

Chapter 2

The Standard $SU(2) \times U(1)$ Weinberg-Salam Model

2.1 Introduction

In this chapter, we shall first provide in Section 2.2 a brief review of electroweak interaction theories in gauge models where we outline the essentials of the gauge theory of Standard Model. Then in Section 2.3, we shall present the full Lagrangian and Feynman rules in the 't Hooft-Feynman gauge.

2.2 The Standard $SU(2) \times U(1)$ Weinberg-Salam Model for Quarks and Leptons

In constructing the Standard $SU(2) \times U(1)$ Weinberg-Salam Model, the following gauge fields are introduced:

$$W_\mu^1, W_\mu^2, W_\mu^3 \quad \text{for } SU(2) \quad (2.1)$$

$$B_\mu \quad \text{for } U(1). \quad (2.2)$$

The corresponding Lagrangian is

$$L_1 = -\frac{1}{4} F_{\mu\nu}^i F^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \quad (2.3)$$

where the field-strength tensors are

$$F_{\mu\nu}^i F^{i\mu\nu} \equiv \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g \epsilon^{ijk} W_\mu^j W_\nu^k \quad (2.4)$$

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu \quad (2.5)$$

The Lagrangian of Eq. (2.3) contains four massless gauge bosons. There is no mass term due to the constrain by gauge symmetry. To implement the Higgs mechanism, we introduce an $SU(2)$ doublet of complex scalar fields, the Higgs fields:

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \quad (2.6)$$

The Lagrangian for Φ is

$$L_2 = -\left(D_\mu \phi\right)^\dagger \left(D^\mu \phi\right) - V(\phi) \quad (2.7)$$

where the covariant derivative for Φ is

$$D_\mu \Phi(x) = \left(\partial_\mu - ig \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu - i \frac{g'}{2} \cdot B_\mu \right) \Phi(x) \quad (2.8)$$

and the Higgs potential is

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi \right)^2, \quad (2.9)$$

with $\mu^2 < 0$ and $\lambda > 0$.

After spontaneous symmetry breaking of both $SU(2) \times U(1)$ symmetries into a $U(1)$ symmetry which is identified as the electromagnetic gauge group, $V(\Phi)$ acquires a non zero vacuum expectation value

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ v / \sqrt{2} \end{pmatrix} \quad (2.10)$$

with

$$v = \sqrt{-\frac{\mu^2}{\lambda}}. \quad (2.11)$$

Expanding the Lagrangian about the minimum of the Higgs potential in unitary gauge, and introducing

$$W_\mu^\pm = (W_\mu^1 \mp W_\mu^2) / \sqrt{2} \quad (2.12)$$

$$A_\mu = B_\mu \cos \theta_w + W_\mu^3 \sin \theta_w \quad (2.13)$$

$$Z_\mu = -B_\mu \sin \theta_w + W_\mu^3 \cos \theta_w \quad (2.14)$$

with

$$\sin \theta_w = \frac{g'}{\sqrt{g^2 + g'^2}} \quad (2.15)$$

$$\cos \theta_w = \frac{g}{\sqrt{g^2 + g'^2}}, \quad (2.16)$$

we will then find that mass terms for the weak bosons are generated in L_2 , namely

$$M_{W^\pm} = g v / 2, \quad (2.17)$$

$$M_Z = \frac{1}{2} [g^2 + (g')^2]^{1/2} v = M_W / \cos \theta_w \quad (2.18)$$

while leaving the other massless,

$$M_A = 0. \quad (2.19)$$

W_μ^\pm are identified as the massive charged boson fields, Z_μ the neutral weak boson field and A_μ the photon field.

In the Unitary gauge, three degrees of freedom associated with ϕ have been absorbed into W_μ^\pm and Z_μ as their longitudinal components, and the remaining one η , the Higgs scalar, acquires a mass $m_\eta^2 = -2\mu^2$.

