CHAPTER 4

THE OFF-DIAGONAL MASS MATRIX AND DECAY MATRIX ELEMENTS

4.1 INTRODUCTION

In this chapter, we calculate the off-diagonal mass matrix and decay matrix elements, $M_{12} + {}^{i}\!/_{2}\Gamma_{12}$ for the $P^0 \leftrightarrow \overline{P}^0$ transition. In Section 4.2, we demonstrate how M_{12} and Γ_{12} can be obtained from the effective Hamiltonian derived in Chapter 3. In Sections 4.3 and 4.4 we derive the analytical expressions for Γ_{12} and M_{12}. Finally, in Section 4.4, we present the behaviour of the form factors.

4.2 DERIVATION OF M_{12} AND Γ_{12} FROM THE EFFECTIVE HAMILTONIAN

The off diagonal mass matrix elements, M_{12} and Γ_{12} can be obtained by 'sandwiching' the effective Hamiltonian, H_{eff} obtained in Eq. (3.18) between the P^0 and $\overline{P}^0$ states, using a strategy originally due to Gaillard and Lee [4],

$$\langle P^0 | H_{eff} | \overline{P^0} \rangle = M_{12} - {}^{i}\!/_{2}\Gamma_{12}$$

$$= \frac{G_F^2 M_W^2}{16\pi^2} \sum_{i,j} \lambda_i \lambda_j \left\{ \langle P^0 | (V-A)^2 | \overline{P^0} \rangle B_{ij} + \langle P^0 | (S+P)^2 | \overline{P^0} \rangle C_{ij} \right\} \tag{4.1}$$

where

$$V - A = \bar{q}\gamma_\mu (1-\gamma_5) Q \text{ and } S + P = \bar{q}(1+\gamma_5) Q$$

One aspect of this calculation involves the estimation of the matrix elements *for the $V-A$ and $S+P$ type of operators appearing in Eq. (4.1). A precise calculation is* not feasible yet because of the traditional difficulties arising from the hadron structure and strong interactions. The simplest approximation is the vacuum-saturation method where one formally inserts a complete set of intermediate states in all possible ways and then assumes that the vacuum saturates [41]. Using the vacuum saturation method and the partial conservation of axial-vector current, we obtain

$$\Phi_{V-A} = \langle P^0 | (V-A)^2 | \overline{P}^0 \rangle = \frac{8}{3} f_P^2 m_P^2 \frac{B_P}{2m_P} \tag{4.2}$$

$$\Phi_{S+P} = \langle P^0 | (S+P)^2 | \overline{P}^0 \rangle = \frac{-5}{3} f_P^2 m_P^2 \frac{B_P}{2m_P} \tag{4.3}$$

where f_P is the meson decay constant and m_P is the meson mass. Here, we have put in the factor B_P, which is the is the bag parameter. In the vacuum saturation method, B_P is equal to unity. In Eqs. (4.2) and (4.3), the $(2m_P)^{-1}$ factor arises from the normalization of the state, and the factor $8/3$ corresponds to the 4 ways of Wick contraction multiplied by a colour factor of $2/3$.

However, in principle one must use a complete set of low lying intermediate states which are not accounted for by vacuum saturation. Deviation of B_P from one indicates the importance of low-energy intermediate states beyond the vacuum. Theoretically, the value of B_p can be determined using the Bag model, and may assume different values for Eqs. (4.2) and (4.3). For simplicity, we shall assume that B_p takes the same value. This is of minor consequence because the role of Γ_{12} for the calculation of the mixing parameters, as we shall see, is a small one [3].

The off diagonal mass matrix elements can then be obtained using Eqs. (4.1), (4.2) and (4.3) with M_{12} and Γ_{12} being the dispersive (d) and absorptive(a) parts respectively and are given by

$$M_{12} = \frac{G_F^2 M_W^2}{16\pi^2} \sum_{i,j} \lambda_i \lambda_j \left[\Phi_{V-A} B_{ij}^{(d)} + \Phi_{S+P} C_{ij}^{(d)}\right] \tag{4.4}$$

$$\Gamma_{12} = \frac{G_F^2 M_W^2}{16\pi^2} \sum_{i,j} \lambda_i \lambda_j \left[\Phi_{V-A} B_{ij}^{(a)} + \Phi_{S+P} C_{ij}^{(a)}\right] \tag{4.5}$$

