CHAPTER 4

DATA ANALYSIS

4.1 Introduction

The written responses of five students to three separate tasks with respect to their understanding were analyzed. Any uncertainties that arose were referred and clarified during the interviews. The tasks were then further analyzed by examining the similarities and differences to determine the understanding of quadratic equations of these students.

4.2 Analysis of students' understanding

Students' understanding of quadratic equations was inferred from their responses to the five tasks. The analysis considered responses to each task. For consistent reference and reporting in the analysis of each of the task, the five students were referred to as Student 1 (S1), Student 2 (S2), Student 3 (S3), Student 4 (S4) and Student 5 (S5).

4.3 Analysis of Task 1

In Task 1, the student was told that her teacher had just completed the topic of quadratic equations and roots of equations. Her good friend was absent from the class that week as she was attending athletic practice in a neighboring state. This friend had requested the student to explain those two topics to her.

The task was for the student to write a letter to her friend to explain the meaning of quadratic equations, its application and other items which may be important in the topic of quadratic equations and the roots of quadratic equations.

In viewing all the responses for this task, the writings focused first on the explanation of quadratic equations, followed by that of roots of quadratic equations. This flow of thought forms the format of this report under the headings Quadratic Equations, Roots of Quadratic Equations and Understanding Inferred from these Responses.

4.3.1 Student 1

Quadratic Equations

Student 1 (S1) explained that the quadratic equation had a fixed form of $ax^2 + bx + c = 0$. She elaborated that the quadratic equation had to have an unknown, which was squared, as well as another term of the same unknown without the square. Furthermore, S1 explained that the quadratic equation must be equated to zero. S1 also gave examples of quadratic equations:

$2x^2 - 4x + 2 = 0$

$3x^2 - 5x = 0$

$4x^2 - 5x + 2 = 0$

S1 also gave an example of a non-quadratic equation that is, $1/x^2 + 5x - 2 = 0$. Her explanation was that the equation's first term, $1/x^2$ could be written as x^{-2}. As x^{-2} was not the same as x^2, the equation was not quadratic.

Roots of Quadratic Equations

S1 explained that quadratic equations were normally given in questions where the value of their roots was required. The equation could have two different roots and these roots could be deduced by factorization, or by using the general formula.

In order to determine the roots of an equation by factorization, the positive or negative value of the factors were determined. S1 drew Table 1 (Appendix 3 – 1- a) to show the two possibilities that could occur when the equation had a positive sign in front of the constant.

Table 1 : Possibilities of signs of factors for Task 1.

+	+, +@ -, -	*sama*
-	+, - @ -, +	*beza*

S1 then gave an example of a quadratic equation, $x^2 - 4x - 5 = 0$ and solved it. First, she determined that the two factors of the constant term had different signs. This was because the constant in the given equation had a negative sign in front of it. The factors of the first term, x^2, and the last term, constant 5, were put into brackets: $(x - 5) \times (x + 1)$. She then explained that as the middle term was negative ($-4x$), the negative sign was put into the pair of brackets which had the factor with the larger value. The equation was then solved.

→ $(x -5) \times (x + 1) = 0$

then $(x -5) = 0$ or $(x + 1) = 0$

$x = 5$ or $x = -1$

To solve an equation that had been factorized, S1 equated the product of the factors to zero. Then, each factor could be equated to zero. The values of x in the factors were then obtained through transposition of the terms (Appendix 3 – 1 – b).

Next, S1 showed that the general formula could be used to solve quadratic equations. She wrote the general formula and identified $a = 1$, $b = -4$ and $c = -5$ from the equation $x^2 - 4x - 5 = 0$. These values were then substituted into the general formula and she obtained $\frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2}$.

This gave $\frac{4 \pm \sqrt{36}}{2}$ followed by $\frac{4+6}{2}$ or $\frac{4-6}{2}$. Finally, two values of x were obtained, $x = -1$ and $x = 5$. These were the same values as the answer obtained by the earlier method.

Inferred Understanding of Student S1

S1 took quadratic equations to be in the form of $ax^2 + bx + c = 0$ but not $ax^2 + c = 0$, and stressed on the importance of equating to zero. She also noted that the presence of the term $1/x^2$ meant that the equation was not quadratic. S1 did give some examples of quadratic equations and tried to identify the conditions where the equation was non-quadratic. Furthermore, applied her understanding of the conditions of quadratic equations to this task with a counter-example. When she stated the purpose of using quadratic equations, she had made a connection between roots and quadratic equations. Generally, S1's written response was theoretical in the aspect of mathematics, as defined by van Dormolen (1985) as she had given several examples and attempted to make connections where possible.

S1 understands that there are two methods of finding the roots of equations; factorization and the general formula. In the process of factorization, she had considered two different cases where the signs in front of the constant in the quadratic equation determined the signs in the factors. However, her explanation for both methods was algorithmic in nature. S1 demonstrated instrumental understanding.

