CHAPTER 5

APPLICATIONS

5.1 INTRODUCTION

In this chapter, we apply the results of the previous chapter to study the mixing phenomenon in the $B_d^0 - \overline{B}_d^0$ and $B_s^0 - \overline{B}_s^0$ systems. To begin with, we discuss in Section 5.2, the formalism required to understand mixing. In Section 5.3, we analyse in detail the behaviour of the dispersive and absorptive parts. Finally, Section 5.4, we calculate the mixing parameters in the $B_d^0 - \overline{B}_d^0$ and $B_s^0 - \overline{B}_s^0$ systems.

5.2 FORMALISM FOR THE MASS AND WIDTH DIFFERENCE AND THE PARAMETERS OF MIXING

A description of the $P^0 - \overline{P}^0$ system is provided by the mass matrix, $M - i\Gamma$. The phenomenological Hamiltonian, H, in the flavour eigenbasis, can be written as

$$H\begin{bmatrix} P^0 \\ \overline{P}^0 \end{bmatrix} = \begin{bmatrix} M - \frac{i\Gamma}{2} & M_{12} - \frac{i\Gamma_{12}}{2} \\ M_{12}^* - \frac{i\Gamma_{12}^*}{2} & M - \frac{i\Gamma}{2} \end{bmatrix}\begin{bmatrix} P^0 \\ \overline{P}^0 \end{bmatrix} \tag{5.1}$$

where both M and Γ are 2×2 Hermitian matrices. Due to the $P^0 \leftrightarrow \overline{P}^0$ transition, the original P^0, $\overline{P}^0$ states are no longer the physical states. Diagonalizing the matrix in Eq. (5.1), one obtains the mass eigenstates which are given by [2]

$$|P_{1,2}\rangle = \frac{1}{\sqrt{2\left(1+|\varepsilon|^2\right)}}\left[(1+\varepsilon)|P^0\rangle \pm (1-\varepsilon)|\overline{P}^0\rangle\right] \tag{5.2}$$

with mass $M_{\pm}(P)$. ε is the CP violation parameter which vanishes if $|P_{\pm}^0\rangle$ are CP eigenstates. The mass and width differences are respectively given by

$$\Delta M_p = M_+(P) - M_-(P) = 2\,\mathrm{Re}(Q) \tag{5.3}$$

$$\Delta\Gamma_p = \Gamma_+ - \Gamma_- = -4\,\mathrm{Im}\,Q \tag{5.4}$$

where $M_{\pm}(P)$ and $\Gamma_{\pm}(P)$ are, respectively , the mass and width of $|P_{\pm}^0\rangle$ and

$$Q = \sqrt{\left(M_{12} - \frac{i\Gamma_{12}}{2}\right)\left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right)} \tag{5.5}$$

In Eq. (5.5), M_{12} and Γ_{12} are respectively the dispersive and absorptive parts of the $P^0 - \overline{P}^0$ transition amplitude. Mixing in the $P^0 - \overline{P}^0$ system is described by the parameters r and $\bar{r}$ which are defined respectively as [42]

$$r_P = \frac{\Pr\ \left(P^0 \to \overline{P}^o\right)}{\Pr\ \left(P^0 \to P^0\right)} = \eta\frac{x_P^2 + y_P^2}{2 + x_P^2 - y_P^2} \tag{5.6}$$

and

$$\bar{r}_P = \frac{\Pr\ \left(\overline{P}^0 \to P^0\right)}{\Pr\ \left(\overline{P}^0 \to \overline{P}^0\right)} = \eta^{-1}\frac{x_P^2 + y_P^2}{2 + x_P^2 - y_P^2} \tag{5.7}$$

