Chapter 4

Results Presentation and Discussions

4.1: Introduction

This chapter presents the results obtained by numerical methods as discussed in chapter 3. The wave phenomena investigated in this study are the interference of two beams and that of a diffraction grating. The objective in this investigation will be focused on the formation of the fringes from the perspective of energy redistribution of the rays.

A mathematical proof will first be shown that the interference or superposition of two beams is self-consistently defined in the eikonal formalism. The numerical results are then presented in the subsequent sections. The ray traces are also presented. It will be the main focus on how the ray leads to the occurrence of the first minimum.

Before investigating the interference effect, the numerical accuracy of the numerical method is studied by the comparison with the analytical results of diffraction of a Gaussian beam.

The last part is on the interference of the multiple slits. This can be considered as the generalization of the interference effect from the two beams. The diffraction patterns are obtained for different number of slits on the grating. The changes in the diffraction pattern when the parameters are changed are also shown. A ray trace of a five slit interference is shown. The light rays give an indication of the redistribution of the effect of interference from five slits.

4.2: Linear Superposition of Waves

In this section, the proof that interference effect is self consistently defined in the eikonal formalism will be discussed. We will prove that if $\Phi_1 = \phi_1 e^{ikL_1}$ and $\Phi_2 = \phi_2 e^{ikL_2}$ satisfy the generalized eikonal equation and the equation of continuity, then $\Phi_3 = \phi_3 e^{ikL_3} = \phi_1 e^{ikL_1} + \phi_2 e^{ikL_2}$ also fulfills the generalized eikonal equation and the equation of continuity

First, consider $\Phi_{1,2} = \phi_{1,2} e^{ikL_{1,2}}$ fulfills

$$(\nabla L_{1,2})^2 = n_0^2 + \frac{1}{k_0^2 \phi_{1,2}} \nabla^2 \phi_{1,2} \tag{4.1}$$

$$\nabla \cdot \left(\phi_{1,2}{}^2 \nabla L_{1,2}\right) = 0 \qquad (4.2).$$

Then, it is easy to show that

$$\nabla^2 \Phi_3 = \nabla^2 \left(\phi_1 e^{ikL_1} + \phi_2 e^{ikL_2}\right)$$

$$\begin{aligned} &= \left(\nabla^2 \phi_1\right) e^{ikL_1} + \left(2ik\nabla L_1 \cdot \nabla \phi_1 - k^2 \phi_1 (\nabla L_1)^2 + ik\nabla^2 L_1 \phi_1\right) e^{ikL_1} \\ &+ \left(\nabla^2 \phi_2\right) e^{ikL_2} + \left(2ik\nabla L_2 \cdot \nabla \phi_2 - k^2 \phi_2 (\nabla L_2)^2 + ik\nabla^2 L_2 \phi_2\right) e^{ikL_2} \end{aligned}$$

$$\begin{aligned} &= \left(\nabla^2 \phi_1\right) e^{ikL_1} - k^2 \phi_1 (\nabla L_1)^2 e^{ikL_1} + \frac{ik}{\phi_1}\left(\nabla \cdot \left(\phi_1{}^2 \nabla L_1\right)\right) e^{ikL_1} \\ &+ \left(\nabla^2 \phi_2\right) e^{ikL_2} - k^2 \phi_2 (\nabla L_2)^2 e^{ikL_2} + \frac{ik}{\phi_2}\left(\nabla \cdot \left(\phi_2{}^2 \nabla L_2\right)\right) e^{ikL_2} \end{aligned}$$

Since $\phi_1 \neq 0$ and $\phi_2 \neq 0$, from equation (4.2)

$$\nabla^2 \Phi_3 = \left(\nabla^2 \phi_1\right) e^{ikL_1} - k^2 \phi_1 (\nabla L_1)^2 e^{ikL_1} + \left(\nabla^2 \phi_2\right) e^{ikL_2} - k^2 \phi_2 (\nabla L_2)^2 e^{ikL_2}$$

