{t} X_r^0 dH_r^0 + \int_{s}^{t} X_r^1 dH_r^1 \qquad (2.44)
$$

$$
= \int_{[s,t_i)} X_r^0 dH_r^0 + \int_{[s,t_i)} X_r^1 dH_r^1 + \int_{(t_i,t]} X_r^0 dH_r^0 + \int_{(t_i,t]} X_r^1 dH_r^1 \qquad (2.45)
$$

$$
+ X_{t_i}^0 (H_{t_i}^{0+} - H_{t_i}^0) + X_{t_i}^1 (H_{t_i}^{1+} - H_{t_i}^1). \qquad (2.46)
$$

The first two terms are 0 by the induction hypothesis. We claim that the third and fourth term are 0 as well. Indeed, on $[t_i, t_{i+1})$, we have $H^0 = Z_t^{0,x} + E_{t_i}$, and $H^1 = Z_t^{1,x}$, so that for $t \in [t_i, t_{i+1})$,

$$
dH_t^0 = dZ_t^{0,x} \ ; \ dH_t^1 = dZ_t^{1,x},
$$

and thus

$$\int_{(t_i,t]} X_r^0 dH_r^0 + \int_{(t_i,t]} X_r^1 dH_r^1 = \int_{(t_i,t]} X_r^0 dZ_r^0 + \int_{(t_i,t]} X_r^1 dZ_r^1 \tag{2.47}$$

$$= 0, \tag{2.48}$$

by (FBSDE 2).

Now, for the last two terms, we note that $X_{t_i}^0 H_{t_i}^0 + X_{t_i}^1 H_{t_i}^1 = V_{t_i}$ almost surely, so

$$\int_s^t X_r^0 dH_r^0 + \int_s^t X_r^1 dH_r^1 \tag{2.49}$$

$$= X_{t_i}^0 H_r^{0+} + X_{t_i}^1 H_r^{1+} - V_{t_i}(H) \tag{2.50}$$

$$= X_{t_i}^0 \big[Z_{t_i}^{0,x} + \frac{1}{X_{t_i}^0}(V_{t_i}(H) - Z_{t_i}^{0,x} X_{t_i}^0 - Z_{t_i}^{1,x} X_{t_i}^1)\big] + X_{t_i}^1 Z_{t_i}^{1,x} - V_{t_i}(H) \tag{2.51}$$

$$= X_{t_i}^0 Z^{0,x} + V_{t_i}(H) - X_{t_i}^0 Z_{t_i}^{0,x} - X_{t_i}^1 Z_{t_i}^{1,x} + X_{t_i}^1 Z_{t_i}^{1,x} - V_{t_i}(H) \tag{2.52}$$

$$= 0. \tag{2.53}$$

Next, assume $s \in (t_i, t_{i+1}]$. Then we simply have

$$\int_s^t X_r^0 dH_r^0 + \int_s^t X_r^1 dH_r^1 = \int_s^t X_r^0 dZ_r^0 + \int_s^t X_r^1 dZ_r^1 \tag{2.54}$$

$$= 0. \tag{2.55}$$

Thus $X_t^0 dH_t^0 + X_t^1 dH_t^1 = 0$ for $t \in [0, t_{i+1}]$, and so we verify the self financing condition (2.32).

It remains to check that the terminal condition $V_{t_{i+1}} \geq J_{i+1}$ holds almost surely. But by construction, $V_{t_{i+1}} \geq J_{i+1}$, $\mu(\omega, \cdot)$-almost surely for each ω, so we have, denoting by F the event $\{V_{t_{i+1}} \geq J_{i+1}\}$,

$$\mathbb{P}(F) = \int_\Omega \xi(\omega, F)\, \mathbb{P}_{|\mathcal{F}_{t_i}}(d\omega) = \int_\Omega 1\, \mathbb{P}_{|\mathcal{F}_{t_i}}(d\omega) = 1 \tag{2.56}$$

where $\mathbb{P}_{|\mathcal{F}_{t_i}}$ denotes the pushforward measure $I_*\mathbb{P}$ of $\mathbb{P}$ under the identity map $I : (\Omega, \mathcal{F}_{t_i}) \to (\Omega, \mathcal{F})$.

Hence we conclude that the FBSDE holds on the interval $[0, t_{i+1}]$.

Inductively, we obtain a solution (V, H^0, H^1) on the whole interval $[0, T]$.

To see that this is a minimal hedge, we note that non negativity holds since on each interval $[t_i, t_{i+1}]$ the value process is the sum of the two non negative portfolios (Z^0, Z^1) and $\left(e^{-rt_i}(V_{t_i}(H) - Z^0_{t_i} X^0_{t_i} - Z^1_{t_i} X^1_{t_i}), 0\right)$, where non negativity of the second portfolio follows from the fact that $V_{t_i}(H) \geq J_{t_i}$.

Further, the hedging property holds since $V_{t_i}(H) \geq J_i \geq b(X^1_{t_i})$ for all i, so $V_\tau(H) \geq b(X_\tau)$ almost surely.

And finally since $V_{t_1} = J_1$, we have that the initial investment V_0 is

$$V_0 = D_0(J_1) = D_0 \ldots D_n(U). \tag{2.57}$$

So the hedge achieves the infimal hedging price, and is thus a minimal hedge.

2.4 A simple example

In order to illustrate the results we have just obtained, we derive an explicit expression for the minimal super-replicating hedge and hedging price associated to a particular life contingent option. Due to the iterated conditional expectations in Eq. (2.9), it is in general difficult, or impossible to obtain closed form solutions for the super-replication price, and minimal hedge. However, we will be able to do so here for a very simple case.

Consider a life contingent option $f_\tau = b(X_\tau)1_{\{\tau<T\}} + c(X_\tau)1_{\{\tau=T\}}$ with only two exercise times $0 < t_1 < t_2 = T$, where we assume $\mathbb{P}(\tau = t_i) \neq 0$ for each $i = 1, 2$. We suppose that the payoffs b, c are given by $b(x) = \max(K, x)$ and $c(x) = x$ respectively, where $K > 0$ is some strike price. We note that $b(x) = x + (K - x)^+$, that is, it is the combination of a long position in the stock and a put option in the stock.

Thus the life contingent option pays off one unit of the stock plus the payoff of a put option in the stock if death occurs before expiry, otherwise it just pays off one unit of the stock.

Due to the iterated conditional expectations in the expression for the super-replication price in Theorem 1, it is often difficult or impossible to obtain closed form expressions for the super-replication price and super-replicating portfolio. Below we present a simple example for which we will be able to give explicit expressions for both.

Proposition 5. *Let the market model as in Section 2. Consider a life contingent option $f_\tau = b(X_\tau)1_{\{\tau<T\}} + c(X_\tau)1_{\{\tau=T\}}$ with only two exercise times $0 < t_1 < t_2 = T$, where we assume $\mathbb{P}(\tau = t_i) \neq 0$ for each $i = 1, 2$. We suppose that the payoffs b, c are given by $b(x) = \max(K, x)$ and $c(x) = x$ respectively, where $K > 0$ is some strike price. We note that $b(x) = x + (K - x)^+$, that is, it is the combination of a long position in the stock and a put option in the stock.*

Then the minimal initial investment for a super-replicating portfolio, π_0 is given by

$$\pi_0 = X_0^1 + \mathbb{E}^{\mathbb{Q}}[e^{-rt_1}(K - X_{t_1}^1)^+], \tag{2.58}$$

where we recall that $\mathbb{Q}$ is an equivalent probability measure under which the discounted asset prices are martingales.

A minimal super-replicating hedge $H := (H^0, H^1)$ is given by

$$H_t^0 := \begin{cases} Ke^{-rt_1}\left(1-\Phi\left(\frac{\log(\frac{X_t^1}{K})+(t_1-t)(r-\frac{\sigma^2}{2})}{\sigma\sqrt{t_1-t}}\right)\right) & \text{for } 0 \le t \le t_1, \\ e^{-rt_1}(K-X_{t_1}^1)^+ & \text{for } t_1 < t \le T, \end{cases} \tag{2.59}$$

$$H_t^1 := \begin{cases} \Phi\left(\frac{\log(\frac{X_t^1}{K})+(t_1-t)(r+\frac{\sigma^2}{2})}{\sigma\sqrt{t_1-t}}\right) & \text{for } 0 \le t \le t_1, \\ 1 & \text{for } t_1 < t \le T, \end{cases} \tag{2.60}$$

where $\Phi(y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}z^2}\,dz$ denotes the cumulative distribution function of the standard normal.

Proof. We recall the notation

$$D_k(Y) := \max\left(b(X_{t_k}^1), \mathbb{E}^{\mathbb{Q}}\left[e^{-r(t_{k+1}-t_k)}Y\middle|\mathcal{F}_{t_k}\right]\right), \tag{2.61}$$

$$D_0(Y) := \mathbb{E}^{\mathbb{Q}}[e^{-rt_1}Y], \tag{2.62}$$

for any random variable Y. By Theorem 1, the minimal super-replication price π_0 is then given by

$$\pi_0 = D_0 D_1[c(X_T^1)]. \tag{2.63}$$

First, note that

$$
\begin{aligned}
D_1[c(X_T^1)] &= D_1[X_T^1] \\
&= \max\left(b(X_{t_1}^1), \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t_1)}X_T^1 \middle| \mathcal{F}_{t_1}\right]\right) \\
&= \max\left(b(X_{t_1}^1), X_{t_1}^1\right) \\
&= \max\left(\max\left(K, X_{t_1}^1\right), X_{t_1}^1\right) \\
&= \max(K, X_{t_1}^1),
\end{aligned}
\tag{2.64}
$$

where in the third equality we have used the martingale property of the discounted stock price under $\mathbb{Q}$.

To compute π_0, we have

$$
\begin{aligned}
\pi_0 &= D_0 D_1[c(X_T^1)] \\
&= D_0[\max(K, X_{t_1}^1)] \\
&= E^{\mathbb{Q}}[e^{-rt_1}\max(K, X_{t_1}^1)] \\
&= E^{\mathbb{Q}}[e^{-rt_1}X_{t_1}^1 + e^{-rt_1}(K - X_{t_1}^1)^+] \\
&= X_0^1 + E^{\mathbb{Q}}[e^{-rt_1}(K - X_{t_1}^1)^+],
\end{aligned}
\tag{2.65}
$$

where again in the second last equality, we have used the martingale property of the discounted stock price. Note that the minimal super-replication portfolio can be viewed as a combination of a long position in the underlying and a long position of an European put option with exercise time t_1.

Next, we derive the expression for the hedge. First we show that the given portfolio $H := (H^0, H^1)$ is well defined, that is, it satisfies assumption (C1) and (C2) in Definition 2.2.2. To this end, consider a European put option on the underlying with exercise time t_1, and strike price K, with corresponding payoff $g(X_{t_1}^1) = (K - X_{t_1}^1)^+$.

By standard results (see Elliott and Kopp (2013), Theorem 7.6.2), a replicating portfolio $R := (R^0, R^1)$ of the European put option with exercise time t_1 for $t \in [0, t_1]$ is given by

$$R_t^0 := Ke^{-rt_1}\left(1 - \Phi\left(\frac{\log(\frac{X_t^1}{K}) + (t_1 - t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{t_1 - t}}\right)\right), \tag{2.66}$$

$$R_t^1 := \Phi\left(\frac{\log(\frac{X_t^1}{K}) + (t_1 - t)(r + \frac{\sigma^2}{2})}{\sigma\sqrt{t_1 - t}}\right) - 1. \tag{2.67}$$

Noting this, we may write our portfolio H as

$$H_t^0 := \begin{cases} R_t^0 + J_t^0 & \text{for } 0 \leq t \leq t_1, \\ e^{-rt_1}(K - X_{t_1}^1)^+ & \text{for } t_1 < t \leq T, \end{cases} \tag{2.68}$$

$$H_t^1 := \begin{cases} R_t^1 + J_t^1 & \text{for } 0 \leq t \leq t_1, \\ 1 & \text{for } t_1 < t \leq T, \end{cases} \tag{2.69}$$

where $J_t^0 = 0, J_t^1 = 1$. Now we check that assumption (C2) holds. Indeed, H is clearly nonnegative on $(t_1, T]$, since the holdings in both the riskless and underlying asset are nonnegative, while on $(0, t_1)$ it is the sum of the two nonnegative portfolios R and $J = (J^0, J^1) := (0, 1)$.