4.3.2 Student 2

Quadratic Equations

Student 2 (S2) explained that the quadratic equation had the form $ax^2 + bx + c = 0$. She then showed with the example $x^2 = 4x + 3$ that an equation of the non-general form could be changed to the general form by transposition. S2 then explained that quadratic equations could be solved by three methods: factorization, completing the square and with the general formula. She suggested the last two methods be used when the quadratic equation could not be solved by factorization. She added that in the third method, the general formula, the answer, depending on the requirement of the question, could be given in the surd form $1 \pm \sqrt{2}$.

During the interview (Appendix 2 – 2), S2 gave the reason for her answer in the decimal form was that she could obtain the values easily with the calculator. Furthermore, she considered the answer in the decimal form was easier than in the surd form because it was not easy to simplify surds.

Roots of Quadratic Equations

S2 explained the three types of roots of equations and their conditions (Appendix 3 – 2 - a). For equations to have real roots, the condition $b^2 - 4ac \geq 0$ applied.

During the interview (Appendix 2 – 2), S2 was asked about the kinds of questions that required the condition for real roots, $b^2 - 4\,ac \geq 0$. She explained that she did not know but she would have to observe the requirement of the question. When the equation had real roots, she would use the condition for real roots.

Inferred Understanding of Student S2

By changing the non-general to the general form of the quadratic equation, S2 had attempted to link a new concept, the non-general form to the existing general form. This is considered as reflecting a higher level of understanding according to Leinhart's (1987) feature of explanation. S2 had only stated the three methods of solving quadratic equations but elaborated only on the last method, the general formula method. She explained that the answer could be stated in the form of a surd or a decimal. Again, S2's explanation was theoretical as she had mainly stated definitions. The understanding she demonstrated is considered instrumental as she was mainly applying the rules that she had learnt.

Similarly, for the types of roots of equations, she did not elaborate on their uses nor give any example of questions or roots. Her written response did not indicate that she had a clear goal and she did not apply the conditions that she had stated for the type of roots. This would indicate a lack of understanding in roots of quadratic equations.

4.3.3 Student 3

Quadratic Equations

Student 3 (S3) explained the importance of equating quadratic equations to zero. She then gave several examples to show the characteristics of quadratic equations. She explained that quadratic equations had to have a square term, as in the example $x^2 - 4x - 5 = 0$. Next, quadratic equations had to be equated to zero, as in the example $1x^2 + 3x + 8 = 0$. She explained that a quadratic equation did not have any fractions by an example $x^2 + 3/2\ x + 1/8 = 0$. Finally, S3 explained that an equation, which had a term with a power higher than two, was also not a quadratic equation, for example, $x^3 + 6x - 3 = 0$. In addition, when she was interviewed (Appendix 2 – 3), S3 identified equations which had negative powers, example $1/x^2 + 12x + 1 = 0$ and $x^2 + 1/x + 1 = 0$ as non-quadratic equations.

S3 explained that quadratic equations were used when their roots, x, were required. In addition, roots could only be obtained from quadratic equations and through three methods: completing the square, the general formula and factorization.

Roots of Quadratic Equations

S3 explained that x was the root in a quadratic equation. When the roots or values of x were substituted in the quadratic equation, *"it"* would be equal to zero. S3 then stated the three ways in which roots could be obtained. She then showed two examples for obtaining the value of roots by factorization. Both the quadratic equations were first factorized and equated to zero. The roots, which were values of x, were then stated (Appendix 3 – 3 – a).

(i) $x^2 + 7x + 6 = 0$	(ii) $x^2 + 2x - 8 = 0$.
$(x + 1)(x + 6) = 0$	$(x + 4)(x - 2) = 0$
$x = -1, x = -6$	$x = -4, x = 2$

The general formula was the alternative method used to obtain the values of the roots. In the example the values of a, b and c from the equation $2x^2 + 3x - 6 = 0$ were substituted and the answer computed. The values of x were left in the surd form $\frac{-9 + \sqrt{7}}{4}$, $\frac{-9 - \sqrt{57}}{4}$. S3 added that the answer could be further simplified with the use of a calculator.

During the interview (Appendix 2 – 3), S3 was asked to explain the reason she disliked the answer in the surd form. Her explanation was that surds were too complicated, with too many signs. She preferred to leave the answer in decimal as the answer in this form would be just as accurate as in the surd form.

S3 used completing the square to solve $x^2 + 8x - 2 = 0$ (Appendix 3 – 3 - b). First, she added $(8/2)^2$ to both sides of the equation $x^2 + 8x = 2$ and then completed the square on the left side to form $(x + 4)^2$. The square root of both sides to obtain $x + 4 = \sqrt{18}$ and the values of x were determined in decimal form.

S3 then showed, with an example that when a pair of roots were given, they could be written as a product of factors and equated to zero. The roots of x, –5 and 4, were written as $(x - 5)(x + 4)= 0$. The values of x were then obtained after transposition as $x = 5$, $x = -4$.