By the above mechanism, we have succeeded in construction an $SU(2) \times U(1)$ gauge theory in which three gauge bosons (W_μ^\pm and Z_μ) are massive while the remaining one, the photon field A_μ , is massless. L_3 and L_2 are invariant but the ground state solution varies with the gauge (i. e. breaks gauge symmetry spontaneously).

For three generation of fermions, the physical lepton fields relevant to weak interaction form a left-handed $SU(2)$ doublet

$$L_l = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \quad (2.20)$$

and the right-handed fermions form a $SU(2)$ singlet

$$R_l = e_R, \mu_R, \tau_R. \quad (2.21)$$

The hadronic sector of the theory is built upon a single left-handed weak isospin doublets

$$L_q = \begin{pmatrix} u \\ d' \end{pmatrix}_L, L_c = \begin{pmatrix} c \\ s' \end{pmatrix}_L, L_t = \begin{pmatrix} t \\ b' \end{pmatrix}_L \quad (2.22)$$

and right-handed weak isospin singlets

$$R_u = u_R, R_c = c_R, R_t = t_R \quad (2.23a)$$

$$R_d = d_R, R_s = s_R, R_b = b_R. \quad (2.23b)$$

The down-type quarks are mixed states related by a general Kobayashi-Maskawa¹⁹ rotation, described by the matrix V through

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad (2.24)$$

where $c_j = \cos \theta_j$, $s_j = \sin \theta_j$, $j = 1, 2, 3$ is the family index. The fermion Lagrangian in the lepton sector is therefore given by

$$L_3 = -\bar{R}\gamma_\mu [\partial_\mu + ig' B_\mu] R - \bar{L}\gamma_\mu [\partial_\mu - ig\vec{\tau} \cdot \vec{W}_\mu / 2 + ig' B_\mu / 2] L. \quad (2.25)$$

The coupling constant for the weak-isospin group $SU(2)_L$ is g , and the coupling constant for the weak-hypercharge group $U(1)$ is denoted as $g'/2$.

To generate the lepton masses, we include the following $SU(2) \times U(1)$ gauge invariant interaction term, which involve Yukawa couplings of the scalar to the leptons

$$L_4^L = - \sum_{l=\mu,\nu,\tau} G_l \left[(\bar{L}_l \phi) R_l + h.c. \right]. \quad (2.26)$$

Whereas for the scalar-quarks interaction Lagrangian, we have

$$L_4^Q = \sum_{i,j=1}^3 -Y_{ij}^d (\bar{u}_i \quad \bar{d}_i)_L \phi d_{jR} - Y_{ij}^u (\bar{u}_i \quad \bar{d}_i)_L \phi_c d_{jR} + h.c. \quad (2.27)$$

with charge-conjugated Higgs doublet

$$\phi_c = -i\tau_2 \phi^* = \begin{pmatrix} -\bar{\phi}^0 \\ \bar{\phi}^- \end{pmatrix}. \quad (2.28)$$

Here, G_l , Y_{ij}^d and Y_{ij}^u being some arbitrary Yukawa couplings that are chosen so that the leptons and quarks are mass eigen states with the correct masses. Putting everything together, the final Lagrangian after spontaneous symmetry breaking in Unitary gauge is summarised as below:

Gauge bosons kinetic energies and interactions:

$$\begin{aligned} L_1 &= L_0^G + L_1^{GG} \\ &= - \left\{ F_{\mu\nu} F^{\mu\nu} + Z^{\mu\nu} Z_{\mu\nu} + 2W_{\mu\nu}^+ W^{-\mu\nu} \right\} / 4 \\ &\quad - ie \left[F^{\mu\nu} W_{\mu}^+ W_{\nu}^- + \left(W_{\mu\nu}^+ W^{\mu-} - W_{\mu\nu}^- W^{\mu+} \right) A^{\mu} \right] - ie \cot \theta_w \times \\ &\quad \times \left\{ Z^{\mu\nu} W_{\mu}^- W_{\nu}^+ + \left(W_{\mu\nu}^+ W^{\mu-} - W_{\mu\nu}^- W^{\mu+} \right) \right\} Z^{\nu} - e^2 \left\{ W_{\mu}^+ W^{\mu-} \left(A^{\nu} + \cot \theta_w Z^{\nu} \right) \times \right. \\ &\quad \times \left(A_{\nu} + \cot \theta_w Z_{\nu} \right) - W_{\mu}^+ W_{\nu}^- \left(A^{\mu} + \cot \theta_w Z^{\mu} \right) \left(A^{\nu} + \cot \theta_w Z^{\nu} \right) \left. \right\} \\ &\quad - \frac{1}{2} \left(e / \sin \theta_w \right)^2 \left(W_{\mu}^+ W^{\mu-} W_{\nu}^+ W^{\nu-} - W_{\mu}^+ W^{\mu+} W_{\nu}^- W^{\nu-} \right) \end{aligned} \quad (2.29)$$