from equation (4.1),

$$\nabla^2 \Phi_3 = -n_0^2 (\Phi_1 + \Phi_2) = -n_0^2 \Phi_3$$

or

$$\nabla^2\Phi_3 + n_0^2\Phi_3 = 0 \tag{4.3}$$

Since Φ_3 can be expanded, as $\phi_3 e^{ikL_3}$ therefore it is solution to

$$(\nabla L_3)^2 = n_0^2 + \frac{1}{k_0^2\phi_3}\nabla^2\phi_3 \tag{4.4}$$

$$\nabla\cdot\left(\phi_3^{\,2}\nabla L_3\right) = 0 \tag{4.5}$$

It has been shown that any complex solution that fulfills the stationary wave equation also fulfills the generalized eikonal equation and the continuity equation. If Φ_1 and Φ_2 both are solutions of equations (4.1, 4.2) then according to the Principle of Superposition, $\Phi_3 = \Phi_1 + \Phi_2$ must be a solution of equation (4.4) and equation (4.5). Based on this deduction, the principle of linear superposition is shown to be self consistently defined in the generalized eikonal formalism (GEF) even though its form appears to be non-linear.

4.3: Accuracy of Numerical Scheme

The diffraction of a Gaussian beam is one of those few cases where it can be solved analytically or by using the Fresnel integral. Even though it had been solved using the eikonal formalism before[4, 5], it is solved again to ensure the numerical accuracy of the present numerical scheme.

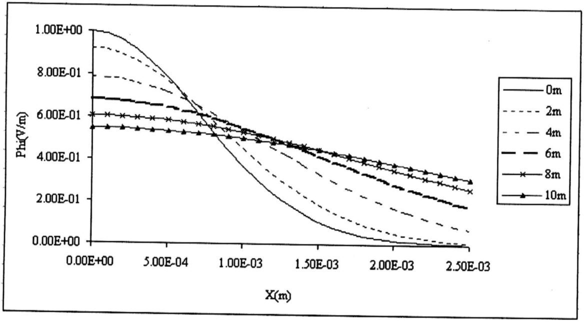

Figure 4.1: Diffraction effect on a Gaussian pulse as it propagates from 0m to 10m.

The figure above shows the diffraction effect of a Gaussian beam, $\phi = A\exp\left(-\frac{x^2}{\sigma_0^{\ 2}}\right)$ where A is the amplitude of the beam initially. In this case, the value of A is assigned as unity. As the beam propagates from $z = 0$ to $z = 10m$, it can be observed that diffraction effect takes place, indicated by the spread of the beam as it propagates forward. As the beam spreads out, the amplitude at x = 0m reduces from unity.

The accuracy of this formalism at z = 1m gives a relative error of the order of 10^{-5} when compared to the results obtained using the Fresnel integral (table 4.1). The parameters used in this case are given as $n_0 = 1.0$, $\sigma_0 = 1mm$ at wavelength of $1\mu m$.

X(m)	Amplitude, (V/m)		Relative Error
	Fresnel integral	GEF	
0.0	0.9761611	0.9761633	2.23E-06
0.0002	0.9413431	0.9413450	2.07E-06
0.0004	0.8441647	0.8441658	1.36E-06
0.0006	0.7039783	0.7039782	-1.69E-07
0.0008	0.5459391	0.5459377	-2.66E-06
0.0010	0.3937150	0.3937126	-6.16E-06
0.0012	0.2640418	0.2640389	-1.07E-05
0.0014	0.1646706	0.1646679	-1.65E-05
0.0016	0.0955020	0.0954997	-2.35E-05
0.0018	0.0515064	0.0515048	-3.18E-05
0.0020	0.0258323	0.0258313	-4.17E-05
0.0022	0.0120481	0.0120475	-5.30E-05
0.0024	0.0052255	0.0052251	-6.61E-05

Table 4.1: Comparison of diffraction amplitude obtained by GEF and Fresnel integral.

The ray trace of the diffraction effect has been obtained. The figure below shows that the light rays are initially uniformly spaced at $z = 0$ to being diffracted out from the center to the side at 10m. The light rays from the side started to diffract out as it moves along the z-axis. This effect moves in from the side causing the light ray on the inside to spread out to the side as it propagates forward. This indicated that the energy spreads out as the beam propagates forward.