Next, S3 explained that a quadratic equation could be obtained from a given pair of roots by adding and multiplying the roots. In her example, she first wrote the equation of the form $x^2 - (a + b)x + (ab) = 0$. She then added and multiplied the roots that she had used earlier, 5 and –4, and obtained the quadratic equation: $x^2 - 1x + 20 = 0$. In another example, S3 used the roots 1 and 2. She obtained, separately, the sum of the roots, 3, and the product, 2, before writing $x^2 - (3 + 2)x + (3 \times 2) = 0$. Finally, the quadratic equation $x^2 - 5x + 6 = 0$ was written.

Inferred Understanding of Student S3

S3's understanding was that quadratic equations had a square term such as x^2, were equated to zero and did not have any fractions. However, she did not consider that some quadratic equations need not be equated to zero first, example $x^2 + 8x = 2$. Furthermore, the example given, $x^2 + 3/2x + 1/8 = 0$ was in actual fact a quadratic equation as quadratic equations do have terms with fractions. On the other hand, she correctly identified that equations with negative powers, example $1/x^2 + 12x + 1 = 0$, were not quadratic equations.

According to S3, the purpose of quadratic equations were only for finding the roots of the equation, x. She explained that when the roots or values of x were substituted in a quadratic equation, "it" equaled zero. It is assumed that she meant that it formed an identity, as the quadratic equation equaled zero. S3 had attempted to link quadratic equations to the meaning of roots of equations and this reflected a higher level of understanding. S3 had given many examples to elaborate each case. Again, her explanation was theoretical.

When S3 used factorization to obtain the roots of quadratic equations, she gave two examples, one with a positive constant and one with a negative constant: $x^2 + 7x + 6 = 0$ and $x^2 + 2x - 8 = 0$. She had done this to show the difference in the signs of the factors when the signs in front of the constant in the equation differed. This attempt demonstrated two parallel representations according to Leinhart (1987) which indicates a higher level of understanding. Nevertheless, the explanation was algorithmic, as the student only computed the answers. Her answer was stated in the surd form even though she had explained her preference for the decimal form. She was not aware that the surd form was more accurate.

S3 understood that the roots of the equation could be used to form an equation. However, she did not seem to realize that the roots of the equation were actually the values of x. She had formed the quadratic equation, incorrectly in terms of the factors (x – 5)(x + 4) from the given pair of roots. As a result, the values of x that followed were not the same as the original roots of the equation.

When S3 elaborated that the addition and multiplication of roots could give a quadratic equation, she first used one positive and one negative root, 5 and –4. In the second example, she used two positive roots, probably to show the difference in the equations that were obtained when roots of different signs were used. However, in the second example, she had performed the addition and multiplication a second time on the sum and product of roots and had incorrectly obtained the quadratic equation.

S3 was the only student who had explained how the quadratic equation could be obtained from a pair of given roots in Task 1. She had also considered the differences of the signs of the roots might make to the quadratic equations formed by giving parallel representations in her explanation. Her explanation of roots of equation has shown relational understanding.

4.3.4 Student 4

Quadratic Equations

Student 4 (S4) explained that quadratic equations were similar to other equations that had been learnt except that quadratic equations had an unknown term to the power of two, or higher. During the interview (Appendix 2 – 4), S4 chose equations to the third and fourth power as examples of quadratic equations. However, she did not consider equations with two terms as quadratic equations even though there was a square, example $x^2 - 3 = 0$. S4 explained that quadratic equations contained at least three terms. On the other hand, she considered $x^3 - 2x = 0$ as a quadratic equation even though it had two terms.

S4 explained the three methods for obtaining solutions of quadratic equations; factorization, completing the square and the general formula. She explained that these three methods could be used to find the roots of the quadratic equation. And then solved an example of a quadratic equation, $x^2 - 4x + 4 = 0$, by factorization (Appendix 3 – 4 - a). After the product of the factors were obtained and equated to zero, each factor was then equated to zero and hence the values of x. In this example the values of x were both the same, x = 2. S4 explained that when both roots were equal, they were described as "equal and different roots" and had the condition $b^2 - 4ac = 0$.

S4 next solved the same equation by completing the square. The equation in the form $x^2 - 4x + 4 = 0$ was first changed to $x^2 - 4x = - 4$. The left side of the equation was made a complete square as there was a new term added on the right, $(4/2)^2$. Finally, the complete square, $(x - 2)^2 = 0$ was written as a product of two factors which was $(x - 2)(x - 2) = 0$. S4 then obtained a pair of equal roots, x = 2.

S4 also used the general formula to solve $x^2 - 4x + 4$ (Appendix 3 – 4 - b). She identified the values for a, b and c and directly substituted into the general formula as $\frac{-(-4) \pm \sqrt{(-4)^2 + 4(1)(4)}}{2(1)}$. During computation, she simplified the answer in the surd form from $\frac{4 \pm 2\sqrt{6}}{2}$ to $2 \pm 2\sqrt{6}$.