Next, we check the self financing condition (C1). On $(0, t_1]$, it is the sum of the two self financing portfolios R and J, and thus is self financing on this interval. On the other hand, on the interval $(t_1, T]$, $dH^0 = dH^1 = 0$, and thus we need only check the self financing condition for times s, t with $s < t_1 < t$. To this end, we compute

$$\begin{aligned}
&\int_s^t X_t^0 dH_t^0 + \int_s^t X_t^1 dH_t^1 \\
&= \int_s^{t_1} X_t^0 dH_t^0 + \int_s^{t_1} X_t^1 dH_t^1 + X_{t_1}^0 (H_{t_1}^{0+} - H_{t_1}^0) + X_{t_1}^1 (H_{t_1}^{1+} - H_{t_1}^1) \\
&= X_{t_1}^0 (H_{t_1}^{0+} - H_{t_1}^0) + X_{t_1}^1 (H_{t_1}^{1+} - H_{t_1}^1) \\
&= X_{t_1}^0 H_{t_1}^{0+} + X_{t_1}^1 H_{t_1}^{1+} - V_{t_1}(H) \\
&= e^{rt_1}(e^{-rt_1}(K - X_{t_1}^1)^+) + X_{t_1}^1 - [(K - X_{t_1}^1)^+ + X_{t_1}^1] \\
&= 0,
\end{aligned} \tag{2.70}$$

where we have written $H_{t_1}^{0+}$ to denote $\lim_{t \to t_1^+} H_t^0$ and likewise for $H_{t_1}^{1+}$.

Thus the portfolio H is a self financing portfolio. Now we check that it is indeed a super-replicating portfolio for the lifetime contingent option. It will suffice to check that $V_{t_i}(H) \geq f_{t_i}$ a.s. for $i = 1, 2$. But for $i = 1$, we see that

$$\begin{aligned}
V_{t_1}(H) &= V_{t_1}(R) + V_{t_i}(J) \\
&= (K - X_{t_1}^1)^+ + X_{t_1}^1 \\
&= \max(K, X_{t_1}^1) \\
&= f_{t_1},
\end{aligned} \tag{2.71}$$

while for $i = 2$, i.e. at time $t_2 = T$, we have

$$\begin{aligned}
V_T(H) &= X_T^1 + X_T^0 e^{-rt_1}(K - X_T^1)^+ \\
&\geq X_T^1 \\
&= f_T.
\end{aligned} \tag{2.72}$$

This shows that the portfolio is super-replicating, as desired. Finally, we check that H is a minimal super replicating portfolio. It will suffice to check that H achieves the minimal super-replication price, which by Thoerem 1 we know to be

$$\pi_0 := X_0^1 + \mathbb{E}^{\mathbb{Q}}[e^{-rt_1}(K - X_{t_1}^1)^+]. \tag{2.73}$$

But by writing

$$\begin{aligned} V_0(H) &= V_0(J) + V_0(R) \\ &= X_0^1 + \mathbb{E}^{\mathbb{Q}}[e^{-rt_1}(K - X_{t_1}^1)^+] \\ &= \pi_0. \end{aligned} \tag{2.74}$$

we verify this immediately. This concludes the proof.

2.5 Existence of a replicating portfolio

Given that a super-replication trading strategy exists, a natural question to ask is - when is this super-replication portfolio a *replication portfolio*? That is, the payoff of the portfolio is exactly the same as the option payoff at the exercise time. To make the above precise, we record here a few initial definitions.

Definition 2.5.1 (Replication portfolio)**.** *A replication portfolio for the life contingent option f is a self financing portfolio (H^0, H^1) with associated value process V such that $V_\tau(H) = f_\tau$ almost surely.*

Definition 2.5.2 (Attainable)**.** *We say that replication of the life contingent option f_τ is attainable if for all stopping times τ independent of the process taking finitely many values, there exists a replication portfolio (H^0, H^1) for f.*

We first state a lemma that says that martingales remain martingales under enlargement of the underlying filtration by an independent σ-algebra. Since the proof is technical but straightforward, we relegate it to the end of the chapter so as to not interrupt the flow of discussion.

Lemma 2. *Let M be a martingale under $\mathbb{Q}$ with respect to some filtration $\{\mathcal{F}_t\}$, and suppose $\mathcal{H}$ is a σ-algebra independent of $\{\mathcal{F}_t\}$. Then M is a martingale under $\mathbb{Q}$ with respect to $\mathcal{F}_t \vee \mathcal{H}$.*

Now we state the main theorem of this section.

Theorem 2. *Replication is attainable if and only if the discounted option price process $\tilde{f}_t := \mathbf{1}_{\{t<T\}} e^{-rt} b(X_t^1) + \mathbf{1}_{\{t=T\}} e^{-rT} c(X_T^1)$ is a $\mathcal{F}_t$ martingale on $(0,T]$ under the equivalent martingale measure $\mathbb{Q}$.*

Proof. Let τ be an arbitrary stopping time taking finitely many values $0 < t_1 < \cdots < t_n < t_{n+1} = T$; independently of the asset prices, and suppose the discounted option price process $\tilde{f}_t$ were a martingale under $\mathbb{Q}$. As usual, we set $t_0 = 0$. We choose τ such that $\mathbb{P}(\tau = T) \neq 0$.

We recall the notation

$$D_k(Y) := \max\left(b(X_{t_k}^1), \mathbb{E}^{\mathbb{Q}}\left[e^{-r(t_{k+1}-t_k)} Y \middle| \mathcal{F}_{t_k}\right]\right) \tag{2.75}$$

for an L^1 random variable Y.

We claim that

$$D_k \ldots D_n (U) = b(X_{t_k}^1) \tag{2.76}$$

for all $1 \leq k \leq n$, where U is as defined in Theorem 1.

We prove by backward induction on k. For the base case $k = n$, since $\tilde{f}_t$ is a martingale under $\mathbb{Q}$, we have

$$\mathbb{E}^{\mathbb{Q}}[\tilde{f}_T|\mathcal{F}_{t_n}] = \tilde{f}_{t_n} \tag{2.77}$$

$$\implies \mathbb{E}^{\mathbb{Q}}[e^{-rT}c(X_T^1)|\mathcal{F}_{t_n}] = e^{-rt_n}b(X_{t_n}^1)$$

$$\implies \mathbb{E}^{\mathbb{Q}}[e^{r(t_n-T)}c(X_T^1)|\mathcal{F}_{t_n}] = b(X_{t_n}^1),$$

so that

$$\begin{aligned} D_n(U) &= \max\left(b(X_{t_n}^1), \mathbb{E}^{\mathbb{Q}}\left[e^{r(t_n-T)}c(X_T^1)\middle|\mathcal{F}_{t_n}\right]\right) \\ &= b(X_{t_n}^1), \end{aligned} \tag{2.78}$$

which proves the base case.

Now assume inductively

$$D_k \dots D_n(U) = b(X_{t_k}^1),$$

for some $1 \leq k \leq n$.

Then we compute

$$\begin{aligned} D_{k-1} \dots D_n(U) &= D_{k-1}\big(b(X_{t_k}^1)\big) \\ &= \max\Big(b(X_{t_{k-1}}^1), \mathbb{E}^{\mathbb{Q}}\big[e^{r(t_{k-1}-t_k)}b(X_{t_k}^1)|\mathcal{F}_{t_{k-1}}\big]\Big) \\ &= b(X_{t_{k-1}}^1), \end{aligned} \tag{2.79}$$

where in the last step we have used the martingale property of $\tilde{f}_t$, i.e

$$\mathbb{E}^{\mathbb{Q}}\left[e^{r(t_{k-1}-t_k)}b(X^1_{t_k})|\mathcal{F}_{t_{k-1}}\right] = b(X^1_{t_{k-1}}). \tag{2.80}$$

This proves the induction step.

Now let $H = (H^0, H^1)$ be the super-replication trading strategy constructed in the proof of Theorem 1. Then by construction, the value process $V(H)$ satisfies

$$V_{t_1}(H) = D_1 \ldots D_n(U) = b(X^1_{t_1}), \tag{2.81}$$

almost surely.

We claim this means that $H^0 = Z^{0,X^1_{t_1}}$, $H^1 = Z^{1,X^1_{t_1}}$, $\mathbb{P}$-almost surely on $[t_1, t_2]$, where Z is as defined in the proof of Theorem 1.

Indeed, by similar reasoning to the beginning of the proof of Proposition 2, $V_{t_1}(H) = b(X^1_{t_1})$ almost surely. Thus the second term in the definition of H^0 in the proof of Theorem 1 vanishes almost surely, which proves that $H^0 = Z^{0,X^1_{t_1}}$, $H^1 = Z^{1,X^1_{t_1}}$, $\mathbb{P}$-almost surely on $[t_1, t_2]$, as claimed.

We conclude

$$V_{t_2} = Z^{0,X^1_{t_1}}_{t_2} X^0_{t_2} + Z^{1,X^1_{t_1}}_{t_2} X^1_{t_2} = b(X^1_{t_2}), \tag{2.82}$$

$\mathbb{Q}$ (and hence $\mathbb{P}$)-almost surely.

That the above holds $\mathbb{P}$ almost surely as well is due to the fact that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures.

Similarly, we can prove inductively that $V_{t_k}(H) = b(X^1_{t_k})$ for all $3 \le k \le n$, and $V_T = c(X^1_T)$. Thus $V_\tau(H) = f_\tau$ almost surely, and so the minimal hedge is a replicating portfolio.

This concludes the "if" direction.

We prove the “only if" direction by contradiction. Let f_t denote the option payoff at time t. Then any replication portfolio for f_τ must have initial investment $\mathbb{E}^{\mathbb{Q}}(e^{-r\tau}V_\tau)$.

Indeed, since $\mathbb{Q}$ is an equivalent martingale measure, the discounted value process $\tilde{V}_t(H) := e^{-rt}V_t(H)$ of the replicating portfolio H is a $\mathcal{F}_t$ martingale under $\mathbb{Q}$. By Lemma 1, it is also a $\mathcal{F}_t \vee \mathcal{G}_\tau$ martingale under $\mathbb{Q}$, where we recall that $\mathcal{G}_\tau$ is the σ-algebra generated by the stopping time τ.

By the optional stopping theorem, we then have

$$\hat{V}_0(H) = \mathbb{E}^{\mathbb{Q}}[\hat{V}_0] = \mathbb{E}^{\mathbb{Q}}[\hat{V}_\tau(H)] = \mathbb{E}^{\mathbb{Q}}[e^{-r\tau}V_\tau(H)], \tag{2.83}$$

as claimed.

We note that, denoting by $f_t := \mathbf{1}_{\{t<T\}}b(X^1t) + \mathbf{1}_{\{t=T\}}c(X_T^1)$ the undiscounted option payoff process, any replicating portfolio must thus have initial investment $\mathbb{E}^{\mathbb{Q}}(e^{-r\tau}\tilde{f}_\tau)$.

Now assume that $\tilde{f}_t$ is not a martingale on $(0, T]$. Then there exist times $s, t \in (0, T]$ with $s < t$ such that $\mathbb{E}^{\mathbb{Q}}[\tilde{f}_t|\mathcal{F}_s] \neq \tilde{f}_s$. Thus either $\mathbb{E}^{\mathbb{Q}}[\tilde{f}_t|\mathcal{F}_s] > \tilde{f}_s$ with positive probability, or $\tilde{f}_s > \mathbb{E}^{\mathbb{Q}}[\tilde{f}_t|\mathcal{F}_s]$ with positive probability.

Take τ to be a stopping time equal to s or t with probability $\frac{1}{2}$ each, independently of the asset filtration. Suppose for contradiction that there existed a replication portfolio $H = (H^0, H^1)$ for the tuple (b, c, τ) with associated value process V.

In the case where

$$\mathbb{E}^{\mathbb{Q}}[\tilde{f}_t|\mathcal{F}_s] > \tilde{f}_s, \tag{2.84}$$

with positive probability, we have that

$$\mathbb{E}^{\mathbb{Q}}[e^{-rt}f_t|\mathcal{F}_s] > e^{-rs}f_s, \tag{2.85}$$

with positive probability, which implies that

$$\max(f_s, \mathbb{E}^{\mathbb{Q}}[e^{r(s-t)} f_t|\mathcal{F}_s]) > f_s, \tag{2.86}$$

with positive probability.

Writing for convenience D_1, D_2 as D_s, D_t respectively, by Proposition 1 we have that

$$V_s \geq D_s D_t(f_T) = D_s\left(\max\left(f_t, \mathbb{E}^{\mathbb{Q}}\left[e^{r(t-T)} f_T|\mathcal{F}_t\right]\right)\right) \geq D_s(f_t) \tag{2.87}$$

$$= \max\left(f_s, \mathbb{E}^{\mathbb{Q}}[e^{r(s-t)} f_t|\mathcal{F}_s]\right) > f_s, \tag{2.88}$$

with positive probability, contradicting the fact that (H^0, H^1) is a replicating portfolio.

On the other hand, if $\tilde{f}_s > \mathbb{E}[\tilde{f}_t|\mathcal{F}_s]$ with positive probability, we have

$$\max(f_s, \mathbb{E}^{\mathbb{Q}}[e^{r(s-t)} f_t|\mathcal{F}_s]) > \mathbb{E}^{\mathbb{Q}}[e^{r(s-t)} f_t|\mathcal{F}_s]), \tag{2.89}$$

with positive probability.