From Eqs. (4.2), (4.3), (4.4) and (4.5), we observe that the form factors come in the following linear combination:

$$A_{ij}^{(a,d)} = B_{ij}^{(a,d)} - \frac{5}{8} C_{ij}^{(a,d)} \tag{4.6}$$

4.3 THE ABSORPTIVE PART

To calculate the absorptive part, we first note that the functions $\Lambda_k, k = 1,2,3,4$ defined in Eq. (3.22) are of the quadratic form $f(x) = Ax^2 + Bx + C$, with A and C positive and can be written as $(x - x_0)(x - x_1)$ where

$$x_0 = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{4.7}$$

and

$$x_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{4.8}$$

The calculation then involves the computation of integrals of the following type:

$$I_n = \int_0^1 x^n \ln\left(Ax^2 + Bx + C\right) dx, \; n = 0,1,2.$$

The integral above can be rewritten it in the following form:

$$I_n' = \int_0^{x_0} x^n \ln\left(Ax^2 + Bx + C\right) dx + \int_{x_0}^{x_1} x^n \ln\left(Ax^2 + Bx + C\right) dx$$

$$+ \int_{x_1}^{1} x^n \ln\left(Ax^2 + Bx + C\right) dx \tag{4.9}$$

When x_o and x_1 fall outside the range (0,1), we find that $Ax^2 + Bx + C$ is always positive. On the other hand, when x_o and x_1 fall in the range and when $Ax^2 + Bx + C$ becomes negative an absorptive part is generated. Since $\ln(-1) = i\pi$, an imaginary part is now introduced to the integral. As a result, Eq. (4.9) can now be written as

$$I_n = \int_0^{x_0} x^n \ln\left(Ax^2 + Bx + C\right) dx + \int_{x_0}^{x_1} x^n \ln\left(Ax^2 + Bx + C\right) dx$$

$$+ \int_{x_1}^{1} x^n \ln\left(Ax^2 + Bx + C\right) dx + i\pi \int_{x_0}^{x_1} x^n dx \tag{4.10}$$

where $i\pi \int_{x_0}^{x_1} x^n dx$ is the absorptive part we are interested in.

Inspection of the behaviour of Λ_k in the range (0,1) shows that only Λ_1 is negative in this range when $i, j = u,c$. Then, from Eq. (4.10) an absorptive part which is equal to $i\pi\int_{x_0}^{x_1} x^n dx$ is generated. On the other hand, when $i,j = t$, Λ_1 is always positive and no absorptive part is generated. The values of x_o and x_1 can be found using Eqs. (4.7) and (4.8) with $A = x_h$, $B = x_i - x_j - x_h$ and $C = x_j$ for $i,j = u,c$.

Using the method outlined above and carrying out the integrations in Eqs. (3.20) and (3.21), we easily obtain the following analytical expressions for $B_{ij}^{(a)}$ and $C_{ij}^{(a)}$:

$$B_{ij}^{(a)} = \frac{i\pi\sqrt{(x_i - x_j)^2 + x_h^2 - 2x_h(x_i + x_j)}}{6x_h^2(1-x_i)(1-x_j)}\left[\left(1+\frac{x_i x_j}{4}\right)\left(4x_h[x_i + x_j] - 2x_h^2 - 2[x_i - x_j]^2\right)\right.$$

$$\left. + 3x_h(x_i + x_j)(x_h - x_i - x_j)\right] \tag{4.11}$$

$$C_{ij}^{(a)} = \frac{i\pi(4 + x_i x_j)\sqrt{(x_i - x_j)^2 + x_h^2 - 2x_h(x_i + x_j)}}{6x_h^2(1-x_i)(1-x_j)}\left(x_h[x_i + x_j] + x_h^2 - 2[x_i - x_j]^2\right) \tag{4.12}$$

From Eqs. (4.6),(4.11) and (4.12), we finally obtain:

$$A_{ij}^{(a)} = -\frac{\pi\sqrt{(x_i - x_j)^2 + x_h^2 - 2x_h(x_i + x_j)}}{4x_h^2(1-x_i)(1-x_j)}\left[\left(1+\frac{x_i x_j}{4}\right)\left(3x_h^2 - x_h[x_i + x_h]\right.\right.$$

$$\left.\left. - 2[x_i - x_j]^2\right) + 2x_h(x_i + x_j)(x_i + x_j - x_h)\right] \quad i,j = u,c. \tag{4.13}$$

Our analytical expression for $A_{ij}^{(a)}$ is symmetric for i and j . We have compared our result and it is in agreement with that of Buras, Slominski and Staeger [9].

4.4 THE DISPERSIVE PART

The computation of the dispersive part is more complicated. The dispersive part occurs when the quadratic function in the logarithms in Eqs. (3.20) and (3.21) is positive in the range (0,1). The integrals encountered for the computation of the dispersive part are of the form $\int_0^1 x^n \ln\left(Ax^2 + Bx + C\right)dx,\ n = 0,1,2$. We shall first calculate $\int_0^1 x^n \ln \Lambda_1 dx$ analytically and then show how the remaining integrals $\int_0^1 x^n \ln \Lambda_2 dx$, $\int_0^1 x^n \ln \Lambda_3 dx$ and $\int_0^1 x^n \ln \Lambda_4 dx$ can be calculated. Depending on the values of x_i and x_j, the evaluation of the integral, $\int_0^1 x^n \ln \Lambda_1 dx$, can be divided into two cases:

(I) $D\left(x_i, x_j\right) > 0$

(II) $D\left(x_i, x_j\right) < 0$

where $D\left(x_i, x_j\right) = \left(x_i - x_j - x_h\right)^2 - 4x_h x_j$. Case (I) arises when $i, j = u,c,t$ but excludes the case when $i, j = t$. Case (II) is for $i, j = t$. After some algebra, we obtain the following results:

CASE (I): $D\left(x_i, x_j\right) > 0$

$$\int_0^1 \ln \Lambda_1 dx = -2 + \frac{\left(x_i - x_j\right)}{2x_h} \ln \frac{x_i}{x_j} + \frac{1}{2} \ln x_i x_j - \frac{1}{2x_h} \sqrt{\Delta\left(x_i, x_j\right)} \ln\left|\Omega\left(x_i, x_j\right)\right| \tag{4.14}$$

$$\int_0^1 x \ln \Lambda_1 dx = -1 + \frac{x_i - x_j}{2x_h} + \frac{1}{2x_h}\left(x_i \ln x_i - x_j \ln x_j\right) - \frac{1}{4}\left(\left[\frac{x_i - x_j}{x_h}\right]^2 - 1\right) \ln x_i$$

$$+ \frac{\left(x_i - x_j - x_h\right)^2}{4x_h^2} \ln x_j + \frac{\left(x_i - x_j - x_h\right)}{4x_h^2} \sqrt{\Delta\left(x_i, x_j\right)} \ln\left|\Omega\left(x_i, x_j\right)\right| \tag{4.15}$$

$$\int_0^1 x^2 \ln \Lambda_1 dx = -\frac{13}{18} + \frac{5x_i - x_j}{6x_h} - \frac{1}{3}\left(\frac{x_i - x_j}{x_h}\right)^2 - \frac{x_j\left(x_i - x_j - x_h\right)}{2x_h^2} \ln\left(\frac{x_i}{x_j}\right)$$

$$+ \frac{1}{6}\left(2 + \left[\frac{x_i - x_j - x_h}{x_h}\right]^3\right) \ln x_i - \frac{1}{6}\left(\frac{x_i - x_j - x_h}{x_h}\right)^3 \ln x_j$$

$$- \frac{\left[\left(x_i - x_j - x_h\right)^2 - x_j x_h\right]}{6x_h^3} \sqrt{\Delta\left(x_i, x_j\right)} \ln\left|\Omega\left(x_i, x_j\right)\right| \tag{4.16}$$

where

$$\Delta\left(x_i, x_j\right) = \left(x_i - x_j - x_h\right)^2 - 4x_h x_j \tag{4.17}$$

$$\Omega\left(x_i, x_j\right) = \frac{x_i + x_j - x_h + \sqrt{\Delta\left(x_i, x_j\right)}}{x_i + x_j - x_h - \sqrt{\Delta\left(x_i, x_j\right)}} \tag{4.18}$$

CASE (II): $D(x_i, x_j) < 0$

$$\int_0^1 \ln \Lambda_1 dx = -2 + \frac{(x_i - x_j)}{2x_h} \ln \frac{x_i}{x_j} + \frac{1}{2} \ln x_i x_j$$

$$+ \frac{\sqrt{-\Delta(x_i, x_j)}}{x_h} \left\{ \tan^{-1}\left(\frac{x_h + x_i - x_j}{\sqrt{-\Delta(x_i, x_j)}} \right) - \tan^{-1}\left(\frac{x_i - x_j - x_h}{\sqrt{-\Delta(x_i, x_j)}} \right) \right\} \tag{4.19}$$

$$\int_0^1 x \ln \Lambda_1 dx = -1 + \frac{x_i - x_j}{2x_h} + \frac{1}{2x_h}(x_i \ln x_i - x_j \ln x_j) - \frac{1}{4}\left(\left[\frac{x_i - x_j}{x_h} \right]^2 - 1 \right) \ln x_i$$

$$+ \frac{(x_i - x_j - x_h)^2}{4x_h^2} \ln x_j - \frac{(x_i - x_j - x_h)}{2x_h^2} \sqrt{-\Delta(x_i, x_j)} \left\{ \tan^{-1}\left(\frac{x_h + x_i - x_j}{\sqrt{-\Delta(x_i, x_j)}} \right) \right.$$

$$\left. - \left(\frac{x_h + x_i - x_j}{\sqrt{-\Delta(x_i, x_j)}} \right) \right\} \tag{4.20}$$

$$\int_0^1 x^2 \ln \Lambda_1 dx = -\frac{13}{18} + \frac{5x_i - x_j}{6x_h} - \frac{1}{3}\left(\frac{x_i - x_j}{x_h} \right)^2 - \frac{x_j(x_i - x_j - x_h)}{2x_h^2} \ln\left(\frac{x_i}{x_j} \right)$$