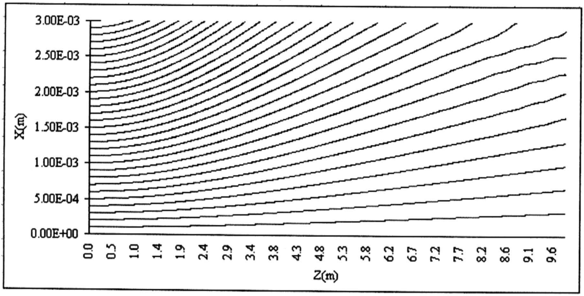

Figure 4.2: Ray trace of diffraction effect.

The generalized eikonal equation as a nonlinear equation, had been shown to be able to produce diffraction effect. Two Gaussian beams of the same nature were allowed to propagate, separately, and their phases and amplitudes were obtained at different distances from the origin. These two beams were then linearly superposed according to the principle of linear superposition. The resultant disturbance, in this case, was not only the sum of the intensities of the two separate disturbances but takes on the value of the magnitude of y_3. These two disturbances were then shown to have interfered.

However, the accuracy of this method, as shown in table 4.2, when compared to that of the Fresnel integral is of the order of 10^{-3} in relative errors. The accuracy is not very high but the method employed was able to describe the linear superposition effects and thus, the interference of the two beams.

Amplitude at z = 4m from origin			
X(m)	Fresnel Integral	Linear Superposition by the GEF	Relative Error
0.0E+00	0.666203	0.666195	-1.19E-05
2.0E-04	0.646448	0.646456	1.32E-05
4.0E-04	0.601700	0.601732	5.27E-05
6.0E-04	0.569583	0.569585	3.98E-06
8.0E-04	0.584809	0.584820	1.92E-05
1.0E-03	0.644074	0.644091	2.69E-05
1.2E-03	0.713947	0.713944	-3.35E-06
1.4E-03	0.764590	0.764587	-4.22E-06
1.6E-03	0.782197	0.782926	9.32E-04
1.8E-03	0.765739	0.759392	-8.29E-03
2.0E-03	0.721026	0.714425	-9.15E-03
2.4E-03	0.578790	0.576994	-3.10E-03
2.6E-03	0.495682	0.495333	-7.04E-04

Table 4.2: Comparison of amplitudes obtained for the two-beam interference based on linear superposition of two beams each generated separately by the GEF method.

4.4: Interference of Two Beams

In the case of interference, the initial condition of the Gaussian beam is modified to give out two beams that serve as the two slits used in interference experiments. The initial condition is modified to $\phi = A\left[\exp\left(-\frac{(x-x_0)^2}{\sigma_0^2}\right)+\exp\left(-\frac{(x+x_0)^2}{\sigma_0^2}\right)\right]$. The amplitudes of both beams, A, is again kept at unity while x_0 refers to the position of the beam from the origin. In this study, $x_0 = 0.0015m$. The interference effect is achieved by placing two beams at a certain distance, $2x_0$, from each other. As the beams propagate forward along the z-axis, a spreading of each beam can be observed. The redistribution of energy by each beam will have an effect on the beam next to it, thus giving rise to interference effect.

The accuracy of this formalism (table 4.3) when it is used to generate the interference effect gives a relative error of the order of 10^{-3} at 4 meters from origin when compared with the Fresnel integral. It is comparable to the relative error of linear superposition of two beams by the generalized eikonal formalism, which is also of that order of magnitude. The interference effect by generalized eikonal formalism is then established.

Amplitude at z = 4m from origin			
X(m)	Fresnel Integral	Interference by GEF	Relative Error
0.0E+00	0.666203	0.666220	2.55E-05
2.0E-04	0.646448	0.645508	-1.45E-03
4.0E-04	0.601700	0.597994	-6.16E-03
6.0E-04	0.569583	0.564640	-8.68E-03
8.0E-04	0.584809	0.584015	-1.36E-03
1.0E-03	0.644074	0.648835	7.39E-03
1.2E-03	0.713947	0.719103	7.22E-03
1.4E-03	0.764590	0.767291	3.53E-03
1.6E-03	0.782197	0.781923	-3.49E-04
1.8E-03	0.765739	0.763622	-2.76E-03
2.0E-03	0.721026	0.718582	-3.39E-03
2.4E-03	0.578790	0.577881	-1.57E-03
2.6E-03	0.495682	0.495631	-1.03E-04

Table 4.3: Comparison of amplitudes obtained for the two-slit interference by the GEF and the Fresnel integral.