where

$$e = g \sin \theta_w \quad (2.30)$$

$$F_{\mu\nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \quad (2.31)$$

$$Z_{\mu\nu} \equiv \partial_{\mu} Z_{\nu} - \partial_{\nu} Z_{\mu} \quad (2.32)$$

$$W_{\mu\nu}^{\pm} \equiv \partial_{\mu} W_{\nu}^{\pm} - \partial_{\nu} W_{\mu}^{\pm}. \quad (2.33)$$

Higgs kinetic energy, mass and couplings with weak bosons and boson masses:

$$L_2 = L^H + L_I^{HG} + L_I^{HH} + L_M^G + \mu^4 / 4\lambda \quad (2.34)$$

$$L^H = \partial^\mu \eta \partial_\mu \eta / 2 - m_\eta^2 \eta^2 / 2 \quad (2.35)$$

$$L^{HG} + L^{HH} = -\frac{\lambda}{4} (\eta^4 + 4v\eta^3) - \frac{1}{8} [(2v\eta + \eta^2)] \left\{ 2g^2 W_\mu^+ W^{\mu-} + [g^2 + (g')^2] Z_\mu Z^\mu \right\} \quad (2.36)$$

$$L_M^G = -\frac{1}{8} v^2 \left\{ 2g^2 W_\mu^+ W^{\mu-} + [g^2 + (g')^2] Z_\mu Z^\mu \right\}. \quad (2.37)$$

Lagrangian for Lepton and quark kinetic energies, and their weak interactions with gauge bosons:

$$\begin{aligned} L_3 &= L_0^L + L_0^Q + L_I^{LQ} + L_I^{QQ} \\ &= -\sum_{l=e,\mu,\tau} \bar{e}_l \partial_\mu \gamma^\mu e_l + \bar{\nu}_l \partial_\mu \gamma^\mu \nu_l - \sum_{j=1,2,3} \bar{u}_j \partial_\mu \gamma^\mu u_j + \bar{d}_j \partial_\mu \gamma^\mu d_j \\ &\quad + \frac{1}{2\sqrt{2}} \frac{e}{\sin \theta_w} \left\{ \tilde{J}_\mu^{(-)}(x) W^\mu(x) + h.c. \right\} + \frac{1}{2} \frac{e}{\sin \theta_w \cos \theta_w} Z_\mu(x) \tilde{N}^\mu(x) \\ &\quad - e \sum_{l=e,\mu,\tau} \bar{e}_l \gamma^\mu e_l A_\mu \end{aligned} \quad (2.38)$$

with

$$e = g \sin \theta_w = g' \cos \theta_w \quad (2.39)$$

$$\tilde{J}_\mu^{(-)} = J_\mu^{(-)} + J_\mu^{(-)} \quad (2.40)$$

and

$$\tilde{N}_\mu^{(-)} = I_\mu^{(0)} + N_\mu. \quad (2.41)$$

Here we have

$$I_\mu^{(-)} = i\bar{e}\gamma_\mu(1+\gamma_5)v_e + i\bar{\mu}\gamma_\mu(1+\gamma_5)v_\mu + i\bar{\tau}\gamma_\mu(1+\gamma_5)v_\tau \quad (2.42)$$