$$+ \frac{1}{6}\left(2 + \left[\frac{x_i - x_j - x_h}{x_h} \right]^3 \right) \ln x_i - \frac{1}{6}\left(\frac{x_i - x_j - x_h}{x_h} \right)^3 \ln x_j$$

$$+ \frac{\left[(x_i - x_j - x_h)^2 - x_j x_h \right]}{3x_h^3} \sqrt{-\Delta(x_i, x_j)} \left\{ \tan^{-1}\left(\frac{x_h + x_i - x_j}{\sqrt{-\Delta(x_i, x_j)}} \right) \right.$$

$$-\left(\frac{x_h + x_i - x_j}{\sqrt{-\Delta(x_i, x_j)}}\right)\Bigg\} \qquad (4.21)$$

We will now demonstrate how the three remaining integrals, $\int_0^1 x^n \ln \Lambda_n dx$, $n = 2,3,4$ can be evaluated. By writing $\Lambda_1 = \Lambda_1(x_i, x_j)$,we are able to express them in terms of $\int_0^1 x^n \ln \Lambda_1(x_i, x_j) dx$ as follows:

$$\int_0^1 x^n \ln \Lambda_2 dx = \int_0^1 x^n \ln \Lambda_1(x_i = 1, x_j = 1) dx \qquad (4.22)$$

$$\int_0^1 x^n \ln \Lambda_3 dx = \int_0^1 x^n \ln \Lambda_1(x_i = 1, x_j) dx \qquad (4.23)$$

$$\int_0^1 x^n \ln \Lambda_4 dx = \int_0^1 x^n \ln \Lambda_1(x_i, x_j = 1) dx \qquad (4.24)$$

Depending on whether $D(x_i, x_j) = (x_i - x_j - x_h)^2 - 4x_h x_j$ is greater than or less than zero, the integrals in Eqs. (4.22)- (4.24) will be evaluated using the results obtained for $\int_0^1 x^n \ln \Lambda_1 dx$ under Case (I) or Case (II). For Eq.(4.22), since $D(x_i = 1, x_j = 1) < 0$, $\int_0^1 x^n \ln \Lambda_2 dx$ will be calculated using the results obtained under Case (II). On the other hand, for the integrals $\int_0^1 x^n \ln \Lambda_3 dx$ and $\int_0^1 x^n \ln \Lambda_4 dx$, $D(x_i = 1, x_j)$ and $D(x_i, x_j = 1)$ are both greater than zero for $i, j = u,c,t$ and these integrals will be calculated using the results obtained under Case (I). Performing the integrations in Eqs. (3.20)-(3.21) using

Eqs. (4.22)-(4.24), we obtain the following expressions for $B_{ij}^{(d)}$ and $C_{ij}^{(d)}$ for 2 cases: (1) $D(x_i, x_j) > 0$ and (11) $D(x_i, x_j) < 0$:

CASE (I): $D(x_i, x_j) > 0$

$$B_{ij}^{(d)} = \frac{1}{(1-x_i)(1-x_j)}\left\{\left(2+\frac{x_i x_j}{2}\right)B_1 + B_2\right\} \tag{4.25}$$

where

$$B_1 = \frac{4-x_h}{3}\sqrt{\frac{4}{x_h}-1}\left(\tan^{-1}\frac{1}{\sqrt{\frac{4}{x_h}-1}}\right) + \frac{1}{12x_h^2}\left(\left[x_i - x_j - x_h\right]^3 - \left[1 - x_j - x_h\right]^3\right)\ln x_j$$

$$+\frac{1}{12x_h^2}\left(\left[x_i - i - x_h\right]^3 - \left[x_i - x_j - x_h\right]^3\right)\ln x_i + \frac{1}{2x_h}\left(1 - x_j^2\right)\ln x_i - \frac{x_i}{x_h}\left(1 - x_j\right)\ln x_i$$

$$+\frac{1}{6x_h}\left(x_i - x_j\right)^2 + \left(\frac{x_i\left[1 - x_j\right]\ln x_i}{2x_h} + (i \leftrightarrow j)\right) - \frac{1}{6x_h}\left(\left[1 - x_i\right]^2 + \left[i \leftrightarrow j\right]\right)$$

$$+\frac{1}{12x_h^2}\Delta(x_i, x_j)\sqrt{\Delta(x_i, x_j)}\ln\left|\Omega(x_i, x_j)\right| - \frac{1}{12x_h^2}\left(\Delta(x_i, 1)\sqrt{\Delta(x_i, 1)}\ln\left|\Omega(x_i, 1)\right| + \left[i \leftrightarrow j\right]\right)$$