After computation, two different roots in the surd form were obtained $2 + 2\sqrt{6}$ and $2 - 2\sqrt{6}$ (Appendix 3 – 4 - b).

During the interview (Appendix 2 –4 - c), when S4 was asked to simplify $\frac{4 + 2\sqrt{6}}{2}$, she first factorized the numerator. The common factor 2 was written outside the brackets. Both the numerator and the denominator of the surd were then divided by two, and the answer obtained was $2 + \sqrt{6}$.

Roots of Quadratic equations

S4 explained that there were three different types of roots as real roots which were equal and different, real and equal roots, and imaginary roots. She then stated the conditions for each of these roots. However, during the interview (Appendix 2 – 4 - a), S4 identified the three types of roots as real and different, real and equal, and imaginary. The condition $b^2 - 4ac > 0$ was for real and different roots, $b^2 - 4ac = 0$ was for real and equal roots, and $b^2 - 4ac < 0$ was for imaginary roots.

An example of a quadratic equation which had imaginary roots, $2x^2 + 8x + 12 = 0$ was given. The equation was then simplified to $x^2 + 4x + 6 = 0$ after being divided by 2. Then, the values of a, b and c were identified and substituted into the inequality $b^2 - 4ac < 0$. Finally, the inequality was expressed as *"-8 < 0"*. During the interview, S4 used the same method to identify the type of roots. She substituted values of a, b and c from the quadratic equation into $b^2 - 4ac > 0$ and obtained the following:

$$3^2 - 4(2)(2) > 0,$$

$$9 - 16 > 0$$

$$-7 > 0.$$

S4 explained that this equation had imaginary roots because –7, which was on the left side of the final inequality, was less than zero. Even though she had obtained the inequality "$-7 > 0$" in her answer, she observed only the left side of the inequality to make her conclusion. Since it was negative, she concluded that it satisfied the condition for imaginary roots.

When asked the meaning of roots of equations (Appendix 2 – 4 - a), S4 explained that roots of equations were numbers that could replace the value of an unknown algebraic term. She explained further that roots could be determined when an equation was simplified and its factors were equated to zero. She then explained that the values of the unknown x could then be found and that these values were the roots of the equation.

<u>Inferred Understanding of Student S4</u>

From the interview and the written responses, S4 had taken quadratic equations to have terms to the power of two and higher, and have at least 3 terms in the equation. An equation with a cube and without a constant term, example $x^3 - 2x = 0$, was also quadratic.

Roots of equations could be obtained by three methods. In the first method, factorization, the product of the factors was first equated to zero before the values of x were then obtained. In the second method, completing the square, the non-general form of the equation was used and the term $(4/2)^2$ was added to the right side of the equation to form a complete square. The value of x could then be obtained from the completed square. When the roots were the same, S4 identified them as equal roots. She had made a link with the solutions of the roots in this method to the types of roots reflecting a higher level of understanding.

In the last method, S4 identified the values of a, b and c from the quadratic expression and not the equation. She did not differentiate between the quadratic expression and quadratic equation when solving for the roots using this method. S4 had then used the general formula of the form $\frac{-b \pm \sqrt{(-b)^2 - 4(a)(c)}}{2a}$. However, there was an error in S4's computation when she cancelled the denominator with only one term in the numerator which caused $\frac{4 \pm 2\sqrt{6}}{2}$ to become "$2 \pm 2\sqrt{6}$".

However, during the interview, she factorized the numerator first before "canceling" it correctly with the denominator. S4 had the tendency to have cancellation errors (Kaur and Sharon, 1997), although she later corrected herself.

In finding the solution of quadratic equations, S4's explanation was algorithmic. She did not explain the steps she had taken but had only computed the values. Though S4 had used the same equation for all three methods, she failed to notice that the value of x in the last method was different from that in the previous two. This also meant that she did not link the three methods. Although the three methods were different, they all had the same purpose, which is to determine the solution of the equation. All three solutions should have been the same, but were not.

In order to identify the type of root, S4 had substituted the values of a, b and c in the quadratic equation into the inequality $b^2 - 4ac < 0$ or $b^2 - 4ac > 0$ and not into the expression $b^2 - 4ac$. After simplifying the inequalities, she could conclude on the type of root an equation had by observing the left side of the inequality. She had attempted to apply the condition for identifying the roots of the equation but was not careful in using the condition. S4's explanation demonstrated instrumental understanding.

4.3.5 Student 5

Quadratic Equations

The fifth student (S5) explained that quadratic equations had unknowns with the highest power of two. She then gave two examples, one which had three terms and the other with two terms: $9p^2 - 6p + 1 = 0$ and $x^2 - 25 = 0$. S5 also gave counter-examples, one with a term with a power higher than two, $x^4 + 2 = 0$, and another was of a linear form, $x + 2 + 2x = 0$. During the interview, S5 also identified equations such as $1/x^2 - 5x + 3 = 0$ and

$x^2 + 1/x + 1 = 0$ as quadratic. However, she disregarded equations with powers of three and four such as $x^3 - 2x + 3 = 0$ and $x^4 - 2x + 3 = 0$.