$$J_{\mu}^{(-)} = i\bar{d}\gamma_{\mu}(1+\gamma_5)u + i\bar{s}\gamma_{\mu}(1+\gamma_5)c + i\bar{b}\gamma_{\mu}(1+\gamma_5)t \quad (2.43)$$

$$I_{\mu}^{(0)} = \frac{i}{2} \left\{ \bar{\nu}_e \gamma_{\mu}(1+\gamma_5) \nu_e - \bar{e} \gamma_{\mu}(1+\gamma_5) e \right\} + 2 \sin^2 \theta_w i \bar{e} \gamma_{\mu} e \\ + \{ e \rightarrow \mu \} + \{ e \rightarrow \tau \} \quad (2.44)$$

and

$$N_{\mu} = \frac{i}{2} \left[\bar{u} \gamma_{\mu}(1+\gamma_5) u - \bar{d} \gamma_{\mu}(1+\gamma_5) d + \bar{c} \gamma_{\mu}(1+\gamma_5) c - \bar{s} \gamma_{\mu}(1+\gamma_5) s \right. \\ \left. + \bar{t} \gamma_{\mu}(1+\gamma_5) t - \bar{b} \gamma_{\mu}(1+\gamma_5) b \right] - 2 \sin^2 \theta_w J_{\mu}^{em}, \quad (2.45)$$

where the hadronic electromagnetic current J_{μ}^{em} is given by

$$J_{\mu}^{em} = \frac{2}{3} i \bar{u} \gamma_{\mu} u - \frac{1}{3} i \bar{d} \gamma_{\mu} d + \frac{2}{3} i \bar{c} \gamma_{\mu} c - \frac{1}{3} i \bar{s} \gamma_{\mu} s + \frac{2}{3} i \bar{t} \gamma_{\mu} t - \frac{1}{3} i \bar{b} \gamma_{\mu} b. \quad (2.46)$$

The lepton and quark masses and couplings to Higgs, in the canonical form, is given by the Lagrangian

$$L_4 = L_4^L + L_4^Q \\ = (L_M^L + L_I^{LH}) + (L_M^Q + L_I^Q) \\ = - \left\{ \begin{pmatrix} \bar{e} & \bar{\mu} & \bar{\tau} \end{pmatrix} \tilde{m}(e) \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} - \begin{pmatrix} \bar{u} & \bar{c} & \bar{t} \end{pmatrix} \tilde{m}(u) \begin{pmatrix} u \\ c \\ t \end{pmatrix} - \begin{pmatrix} \bar{d}' & \bar{s}' & \bar{b}' \end{pmatrix} \tilde{m}(d) \mathcal{V}^{\dagger} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \right\} \left(1 + \frac{\eta}{\nu} \right) \quad (2.47)$$

where

$$\tilde{m}(e) = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_{\mu} & 0 \\ 0 & 0 & m_{\tau} \end{pmatrix}, \quad \tilde{m}(u) = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad \tilde{m}(d) = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}. \quad (2.48)$$

2.3 Lagrangian and Feynman Rules in the 't Hooft-Feynman Gauge

Since the calculation in this project shall be done in the 't Hooft-Feynman (HF) gauge⁵⁶⁻⁵⁸, it is proper to present the full Lagrangian in this gauge. In this gauge the Higgs doublet is parametrized in a more general form,

$$\phi = \begin{pmatrix} \phi^+ \\ \frac{(v + \eta + i\phi_4)}{\sqrt{2}} \end{pmatrix} \quad (2.49)$$

where ϕ_4 and ϕ^+ are the neutral and charged unphysical scalar boson. The expression in Eq. (2.49) is then substituted into the Lagrangian L_1, L_2, L_3 and L_4 to break the symmetry. The Lagrangian not involving any Higgs field remains explicitly the same as in the Unitary gauge, whereas the Lagrangians involving Higgs fields are modified. List below are the related modified Lagrangians, namely $L^H, L_I^{HO} + L_I^{HH}$ in L_2 and L_I^{LH}, L_I^{QH} in L_4 :