(4.26)

$$B_2 = \sqrt{\frac{4}{x_h}-1}\left(\tan^{-1}\frac{1}{\sqrt{\frac{4}{x_h}-1}}\right)\left(x_h\left[x_i + x_j\right] - 4x_i x_j\right)$$

$$+\frac{1}{4x_h}\left(x_i - 2\left[1+x_i^2\right] + x_j\left[2\left(x_h + x_i^2\right) + x_j\left(x_i + x_j - 1\right) - 1\right]\ln x_i + \left[i \leftrightarrow j\right]\right)$$

$$+\frac{1}{4}\sqrt{\Delta\left(x_i,1\right)}\ln\left|\Omega\left(x_i,1\right)\right|\left(x_i + x_j + \frac{x_i - x_j}{x_h} - \frac{x_i}{x_h}\left[x_i + 3x_j\right] + \left[i \leftrightarrow j\right]\right)$$

$$+\frac{1}{4}\sqrt{\Delta\left(x_i,x_j\right)}\ln\left|\Omega\left(x_i,x_j\right)\right|\left(\frac{\left(x_i - x_j\right)^2}{x_h} + 4\frac{x_i x_j}{x_h} - \left[x_i + x_j\right]\right) \tag{4.27}$$

and for C_{ij}

$$C_{ij} = \frac{1}{\left(1-x_i\right)\left(1-x_j\right)}\left(4 + x_i x_j\right)C_1 \tag{4.28}$$

where

$$C_1 = \frac{1}{3}\left(x_i - x_j\right)^2 - \frac{1}{3x_h}\left(\left[1-x_i\right]^2 + \left[i \leftrightarrow j\right]\right) + \sqrt{\frac{4}{x_h}-1}\left(\frac{2+x_h}{3}\right)\tan^{-1}\frac{1}{\sqrt{\frac{4}{x_h}-1}}$$

$$+\frac{1}{2x_h}\left(\left[1-x_i\right] + \left[1-x_j\right] - \left[x_i - x_j\right]^2 - 2x_i\left[1-x_j\right]\right)\ln x_i$$

$$+\frac{1}{6x_h^2}\left(\left[x_i - x_j - x_h\right]^3 - \left[1 - x_j - x_h\right]^3\right)\ln x_j$$

$$+\frac{1}{6x_h^2}\left(\left[x_i - x_h - 1\right]^3 - \left[x_i - x_j - x_h\right]^3\right)\ln x_i$$

$$+\frac{1}{12x_h^2}\left(2\left[x_i - x_j\right]^2 - x_h\left[x_i + x_j + x_h\right]\right)\sqrt{\Delta(x_i,x_j)}\ln\left|\Omega(x_i,x_j)\right|$$

$$+\frac{1}{4x_h}\left\{\left(2\left[1-x_j\right]\left[x_i + x_h\right] + \left[x_i - x_j\right]^2 - \left[1-x_i\right]^2\right)\ln x_i + (i \leftrightarrow j)\right\}$$

$$+\frac{1}{12x_h^2}\left\{\left(x_h\left[1+x_i+x_h\right] - 2\left[1-x_i\right]^2\right)\sqrt{\Delta(x_i,1)}\ln\left|\Omega(x_i,1)\right| + (i \leftrightarrow j)\right\} \tag{4.29}$$

Collecting all the terms in Eqs. (4.25)-(4.29) and using Eq. (4.6), we finally obtain the following analytical expression for $A_{ij}^{(d)}$:

$$A_{ij}^{(d)} = \frac{1}{(1-x_i)(1-x_j)}\left(\left[2+\frac{x_i x_j}{2}\right]A_1 + A_2\right) \tag{4.30}$$

where

$$A_1 = \frac{1}{4}(2-3x_h)\sqrt{\frac{4}{x_h}-1}\left(\tan^{-1}\left[\frac{4}{x_h}-1\right]^{-1/2}\right) + \frac{1+x_i x_j - x_i - x_j}{2x_h}$$

$$+\frac{1}{16x_h^2}\left\{\left(x_h\left[x_j^2-1\right] + 4x_h^2\left[x_i - 1\right] + 2\left[(1-x_j)^3 - (x_i - x_j)^3\right]\right)\ln x_j + \left[i \leftrightarrow j\right]\right\}$$

$$+\frac{1}{16x_h^2}\left(3x_h^2 - 2\left[x_i - x_j\right]^2 - x_h\left[x_i + x_j\right]\right)\sqrt{\Delta(x_i,x_j)}\ln\left|\Omega(x_i,x_j)\right|$$

$$+\frac{1}{16x_h^2}\left\{\left(2\left[x_i - 1\right]^2 + x_h\left[x_i + 1\right] - 3x_h^2\right)\sqrt{\Delta(x_i,1)}\ln\left|\Omega(x_i,1)\right| + \left[i \leftrightarrow j\right]\right\} \tag{4.31}$$