where Pr stands for probability and

$$\eta = \left|\frac{1-\varepsilon}{1+\varepsilon}\right|^2 \tag{5.8}$$

$$x_P = \frac{\Delta M_P}{\Gamma_P} \tag{5.9}$$

$$y_P = \frac{\Delta\Gamma_P}{2\Gamma_P} \tag{5.10}$$

In Eqs. (5.10) and (5.11), the width Γ_P is

$$\Gamma_P = \frac{1}{2}\left[\Gamma_+(P) + \Gamma_-(P)\right] \tag{5.11}$$

It can be seen from Eqs. (5.6) and (5.7) that when $\varepsilon << 1$, $r = \bar{r}$. Alternatively, mixing can also be described by another parameter χ which is given by

$$\chi = \frac{r}{1+r} \tag{5.12}$$

5.3 THE BEHAVIOUR OF M_{12} AND Γ_{12}

Using Eqs. (4.41) and (4.42) and summing over for $i, j = u,c,t$ we are able to obtain the following expressions for M_{12} and Γ_{12}:

$$M_{12} = \frac{G_F^2 M_W^2 f_p^2 m_p B_p}{12\pi^2}\left[\lambda_c^2 A_{cc}^{(d)} + \lambda_t^2 A_{tt}^{(d)} + \lambda_u^2 A_{uu}^{(d)} + 2\lambda_c\lambda_t A_{ct}^{(d)}\right.$$

$$\left. + 2\lambda_c\lambda_u A_{cu}^{(d)} + 2\lambda_u\lambda_t A_{ut}^{(d)}\ \right] \tag{5.13}$$

$$\Gamma_{12} = \frac{iG_F^2 M_W^2 f_p^2 m_p B_p}{12\pi^2}\left[\lambda_c^2 A_{cc}^{(a)} + \lambda_t^2 A_{tt}^{(a)} + \lambda_u^2 A_{uu}^{(a)} + 2\lambda_c\lambda_t A_{ct}^{(a)}\right.$$

$$\left. + 2\lambda_c\lambda_u A_{cu}^{(a)} + 2\lambda_u\lambda_t A_{ut}^{(a)}\ \right] \tag{5.14}$$

where p is either B_d^0 or B_s^0. By exploiting the unitarity property of the KM matrix,

$$\lambda_u + \lambda_c + \lambda_t = 0 \tag{5.15}$$

in Eqs. (5.15) and (5.16), we finally obtain:

$$M_{12} = \frac{G_F^2 M_W^2 f_p^2 m_p B_p}{12\pi^2}\left[\ \lambda_c^2 U_{cc}^{(d)} + \lambda_t^2 U_{tt}^{(d)} + 2\lambda_c\lambda_t U_{ct}^{(d)}\ \right] \tag{5.16}$$

$$\Gamma_{12} = \frac{iG_F^2 M_W^2 f_P^2 m_P B_P}{12\pi^2}\left[\lambda_c^2 U_{cc}^{(a)} + \lambda_t^2 U_{tt}^{(a)} + 2\lambda_c\lambda_t U_{ct}^{(a)}\right] \tag{5.17}$$

where

$$U_{ij}^{(d,a)} = A_{uu}^{(d,a)} + A_{ij}^{(d,a)} - A_{ui}^{(d,a)} - A_{uj}^{(d,a)} \tag{5.18}$$

It can be seen from Eqs. (5.16) and (5.17) that 3 terms contribute to M_{12} and Γ_{12}, and to find the dominant contribution we need to calculate $U_{ij}^{(d,a)}$ first. To do this, we shall use the results obtained for $A_{ij}^{(a)}$ and $A_{ij}^{(d)}$ in Sect.4.4 and Eq. (5.18). The numerical values for $U_{ij}^{(d,a)}$ are presented in Table 5.1.