As before, the ray trace of the interference effect can be obtained. The ray trace represents the energy flow of the beams when they interfere. Figure 4.4 is obtained by using the same parameters that was used to generate figure 4.3 that is the two beams are separated by 0.003m with sigma taking the value of 0.001m. The density of light rays near to the z-axis is higher when compared to other areas that are further away from the z-axis. It starts building up after 2m, which leads to the building up of the principal maxima, and it keeps on getting denser as the intensity of the maxima increases. This can be observed with each of the light rays bending into the z-axis before paralleling out to propagate along each other near the region of the z-axis.

The initial beam with a peak situated at 0.0015m along the x-axis drops in intensity with the light rays diffracting away to its side. In between 0m to 0.0015m, the light rays diffracted to compress together along the z-axis. At about 7m along the z-axis, the light rays at the right of x = 0.0015m are bent towards the center while light rays at the left are diffracted out. Thus the energy becomes less and it gives a minimum.

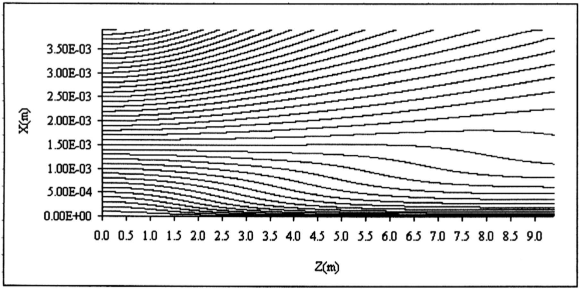

Figure 4.3: Ray trace of interference effect.

From figure 4.4, interference of two beams placed 0.0015m at either side of the origin was allowed to propagate up to 10m. The generalized eikonal formalism was able to show interference effect that has a build up of second maximum as the principal maximum builds up at $x = 0$m axis.

At 4m, the minimum that was formed was at 0.0006m, shifts away from the $x = 0$ axis and reduces in amplitude. The minimum formed becomes more obvious, as the beams propagate. The formation of the minimum at all the distances along z-axis agrees with the calculation that was done based on the parameters used to generate this figure. As was expected from the interference effect, the minima shift away from the origin as the distance in z-axis increases.

It can be seen that the conservation of energy is not violated. The build up of a maximum along $x = 0m$ where the initial intensity is very low is compensated by the drop of intensity where intensity was high when z = 0m, giving a minimum. This was shown in figure 4.4. However, these minima obtained do not go to zero as expected from wave and geometrical optics. This is due to the fact that the initial value used here is a Gaussian profile of finite extent somewhat larger than employed in the Young's two-slit experiment.

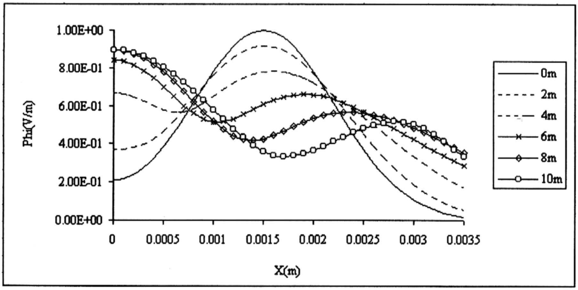

Figure 4.4: Interference pattern of two Gaussian beams, separated by 0.003m

When some of the parameters are changed, other characteristics of interference are also obvious. These observations are made by comparing figures 4.5, 4.6 and 4.7, each with different values of x_0. By varying the distance between the two beams, changes to the size of the fringes formed was expected. As was known, if the slits are closer together, they give widely spaced fringes, whereas slits farther apart give narrower fringes.

By comparing the different values of x_0 used, the interference fringes of $x_0 = 0.001m$ at 6m away from the origin has a minimum at around 0.0018m away from the origin. An arrow on each of the figures marks the minimum location. When x_0 is increased to 0.0012m, the minimum then decreases to 0.0013m and a further decrease is noted for $x_0 = 0.0015m$ at 0.001m. It is found that the minimum position is inversely proportional to the separation of the beams, which is exactly predicted in Young's experiment. The mathematical expression for the location of the minimum, y_{min} in relation to the separation of the beams, a is given as

$$y_{\min} = \left(m + \frac{1}{2}\right)\frac{s\lambda}{a}$$

where s indicates the distance from the observation point to the source

m its value in integer and

λ refers to the wavelength.