$$A_2 = \sqrt{\frac{4}{x_h}-1}\left(\tan^{-1}\frac{1}{\sqrt{\frac{4}{x_h}-1}}\right)\left(x_h\left[x_i + x_j\right] - 4x_i x_j\right)$$

$$+\frac{1}{4x_h}\left\{\left(x_j^2\left[x_i + x_j\right] + \left[x_i - x_j\right] + 2x_i x_j\left[x_i - 1\right] + 2x_j x_h\left[1-x_j\right] - 2x_i^2\right)\ln x_i + \left[i \leftrightarrow j\right]\right\}$$

$$+\frac{1}{4}\left\{-\left(x_i+x_j\right)+\frac{\left(x_i-x_j\right)^2}{x_h}+4\frac{x_i x_j}{x_h}\right\}\sqrt{\Delta\left(x_i,x_j\right)}\ln\left|\Omega\left(x_i,x_j\right)\right|$$

$$+\frac{1}{4}\left\{\left(\left[x_i+x_j\right]+\frac{\left[x_i-x_j\right]}{x_h}-\frac{x_i}{x_h}\left[x_i+3x_j\right]\right)\sqrt{\Delta\left(x_i,1\right)}\ln\left|\Omega\left(x_i,1\right)\right|+\left[i\leftrightarrow j\right]\right\} \tag{4.32}$$

CASE(II), $D\left(x_i,x_j\right)<0$

$$B_{ij}^{(d)}=\frac{1}{\left(1-x_i\right)\left(1-x_j\right)}\left\{\left(2+\frac{x_i x_j}{2}\right)B_3+B_4\right\} \tag{4.33}$$

where

$$B_3=B_1-\frac{1}{12x_h^2}\Delta\left(x_i,x_j\right)\sqrt{\Delta\left(x_i,x_j\right)}\ln\left|\Omega\left(x_i,x_j\right)\right|$$

$$-\frac{1}{6x_h^2}\Delta\left(x_i,x_j\right)\sqrt{-\Delta\left(x_i,x_j\right)}\left(\tan^{-1}\left[\frac{x_i-x_j+x_h}{\sqrt{-\Delta\left(x_i,x_j\right)}}\right]+\left[i\leftrightarrow j\right]\right) \tag{4.34}$$

$$B_4=B_2-\frac{1}{4}\sqrt{\Delta\left(x_i,x_j\right)}\ln\left|\Omega\left(x_i,x_j\right)\right|\left(\frac{\left(x_i-x_j\right)^2}{x_h}+4\frac{x_i x_j}{x_h}-\left[x_i+x_j\right]\right)$$

$$-\frac{1}{2}\sqrt{-\Delta\left(x_i,x_j\right)}\left(\tan^{-1}\left[\frac{x_i-x_j+x_h}{\sqrt{-\Delta\left(x_i,x_j\right)}}\right]+\left[i\leftrightarrow j\right]\right) \tag{4.35}$$

with B_1 and B_2 defined respectively as in Eqs. (4.26) and (4.27).

Similarly, for C_{ij} we obtain :

$$C_{ij}^{(d)} = \frac{1}{(1-x_i)(1-x_j)}(4 + x_i x_j)C_2 \tag{4.36}$$

where

$$\begin{aligned} C_2 = C_1 &- \frac{1}{12x_h^2}\left(2[x_i - x_j]^2 - x_h[x_i + x_j + x_h]\right)\sqrt{\Delta(x_i,x_j)}\ \ln\left|\Omega(x_i,x_j)\right| \\ &- \frac{1}{6x_h^2}\left(2[x_i - x_j]^2 - x_h[x_i + x_j + x_h]\right)\sqrt{-\Delta(x_i,x_j)} \\ &\times\left(\tan^{-1}\left[\frac{x_i - x_j + x_h}{\sqrt{-\Delta(x_i,x_j)}}\right] + [i \leftrightarrow j]\right) \end{aligned} \tag{4.37}$$

with C_1 given by Eq. (4.29). From Eqs. (4.33-4.37) and Eq. (4.6) we obtain

$$A_{ij}^{(d)} = \frac{1}{(1-x_i)(1-x_j)}\left[\left(2 + \frac{x_i x_j}{2}\right)A_3 + A_4\right] \tag{4.38}$$

where

$$\begin{aligned} A_3 = A_1 &- \frac{1}{16x_h^2}\left(3x_h^2 - 2[x_i - x_j]^2 - x_h[x_i + x_j]\right)\sqrt{\Delta(x_i,x_j)}\ \ln\left|\Omega(x_i,x_j)\right| \\ &- \frac{1}{8x_h^2}\left(3x_h^2 - 2[x_i - x_j]^2 - x_h[x_i + x_j]\right)\sqrt{-\Delta(x_i,x_j)} \\ &\times\left(\tan^{-1}\left[\frac{x_i - x_j + x_h}{\sqrt{-\Delta(x_i,x_j)}}\right] + [i \leftrightarrow j]\right) \end{aligned} \tag{4.39}$$