Since the portfolio is assumed to be a replication portfolio, the associated value process satisfies $V_s = f_s$, so by the assumption that $\tilde{f}_s > \mathbb{E}[\tilde{f}_t|\mathcal{F}_s]$, we have $V_s > \mathbb{E}^{\mathbb{Q}}[e^{r(s-t)} f_t|\mathcal{F}_s]$ with positive probability.

Further, we note that by similar arguments to the beginning of the proof of Proposition 1, we must have that $V_t = f_t$ almost surely - that is, the portfolio (H^0, H^1) is a replicating portfolio for the simple European type claim f_t. By Theorem 7.13 in Elliott and Kopp (2013), the replicating portfolio is unique, and further by Lemma 7.5.9 in Pascucci and Agliardi (2011), we have that the discounted value process $\tilde{V}$ satisfies $\tilde{V}_r = \mathbb{E}^{\mathbb{Q}}[\tilde{f}_t|\mathcal{F}_r]$ for $r \in [0, t]$.

In particular, the discounted value process is a martingale on $[0, t]$. Thus we have that

$\mathbb{E}^{\mathbb{Q}}[\tilde{V}_t|\mathcal{F}_s] = \tilde{V}_s$, that is, $\mathbb{E}^{\mathbb{Q}}[e^{r(s-t)}V_t|\mathcal{F}_s] = V_s$, almost surely.

Since the portfolio is replicating, $V_t = f_t$, so we have simultaneously

$$\mathbb{E}^{\mathbb{Q}}[e^{r(s-t)} f_t|\mathcal{F}_s] = V_s, \tag{2.90}$$

almost surely, and

$$\mathbb{E}^{\mathbb{Q}}[e^{r(s-t)} f_t|\mathcal{F}_s] < V_s, \tag{2.91}$$

with positive probability, contradiction.

This concludes the "only if" direction.

2.6 Technical Proofs

Proof. Let $(C, \mathcal{B}(C))$ denote the Wiener space of continuous functions $f : [0, \infty) \to \mathbb{R}$ with $f(0) = 0$. Consider also $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the space of real numbers with its usual Borel sigma algebra.

Further, let $\mathbb{P}_0$ denote the Wiener measure on C; that is, the law of a standard Brownian motion. Let $\mathbb{P}_1$ an arbitrary probability measure on $\mathbb{R}$ supported on finitely many values in $(0, T]$, to be interpreted as the law of the stopping time τ.

Now let

$$(\Omega, \mathcal{F}, \mathbb{P}) := (C \times \mathbb{R}, \mathcal{B}(C) \otimes \mathcal{B}(\mathbb{R}), \mathbb{P}), \tag{2.92}$$

where $\mathbb{P} := \mathbb{P}_0 \times \mathbb{P}_1$,. Then the probability space supports a Brownian motion W and a stopping time τ independent of each other. Indeed, we may set $W(\omega, r) = \omega$, and $\tau(\omega, r) = r$. By construction, W is a Brownian motion, τ is a stopping time with prescribed law $\mathbb{P}_1$, and since $\mathbb{P}$ is a product measure, W and τ are independent of each other.

Finally, we note that Ω, being the product of Radon probability spaces is itself a Radon probability space. This concludes the proof.

Proof. [Proof of Lemma 2:]

The main technical tool here will be the monotone class lemma.

We need to show that for all $s, t \geq 0$ with $s < t$, $\mathbb{E}[M_t|\mathcal{F}_s \vee \mathcal{H}] = M_s$ almost surely. This means that for all events $E \in \mathcal{F}_t \vee \mathcal{H}$,

$$\int_E M_s \, d\mathbb{P} = \int_E M_t \, d\mathbb{P}. \tag{2.93}$$

rewla

First, assume $E \in \mathcal{F}_s$. Then Eq. (2.93) follows from the fact that M is an $\mathcal{F}_t$-martingale. Next, for events $E \in \mathcal{H}$, we compute

$$\begin{aligned}\int_E M_s \, d\mathbb{P} &= \int_\Omega \mathbf{1}_E M_s \, d\mathbb{P} = \int_\Omega \mathbf{1}_E \, d\mathbb{P} \int_\Omega M_s \, d\mathbb{P} \\ &= \int_\Omega \mathbf{1}_E \, d\mathbb{P} \int_\Omega M_t \, d\mathbb{P} = \int_\Omega \mathbf{1}_E M_t \, d\mathbb{P} \\ &= \int_E M_t \, d\mathbb{P}.\end{aligned} \tag{2.94}$$

So Eq. (2.93) holds for events $E \in \mathcal{H}$. Next, assume E is of the form $F \cap H$ for events $F \in \mathcal{F}_t$ and $H \in \mathcal{H}$. Then we compute

$$\begin{aligned}\int_E M_s \, d\mathbb{P} &= \int_{F\cap H} M_s \, d\mathbb{P} = \int_\Omega \mathbf{1}_F \mathbf{1}_H M_s \, d\mathbb{P} = \int_\Omega \mathbf{1}_H \, d\mathbb{P} \int_\Omega \mathbf{1}_F M_s d\mathbb{P} \\ &= \int_\Omega \mathbf{1}_H d\mathbb{P} \int_F M_s \, d\mathbb{P} = \int_\Omega \mathbf{1}_H \, d\mathbb{P} \int_F M_t \, d\mathbb{P} \\ &= \int_\Omega \mathbf{1}_F \mathbf{1}_H M_t \, d\mathbb{P} = \int_{F\cap H} M_t \, d\mathbb{P} \\ &= \int_E M_t \, d\mathbb{P}.\end{aligned} \tag{2.95}$$

Now assume E is of the form $F \cup H$ for events $F \in \mathcal{F}_t$ and $H \in \mathcal{H}$. We note we can write

E as $(F^c \cap H^c)^c$, thus we have

$$\begin{aligned}
\int_E M_s \, d\mathbb{P} &= \int_{(F^c \cap H^c)^c} M_s \, d\mathbb{P} = \int_\Omega M_s \, d\mathbb{P} - \int_{(F^c \cap H^c)} M_s \, d\mathbb{P} \\
&= \int_\Omega M_t \, d\mathbb{P} - \int_{(F^c \cap H^c)} M_t \, d\mathbb{P} = \int_{(F^c \cap H^c)^c} M_s \, d\mathbb{P} \\
&= \int_E M_t \, d\mathbb{P}.
\end{aligned} \tag{2.96}$$

So Eq. (2.93) holds for all events E of the above four types.

We then note that the set S of events of the above four types forms a algebra of sets containing both $\mathcal{F}_s$ and $\mathcal{H}$. Thus by the monotone class lemma, we will have Eq. (2.93) for all events in $\mathcal{F}_s \vee H$ once we prove that Eq. (2.93) is preserved under increasing unions and decreasing intersections. But this follows immediately from the monotone convergence theorem - indeed, if $A_n \in S$ are such that $A_n \uparrow A$ we have

$$\begin{aligned}
\int_A M_s \, d\mathbb{P} &= \int_\Omega \mathbf{1}_A M_s \, d\mathbb{P} = \int_\Omega \lim_{n\to\infty} \mathbf{1}_{A_n} M_s \, d\mathbb{P} \\
&= \lim_{n\to\infty} \int_\Omega \mathbf{1}_{A_n} M_s \, d\mathbb{P} = \lim_{n\to\infty} \int_{A_n} M_s d\mathbb{P} \\
&= \lim_{n\to\infty} \int_{A_n} M_t d\mathbb{P} = \lim_{n\to\infty} \int_\Omega \mathbf{1}_{A_n} M_t d\mathbb{P} \\
&= \int_\Omega \lim_{n\to\infty} \mathbf{1}_{A_n} M_t \, d\mathbb{P} = \int_\Omega \mathbf{1}_A M_t \, d\mathbb{P} \\
&= \int_A M_t \, d\mathbb{P},
\end{aligned} \tag{2.97}$$

where the third, and third to last equalities follow from the monotone convergence theorem.

The proof when A_n decrease to A follows from the above by taking complements, and using the identity $\int_A X \, d\mathbb{P} = \int_\Omega X \, d\mathbb{P} - \int_{A^c} X \, d\mathbb{P}$.

So the set of events for which Eq. (2.93) holds is a monotone class containing $\mathcal{F}_s$ and $\mathcal{H}$, and thus it contains $\mathcal{F}_s \vee \mathcal{H}$.

Thus $\int_E M_s \, d\mathbb{P} = \int_E M_t \, d\mathbb{P}$ for all $E \in \mathcal{F}_t \vee \mathcal{H}$, so that M is a $\mathcal{F}_s \vee \mathcal{H}$ martingale as claimed.

CHAPTER 3: SHORT MATURITY BARRIER ASIAN OPTIONS

3.1 Overview

In this chapter, we investigate barrier Asian options close to maturity. Asian options are unique in the sense that their payoff depends on the entire history of the process before the exercise time. As such, it is often difficult to obtain closed form expressions for the fair price of Asian options. However, with our limiting result, we will be able to obtain closed form asymptotic expressions for the price of short maturity Asian options with barrier. The barrier component means that the option may only be exercised upon the price hitting a certain agreed upon price level, while short maturity implies that the option is priced at a time close to the exercise time.

We work again in a Black-Scholes framework. In order to fix notation and for reading convenience, we recall the market model here. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. The probability measure $\mathbb{P}$ is referred to as the physical measure. We consider the following two processes

$$X^0, X^1 : [0,T] \times \Omega \to \mathbb{R}_+, \tag{3.1}$$

known as the *bond* and *stock* price respectively satisfying the stochastic differential equation (SDE)

$$dX_t^0 = rX_t^0\, dt, \tag{3.2}$$

$$dX_t^1 = X_t^1 \left(\mu\, dt + \sigma\, dW_t\right), \tag{3.3}$$

with $X_0^0 = 1$, $X_0^1 = x_0$ a.s. for some $x_0 \in \mathbb{R}_+$, and $W = \{W_t\}_{t\in[0,T]}$ is a standard Brownian motion with T denoting a fixed and finite time horizon. Here $r, \mu, \sigma \in \mathbb{R}$ are positive constants, known as the *risk free interest rate*, *drift* and *volatility* of the stock respectively. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by the Brownian motion W. We assume $\mathbb{F}$ satisfies the usual conditions, that is, it is right continuous and contains all $\mathbb{P}$-null sets.

Now we define the primary object of our investigation, the barrier Asian option. By definition, the payoff of an Asian call option *without* barrier, with strike price $K > 0$ and maturity time T is given by

$$\left(\frac{1}{T} \int_0^T X_t \, dt - K \right)_+ . \tag{3.4}$$

Thus the option payoff is the positive difference between the average price of the asset over the maturity period and the strike price, agreed upon in advance.

Now we introduce the barrier component. This means that the option can only be exercised if the asset price hits a certain threshold before maturity, agreed upon in advance. In our case, we will take the barrier to be above the current price of the asset, thus our option is a so called up and in option. The payoff of a barrier Asian call option with barrier $B > 0$, strike price $K > 0$ and maturity time $T > 0$ is thus given by

$$\left(\frac{1}{T} \int_0^T X_t \, dt - K \right)_+ \mathbf{1}_{\{\max_{0 \leq t \leq T} X_t \geq B\}} . \tag{3.5}$$

The main problem we will try to answer in this chapter is that of the fair price of the above option in the limit as $T \to 0$. First, we take a short digression into the aspects of stochastic analysis which will be relevant for our work.

3.2 Large Noise Limits

As mentioned in the Introduction chapter, our analysis in this chapter rely crucially on results from stochastic analysis which we term *large noise limits*. In this section we offer an intuitive introduction to this line of study, as well as some motivation and heuristics.

A well studied phenomena in stochastic analysis is the behaviour of the solutions to stochastic differential equations as the noise intensity tends to zero. Typically, as the noise intensity tends to zero, the solution converges, in the sense of uniform convergence in probability on compacts, to the solution of a deterministic ODE driven by the drift coefficient of the SDE. Further, the distribution of the solution satisfies a large deviation principle as the noise tends to zero.

A prototypical example of such a scenario is as follows - consider, for each $\varepsilon > 0$ the solution to the SDE

$$dX_t = \mu(X_t)\, dt + \varepsilon\, dW_t, \tag{3.6}$$

with W a standard d-dimensional Brownian motion, $\mu : \mathbb{R}^d \to \mathbb{R}^d$ a sufficiently regular function, say Lipschitz, and $X_0 = x_0$ for $x_0 \in \mathbb{R}^d$ an arbitrary initial condition.

Then as $\varepsilon \to 0$, the solutions converge, in the sense of uniform convergence in probability on compacts to the solution of the deterministic unperturbed equation

$$dX_t = \mu(X_t)\, dt, \tag{3.7}$$

with the same initial condition $X_0 = x_0$.