Roots of Quadratic Equations

S5 explained that roots of a quadratic equation was a quadratic equation written in the general form of the equation, $ax^2 + bx + c = 0$. This was then solved by either factorization, completing the square or with the general formula. When S5 was asked during the interview (Appendix 2 – 5) on the meaning of a root, she stated that the root was x.

Inferred Understanding of Student S5

S5 understands that quadratic equations were not limited to x terms but could have other unknowns such as p. She had made a parallel representation of a quadratic equation in the x term, thus showing a higher level of understanding. S5 also understands that quadratic equations could have two or three terms with the highest power of two. Equations to the power of four, and of the linear form, were not considered quadratic equations. However, equations with the term $1/x^2$ were considered quadratic.

S5 could not clearly define roots of the equation but she knew that the roots were the values of x. She did not explain the different types of roots. This indicated that she did not have much understanding of roots of quadratic equations. S5's written responses were brief and her understanding of the roots of quadratic equations was mainly instrumental.

4.4 Analysis of Task 2

In Task 2, the identity $x^2 - 4x - 5 = (x - 5)(x + 1)$ was given. The student was asked to explain how the factorization was done and that factorization could be used to solve the quadratic equation $x^2 - 4x - 5 = 0$.

4.4.1 Student 1

S1 explained that there were many methods to perform the factorization. The first method involves drawing a "cross" with arrows to denote multiplication (Appendix 3 – 1 - c). First, S1 explained that the "cross" was drawn in the middle. On the left, the factors of x^2 was found which was $x \times x$. Similarly, factors for the constant, -5 comprising -1×5 or -5×1 were considered. These factors on the right were then multiplied with those on the left. For confirmation of correct choice of factors, the sum of the products should be equal to the middle term in the quadratic equation. After confirmation, S1 wrote the terms $(x - 5)(x + 1)$ as the solution to the quadratic equation.

During the interview (Appendix 2 – 1), S1 elaborated on her choice of 5 and 1 as factors. She reasoned that because the last term in the expression was –5 and as such the factors of –5 were either -5 multiplied 1, or -1 multiplied 5. When asked to explain the reason 5 was placed at the top of the cross on the right and 1 at the bottom, S1 replied that the positions did not matter. She further explained her choice of the negative sign in front of 5 by referring to the middle term of the equation, -4 and placed the negative sign on the larger number in the factor, which was –5. When –5 was added to 1, it gave –4.

The second method in the explanation involved the use of a table (Appendix 3 – 1 - c). S1 explained that the first term contained an "x^2", and the last term, "which did not have an unknown", were written at the bottom of the table. The factors were filled into the rows of the table. Factors along the horizontal were multiplied and the sum of the two products was then written in the space on the right. Table 2 resembled that used by S1.

Table 2 : Factorization using a table.

x	→	-5	-5x
x	→	1	x
x^2		-5	-4x

In the third method, the sign on the last term of the quadratic equation was used to determine the signs in the factors. She then drew Table 3 to show that the last term of the quadratic equation was positive indicating two positives or two negatives. For the last term to be negative, it meant that there was a positive and a negative term, or, a negative and a positive term. S1 explained that after the signs were determined, the position of the positive and the negative signs had to be determined. If the middle term, bx, was negative, then it meant that the larger factor would be negative. Table 3 was used earlier when the student had answered Task 1.

Table 3 : Possibilities of signs of factors for Task 2.

+	+,+ *atau* -, -
-	+, - *atau* -, +

S1 used the third method to factorize the equation. She first explained that the last term in the quadratic equation was negative implying the two factors had to have different signs. The factors were then put into two brackets to give the product (x + 1)(x - 5). As the middle term was negative, the factor with the larger value had to be given a negative sign. The factors were finally left as (x + 1)(x - 5).

In conclusion, S1 explained that in order to determine the factors of x, one had to choose the factors correctly. This could be done by observing the difference or the sum of the factors, and then determining if they were equal to that of the middle term in the equation. She preferred the "cross method" which to her was clearer than the method involving the table as it involved many steps.

Inferred Understanding of Student S1

S1 understood that there were three methods for factorization; using a cross, using a table and observing the signs of the terms in the equation.

In the first method the factors of x^2 and –5 were obtained and put at the top the cross. The positions of factors did not matter. The negative value for the factor 5 was determined from the middle term of the given quadratic equation, -4x as the larger number in the factors was given the same sign as that of the middle term.

In the second method, the first and last term, which S1 defined as "having an x squared" and "being a constant", respectively, were written at the bottom of a table. The factors of the square and of the constant were written in the table above these terms. The factors along the same row of the table were multiplied and the sum of their products was written below, on the right.

The second method differed from that of the first because the product of the factors in the first method were multiplied diagonally, while in the second method, they were multiplied horizontally. However, S1 did not point out any differences or similarities between the first and second method. These two methods actually differed in the way the products of the factors are multiplied in the former, diagonally and horizontally in the latter. In this question, it happens that the factors appear the same as in the first method after factorization. However if an equation of the general form with $a \neq 1$ were used, the answer obtained from both these methods, would be different.