Table 5.1: Numerical values for $U_{ij}^{(d,a)}$

$U_{tt}^{(d)} = 2.77$	$U_{tt}^{(a)} = -6.77\times 10^{-3}$
$U_{ct}^{(d)} = 2.70\times 10^{-3}$	$U_{ct}^{(a)} = -8.40\times 10^{-4}$
$U_{cc}^{(d)} = 2.00\times 10^{-4}$	$U_{cc}^{(a)} = 2.55\times 10^{-5}$

The magnitudes of the KM matrix elements are approximately given by [43]

$$|V| \approx \begin{pmatrix} 1 & 0.22 & 0.003 \\ 0.22 & 1 & 0.04 \\ 0.01 & 0.04 & 1 \end{pmatrix} \tag{5.19}$$

and from Eq. (5.19), we obtain

$$|\lambda_c| = |V_{cb}{}^* V_{cd}| \approx 8.8\times 10^{-3}$$

$$|\lambda_t| = |V_{tb}{}^* V_{td}| \approx 1.0\times 10^{-2} \tag{5.20}$$

for the $B_d^0 - \overline{B}_d^0$ system, and

$$|\lambda_c| = |V_{cb}{}^* V_{cs}| \approx 4.0\times 10^{-2}$$

$$|\lambda_t| = |V_{tb}{}^* V_{ts}| \approx 4.0 \times 10^{-2} \tag{5.21}$$

for the $B_s^0 - \overline{B}_s^0$ system.

Using the magnitudes of the KM matrix elements in Eq. (5.20) and the numerical values in Table 1, we obtain the following values for the $B_d^0 - \overline{B}_d^0$ system:

$$|\lambda_c^2 U_{cc}^{(d)}| \approx 1.55 \times 10^{-8}$$

$$2|\lambda_c \lambda_t U_{ct}^{(d)}| \approx 4.75 \times 10^{-7}$$

$$|\lambda_t^2 U_{tt}^{(d)}| \approx 2.77 \times 10^{-4} \tag{5.22}$$

and

$$|\lambda_c^2 U_{cc}^{(a)}| \approx 1.97 \times 10^{-9}$$

$$2|\lambda_c \lambda_t U_{ct}^{(a)}| \approx 1.48 \times 10^{-7}$$

$$|\lambda_t^2 U_{tt}^{(a)}| \approx 6.77 \times 10^{-7} \tag{5.23}$$

Similarly, for the $B_s^0 - \overline{B}_s^0$ system , we get the following values:

$$|\lambda_c^2 U_{cc}^{(d)}| \approx 3.20 \times 10^{-7}$$

$$2|\lambda_c \lambda_t U_{ct}^{(d)}| \approx 8.64 \times 10^{-6}$$

$$|\lambda_t^2 U_{tt}^{(d)}| \approx 4.43 \times 10^{-3} \tag{5.24}$$

and

$$|\lambda_c^2 U_{cc}^{(a)}| \approx 4.08 \times 10^{-8}$$

$$2|\lambda_c \lambda_t U_{ct}^{(a)}| \approx 2.69 \times 10^{-6}$$

$$|\lambda_t^2 U_{tt}^{(a)}| \approx 1.08 \times 10^{-5} \tag{5.25}$$

From our analysis in Eqs.(5.22)-(5.25), the following inequalities hold:

$$\left|\lambda_t^2 U_{tt}^{(d)}\right| >> \left|\lambda_c^2 U_{cc}^{(d)}\right|,\ 2\left|\lambda_c \lambda_t U_{ct}^{(d)}\right| \tag{5.26}$$

$$\left|\lambda_t^2 U_{tt}^{(a)}\right| > 2\left|\lambda_c \lambda_t U_{ct}^{(a)}\right| >> \left|\lambda_c^2 U_{cc}^{(a)}\right| \tag{5.27}$$

and so the dominant contribution to M_{12} comes from the $\left|\lambda_t^2 U_{tt}^{(d)}\right|$ term, with the internal quarks being the top quarks. Therefore, the dispersive and absorptive parts in both these systems can be represented approximately by

$$M_{12} \approx \frac{G_F^2 M_W^2 f_P^2 m_P B_P}{12\pi^2}\left|\lambda_t^2 U_{tt}^{(d)}\right| \tag{5.28}$$

$$\Gamma_{12} \approx i\frac{G_F^2 M_W^2 f_P^2 m_P B_P}{12\pi^2}\left|\lambda_t^2 U_{tt}^{(a)}\right| \tag{5.29}$$

From Eqs. (5.28) and (5.29), we obtain the ratio of the absorptive part to the dispersive part,

$$\frac{\left|\Gamma_{12}\right|}{\left|M_{12}\right|} \approx \frac{\left|U_{tt}^{(a)}\right|}{\left|U_{tt}^{(d)}\right|} \tag{5.30}$$

and from the values in Table 5.1, it is seen that $\left|\Gamma_{12}\right| << \left|M_{12}\right|$.