We refer to Freidlin et al. (2012) for a proof, as well as the statement of the associated large deviations principle.

A natural counterpart to this line of study is the question of the behaviour of the solutions as the noise intensity grows large, in a suitable sense. One option would be to investigate the scenario in which $\varepsilon \to \infty$ in Eq (3.6).

Motivated by the application to pricing of Asian options, we opt however, to investigate a rather different scenario, namely, the behaviour of solutions conditional on the solution to the SDE itself taking a large value. This question is of interest even for the trajectories of the Brownian motion itself. A simple preliminary result in this direction is the following:

Let W be a standard one dimensional Brownian motion.

For every $\varepsilon > 0$, let A_ε denote the event

$$\{W_T \geq \frac{1}{\varepsilon}\} , \tag{3.8}$$

and let $\mathbb{P}_\varepsilon$ be the probability measure given by

$$\mathbb{P}_\varepsilon(E) = \frac{\mathbb{P}(E \cap A_\varepsilon)}{\mathbb{P}(A_\varepsilon)} , \tag{3.9}$$

for all events E.

We denote by $\mathbb{E}_{\mathbb{P}^\varepsilon}$ the expectation under $\mathbb{P}_\varepsilon$.

Then we have

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mathbb{P}^\varepsilon} \Big[\sup_{0 \leq t \leq 1} |\varepsilon W_t - t| \Big] = 0. \tag{3.10}$$

We provide a proof of the above in the Additional Proofs section at the end of this chapter. Thus already on the level of trajectories of the Brownian motion we observe some nontrivial limiting behaviour - the trajectories of the Brownian motion converge simply to a deterministic straight line path, after a suitable regularisation.

Of even greater interest are the implications of this phenomenon on solutions to SDE driven by the Brownian motion. We show in this chapter that for the geometric Brownian motion, an analogue of the above result holds, with the limiting path being the solution to a deterministic exponential ODE. This result for the geometric Brownian motion will be the first main result of our chapter - we do not state it rigorously here but instead refer the reader to Theorem 4 for a full statement.

There is a small difference in the way we choose to perform the regularisation. We instead condition on the sample paths exceeding a fixed large value, and send the timeframe over which the process is defined to zero. By Brownian scaling this is essentially equivalent the earlier spatial rescaling procedure, however it will prove useful to have in this form for the purposes of our applications to Asian options. Further, the result in this form seems to be more fitting for generalisations to more general SDE.

A more significant difference is that we condition instead on the running maximum of the process, rather than the endpoint, taking a large value. This seems to us to be a stronger result than conditioning on the endpoint value alone, and will indeed be more suited for applications and extensions than the latter. This also places the current work more in line with the existing work on extreme value theory for Levy processes, which we elaborate more about shortly.

We briefly offer some heuristics as to why such a result might be expected. Conditioning on the geometric Brownian motion being large implies that the driving Brownian motion itself will be large - indeed, in the notation of the theorem, it can be seen that the maximum of the Brownian motion will be close to $\frac{\log B}{\sigma}$, where B is the barrier level. By the above reasoning, we have that $W_t \sim (t/T)\frac{\log B}{\sigma}$.
More boldly, we may write heuristically

$$dW_t \sim \frac{1}{T}\left(\frac{\log B}{\sigma}\right) dt. \tag{3.11}$$

Thus the defining SDE will be close to the deterministic ODE

$$dX_t = \left(\mu + \frac{\log B}{T}\right) X_t \, dt. \tag{3.12}$$

In the limit as the timeframe T tends to 0, the contribution from the μX_t term becomes negligible, and we are left with

$$dX_t = \frac{\log B}{T} X_t \, dt. \tag{3.13}$$

This is a standard exponential ODE, which is easily seen to solve to

$$X_t = B^{\frac{t}{T}}, \tag{3.14}$$

which is our limiting result.

As mentioned earlier, related work has been done on specific classes of Levy processes, under the name of extreme value theory. Namely for the classes of regularly varying, and heavy tailed jump Levy processes, conditional again on the maximum being large at some point, the limiting process obeys the so called "law of one jump" - the limiting process is piecewise constant with only one jump from the initial value to the conditional value. This is another manifestation of the large deviation principle, which says that extremely rare events are dominated by a single, "most likely" outcome. A seminal reference for this theory is Hult and Lindskog (2005).

3.3 Pricing of Short Maturity Barrier Asian Options

Here is the main theorem of this chapter. Suppose X is the risky asset price process in the Black-Scholes framework. For convenience, we restate the defining SDE for X:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_0 = 1 \text{ almost surely.} \tag{3.15}$$

Consider an out of the money up-and-in Asian option written on the stock price X with barrier $B > 1$, strike price $K > 0$ and maturity time $T > 0$.

Then as is well known, its fair price $C(B, K, T)$ is given by

$$C(B, K, T) = \mathbb{E}\left[\left(\frac{1}{T} \int_0^T X_t dt - K\right)_+ \mathbf{1}_{\{\max_{0 \le t \le T} X_t \ge B\}} \right]. \tag{3.16}$$

Theorem 3 (Asymptotics for short maturity Asian options)**.** *The fair price $C(B, K, T)$ of the Asian option satisfies the following short time asymptotics as $T \to 0^+$:*

$$C(B, K, T) = P(B, T) \left[\left(\frac{B-1}{\ln B} - K\right)_+ + O(\sqrt{T}) \right] \tag{3.17}$$

where $P(B, T) := \mathbb{P}(\max_{0 \le t \le T} X_t \ge B)$. The implied constant in the O notation depends only on σ, μ, and B.

Remark 2. We remark that $P(B, T)$ may be explicitly computed as

$$P(B, T) = 1 + B^{-1} \Phi\left(\frac{\frac{\sigma^2 T}{2} - \ln B}{\sigma \sqrt{T}} \right) - \Phi\left(\frac{\ln B - \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} \right), \tag{3.18}$$

using the probability density function of the running maximum of a geometric Brownian motion. Here Φ denotes the CDF of the standard normal distribution.

The above result relies crucially on the following *large noise limit* result, which we will in fact spend most of this chapter proving.

Theorem 4 (Large noise limit for geometric Brownian motion)**.** *Let X be the solution to the SDE*

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_0 = 1, \tag{3.19}$$

with W a standard one dimensional Brownian motion, and $\mu, \sigma > 0$ constants.

Let $B > 1$ be arbitrary. For every $T > 0$, let A_T denote the event

$$\{\max_{0 \le t \le T} X_t \ge B\}, \tag{3.20}$$

and let $\mathbb{P}_T$ be the probability measure given by

$$\mathbb{P}_T(E) = \frac{\mathbb{P}(E \cap A_T)}{\mathbb{P}(A_T)}, \tag{3.21}$$

for all events E.

Denote by $\mathbb{E}_{\mathbb{P}_T}$ the expectation under $\mathbb{P}_T$. Then we have

$$\mathbb{E}_{\mathbb{P}_T}\Big[\sup_{0<t<T} |X_t - B^{\frac{t}{T}}|\Big] = O(\sqrt{T}), \tag{3.22}$$

as $T \to 0^+$, where the implied constants in the O notation depend only on μ, σ, B.

Intuitively, the above theorem says that in the limit as $T \to 0$, conditional on hitting the barrier, the risky asset price process behaves almost deterministically - indeed, its price will be very close to the deterministic exponential function. The majority of the remainder of this section will be dedicated to the proof of Theorem 4.

We break the proof of Theorem 4 into a series of four lemmas, followed by the main proof. First we make some preliminary definitions.

For each $M \ge 0$ and $T > 0$, denote by $H_{M,T}$ the event $\{W_T \ge M\}$, and let $\mathbb{Q}_{M,T}$ be the probability measure given by

$$\mathbb{Q}_{M,T}(E) = \frac{\mathbb{P}(E \cap H_{M,T})}{\mathbb{P}(H_{M,T})}, \tag{3.23}$$

for all events E.

Throughout the first three lemmas, we assume that $f : (0, \infty) \to \mathbb{R}$ is a function such that $f(x) = O(x)$ as $x \to 0^+$.

Lemma 3. *We have*

$$\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[|W_T - M|\right] = O(T), \tag{3.24}$$

as $T \to 0^+$, where the implied constant in the O notation depends only on f, M.

Proof. Since W_T is a normal random variable with mean 0 and variance T, for any bounded nonnegative function $r : [0, \infty) \to \mathbb{R}$ we have

$$\begin{aligned}\mathbb{E}[|W_T - r(T)| \,\big|W_T \geq r(T)] &= \frac{(2\pi T)^{-1/2} \int_{r(T)}^{\infty} x e^{-\frac{x^2}{2T}}\, dx}{\mathbb{P}(W_T \geq r(T))} - r(T) \\ &= \frac{(2\pi T)^{-1/2} \int_{r(T)}^{\infty} x e^{-\frac{x^2}{2T}}\, dx}{\mathbb{P}(Z \geq \frac{r(T)}{\sqrt{T}})} - r(T)\end{aligned} \tag{3.25}$$

where in the second line Z is a standard normal random variable. Writing ϕ for the density of the standard normal, noting that $\mathbb{P}(Z \geq x) = (1 + O(\frac{1}{x^2}))\frac{\phi(x)}{x}$ (see for example, (Patel & Read, 1996), Chapter 3), we have

$$\mathbb{E}[|W_T - r(T)| \,\big|W_T \geq r(T)] = \frac{(2\pi T)^{-1/2} \int_{r(T)}^{\infty} x e^{-\frac{x^2}{2T}} dx}{\left(1 + O\left(\frac{T}{r(T)^2}\right)\right) \phi(\frac{r(T)}{\sqrt{T}}) / \frac{r(T)}{\sqrt{T}}} - r(T) \tag{3.26}$$

We find by elementary calculus,

$$\int_{r(T)}^{\infty} x e^{-\frac{x^2}{2T}} dx = T e^{-\frac{r(T)^2}{2T}}.$$

Substituting this into the above, we find

$$\begin{aligned}\mathbb{E}[|W_T - r(T)|\,\big|W_T \geq r(T)] &= \left(\frac{1}{1+O\left(\frac{T}{r(T)^2}\right)}\right)r(T) - r(T) \\ &= \left(1+O\left(\frac{T}{r(T)^2}\right)\right)r(T) - r(T) \\ &= O\left(\frac{T}{r(T)}\right)\end{aligned} \tag{3.27}$$

as $T \to 0^+$. Setting $r(T) = M - f(T)$, we find that

$$\mathbb{E}[|W_T - (M - f(T))|\,\big|W_T \geq M - f(T)] = O(T), \tag{3.28}$$

with the implied constant depending only on M. Applying the triangle inequality, and recalling that $f(T)$ is of order $O(T)$ then concludes the proof. □

Lemma 4. *For any constant $c > 0$, we have*

$$\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[|e^{cW_T} - e^{cM}|\right] = O(\sqrt{T}), \tag{3.29}$$

as $T \to 0^+$, with the implied constant depending only on f, c, M.

Proof. Set $\tau_T := \inf\{t > 0 \,|\, W_t \geq M - f(T)\}$. Then we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[e^{cW_T}\right] &= \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[e^{c(M-f(T))}e^{c(W_T - W_{\tau_T})}\right] \\ &= e^{c(M-f(T))}\mathbb{E}\left[e^{c(W_T - W_{\tau_T})}\right] \\ &= e^{c(M-f(T))}\exp\left(\frac{c^2(T-\tau_T)}{2}\right),\end{aligned} \tag{3.30}$$

which tends to e^{cM} as $T \to 0^+$. In fact, Taylor expanding the exponentials to first order shows that $\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[e^{cW_T}\right] - e^{cM}$ is of order $O(T) + O(f(T)) = O(T)$.