However, the maximum at x = 0 axis will also decrease in its intensity. The intensity formed when $x_0 = 0.001m$ is more than 1V/m but it decreases to less than 1V/m as x_0 increases. This is compensated by the formation of a second maximum that increases in intensity as the initial two beams have a wider separation, in keeping with conservation of energy.

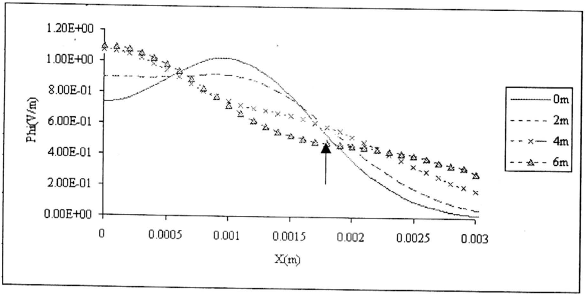

Figure 4.5: Two-beam interference pattern with x_0 at 0.001m.

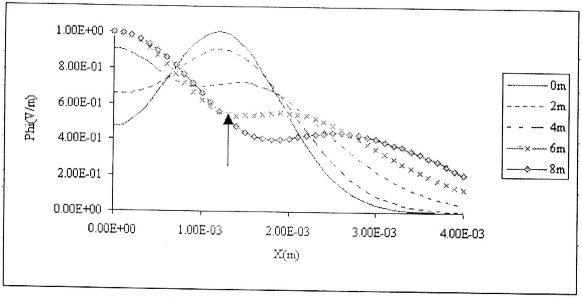

Figure 4.6: Two-beam interference pattern with x_0 at 0.0012m.

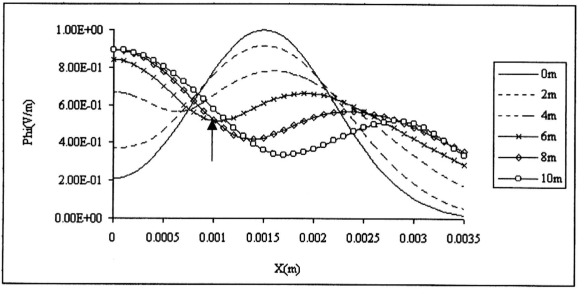

Figure 4.7: Two-beam interference pattern with x_0 at 0.0015m.

As before, with all the other parameters kept at the same values, changes were made to the sigma parameter, varying it from 0.0009m to 0.0014m as shown in figure 4.8. Observation of the changes brought to the fringe pattern was made at 5m away from the origin along the z-axis. Comparison of the fringes shows that the interference effect gives the maxima and the minima at the same points along the x-axis. It can be expected that when the value of sigma was further reduced, the minima formed will be deeper. The amplitude of the minima shows a tendency to approach zero when the beam width is reduced to the size of a slit as in the experiment. It can also be seen that the fringes for a smaller sigma are more pronounced when compared to that of higher valued sigma. The second peak then becomes more obvious. In other words, the smaller the value of sigma, the fringe becomes sharper.

When comparison of the ratio of second maximum to the minimum was made in table 4.4 for different sigma values, it is obvious that the narrower beam has a higher ratio. As was mentioned, it has a sharper fringe pattern. A sharper beam diffracts faster than a less sharp beam thus and it goes through a more significant drop in amplitude. However, there is a general drop in amplitude as sigma decreases in value. This is caused by the total energy of the beams are less.

Sigma	Phi(V/m)		Ratio of maximum over minimum
	Minima	Maxima	
0.0009m	5.74E-01	6.56E-01	1.14E+00
0.0010m	6.25E-01	6.92E-01	1.11E+00
0.0012m	7.34E-01	7.48E-01	1.02E+00
0.0014m	8.42E-01	7.96E-01	9.45E-01

Table 4.4: Comparison of ratio of maximum over minimum for different values of sigma.