$$L^H = \frac{1}{2} \partial^\mu \eta \partial_\mu \eta + \partial^\mu \phi^+ \partial_\mu \phi^- - \frac{1}{2} \partial^\mu \phi^4 \partial_\mu \phi^4 - \frac{1}{2} m_H^2 \eta^2 \quad (2.50)$$

$$L_I^{HH} = -\lambda v (\eta^3 + 2\eta \phi^+ \phi^- + \eta \phi_4^2) - \frac{1}{4} \lambda (4\phi^{+2} \phi^{-2} + 4\phi^+ \phi^- \eta^2 + \eta^4 + \phi_4^4 + 4\phi_4^2 \phi^+ \phi^- + 2\eta^2 \phi_4^2) \quad (2.51)$$

$$L_I^{LH} = \sum_{t=s,\mu,\tau} \left[-\frac{m_t}{\sqrt{2}} \bar{e}_t (1 - \gamma_5) v_t \phi^- - m_t \bar{e}_t e_t \eta - im_t \bar{e}_t \gamma_5 e_t \phi_4 - \frac{m_t}{\sqrt{2}} \bar{v}_t (1 + \gamma_5) e_t \phi^+ \right] \quad (2.52)$$

$$\begin{aligned}
L_1^{\text{GH}} = & -\frac{g}{\sqrt{2}M_w} \left[\bar{u} [V\tilde{m}(d)R - \tilde{m}(u)VL] d\phi^+ + \bar{d} [\tilde{m}(d)V^+L - V^+\tilde{m}(u)R] V\phi^- \right. \\
& \left. + \frac{1}{\sqrt{2}} \bar{d}\tilde{m}(d)d\eta + \frac{1}{\sqrt{2}} \bar{u}\tilde{m}(u)u\eta - \frac{i}{\sqrt{2}} \bar{d}\tilde{m}(d)\gamma_5 d\phi_4 - \frac{i}{\sqrt{2}} \bar{u}\tilde{m}(u)\gamma_5 u\phi_4 \right] \quad (2.53)
\end{aligned}$$

$$\begin{aligned}
\hat{L}_1^{\text{H}\bar{O}} = & gM_w W_\mu^- W^{+\mu} \eta + \frac{gM_z}{2\cos\theta_w} Z_\mu Z^\mu \eta - \frac{gM_w \sin^2\theta_w}{\cos\theta_w} (Z_\mu W^{+\mu} \phi^- + Z_\mu W^{-\mu} \phi^+) \\
& + gM_w W_\mu^- W^{+\mu} (\eta^2 + \phi_4^2) + \frac{g^2}{8\cos^2\theta_w} Z_\mu Z^\mu (\eta^2 + \phi_4^2) + \frac{g^2}{4} \left[\frac{\cos^2 2\theta_w}{\cos^2\theta_w} Z_\mu Z^\mu \right. \\
& \left. + 4\cos 2\theta_w \tan\theta_w Z_\mu A^\mu + 4\sin^2\theta_w A_\mu A^\mu + 2W_\mu^+ W^{-\mu} \right] \phi^+ \phi^- \\
& + \frac{ig\cos 2\theta_w}{2\cos\theta_w} [Z_\mu (\partial^\mu \phi^-) \phi^+ - Z_\mu (\partial^\mu \phi^+) \phi^-] \\
& + ig\sin\theta_w [A_\mu (\partial^\mu \phi^-) \phi^+ - A_\mu (\partial^\mu \phi^+) \phi^-] \\
& + \frac{ig}{2} [W_\mu^+ (\partial^\mu \phi^-) - W_\mu^- (\partial^\mu \phi^+)] \eta - \frac{g^2 \sin^2\theta_w}{2\cos\theta_w} (Z_\mu W^{+\mu} \phi^- + Z_\mu W^{-\mu} \phi^+) \eta \\
& + \frac{g^2}{2} \sin\theta_w (A_\mu W^{+\mu} \phi^- + A_\mu W^{-\mu} \phi^+) \eta - \frac{g}{2} [W_\mu^+ (\partial^\mu \phi^-) + W_\mu^- (\partial^\mu \phi^+)] \phi_4 \\
& + \frac{ig^2 \sin^2\theta_w}{2\cos\theta_w} (Z_\mu W^{-\mu} \phi^+ + Z_\mu W^{+\mu} \phi^-) \phi_4 + \frac{ig^2}{2} \sin\theta_w (A_\mu W^{+\mu} \phi^- - A_\mu W^{-\mu} \phi^+) \phi_4 \\
& + \frac{ig}{2} (W_\mu^- \phi^+ - W_\mu^+ \phi^-) (\partial^\mu \eta) + \frac{g}{2} (W_\mu^- \phi^+ + W_\mu^+ \phi^-) (\partial^\mu \phi_4) \\
& + \frac{g}{2\cos\theta_w} Z_\mu [\phi_4 (\partial^\mu \eta) - \eta (\partial^\mu \phi_4)] . \quad (2.54)
\end{aligned}$$