Next, by the Markov inequality we have, for every $\delta > 0$,

$$\mathbb{Q}_{M-f(T),T}\big[|W_T - M| \geq \delta\big] \leq \frac{\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\big[|W_T - M|\big]}{\delta}. \tag{3.31}$$

Setting $\delta = \sqrt{T}$, and recalling Lemma 3, we thus obtain that

$$\mathbb{Q}_{M-f(T),T}\big[|W_T - M| \geq \sqrt{T}\big] = O(\sqrt{T}). \tag{3.32}$$

Now we compute

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}[|e^{cW_T} - e^{cM}|] \\
&= \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\big[\mathbf{1}_{\{|W_T-M|<\sqrt{T}\}}|e^{cW_T} - e^{cM}|\big] + \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\big[\mathbf{1}_{\{|W_T-M|\geq\sqrt{T}\}}|e^{cW_T} - e^{cM}|\big] \\
&\leq O(\sqrt{T}) + E_{\mathbb{Q}_{M-f(T),T}}\big[\mathbf{1}_{\{|W_T-M|\geq\sqrt{T}\}}|e^{cW_T} - e^{cM}|\big].
\end{aligned} \tag{3.33}$$

Hence it will suffice to show that the second term above is of order $O(\sqrt{T})$. We write said term as $A_T + B_T$, where

$$A_T := \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\big[\mathbf{1}_{\{W_T-M\geq\sqrt{T}\}}|e^{cW_T} - e^{cM}|\big], \tag{3.34}$$

$$B_T := \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\big[\mathbf{1}_{\{W_T-M\leq-\sqrt{T}\}}|e^{cW_T} - e^{cM}|\big]. \tag{3.35}$$

Observe that $B_T = O(\sqrt{T})$. Indeed,

$$
\begin{aligned}
B_T &= e^{cM}\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\big[\mathbf{1}_{\{W_T-M\le -\sqrt{T}\}}|e^{c(W_T-M)}-1|\big].\\
&\le e^{cM}\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\big[\mathbf{1}_{\{W_T-M\le -\sqrt{T}\}}|e+1|\big].\\
&\le (e+1)e^{cM}\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}[\mathbf{1}_{\{|W_T-M|\ge\sqrt{T}\}}] \\
&= (e+1)e^{cM}O(\sqrt{T})\\
&= O(\sqrt{T}).
\end{aligned} \tag{3.36}
$$

where in the second to last line, we have applied Equation (3.32).

Now we rewrite $A_T + B_T$ as $A_T - B_T + 2B_T$, and note that

$$
\begin{aligned}
A_T - B_T &= \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}[\mathbf{1}_{\{|W_T-M|\ge\sqrt{T}\}}(e^{c(W_T-M)}-1)]\\
&= (\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\big[e^{cW_T}\big] - e^{cM}) - \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\big[\mathbf{1}_{\{|W_T-M|<\sqrt{T}\}}(e^{cW_T}-e^{cM})\big].
\end{aligned} \tag{3.37}
$$

Since the term in brackets is of order $O(T)$ by the earlier discussion, and the latter term is of order $O(\sqrt{T})$, as can be seen by say, Taylor expansion, we obtain that

$$
A_T + B_T = O(T) + O(\sqrt{T}) = O(\sqrt{T}). \tag{3.38}
$$

as desired.

Lemma 5. *We have*

$$
\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}[\sup_{0\le t\le T}|X_t - e^{\frac{t}{T}\sigma M}|] = O(\sqrt{T}), \tag{3.39}
$$

as $T \to 0^+$, with the implied constant depending only on f, M.

Proof. Since X is a geometric Brownian motion, it admits the explicit solution

$$X_t = \exp(Ct + \sigma W_t), \tag{3.40}$$

where for convenience we have written $C := \mu - \frac{\sigma^2}{2}$.

Write $W_t = \frac{t}{T} W_T - B_t$, where

$$B_t := W_t - \frac{t}{T} W_T, \tag{3.41}$$

is a standard Brownian bridge, independent of W_T.

We then have

$$X_t = \exp(Ct - \sigma B_t + \frac{\sigma t}{T} W_T). \tag{3.42}$$

Let D be the event $\{W_T \geq M - f(T)\}$. We compute

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\Big[\sup_{0\leq t\leq T} |X_t - e^{\frac{t}{T}\sigma M}|\Big] \\ &\leq \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\Big[\sup_{0\leq t\leq T} |\exp(Ct - \sigma B_t + \frac{\sigma t}{T} W_T) - e^{(\sigma t/T) W_T}|\Big] \\ &+ \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\Big[\sup_{0\leq t\leq T} |e^{(\sigma t/T) W_T} - e^{\frac{t}{T}\sigma M}|\Big]. \end{aligned} \tag{3.43}$$

Clearly, the supremum in the last term occurs at $t = T$, and hence the last term is of order $O(\sqrt{T})$ by Lemma 4.

For the first term, we claim that

$$\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\Big[\sup_{0\leq t\leq T} |\exp(Ct - \sigma B_t + \frac{\sigma t}{T} W_T) - e^{(\sigma t/T) W_T}|\Big] \tag{3.44}$$

$$\leq \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\Big[\sup_{0\leq t\leq T} |\exp(Ct - \sigma B_t) - 1| e^{(\sigma t/T) W_T}\Big] \tag{3.45}$$

Indeed, we have trivially

$$\sup_{0\le t\le T} \left|\exp\left(Ct - \sigma B_t + \frac{\sigma t}{T} W_T\right) - e^{(\sigma t/T)W_T}\right|$$

$$\le \sup_{0\le t\le T} \sup_{0\le r\le T} \left|\exp\left(Ct - \sigma B_t + \frac{\sigma t}{T} W_T\right) - e^{(\sigma r/T)W_T}\right| \tag{3.46}$$

Since $\sigma > 0$, and $W_T > 0$ on the event D, we have that for all t, the inner supremum is attained at $r = T$, whence Equation 3.45 follows.

Next, using the independence of B_t from W_T (for all $t \in 0 \le t \le T$), denoting by $\mathcal{W}$ the σ-algebra generated by W_T, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\Big[\sup_{0\le t\le T} \left|\exp\left(Ct - \sigma B_t + \frac{\sigma t}{T} W_T\right) - e^{(\sigma t/T)W_T}\right|\Big] \\
&\le \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\Big[\sup_{0\le t\le T} e^{(\sigma t/T)W_T}\left|\exp\left(Ct - \sigma B_t\right) - 1\right|\Big] \\
&\le \frac{\mathbb{E}[\mathbf{1}_D e^{\sigma W_T} \sup_{0\le t\le T} |\exp(Ct - \sigma B_t) - 1|]}{\mathbb{P}(D)} \\
&= \frac{\mathbb{E}[\mathbb{E}[\mathbf{1}_D e^{\sigma W_T} \sup_{0\le t\le T} |\exp(Ct - \sigma B_t) - 1| \,|\, \mathcal{W}]]}{\mathbb{P}(D)} \\
&= \frac{\mathbb{E}[\mathbf{1}_D e^{\sigma W_T} \mathbb{E}[\sup_{0\le t\le T} |\exp(Ct - \sigma B_t) - 1| |\mathcal{W}]]}{\mathbb{P}(D)} \\
&= \frac{\mathbb{E}[\sup_{0\le t\le T} |\exp(Ct - \sigma B_t) - 1|]\, \mathbb{E}[1_D e^{\sigma W_T}]}{\mathbb{P}(D)} \\
&= \mathbb{E}[\sup_{0\le t\le T} |\exp(Ct - \sigma B_t) - 1|]\, \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[e^{\sigma W_T}\right].
\end{aligned} \tag{3.47}$$

We now claim that $\mathbb{E}[|\sup_{0\le t\le T} \exp(Ct - \sigma B_t) - 1|]$ is of order $O(\sqrt{T})$, while $\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[e^{\sigma W_T}\right]$ is of order $O(1)$, whence the result would follow.

To see the first claim, note that we have

$$\begin{aligned}
&\mathbb{E}[|\sup_{0\le t\le T}\exp(Ct-\sigma B_t)-1|]\\
&\le \mathbb{E}[|\sup_{0\le t\le T}\exp(Ct-\sigma B_t)-\exp(-\sigma B_t|)]+\mathbb{E}[|\sup_{0\le t\le T}\exp(-\sigma B_t)-1|] \qquad (3.48)\\
&\le (e^{CT}-1)\mathbb{E}[|\sup_{0\le t\le T}\exp(-\sigma B_t)]+\mathbb{E}[|\sup_{0\le t\le T}\exp(-\sigma B_t)-1|].
\end{aligned}$$

Since the former term tends to 0 as $T\to 0^{+}$, it will thus suffice to show that

$$\mathbb{E}[|\sup_{0\le t\le T}\exp(-\sigma B_t)-1], \qquad (3.49)$$

tends to 0.

We estimate

$$\mathbb{E}[\sup_{0\le t\le T}|\exp(-\sigma B_t)-1|]\le \mathbb{E}[|\exp(\sup_{0\le t\le T}-\sigma B_t)-1|]+\mathbb{E}[|\exp(\inf_{0\le t\le T}-\sigma B_t)-1|]. \qquad (3.50)$$

We show in turn that both terms in Equation (3.50) are of order $O(\sqrt{T})$. For the first term, we note that since

$$B_t = W_t-\frac{t}{T}W_T, \qquad (3.51)$$

we have

$$0\le \sup_{t\in[0,T]}-B_t\le M_T+|W_T|, \qquad (3.52)$$

where

$$M_t := \sup_{t\in[0,T]}-W_t. \qquad (3.53)$$

So by the Cauchy-Schwartz inequality,

$$\begin{aligned}\mathbb{E}[|\sup_{0\le t\le T}\exp(-\sigma B_t)-1] &= \mathbb{E}[\sup_{0\le t\le T}\exp(-\sigma B_t)-1]\\ &\le \sqrt{\mathbb{E}[\exp(2\sigma M_T)]}\sqrt{\mathbb{E}[\exp(2\sigma|W_T|)]}-1.\end{aligned} \tag{3.54}$$

By the reflection principle, $M_T = |W_T|$ in law, so

$$\mathbb{E}[\sup_{0\le t\le T}\exp(-\sigma B_t)-1] \le \mathbb{E}[\exp(2\sigma|W_T|)]-1. \tag{3.55}$$

Letting Φ denote the CDF of a standard normal random variable, by standard formulae, we have

$$\begin{aligned}\mathbb{E}[\exp(2\sigma|W_T|)] &= 2e^{2T\sigma^2}\Phi(2\sigma\sqrt{T})\\ &= (1+O(\sqrt{T}))e^{2T\sigma^2}\\ &= (1+O(\sqrt{T}))(1+O(T))\\ &= 1+O(\sqrt{T}),\end{aligned} \tag{3.56}$$

whence

$$\mathbb{E}[|\exp(\sup_{0\le t\le T}-\sigma B_t)-1|] = O(\sqrt{T}), \tag{3.57}$$

as claimed.

Now we deal with the second term in Equation 3.50.

Since $\inf_{0\le t\le T}-B_t \le 0$ almost surely, we have $\exp(\sigma\inf_{0\le t\le T}-W_t)\le 1$, so the second term is

$$\mathbb{E}[1-\exp(\sigma\inf_{0\le t\le T}-B_t)]. \tag{3.58}$$

Hence, it will suffice to show that

$$\mathbb{E}[\exp(\sigma \inf_{0\le t\le T} -B_t)] \to 1. \tag{3.59}$$

Again, since $B_t = W_t - \frac{t}{T}W_T$, we have

$$m_T - |W_T| \le \inf_{0\le t\le T} -B_t \le 0, \tag{3.60}$$

where $m_T := \inf_{0\le t\le T} -W_t$. So

$$\begin{aligned} E[\exp(\sigma \inf_{0\le t\le T} -B_t)] &\ge E[\exp(\sigma(m_T - |W_T|))] \\ &= \mathbb{E}\left[\frac{1}{\exp(\sigma(-m_T + |W_T|))}\right] \\ &\ge \frac{1}{\mathbb{E}[\exp(\sigma(-m_T + |W_T|))]}, \end{aligned} \tag{3.61}$$

where in the last line we have applied Jensen's inequality. Applying the Cauchy Schwartz inequality, we have

$$\frac{1}{\mathbb{E}[\exp(\sigma(-m_T + |W_T|))]} \ge \frac{1}{\sqrt{\mathbb{E}[\exp(-2\sigma m_T)]}\sqrt{\mathbb{E}[\exp(2\sigma|W_T|)]}}. \tag{3.62}$$

By the reflection principle, $-m_T = |W_T|$ in distribution, so

$$\frac{1}{\sqrt{\mathbb{E}[\exp(-2\sigma m_T)]}\sqrt{\mathbb{E}[\exp(2\sigma|W_T|)]}} \ge \frac{1}{\mathbb{E}[\exp(2\sigma|W_T|)]}. \tag{3.63}$$

Consequently, we have

$$\mathbb{E}[1 - \exp(\sigma \inf_{0\le t\le T} -B_t)] \le 1 - \frac{1}{\mathbb{E}[\exp(2\sigma|W_T|)]}. \tag{3.64}$$

Since

$$\mathbb{E}[\exp(2\sigma|W_T|)] = 1 + O(\sqrt{T}), \tag{3.65}$$

as proven earlier, we deduce

$$\mathbb{E}[1 - \exp(\sigma \inf_{0\le t\le T} -B_t)] \le 1 - \frac{1}{1 + O(\sqrt{T})} = O(\sqrt{T}), \tag{3.66}$$

as claimed.