$$A_4 = A_2 - \frac{1}{4}\left(4\frac{x_i x_j}{x_h} - \left[x_i + x_j\right] + \frac{\left[x_i - x_j\right]^2}{x_h}\right)\sqrt{\Delta\left(x_i, x_j\right)}\ \ln\left|\Omega\left(x_i, x_j\right)\right|$$

$$-\frac{1}{2}\left(\frac{4x_i x_j}{x_h} - \left[x_i + x_j\right] + \frac{\left(x_i - x_j\right)^2}{x_h}\right)\sqrt{-\Delta\left(x_i, x_j\right)}$$

$$\times\left(\tan^{-1}\left[\frac{x_i - x_j + x_h}{\sqrt{-\Delta\left(x_i, x_j\right)}}\right] + \left[i \leftrightarrow j\right]\right) \qquad (4.40)$$

Our final expressions for the dispersive part, $A_{ij}^{(d)}$ for $D\left(x_i, x_j\right) > 0$ and $D\left(x_i, x_j\right) < 0$ are symmetric with respect to i and j. Our final expressions for the dispersive part, $A_{ij}^{(d)}$ for $D\left(x_i, x_j\right) > 0$ and $D\left(x_i, x_j\right) < 0$ are symmetric with respect to i and j. Using Eqs. (4.2)-(4.6), M_{12} and Γ_{12} can now be expressed as

$$M_{12} = \frac{G_F^2 M_W^2 f_p^2 m_p B_p}{12\pi^2} \sum \lambda_i \lambda_j A_{ij}^{(d)} \qquad (4.41)$$

$$\Gamma_{12} = \frac{i G_F^2 M_W^2 f_p^2 m_p B_p}{12\pi^2} \sum \lambda_i \lambda_j A_{ij}^{(a)} \qquad (4.42)$$

with $A_{ij}^{(d)}$ either given by Eq. (4.30) or Eq. (4.38) and $A_{ij}^{(a)}$ by Eq. (4.13).

4.4. THE BEHAVIOUR OF THE FORM FACTORS.

We have shown in previous sections the method used to obtain analytical expressions $A_{ij}^{(d,a)}$. We now display the behaviour of the form factors and we shall use the following set of values for the quark masses involved:

$m_u \approx 0$ GeV

$m_c = 1.3$ GeV

$m_t = 176$ GeV

$m_b = 4.3$ GeV

The form factors demonstrate a strong dependence on the internal quark masses. First, let us look at $A_{jj}^{(d,a)}$, where both of the internal quarks are of the same kind. Fig. 4.1 shows the behaviour of $A_{jj}^{(d)}$ for values of m_j up to 600 GeV. $A_{jj}^{(d)}$ is seen to increase almost linearly as m_j increases from 100 to 600 GeV. When m_j is large, the expression for $A_{jj}^{(d)}$ of Eq. (4.38) reduces to:

$$A_{jj}^{(d)} = \frac{1}{\left(1-x_j\right)^2}\left[\left(3-4x_j\right)x_j + \frac{\left(4+x_j^2\right)\left(1-x_j\right)^2}{4x_h} + \frac{x_j\left(x_j-1\right)\left(x_j^2-4\right)\ln x_j}{8x_h}\right] \tag{4.43}$$

From Eq. (4.13), it is seen that the absorptive part develops whenever $x_h^2 > 4x_h x_j$. The behaviour of $A_{jj}^{(a)}$ as a function of m_j is displayed in Fig. 4.2. It can be seen that $A_{jj}^{(a)}$ decreases in magnitude as m_j increases. When $m_j \approx 2.1$ GeV, $A_{jj}^{(a)}$ becomes zero. Putting $i = j$ it is straight forward to see that Eq. (4.13) reduces to the following form:

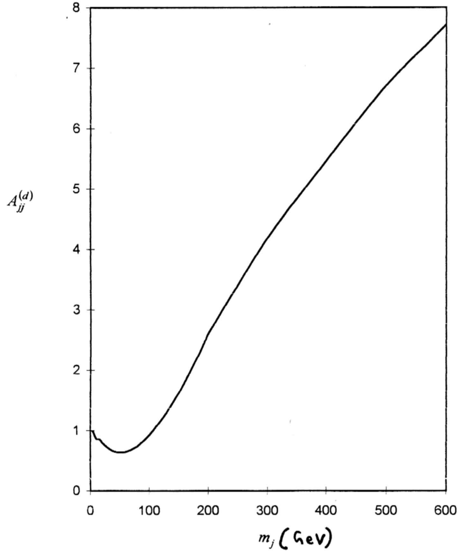

Fig. 4.1. The behaviour of $A_{jj}^{(d)}$ as a function of m_j

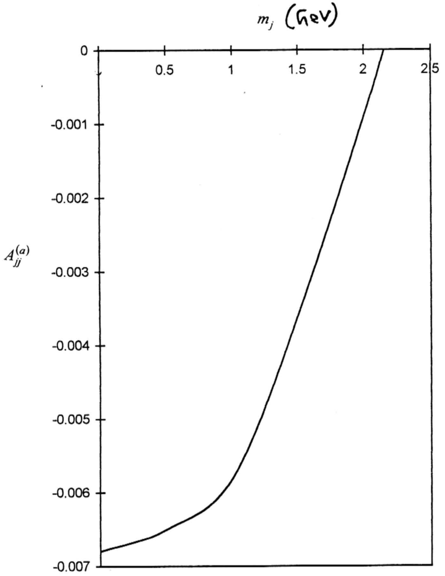

Fig. 4.2. The behaviour of $A_{jj}^{(a)}$ as a function of m_j

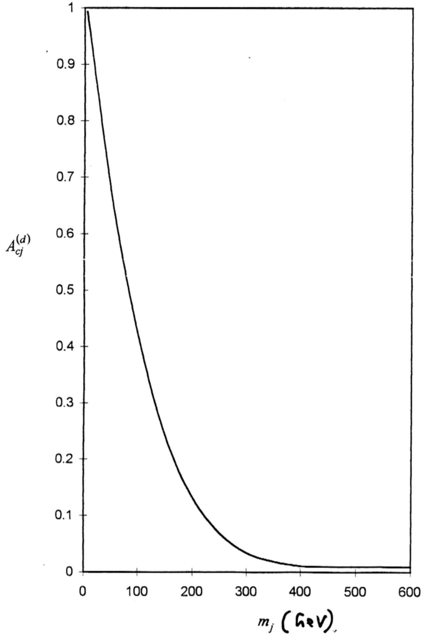

Fig. 4.3. The behaviour of $A_{cj}^{(d)}$ as a function of m_j

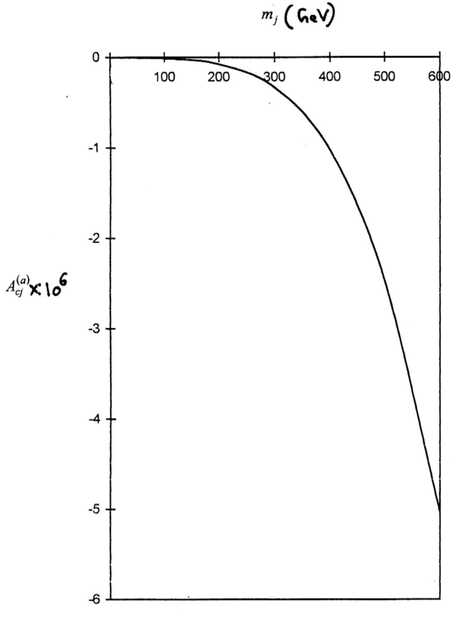

Fig. 4.4. The behaviour of $A_{cj}^{(a)}$ as a function of m_j

$$A_{ij}^{(a)} \approx \frac{-\pi x_h}{4}\left(1-4\frac{x_j}{x_h}\right)^{\frac{1}{2}}\left(3-2\frac{x_j}{x_h}\right) \tag{4.44}$$

We now turn to the behaviour of the form factors when the two internal quarks are different. To see a typical behaviour, we display $A_{cj}^{(d)}$ and $A_{cj}^{(a)}$ versus m_j in Figs. 4.3 and 4.4 respectively. It is observed that $A_{cj}^{(d)}$ drops sharply as m_j increases whereas $A_{cj}^{(a)}$ increases in magnitude.

The values of the form factors at physical masses of the internal quarks are given in Table 4.1:

Table 4.1: Numerical values for $A_{ij}^{(d,a)}$

$A_{uu}^{(d)} = 1.0012$	$A_{uu}^{(a)} = -6.77 \times 10^{-3}$
$A_{cc}^{(d)} = 1.0012$	$A_{cc}^{(a)} = -5.93 \times 10^{-3}$
$A_{tt}^{(d)} = 2.1241$	$A_{uc}^{(a)} = -5.07 \times 10^{-3}$
$A_{uc}^{(d)} = 1.0011$	
$A_{ut}^{(d)} = 0.1775$	
$A_{ct}^{(d)} = 0.1801$	

The values in Table 4.1 were obtained using a computer program called FORTRAN. In order to check the correctness of these values, we recalculated $A_{ij}^{(d,a)}$ analytically, and obtained results that are consistent with that given by Table 4.1.

In the next chapter, we shall analyse the behaviour of M_{12} and Γ_{12} using the numerical values in Table 4.1 and calculate the mixing parameters which account for particle antiparticle mixing in the $B_d^o \leftrightarrow \bar{B}_d^o$ and $B_s^o \leftrightarrow \bar{B}_s^o$ systems.