In addition to the above Lagrangian, there emerges another term L_{mixing} in this gauge, given by

$$L_{\text{mixing}} = iM_W [W_\mu^+ \partial^\mu \phi^- - W_\mu^- \partial^\mu \phi^+] - M_Z Z_\mu \partial^\mu \phi_4 \quad (2.55)$$

The presence of the mixing terms in L_{mixing} show that W_μ^\pm , ϕ^\pm , Z_μ and ϕ_4 are not independent normal coordinates. They can be removed by fixing the gauge, i. e. by adding a

gauge fixing term to the Lagrangian. This gauge fixing term also gives rise to mass term of M_W and M_Z to the unphysical Higgs fields ϕ^\pm and ϕ_4 respectively.

We list below the set of Feynman rules in the 't Hooft-Feynman gauge, relevant to our subsequent calculation of the flavour-changing $Zq_1\bar{q}_2$ vertex:

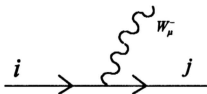
Propagators:

quark:
$$\frac{i}{q \cdot \gamma - m}$$

W^\pm boson:
$$\frac{-ig_{\mu\nu}}{p^2 - M_W^2}$$

unphysical ϕ^\pm boson:
$$\frac{i}{p^2 - M_W^2}$$

Vertices:



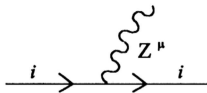
$$-i \frac{g}{\sqrt{2}} \gamma_\mu \frac{1}{2} (1 - \gamma_5) V_{ij}$$



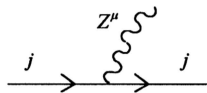
$$\frac{ig}{\sqrt{2}M_W} (m_{jR} - m_{iL}) V_{ij}$$



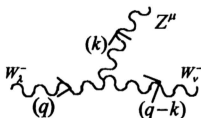
$$\frac{ig}{\sqrt{2}M_w}(m, L-m, R)V_\mu$$



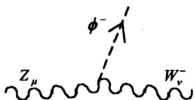
$$\frac{ig\gamma_\mu}{\cos\theta_w}\left(L-\frac{2}{3}\sin^2\theta_w\right)$$



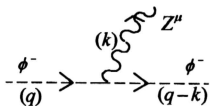
$$-\frac{ig\gamma_\mu}{2\cos\theta_w}\left(L-\frac{4}{3}\sin^2\theta_w\right)$$



$$ig\cos\theta_w\left[(k-2q)_\mu g_{\nu\lambda} + (q+k)_\nu g_{\lambda\mu} + (q-2k)_\lambda g_{\mu\nu}\right]$$



$$-\frac{ig\sin^2\theta_w}{\cos\theta_w}M_w g_{\mu\nu}$$



$$\frac{ig \cos 2\theta_w}{2 \cos \theta_w} (2q - k)^\mu$$

In the vertices shown above, m_i and m_j refer to the masses of quark i and j respectively, where $i = d, s, b; j = u, c, t$.