On the other hand, the second claim follows from a stopping time argument and standard estimates. Indeed, write

$$\tau_T := \inf\{t > 0 \,|\, W_T \ge M - f(T)\}. \tag{3.67}$$

We have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[e^{\sigma W_T}\right] &= \frac{\mathbb{E}\left[1_D e^{\sigma W_T}\right]}{\mathbb{P}(D)} \\ &= \frac{\mathbb{E}\left[\mathbb{E}[\left[1_D e^{\sigma W_T}\middle|\mathcal{F}_\tau\right]]\right.}{\mathbb{P}(D)} \\ &= \frac{\mathbb{E}\left[1_D \mathbb{E}[\left[e^{\sigma W_T}\middle|\mathcal{F}_\tau\right]]\right.}{\mathbb{P}(D)}. \end{aligned} \tag{3.68}$$

On D, we have $\tau_T \le T$ almost surely. Thus by the strong Markov property, conditional on $\mathcal{F}_\tau$, $R_t := W_{t+\tau}$ is a Brownian motion with initial value $R_0 = W_\tau = M - f(T)$. Thus

$$\begin{aligned} \mathbb{E}\left[e^{\sigma W_T}\middle|\mathcal{F}_\tau\right] &= \mathbb{E}\left[e^{\sigma R_{t-\tau}}\middle|\mathcal{F}_\tau\right] \\ &= \mathbb{E}\left[e^{\sigma R_{t-r}}\right]\Big|_{r=\tau}, \end{aligned} \tag{3.69}$$

where in the last equality we have applied the freezing lemma.

We recognise $e^{\sigma R_{t-r}}$ as a log normal random variable with mean $\exp(M - f(T) + \frac{t-r}{2}) \le \exp(M + |f(T)| + \frac{T}{2}) < \exp(M + 1) := C$ for all small enough T, uniformly over all

$0 \le r \le t$. Thus

$$\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[e^{\sigma W_T}\right] = \frac{\mathbb{E}[1_D \mathbb{E}\left[e^{\sigma R_{t-r}}\right]\big|_{r=\tau}]}{\mathbb{P}(D)}$$
$$\le C\frac{\mathbb{E}[1_D]}{\mathbb{P}(D)} \tag{3.70}$$
$$= C.$$

Thus $\mathbb{E}_{\mathbb{Q}_{M-f(T),T}}\left[e^{\sigma W_T}\right]$ is of order $O(1)$ as claimed, and this concludes the proof. □

Lemma 6. *Let* $\tau = \inf\{t > 0 \,|\, X_t = B\}$. *Then we have*

$$\mathbb{P}(\tau \ge (1 - T^{1/2})T \,\big|\, \tau \le T) \to 1, \tag{3.71}$$

as $T \to 0^+$.

Proof. Using that the density f_τ of τ is given by

$$f_\tau(t) = \frac{R \exp\left(\frac{-(R+(\frac{\sigma^2}{2}-\mu)t)^2}{2t}\right)}{\sqrt{2\pi t^3}}, \tag{3.72}$$

we may write

$$\mathbb{P}(\tau \ge (1 - T^{1/2})T \,\big|\, \tau \le T) = \frac{\int_{(1-T^{1/2})T}^{T} f_\tau \, dt}{\int_0^T f_\tau \, dt}$$
$$= \frac{\int_{(1-T^{1/2})T}^{T} f_\tau \, dt}{\int_{(1-T^{1/2})T}^{T} f_\tau + \int_0^{(1-T^{1/2})T} f_\tau \, dt} \tag{3.73}$$
$$=: \frac{A_1}{A_1 + A_2}$$
$$= \frac{1}{1 + A_2/A_1},$$

with $A_1 := \int_{(1-T^{1/2})T}^{T} f_\tau \, dt$ and $A_2 := \int_0^{(1-T^{1/2})T} f_\tau \, dt$. Hence we may conclude that $\mathbb{P}(\tau \geq (1-T^{1/2})T \,|\tau \leq T) \to 1$ if we can show that $\lim_{T\to 0_+} \frac{A_2}{A_1} = 0$.

Now we have

$$\frac{df_\tau}{dt} = \frac{Re^{-(2R+t(\sigma^2-2\mu))^2/8t}(4R^2 - t(t(2\mu-\sigma^2)^2) + 12)}{8\sqrt{2\pi}t^{7/2}}, \tag{3.74}$$

which is positive on $[0,T]$ for all small enough $T>0$, so f_τ is increasing on this interval. Thus we may estimate

$$\begin{aligned} A_2 &\leq \int_0^{(1-T^{1/2})T} \frac{R\exp\left(\frac{-(R+(\frac{\sigma^2}{2}-\mu)((1-T^{1/2})T)^2)}{2(1-T^{1/2})T}\right)}{\sqrt{2\pi((1-T^{1/2})T)^3}} \, dt \\ &\leq \frac{TR\exp\left(\frac{-(R+(\frac{\sigma^2}{2}-\mu)((1-T^{1/2})T)^2)}{2(1-T^{1/2})T}\right)}{\sqrt{2\pi((1-T^{1/2})T)^3}}, \end{aligned} \tag{3.75}$$

where we have used the fact that f_τ is increasing on $[0,T]$ for small enough T. Similarly,

$$\begin{aligned} A_1 &\geq \int_{(1-\frac{T^{1/2}}{2})T}^{T} f_\tau \, dt \\ &\geq \int_{(1-\frac{T^{1/2}}{2})T}^{T} \frac{R\exp\left(\frac{-(R+(\frac{\sigma^2}{2}-\mu)(1-\frac{T^{1/2}}{2})T)^2}{(2(1-\frac{T^{1/2}}{2})T)}\right)}{\sqrt{2\pi(2(1-\frac{T^{1/2}}{2})T)/3)^3}} \, dt \\ &= \left(\frac{T^{3/2}}{2}\right)\left(\frac{R\exp\left(\frac{-(R+(\frac{\sigma^2}{2}-\mu)(1-\frac{T^{1/2}}{2})T)^2}{(2(1-\frac{T^{1/2}}{2})T)}\right)}{\sqrt{2\pi(2(1-\frac{T^{1/2}}{2})T)/3)^3}}\right), \end{aligned} \tag{3.76}$$

so that, after dividing the above two equations we obtain

$$\frac{A_2}{A_1} \leq T^{-1/2}C_0\exp\left(-\frac{C_1}{T^{1/2}} + C_2 + C_3T\right), \tag{3.77}$$

where $C_0,\ldots,C_3$ are constants with $C_0, C_1 > 0$ that do not depend on T. We use the simple estimate

$$
\begin{aligned}
\frac{A_2}{A_1} &\leq T^{-1/2} C_0 \exp\left(-\frac{C_1}{T^{1/2}} + C_2 + C_3\right) \\
&= C_4 T^{-1/2} \exp\left(-\frac{C_1}{T^{1/2}}\right),
\end{aligned}
\tag{3.78}
$$

for all $T < 1$, say, which tends to 0 as $T \to 0^+$, as desired. □

We are now ready to give the proof of Theorem 4.

Proof. [Proof of Theorem 4]

First we show that

$$
\mathbb{E}_{\mathbb{P}_T}\big[\sup_{0<t<T} |X_t - B^{\frac{t}{T}}|\big], \to 0 \tag{3.79}
$$

and later refine our analysis to achieve the $O(\sqrt{T})$ convergence rate.

To this end, let Y_T be the event $\{W_T \geq G - \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right) T\}$. We recall that $G := \frac{\log B}{\sigma}$ and A_T is the event $\{\max_{0\leq t\leq T} X_t \geq B\}$.

Note that if $W_T \geq G - (\frac{\mu}{\sigma} T)$, then $X_T = \exp(\frac{\mu}{\sigma} - \frac{\sigma}{2})T + \sigma W_T \geq B$, and thus Y_T is a subset of A_T.

We then have

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}_T}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] \\
&= \mathbb{E}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\,\big|\,A_T\big] \\
&= \mathbb{E}\Big[\mathbf{1}_{Y_T}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\,\Big|\,A_T\Big] + \mathbb{E}\Big[\mathbf{1}_{Y_T^c}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\,\Big|\,A_T\Big] \\
&= \frac{\mathbb{E}\big[\mathbf{1}_{Y_T}\mathbf{1}_{A_T}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} + \frac{\mathbb{E}\big[\mathbf{1}_{Y_T^c}\mathbf{1}_{A_T}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} \\
&= \left(\frac{\mathbb{E}\big[\mathbf{1}_{Y_T}\mathbf{1}_{A_T}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(Y_T)}\right)\left(\frac{\mathbb{P}(Y_T)}{\mathbb{P}(A_T)}\right) + \frac{\mathbb{E}\big[\mathbf{1}_{Y_T^c}\mathbf{1}_{A_T}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} \\
&\leq \mathbb{E}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\,|\,Y_T\big] + \frac{\mathbb{E}\big[\mathbf{1}_{Y_T^c}\mathbf{1}_{A_T}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} \\
&= \mathbb{E}_{\mathbb{Q}_{G-(\frac{\mu}{\sigma}-\frac{\sigma}{2})T,T}}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] + \frac{\mathbb{E}\big[\mathbf{1}_{Y_T^c}\mathbf{1}_{A_T}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)},
\end{aligned}
\tag{3.80}
$$

where in the last two lines we have applied the fact that Y_T is a subset of A_T, hence $\mathbf{1}_{Y_T}\mathbf{1}_{A_T} = \mathbf{1}_{Y_T}$ and $\frac{\mathbb{P}(Y_T)}{\mathbb{P}(A_T)} \leq 1$.

We now examine the second term. Writing $\tau := \inf\{t > 0 \,|\, X_\tau = B\}$, we have

$$
\begin{aligned}
&\frac{\mathbb{E}\big[\mathbf{1}_{Y_T^c}\mathbf{1}_{A_T}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} \\
&\leq \frac{\mathbb{E}\big[\mathbf{1}_{Y_T^c}\mathbf{1}_{A_T}\sup_{0<t<\tau}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} + \frac{\mathbb{E}\big[\mathbf{1}_{Y_T^c}\mathbf{1}_{A_T}\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} \\
&= \frac{\mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}_{Y_T^c}\mathbf{1}_{A_T}\sup_{0<t<\tau}|X_t - B^{\frac{t}{T}}|\,\big|\,\mathcal{F}_\tau\big]\Big]}{\mathbb{P}(A_T)} + \frac{\mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}_{Y_T^c}\mathbf{1}_{A_T}\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}|\,\big|\,\mathcal{F}_\tau\big]\Big]}{\mathbb{P}(A_T)} \\
&= \frac{\mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}_{Y_T^c}\big|\mathcal{F}_\tau\big]\mathbf{1}_{A_T}\sup_{0<t<\tau}|X_t - B^{\frac{t}{T}}|\Big]}{\mathbb{P}(A_T)} + \frac{\mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}_{Y_T^c}\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}|\,\big|\,\mathcal{F}_\tau\big]\mathbf{1}_{A_T}\Big]}{\mathbb{P}(A_T)} \\
&\leq \frac{\mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}_{Y_T^c}\big|\mathcal{F}_\tau\big]\mathbf{1}_{A_T}\sup_{0<t<\tau}|X_t - B^{\frac{t}{T}}|\Big]}{\mathbb{P}(A_T)} + \frac{\mathbb{E}\Big[\mathbb{E}\big[\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}|\,\big|\,\mathcal{F}_\tau\big]\mathbf{1}_{A_T}\Big]}{\mathbb{P}(A_T)} \\
&= \frac{\mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}_{Y_T^c}\big|\mathcal{F}_\tau\big]\mathbf{1}_{A_T}\sup_{0<t<\tau}|X_t - B^{\frac{t}{T}}|\Big]}{\mathbb{P}(A_T)} + \mathbb{E}\big[\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}|\,\big|\,\mathcal{A}_T\big],
\end{aligned}
\tag{3.81}
$$

where we have performed an intermediate conditioning on $\mathcal{F}_\tau$, and made use of the

$\{\mathcal{F}_t\}$-adaptedness of X to bring terms outside the conditional expectation.

We now make two claims - the first is that

$$\mathbb{E}\big[\mathbf{1}_{Y_T^c}\big|\mathcal{F}_\tau\big]$$

is almost surely bounded away from 1 as $T \to 0$ - that is, there exists some $0 < C < 1$ and $T_0 > 0$ such that

$$\mathbb{E}\big[\mathbf{1}_{Y_T^c}\big|\mathcal{F}_\tau\big] \leq C, \tag{3.82}$$

almost surely whenever $T < T_0$.

The second is that

$$\mathbb{E}\big[\sup_{\tau<t<T}|X_t - B^{\frac{t}{T}}|\big|\mathcal{A}_T\big] \to 0, \tag{3.83}$$

as $T \to 0^+$.