In the third method, S1 explained that the sign before the last term of the equation was important. The signs of the factors were determined from the sign of the constant in the equation according to the table given. When the constant was negative it meant that one factor was positive and the other negative. Next, by observing that the middle term in the equation was bx, the factor with the larger value would be negative.

The methods that S1 had explained are basically the same. They involve listing the factors of the first and last terms, determining the signs and checking whether the sum of the product of the factors were the same as the term in x in the quadratic equation. Her explanation was algorithmic, as she had only explained the procedure. However, she did attempt to justify the importance in the sign of factors. There is relational understanding as defined by Skemp (1979), as the student had deduced a rule for determining the signs of the factors.

4.4.2 Student 2

S2 explained that cross multiplication was used in the factorization by showing that x^2 consisting of the factors, x multiplied with a number in the form of a cross (Appendix 3 – 2 - b). The "number" that she referred to was a general number. S2 also gave examples of how the factors of $2x^2$ and $4x^2$ were represented in a similar manner, that is $2x^2$ as 2x and x and $4x^2$ for 2x and 2x, or x and 4x.

In the factorization of $x^2 - 4x - 5 = (x + 1)(x - 5)$, S2 explained that the last term in the quadratic equation, 5, was a product of 1 and 5. When 1 and 5 were multiplied diagonally by x, the products were summed as -4x. S2 elaborated further that the negative sign of 4x was determined by the signs attached to 5x and x. During the interview (Appendix 2 - 2 - a) she was asked whether the products must always be added, or subtracted. She explained that it was not necessary to use only addition as the products could be subtracted.

In that case, the larger number was assigned the positive sign and subtracted with the smaller number, to obtain 4x. As the middle term was –4x, the larger number was then assigned the negative sign.

To solve the equation, S2 equated the product of the factors to zero. Following that each individual factor was equated to zero, which were (x - 5) = 0 and (x + 1) = 0. The values of x were transposed to obtain x = 5 and x = -1. She had written the observation that the negative sign had changed to positive, referring to -5, and the positive had changed to negative, referring to 1.

During the interview (Appendix 2 – 2 - a), S2 explained that the factors had to be equated to zero for finding x. She accepted that the equation could be equated to a value other than zero. The value would then have to be transposed to the right side and equated to zero first, before solving for x.

<u>Inferred Understanding of Student S2</u>

S2 showed the factorization of $2x^2$ and $4x^2$, the first terms, even though the task only required an equation with the term in x^2, to be factorized. She had extended her explanation to demonstrate other representations of the first term. Then, the factors of the positive value of the constant were noted and their signs were determined by observing the middle term in the equation. Her explanation approached relational understanding (Skemp, 1979) by deducing the procedure for determining the signs in the factors. However, when S2 had applied the Product Rule to obtain the value of each factor, without knowing the reason, this indicated only instrumental understanding.

4.4.3 Student 3

S3 began with the explanation of the need to determine if the equation was quadratic (Appendix 3 – 3 – d). She wrote the general form of the equation which she says was the basis for the equation. The values of a, b and c were numbers that were "added in" while x was a fixed value. She then determined the values for a, b and c from the equation $x^2 - 4x - 5 = 0$ before she factorized the equation.

S3 first explained that when the coefficient a was equal to 1 and that there was a "square" next to x, it meant $x \times x$. Both the factors, x were then put on the left side of the cross. She further explained that on the right were the numbers that could be multiplied to obtain the constant, c. When "cross-multiplication" was performed, 1×1 gave 1 and 5×1 gave 5. The products 1 and 5 would be used to determine b, and the signs for these products would then be determined. She elaborated that one had to follow the sign in front of the constant, c, in the equation for addition or subtraction.

During the interview (Appendix 2 – 3 - a), S3 wrote the factors for x and of the constant term, 5. She then multiplied along the cross to obtain the products x and 5x. The negative sign was later assigned to 5x by considering the sign in front of 4 and the middle term in the quadratic equation. S3 also used boxes to contain the factors to remind her to write the factors (x + 1) and (x - 5). By trial and error, S3 determined that the factors were correct. She cautioned that one had to be careful when writing the factors to refer to them "across" and not "diagonally".

Inferred Understanding of Student S3

In her explanation, S3 first determined whether an equation was quadratic by first checking for the power of x in the term ax. The factors of $1x^2$ and the constant were determined next and written on the right side of the cross. "Cross multiplication" was then performed, which according to S3 meant that the factors of 1x and the constant were multiplied diagonally across the cross.

These products were added or subtracted according to the sign in front of the constant term in the quadratic equation. If the sign in front of the constant was negative, the products would have to be subtracted. When factorizing the equation, she had checked the expansion by multiplying the factors. However, S3 stopped after performing the factorization. She did not continue to solve for x, showing the lack of understanding of the process of solving quadratic equations. Her explanation was mainly algorithmic, even though she had tried to justify certain steps and her understanding was instrumental.