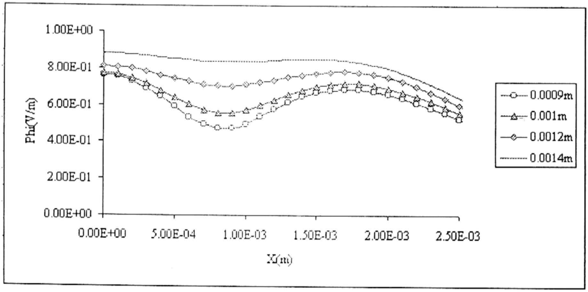

Figure 4.8: Comparison of interference pattern at 5m when the value of sigma is varied.

4.5: Interference of Multiple Slits

The pattern formed from the interference of multiple slits is very similar to that of the interference pattern of the double slits mentioned earlier. In this sense, the double slits can be considered as an elementary grating. The pattern produced by multiple slits can be considered as an extension to that of a two-beam interference.

The initial condition for multiple slits still uses Gaussian beam but it is modified to include a number of beams at different distances from the origin. The initial condition can be written as $\phi = A\exp\left(-\frac{x^2}{\sigma_0^2}\right) + A\sum_{n=1}^{N}\left\{\exp\left[-\frac{(x-nx_0)^2}{\sigma_0^2}\right] + \exp\left[-\frac{(x+nx_0)^2}{\sigma_0^2}\right]\right\}$ where (N + 1) indicates the number of slits present in the initial condition while x_0 represents the separation between two slits.

Employing the same method as discussed previously, the multiple slits were obtained with only a modification of the initial condition as mentioned above. As the number of slits is increased to three and later to five, a conclusion can be made that the number of maxima formed will be one more than the number of slits.

The ray trace figure 4.9 was obtained from the interference of five beams separated from each other at a distance of 0.003m. The figure shows a good correlation between the intensity of light rays and that of the building up of the maxima as the beams propagate. The maxima start to form at z = 2m are at points x = 0.0015m and x = 0.0045m from the origin. The light rays are more widely spaced on the outer side of these maxima. This relates to the build up of lower amplitude maxima.

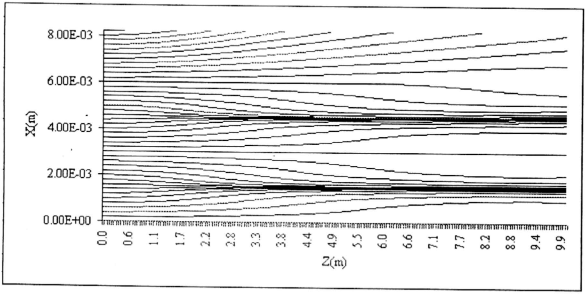

Figure 4.9: Ray trace of a five-slit interference pattern with a separation of 0.003m.

From figure 4.10, three slits are placed so that it is symmetrical along the z-axis. At 8m from the origin, four maxima developed as a result of the interference between beams. The intensity of the maximum nearest to the center is much higher compared to the other maxima formed at the sides. The minimum at the z-axis at 8m has the lowest intensity compared to the second minima.

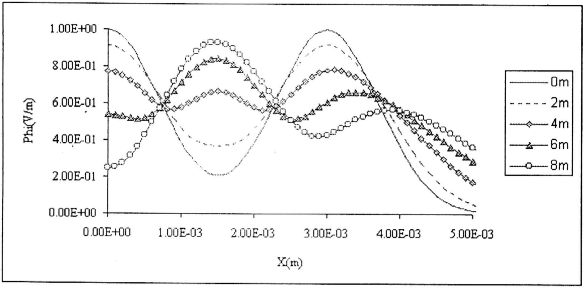

Figure 4.10: Interference pattern formed from three slits.

When changes are made to the separation of the beams, the maximum is observed to have shifted away from the z-axis as it propagates along the z-axis. This happens when the separations of the beams are closer to each other. As the separations between the beams are increased, the maxima that are formed from the minima of the initial condition keeps on increasing in intensity without an obvious shifting from the z-axis. The fringe pattern is sharper with narrower fringes replacing those of wide fringe pattern formed from closely placed beams. This characteristic can also be observed when the value of sigma is changed from 0.0009m to 0.0014m when the beam separations are kept at a constant as shown in table 4.5. The same effect as in interference is observed for the comparison of different sigma values. The ratio however is much higher for the interference of three slits when compared to the interference of two slits.