Admitting for now these two claims, letting $\varepsilon > 0$ be arbitrary, we have

$$\mathbb{E}_{\mathbb{P}_T}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big]$$

$$\leq \mathbb{E}_{\mathbb{Q}_{G-(\frac{\mu}{\sigma}-\frac{\sigma}{2})T,T}}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] + \frac{\mathbb{E}\big[\mathbb{E}\big[\mathbf{1}_{Y_T^c}\big|\mathcal{F}_\tau\big]\mathbf{1}_{A_T}\sup_{0<t<\tau}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} \tag{3.84}$$

$$+\,\mathbb{E}\big[\sup_{\tau<t<T}|X_t - B^{\frac{t}{T}}|\big|\mathcal{A}_T\big]$$

$$\leq \mathbb{E}_{\mathbb{Q}_{G-(\frac{\mu}{\sigma}-\frac{\sigma}{2})T,T}}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] + \frac{\mathbb{E}\big[C\mathbf{1}_{A_T}\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} \tag{3.85}$$

$$+\,\mathbb{E}\big[\sup_{\tau<t<T}|X_t - B^{\frac{t}{T}}|\big|\mathcal{A}_T\big]$$

$$= \mathbb{E}_{\mathbb{Q}_{G-(\frac{\mu}{\sigma}-\frac{\sigma}{2})T,T}}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] + C\,\mathbb{E}_{\mathbb{P}_T}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] + \varepsilon \tag{3.86}$$

$$= C\,\mathbb{E}_{\mathbb{P}_T}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] + 2\varepsilon,$$

for all small enough T, where in the third to last line we have applied claim 3.82, in the second to last line we have applied claim 3.83, and in the last line we have applied Lemma 5. Thus

$$(1-C)\mathbb{E}_{\mathbb{P}_T}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] \leq 2\varepsilon, \tag{3.87}$$

which implies

$$\mathbb{E}_{\mathbb{P}_T}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] \leq \frac{2\varepsilon}{1-C}. \tag{3.88}$$

Since ε was arbitrary, we conclude

$$\mathbb{E}_{\mathbb{P}_T}\big[\sup_{0<t<T}|X_t - B^{\frac{t}{T}}|\big] \tag{3.89}$$

tends to 0 as $T \to 0^+$ as required.

It remains only to prove the earlier two claims in Eqs (3.82) and (3.83). For the first claim, we note that

$$\mathbb{E}[\mathbf{1}_{Y_T^c} \mid F_\tau] = 1 - \mathbb{E}[\mathbf{1}_{Y_T} \mid F_\tau]. \tag{3.90}$$

Hence it will suffice to show that there is some $C > 0$ such that $\mathbb{E}[\mathbf{1}_{Y_T} \mid F_\tau] > C$ almost surely for all small enough T. To this end, we estimate

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{Y_T} \mid F_\tau] &= \mathbb{P}\left(W_T \geq G - \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}T\right) \big|\mathcal{F}_\tau\right) \\
&= \mathbb{P}\left(W_\tau + W_T - W\tau \geq G - \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right) T \big|\mathcal{F}_\tau\right) \\
&= \mathbb{P}\left(W_T - W\tau \geq \left(\frac{\sigma}{2} - \frac{\mu}{\sigma}\right)(T-\tau) \big|\mathcal{F}_\tau\right).
\end{aligned} \tag{3.91}$$

Recalling that $W_T - W_\tau$ is a normal random variable with variance $T - \tau$, we have

$$\mathbb{P}\left(W_T - W\tau \geq \left(\frac{\sigma}{2} - \frac{\mu}{\sigma}\right)(T-\tau) \big|\mathcal{F}_\tau\right) = \mathbb{P}\left(Z \geq \left(\frac{\sigma}{2} - \frac{\mu}{\sigma}\right)\sqrt{T-\tau}\right), \tag{3.92}$$

where Z is a standard normal random variable. The above tends to $\mathbb{P}(Z \geq 0)$ as $T \to 0$, uniformly in ω, and so any $0 < C < \frac{1}{2}$ will satisfy the required inequality, say $C = \frac{1}{3}$. This proves claim (3.82).

For the second claim (3.83), we estimate, for any $\delta > 0$,

$$\mathbb{E}\big[\sup_{\tau<t<T}|X_t - B^{\frac{t}{T}}|\big|\mathcal{A}_T\big] = \mathbb{E}\big[\mathbf{1}_{\tau<(1-\delta)T}\sup_{\tau<t<T}|X_t - B^{\frac{t}{T}}|\big|\mathcal{A}_T\big] \tag{3.93}$$

$$+\mathbb{E}\big[\mathbf{1}_{\tau\geq(1-\delta)T}\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}|\,\big|\mathcal{A}_T\big]. \tag{3.94}$$

The first term above is equal to

$$\begin{aligned}
&\frac{\mathbb{E}\big[\mathbf{1}_{A_T}\mathbf{1}_{\{\tau<(1-\delta)T\}}\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} \\
&= \frac{\mathbb{E}\big[\mathbb{E}[\mathbf{1}_{A_T}\mathbf{1}_{\{\tau<(1-\delta)T\}}\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}||\mathcal{F}_\tau]\big]}{\mathbb{P}(A_T)} \\
&= \frac{\mathbb{E}\big[\mathbf{1}_{A_T}\mathbf{1}_{\{\tau<(1-\delta)T\}}\mathbb{E}[\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}||\mathcal{F}_\tau]\big]}{\mathbb{P}(A_T)}
\end{aligned} \tag{3.95}$$

Applying the strong Markov property and the freezing lemma, we have

$$\mathbb{E}[\sup_{\tau<t<T}|X_t - B^{\frac{t}{T}}||\mathcal{F}_\tau] = \mathbb{E}[\sup_{0\leq s<T-r}|R_s - B^{\frac{r+s}{T}}|]|_{r=\tau}, \tag{3.96}$$

where $R_s := X_{\tau+s}$ is a geometric Brownian motion independent of $\mathcal{F}_\tau$. Hence

$$\begin{aligned}
&\frac{\mathbb{E}\big[\mathbf{1}_{A_T}\mathbf{1}_{\{\tau<(1-\delta)T\}}\sup_{\tau\leq t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} \\
&= \frac{\mathbb{E}\big[\mathbf{1}_{A_T}\mathbf{1}_{\{\tau<(1-\delta)T\}}\mathbb{E}[\sup_{0\leq s<T-r}|R_s - B^{\frac{r+s}{T}}|]|_{r=\tau}\big]}{\mathbb{P}(A_T)} \\
&\leq \frac{\mathbb{E}\big[\mathbf{1}_{A_T}\mathbf{1}_{\{\tau<(1-\delta)T\}}\mathbb{E}[\sup_{0\leq s<T-r}|R_s + B^{\frac{r+s}{T}}|]|_{r=\tau}\big]}{\mathbb{P}(A_T)}.
\end{aligned} \tag{3.97}$$

We note that

$$\mathbb{E}[\sup_{0\le s<T-r}|R_s + B^{\frac{r+s}{T}}|]|_{r=\tau} \le \mathbb{E}[\sup_{0\le s\le T}|R_s + B|] \\ \le \mathbb{E}[\sup_{0\le s\le 1}|R_s + B|], \tag{3.98}$$

for all small enough T. Since $\sup_{0\le s\le 1} R_s$ is an L^1 random variable, we deduce that for all small enough T, $\mathbb{E}[\sup_{0\le s<T-r}|R_s + B^{\frac{r+s}{T}}|]|_{r=\tau}$ is almost surely bounded above by some C depending not on T or τ. Thus,

$$\frac{\mathbb{E}\big[\mathbf{1}_{A_T}\mathbf{1}_{\{\tau<(1-\delta)T\}}\sup_{\tau\le t<T}|X_t - B^{\frac{t}{T}}|\big]}{\mathbb{P}(A_T)} = O(1)\mathbb{E}[\mathbf{1}_{\{\tau<(1-\delta)T\}}|A_T] \\ = O(1)\mathbb{P}[\tau < (1-\delta)T|\tau \le T] \\ \to 0. \tag{3.99}$$

as $T \to 0$ by Lemma 4.

On the other hand, we estimate

$$
\begin{aligned}
&\mathbb{E}\big[\mathbf{1}_{\{\tau\geq(1-\delta)\}}\sup_{\tau<t<T}|X_t - B^{\frac{t}{T}}|\,\big|\,A_T\big] \\
&= \mathbb{E}\big[\mathbf{1}_{\{\tau\geq(1-\delta)T\}}\sup_{\tau<t<T}|X_t - B^{\frac{t}{T}}|\,\big|\,A_T\big] \\
&= \mathbb{E}\big[\mathbf{1}_{\{\tau\geq(1-\delta)T\}}\sup_{\tau<t<T}|X_\tau + X_T - X_\tau - B^{\frac{t}{T}}|\,\big|\,A_T\big] \\
&= \mathbb{E}\big[\mathbf{1}_{\{\tau\geq(1-\delta)T\}}\sup_{\tau<t<T}|X_T - X_\tau + B - B^{\frac{t}{T}}|\,\big|\,A_T\big] \\
&\leq \mathbb{E}\big[\mathbf{1}_{\{\tau\geq(1-\delta)T\}}\sup_{\tau<t<T}|X_T - X_\tau|\,\big|\,A_T\big] + \mathbb{E}\big[\mathbf{1}_{\{\tau\geq(1-\delta)T\}}\sup_{\tau<t<T}|B - B^{\frac{t}{T}}|\,\big|\,\mathcal{A}_T\big]. \qquad (3.100) \\
&= \mathbb{E}\big[\mathbf{1}_{\{\tau\geq(1-\delta)T\}}\sup_{\tau<t<T}|X_T - X_\tau|\,\big|\,A_T\big] + \mathbb{E}\big[\mathbf{1}_{\{\tau\geq(1-\delta)T\}}|B - B^{\frac{\tau}{T}}|\,\big|\,A_T\big]. \\
&\leq \frac{\mathbb{E}\big[\mathbf{1}_{\{(1-\delta)T\leq\tau\leq T\}}\sup_{\tau<t<T}|X_T - X_\tau|\,\big]}{\mathbb{P}(A_T)} + \frac{\mathbb{E}\big[\mathbf{1}_{\{(1-\delta)T\leq\tau\leq T\}}|B - B^{1-\delta}|\,\big]}{\mathbb{P}(A_T)}. \\
&= \frac{\mathbb{E}\big[\mathbf{1}_{\{(1-\delta)T\leq\tau\leq T\}}\sup_{\tau<t<T}|X_T - X_\tau|\,\big]}{\mathbb{P}(A_T)} + |B - B^{1-\delta}|\frac{\mathbb{E}\big[\mathbf{1}_{\{(1-\delta)T\leq\tau\leq T\}}\big]}{\mathbb{P}(A_T)}. \\
&\leq \frac{\mathbb{E}\big[\mathbf{1}_{\{(1-\delta)T\leq\tau\leq T\}}\sup_{\tau<t<T}|X_T - X_\tau|\,\big]}{\mathbb{P}(A_T)} + |B - B^{1-\delta}|.
\end{aligned}
$$

To estimate the first term above, we write $R_t := X_{\tau+t}$ and note that by the strong Markov property of SDEs, R_t is a geometric Brownian motion independent of $\mathcal{F}_\tau$ with the same parameters μ, σ as X and initial condition $R_0 = B$. Noting also that $X_\tau = B$, the first term reads

$$
\begin{aligned}
&\frac{\mathbb{E}\big[\mathbf{1}_{\{(1-\delta)T\leq\tau\leq T\}}\mathbb{E}[\sup_{0\leq t\leq T-\tau}|R_t - B|\,]\big]}{\mathbb{P}(A_T)} \\
&\leq \frac{\mathbb{E}\big[\mathbf{1}_{\{(1-\delta)T\leq\tau\leq T\}}\mathbb{E}[\sup_{0\leq t\leq \delta T}|R_t - B|\,\big]}{\mathbb{P}(A_T)} \qquad (3.101) \\
&\leq \mathbb{E}\Big[\sup_{0\leq t\leq\delta T}|R_t - B|\,\Big],
\end{aligned}
$$

which tends to 0 as $T \to 0$ by standard estimates on SDE (see, for example Baldi (2017), Theorem 9.1).

Thus we have, for any $\delta > 0$,

$$\lim_{T\to 0^+} \mathbb{E}\Big[\sup_{\tau<t<T} |X_t - B^{\frac{t}{T}}| \Big| \mathcal{A}_T\Big] \leq |B - B^{1-\delta}|. \tag{3.102}$$

which tends to 0 as $\delta \to 0$. Thus sending δ to 0, we obtain the desired claim (3.83). This completes the proof of Eq (3.79).

Now we prove the $O(\sqrt{T})$ convergence rate.