4.3.4 Student 4

S4 remarked that expansion of an expression was the foundation for factorization (Appendix 3 – 4 - c). She gave an example of a set of factors (x - 2)(x + 4), and its expansion as $x^2 + 2x - 8$. Continuing the task, she stated that the expression $x^2 + 2x - 8$ could be factorized to obtain (x - 2)(x + 4).

S4 then proceeded to factorize $x^2 - 4x - 5 = 0$ and first took x^2 and put it in the form of a cross. Similar to the other students, the factors of 5 were associated with 1 and 5 and put into the "cross" beside x by multiplying diagonally. The positive and negative "symbols" were added to the products, x and 5x to obtain the sum, –4x. According to S4, the signs for the factors were determined by trial and error. S4 had also explained that the + and – symbols were written between x, 1, and 5. Two expressions were then obtained that is (x + 1)(x – 5) and equated to zero.

S4 had used the term "cross multiplication" when factorizing. She explained that cross multiplication was used in solving equations with fractions, such as a/5 = 2/3 as well as in factorization. S4 continued to solve the equation by first equating the product of the factors to zero. Next, each expression within the brackets was equated to zero; x + 1 = 0 and x - 5 = 0 to obtain the values x = -1 and x = 5. These values were the roots of the equation. When the roots were obtained, she considered the equation was solved.

Inferred Understanding of Student S4

S4 was the only student who related factorization to expansion. However, S4 had emphasized the importance of expansion, she did not explain the procedure involved. Nevertheless, by relating factorization to expansion, S4 was making a link with a previously learnt topic, indicating a higher level of understanding.

The factors of the first term, x^2 and the constant, 5 were written along a cross sign and "cross-multiplied". S4 considered this as cross-multiplication was used for solving simple equations with fractions.

The negative and positive symbols for the factors were determined by looking at the sum of the product of factors, and by trial and error. She equated the product of factors to zero and after that, she equated each factor to zero. However, her explanation was mainly algorithmic, as she did not explain the reason it was so. By explaining that the solution, x, of a quadratic equation were its roots, S4 had linked the solution of quadratic equations and roots of the equation. S4 showed relational understanding as she tried to deduce the roots when an equation was given.

4.4.4 Student 5

S5 explained that factorization was used when there were "same" algebraic terms in an expression for example x + 2x + 3x (Appendix 3 – 5 – b). This expression is factorized to x (1 + 2 + 3) as x was the common factor. Factorization enabled one to "see and count easily", and is not limited to unknown "algebras" as numbers could also be factorized. For example, $3a + 6b$ could be written as $3a + 2 \times 3b$ to give $3(a + 2b)$.

She used factorization to find the value of x. S5 explained that factorization had three terms when she gave the example $x^2 - 4x - 5$ and she said that the answer might have one or two expressions. The subject meant that the expression to be factorized had 3 terms while the factorized expression may contain one or two expressions.

S5 then solved the expression $x^2 + 7x + 10$ by first obtaining the factors of x^2 as $x \times x$, 10 as 1×10 and 2×5. The factors, 2 and 5, were then written in boxes and the product of x and the factors, 2x and 5x, written by the side (Appendix 3 – 5 - b). These factors were summed as +7x. Hence $x^2 + 7x + 10$ equaled $(x + 2)(x + 5)$.

During the interview (Appendix 2 – 5 - a), S5 showed the factorization of $x^2 + 7x + 10$ by taking 2 and 5 as the factors of 10 and –10. She also explained that if the number was positive, the two factors, 2 and 5 would be both positive. If the factors were negative, the middle term 7 would be negative. She explained that the products 2x and 5x were obtained through "cross multiplication" and were then added to obtain 7x.

S5 then elaborated on the method of obtaining the solution by considering $x^2 - 4x - 5 = 0$. Then she proceeded to make $(x - 5)(x + 1) = 0$, followed by $(x - 5) = 0$ and $(x + 1) = 0$ and the values $x = 5$ and $x = -1$.

Inferred Understanding of Student 5

S5's understanding of factorization included considering common factors of expressions like x + 2x + 3x and 3a + 6b. S5 seemed to think that the expression to be factorized should have 3 terms, for example $x^2 - 4x - 5$, and that the answer might contain one or two expressions.

S5 gave her own example of a quadratic equation $x^2 + 7x + 10$ and factorized it by first listing the factors of x^2 and 10. She termed the process of obtaining the products x and the factors, 2 and 5, as "cross multiplication". When solving the equation $x^2 - 4x - 5 = 0$, S5 equated the product of the roots to zero before obtaining x. Though S5 had gone into details on how the equation could be factorized, her explanation was algorithmic. The understanding shown was instrumental as she only applied the rules.

4.5 Analysis of Task 3

In Task 3 the student was told that they had learnt many different methods of solving quadratic equations, other than factorization. She was then required to choose one method and explain how this method would enable her to solve the quadratic equation $x^2 - 2x - 1 = 0$.