5.4 CALCULATION OF THE MIXING PARAMETERS

As $\left|\Gamma_{12}\right| << \left|M_{12}\right|$, the expression for the mass difference given by Eq. (5.3) reduces to

$$\Delta M_P = 2\,\mathrm{Re}\,M_{12} \approx \frac{G_F^2 M_W^2 f_P^2 m_P B_P}{6\pi^2}\left|\lambda_t^2 U_{tt}^{(d)}\right| \tag{5.31}$$

and from Eq.(5.4), the width difference becomes $\Delta\Gamma_P \approx 0$. For our calculations, we shall use the input parameters given in Table 5.2

TABLE 5.2: Input parameters

	B_d^0 meson	B_s^0 meson
f_P	173 ± 40 MeV	200.68 ± 4.00 MeV
B_P	1.2 ± 0.2	1.2 ± 0.2
m_P	5279.2 ± 1.8 MeV	5369.6 ± 2.4 MeV
t_P	$(1.56 \pm 0.06) \times 10^{-12}$ s	$(1.61 + 0.10 - 0.09) \times 10^{-12}$ s

We shall now take into account the QCD corrections to ΔM_P. The experimental value of the QCD correction is $\eta_B = 0.55 \pm 0.01$ [44]. Eq. (5.31) now becomes:

$$\Delta M_P = 2\,\mathrm{Re}\, M_{12} \approx \frac{G_F^2 M_W^2 f_P^2 m_P B_P \eta_B}{6\pi^2} \left| \lambda_t^2 U_{tt}^{(d)} \right| \qquad (5.32)$$

Let us calculate the mixing parameters in the $B_d^0 - \overline{B}_d^0$ system first. Using Eq. (5.32) and the input parameters in Table 5.2, we obtain

$$\Delta M_d = 4.27 \times 10^{-13} \text{ GeV} \qquad (5.33)$$

The mixing parameters, r_d can χ_d now be determined. Noting that the width, Γ_P is the inverse of the mean life time, t_p and using the value of ΔM_d from Eq. (5.33), we obtain from Eq. (5.6):

$$r_d = 0.34 \qquad (5.34)$$

Using Eq. (5.12), we get

$$\chi_d = 0.25 \qquad (5.35)$$

The experimental values are [32]

$$\chi_d = 0.175 \pm 0.016 \text{ and } \Delta M_d = (3.121 \pm 0.204) \times 10^{-13} \text{ GeV}.$$

The calculated values for ΔM_d and χ_d are larger than the experimental values. There are several reasons for this. Firstly, certain parameters like f_p and B_p cannot be obtained from experiment and need to be estimated theoretically, for example from lattice QCD calculations. Secondly, the V_{td} and V_{ts} matrix elements, which were used in calculating the dominant contribution to M_{12} are not known accurately.

A similar treatment for the $B_s^0 - \overline{B}_s^0$ system gives

$$\Delta M_s = 9.35 \times 10^{-12} \text{ GeV} \tag{5.36}$$

$$r_s \approx 1 \tag{5.37}$$

$$\chi_s \approx 0.5 \tag{5.38}$$

From experiment, $0.4975 < \chi_s < 0.5$, and $\Delta M_s > 3.883 \times 10^{-12}$ GeV. The value of r_s in Eq. (5.37) indicates that the effect of mixing in the $B_s^0 - \overline{B}_s^0$ system is close to the maximum value of one, whereas the corresponding value for the $B_d^0 - \overline{B}_d^0$ system is 0.42. From these values we may conclude that mixing is more substantial in the $B_s^0 - \overline{B}_s^0$ system than in the $B_d^0 - \overline{B}_d^0$ system.