From Eq (3.86), we have

$$(1-K)\mathbb{E}_{\mathbb{P}_T}\Big[\sup_{0<t<T} |X_t - B^{\frac{t}{T}}|\Big] \leq \mathbb{E}_{\mathbb{Q}_{G-(\frac{\mu}{\sigma}-\frac{\sigma}{2})T,T}}\Big[\sup_{0<t<T} |X_t - B^{\frac{t}{T}}|\Big]$$

$$+\,\mathbb{E}\Big[\sup_{\tau<t<T} |X_t - B^{\frac{t}{T}}| \Big| \mathcal{A}_T\Big]. \tag{3.103}$$

for some fixed $0 < K < \frac{1}{2}$. By Lemma 5, the first term on the right hand side above is of order $O(\sqrt{T}) + |\left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)T| = O(\sqrt{T})$. Hence to prove the proposition, it will suffice to show that

$$\mathbb{E}\Big[\sup_{\tau<t<T} |X_t - B^{\frac{t}{T}}| \Big| \mathcal{A}_T\Big] = O(\sqrt{T}). \tag{3.104}$$

To this end, we write

$$\mathbb{E}\Big[\sup_{\tau<t<T} |X_t - B^{\frac{t}{T}}| \Big| \mathcal{A}_T\Big] = \mathbb{E}\Big[\mathbf{1}_{\{\tau<(1-T^{1/2})T\}} \sup_{\tau<t<T} |X_t - B^{\frac{t}{T}}| \Big| \mathcal{A}_T\Big]$$

$$+\,\mathbb{E}\Big[\mathbf{1}_{\tau\geq(1-T^{1/2})T} \sup_{\tau\leq t<T} |X_t - B^{\frac{t}{T}}| \Big| \mathcal{A}_T\Big]. \tag{3.105}$$

Similarly as to the estimate of the first term in (3.94), we deduce that

$$\mathbb{E}\Big[\mathbf{1}_{\{\tau<(1-T^{1/2})T\}} \sup_{\tau<t<T} |X_t - B^{\frac{t}{T}}| \Big| \mathcal{A}_T\Big] = O(1)\mathbb{P}[\tau < (1-T^{1/2})T \mid \tau \leq T]. \tag{3.106}$$

The proof of Lemma 6 shows that

$$\mathbb{P}[\tau < (1 - T^{1/2})T|\tau \leq T] = O\left(T^{-1/2}\exp\left(-\frac{C_1}{T^{1/2}}\right)\right), \tag{3.107}$$

which is certainly of order $O(\sqrt{T})$. Hence it is left to show that

$$\mathbb{E}\big[\mathbf{1}_{\tau\geq(1-T^{1/2})T} \sup_{\tau\leq t<T} |X_t - B^{\frac{t}{T}}|\,\big|\mathcal{A}_T\big] = O(\sqrt{T}). \tag{3.108}$$

But similar to the handling of the second term in (3.94), we may estimate

$$\mathbb{E}\big[\mathbf{1}_{\tau\geq(1-T^{1/2})T} \sup_{\tau\leq t<T} |X_t - B^{\frac{t}{T}}|\,\big|\mathcal{A}_T\big] \leq \mathbb{E}\big[\sup_{0\leq t\leq T^{3/2}} |R_t - B|\,\big] + |B - B^{1-T^{1/2}}|, \tag{3.109}$$

where again $R_t := X_{t+\tau}$. The first term above is of order $O(T^{3/4})$ by standard estimates on solutions to SDE (see reference, Baldi, Theorem 9.1), and hence a fortiori of order $O(\sqrt{T})$. On the other hand, we have

$$\begin{aligned} |B - B^{1-T^{1/2}}| &= B^{1-T^{1/2}}(B^{T^{1/2}} - 1) \\ &\leq B(e^{T^{1/2}\ln B} - 1) \\ &= B(1 + (\ln B)T^{1/2} + o((\ln B)T^{1/2}) - 1) \\ &= O(\sqrt{T}), \end{aligned} \tag{3.110}$$

where we have applied a Taylor expansion in the second to last equality. Combining the two estimates above gives

$$\mathbb{E}\big[\mathbf{1}_{\tau\geq(1-T^{1/2})T} \sup_{\tau\leq t<T} |X_t - B^{\frac{t}{T}}|\,\big|\mathcal{A}_T\big] = O(\sqrt{T}), \tag{3.111}$$

which concludes the proof.

With Theorem 4 in hand, we can now prove the main result of the chapter, Theorem 3.

Proof of Theorem 3. We first note that by elementary calculus, we have

$$\frac{1}{T}\int_0^T B^{t/T}\,dt = \frac{B-1}{\ln B}. \tag{3.112}$$

Next, we estimate

$$\begin{aligned} C(B,K,T) &= \mathbb{E}\left[\big(\frac{1}{T}\int_0^T X_t dt - K\big)_+ \mathbf{1}_{\{\max\limits_{0\le t\le T} X_t \ge B\}}\right]. \\ &= \mathbb{P}\left(\max_{0\le t\le T} X_t \ge B\right)\mathbb{E}_{\mathbb{P}_T}\left[\big(\frac{1}{T}\int_0^T X_t dt - K\big)_+\right]. \end{aligned} \tag{3.113}$$

To estimate the last term above, we note that

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}_T}\left[\big(\frac{1}{T}\int_0^T X_t dt - K\big)_+\right] - \big(\frac{B-1}{\ln B} - K\big)_+ \\ &\le \mathbb{E}_{\mathbb{P}_T}\left[\left|\big(\frac{1}{T}\int_0^T X_t\,dt - K\big)_+ - \big(\frac{B-1}{\ln B} - K\big)_+\right|\right] \\ &= \mathbb{E}_{\mathbb{P}_T}\left[\left|\big(\frac{1}{T}\int_0^T X_t\,dt - \frac{1}{T}\int_0^T B^{t/T}\,dt + \frac{B-1}{\ln B} - K\big)_+ - \big(\frac{B-1}{\ln B} - K\big)_+\right|\right] \\ &= \mathbb{E}_{\mathbb{P}_T}\left[|(Y_T+G)_+ - G_+|\right], \end{aligned} \tag{3.114}$$

where we have written

$$Y_T := \big(\frac{1}{T}\int_0^T X_t\,dt - \frac{1}{T}\int_0^T B^{t/T}\,dt\big), \tag{3.115}$$

$$G := \frac{B-1}{\ln B} - K \tag{3.116}$$

Direct computation gives that for $G \leq 0$,

$$
\begin{aligned}
|(Y_T + G)_+ - G_+| &= (Y_T - |G|)_+ \\
&= \begin{cases} |Y_T - |G|| & \text{if } \ Y_T \geq |G|. \\ 0 & \text{if } \ Y_T < |G|. \end{cases}
\end{aligned} \tag{3.117}
$$

On the other hand, for $G > 0$, we have

$$
\begin{aligned}
|(Y_T + G)_+ - G_+| &= |(Y_T + G)_+ - G| \\
&= |Y_T|.
\end{aligned} \tag{3.118}
$$

In either case, we have that

$$
|(Y_T + G)_+ - G_+| \leq |Y_T|, \tag{3.119}
$$

hence

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}_T}\left[|(Y_T + G)_+ - G|\right] &\leq \mathbb{E}_{\mathbb{P}_T}\left[|Y_T|\right] \\
&= \mathbb{E}_{\mathbb{P}_T}\left[\Big|\frac{1}{T}\int_0^T X_t \, dt - \frac{1}{T}\int_0^T B^{t/T} \, dt\Big|\right] \\
&= \frac{1}{T}\Big|\int_0^T \mathbb{E}_{\mathbb{P}_T}[X_t - B^{t/T}] \, dt\Big| \\
&\leq \frac{1}{T}\int_0^T \mathbb{E}_{\mathbb{P}_T}\int_0^T \sup_{0\leq s\leq T} |X_s - B^{s/T}| \, dt \\
&= \mathbb{E}_{\mathbb{P}_T}\int_0^T \sup_{0\leq s\leq T} |X_s - B^{s/T}| \\
&= O(\sqrt{T}),
\end{aligned} \tag{3.120}
$$

by Proposition 4.

We conclude that

$$E_{\mathbb{P}_T}\left[(\frac{1}{T}\int_0^T X_t dt - K)_+\right] = (\frac{B-1}{\ln B} - K)_+ + O(\sqrt{T}), \tag{3.121}$$

and hence

$$C(B,K,T) = P(B,T)\left[(\frac{B-1}{\ln B} - K)_+ + O(\sqrt{T})\right], \tag{3.122}$$

as claimed.

3.4 Additional Proofs

In this section, we prove the result stated in the introduction concerning the limiting behaviour of the Brownian motion conditional on the sample path achieving a high running maximum. We repeat the theorem statement here for convenience.

Proposition 6 (Large noise limit for Brownian motion)**.** *Let W be a standard one dimensional Brownian motion. For every $\varepsilon > 0$, let A_ε denote the event*

$$\{\max_{0\leq t\leq 1} W_t \geq \frac{1}{\varepsilon}\}\ , \tag{3.123}$$

and let $\mathbb{P}^\varepsilon$ be the probability measure given by

$$\mathbb{P}^\varepsilon(E) = \frac{\mathbb{P}(E\cap A_\varepsilon)}{\mathbb{P}(A_\varepsilon)}\ , \tag{3.124}$$

for all measurable events E. Denote by $\mathbb{E}_{\mathbb{P}^\varepsilon}$ the expectation under $\mathbb{P}^\varepsilon$. Then

$$\lim_{\varepsilon\to 0}\mathbb{E}_{\mathbb{P}^\varepsilon}\left[|\varepsilon W_1 - 1|\right] = 0. \tag{3.125}$$

Proof. Write

$$\tau = \min\{t > 0 \, : \, W_t \geq \frac{1}{\varepsilon}\}. \tag{3.126}$$

By the reflection principle, we have

$$\mathbb{P}(\tau \leq 1) = \mathbb{P}(A_\varepsilon) = 2\Phi(-\frac{1}{\varepsilon}), \tag{3.127}$$

where $\Phi(x) := \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-\frac{t^2}{2}} \, dt$ denotes the CDF of the standard normal distribution. Using the strong Markov property at time τ, we have that $|W_1 - W_\tau|$ is a half normal random variable with parameter $\sigma = 1 - \tau$, independent of $\mathcal{F}_\tau$. Thus we compute

$$\begin{aligned}
\mathbb{E}\big[|W_1 - \frac{1}{\varepsilon}| \, \big| \, \tau \leq 1\big] &= \mathbb{E}\big[|W_1 - W_\tau| \, \big| \, \tau \leq 1\big] \\
&= \frac{\mathbb{E}[\mathbf{1}_{\{\tau \leq 1\}} |W_1 - W_\tau|]}{\mathbb{P}(\tau \leq 1)} \\
&= \frac{\mathbb{E}[\mathbf{1}_{\{\tau \leq 1\}} \mathbb{E}[|W_1 - W_\tau| \big| \mathcal{A}(\tau)]]}{\mathbb{P}(\tau \leq 1)} \\
&= \frac{\mathbb{E}\big[\mathbf{1}_{\{\tau \leq 1\}} \sqrt{\frac{2}{\pi}(1 - \tau)}\big]}{\mathbb{P}(\tau \leq 1)} \\
&\leq \sqrt{\frac{2}{\pi}}.
\end{aligned} \tag{3.128}$$

where $\mathcal{A}(\tau)$ denotes the sigma algebra generated by τ. Thus

$$\mathbb{E}_{\mathbb{P}^\varepsilon}[|\varepsilon W_1 - 1|] = \mathbb{E}[|\varepsilon W_1 - 1| \big| A_\varepsilon] \leq \varepsilon \sqrt{\frac{2}{\pi}}, \tag{3.129}$$

which tends to 0, as desired.

□

CHAPTER 4: CONCLUSION

We have investigated the pricing and hedging problem for various types of exotic options. The life contingent option is a key product in insurance and finance, providing a sort of insurance scheme whose payoff is in the form of an option instead of a fixed payoff. From a mathematical perspective, these are interesting as they involve a random terminal time that is not under the user's control, giving rise to a novel stochastic control problem. On the other hand, barrier Asian options are a commonly traded exotic option, thus the fair pricing problem for these is of great interest.

For the life contingent option, we have proven the existence of a minimal super-replicating portfolio, and characterised when replication is possible for all random exercise times. For the barrier Asian option, we have obtained explicit asymptotic expressions for the price of the option in the short maturity regime, along with the asymptotic convergence rate.

We briefly outline some directions for further research. Concerning the life contingent option, it would be of interest to investigate the case in which the stopping time is neither fully dependent nor independent of the asset prices. Further, an extension to more complex market models such as a jump diffusion model or a stochastic volatility model might be a worthwhile extension. Finally, one could generalise the payoff to Asian style payoffs that depend on the entire history of the asset prices.

For the barrier Asian option, an extension to local or stochastic volatility models seems to be a natural step. Further, we note that the main convergence theorem for the geometric Brownian motion may be relevant to other Asian style option prices depending on the history of the process. In this case we considered the arithmetic average payoff, but the theorem applies equally well to any continuous functional of the asset prices.

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