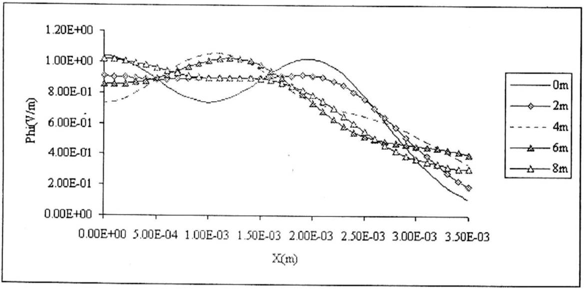

Figure 4.11: Interference of three beams separated by 0.002m.

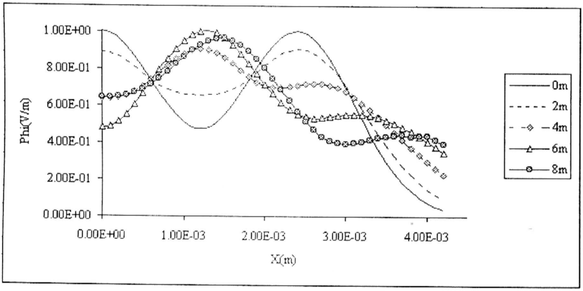

Figure 4.12: Interference of three beams separated by 0.0024m.

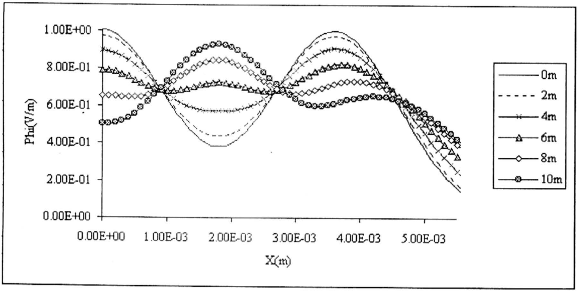

Figure 4.13: Interference of three beams separated by 0.0036m.

Sigma	Phi(V/m)		Ratio of maximum over minimum
	Minima	Maxima	
0.0009m	4.90E-01	7.59E-01	1.55E+00
0.0010m	5.65E-01	7.72E-01	1.37E+00
0.0012m	7.00E-01	8.11E-01	1.16E+00
0.0014m	8.23E-01	8.80E-01	1.07E+00

Table 4.5: Comparison of ratio of maximum over minimum for different values of sigma.

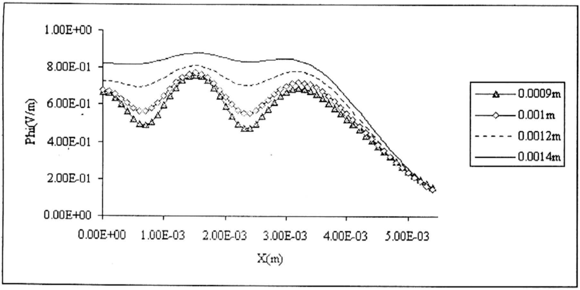

Figure 4.14: Comparison of interference patterns of three slits at 5m from origin when the value of sigma is changed from 0.0009m to 0.0014m.

To further observe the interaction of the beams, the number of slits is increased to five. It gives the expected result that can be deduced by considering the results from interference of two slits and that of three slits. The amplitudes of the maxima formed at 10m decrease as they move away from the z-axis with the maxima furthest away having the smallest amplitude. Figure 4.15 below show the amplitude of an interference pattern from five slits. The corresponding ray tracing for this figure was shown in figure 4.9.

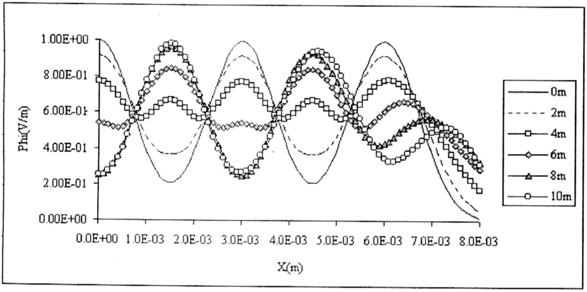

Figure 4.15: Interference pattern formed from five beams separated by 0.003m.