4.5.1 Student 1

S1 chose the general formula to solve quadratic equations (Appendix 3 – 1 – e). She wrote down the general formula and explained that the values of a, b and c from the equation, $ax^2 + bx + c$ could be substituted into the general formula.

From the equation $x^2 - 2x - 1 = 0$, S1 identified a = 1, b = -2 and c = -1 and substituted them into the general formula. The expression was then simplified to $\frac{2 \pm \sqrt{4 + 4}}{2}$, and further to $\frac{2 + 2.8284}{2}$ or $\frac{2 - 2.8284}{2}$.

The final values of x were 2.4142 or – 0.4142.

S1 had used the calculator because she preferred the answers in decimal form as those were more accurate.

Inferred Understanding of Student S1

S1 had used the general formula and identified values of a, b and c from the given equation. The value of a was taken as 1, though nothing was written next to x.

S1 multiplied –(-2) to obtain positive 2, and squared –2 to obtain 4. She had no problems with substitution and multiplication. Her answer was given in decimal form, to four places of decimal, because it was considered more accurate. S1's explanation was algorithmic, as she was more concerned with obtaining the answer then explaining the procedure.

4.5.2 Student 2

S2 had explained that the given equation $x^2 - 2x - 1 = 0$ could not be factorized and hence the general formula had to be used (Appendix 3 – 2 – c). The general formula was then written and the values of a, b and c were identified from the equation $x^2 - 2x - 1 = 0$. As x^2 had no number in front of it, this meant that the value for the coefficient, a was 1.

S2 then wrote the formula with the substituted values. She showed the multiplication of two negative terms with arrows in the formula. Beside it, she explained that $(-2)^2$ was the product of two negative values (-2)(-2), which would give a positive value. S2 simplified the expression for x in the general formula to $\frac{2 \pm \sqrt{4 + 4}}{2}$ and then $1 \pm \sqrt{8/4}$, and finally $1 \pm \sqrt{2}$. She noted that the square root sign was longer for the surd $\sqrt{\frac{8}{4}}$.

During the interview (Appendix 2 – 2), S2 explained that she had simplified the first number by dividing it by 2. This would give the result, 1. Next, the denominator was squared. This meant that $\frac{2 \pm \sqrt{8}}{2}$ was simplified to $1 \pm \sqrt{\frac{8}{4}}$ which could be further simplified to $1 \pm \sqrt{2}$.

Inferred Understanding of Student S2

S2 had chosen the general formula because the equation $x^2 - 2x - 1 = 0$ could not be factorized. She may have attempted to factorize the equation first before coming to the conclusion that it could not be factorized. Thus, she has attempted to justify the choice of the use of the general formula.

S2 had written the general form of the equation and compared with the given equation to obtain the values of a, b and c. She specifically noted that the term x^2 which had no number attached which meant a equaled 1.

S2 also stressed on the multiplication of two negative numbers to give a positive number. S2 gave her answer in the surd form. On simplifying the surd $\frac{2 \pm \sqrt{8}}{2}$ by first separating the terms into $\frac{2}{2} \pm \frac{\sqrt{8}}{2}$ and then dividing them.

The surd term was then written as $\sqrt{8/4}$ and she particularly stressed on the longer square root sign. The answer was then simplified to the simpler surd form $1 \pm \sqrt{2}$. However, as the subject did not simplify further, it could not be deduced whether the final answer was actually one complete answer or whether it was two different answers.

S2 had attempted to make some justifications and explanations on the multiplication of negative numbers. Her written response was mainly algorithmic, but she had attempted to explain the steps wherever possible. This indicated that she had a relational understanding (Skemp, 1979) of the topic.

4.5.3 Student 3

S3 explained that that there were several methods of solving the equation $x^2 - 2x - 1 = 0$ (Appendix 3 – 3 – e). She explained that other than factorization, the methods that could be used were completing the square and using the general formula. To her, the easier method to solve quadratic equations was using the general formula, which could be memorized. After writing the general formula and the equation $x^2 - 2x - 1 = 0$ with the general form $ax^2 + bx + c = 0$, she then explained that to solve the equation one had to substitute the values into the general formula. On simplifying the formula, she obtained $\sqrt{8}$ which she wrote as $\sqrt{4 \times 2}$, and further simplified to $2\sqrt{2}$. The reduction was by canceling the factor 2 in the numerator and denominator as follows :

$$\frac{2 \pm 2\sqrt{2}}{2} = \left(\frac{2}{2} \pm \frac{2\sqrt{2}}{2}\right) = \pm \sqrt{2}$$

During the interview, S3 simplified similar surd expressions by dividing both terms with the denominator of the fraction and obtained the following answers:

(a) $\frac{3 + 6\sqrt{2}}{3}$ (b) $\frac{2 \pm 2\sqrt{2}}{2}$

$= 1 + 2\sqrt{2}$ $= 1 \pm \sqrt{2}$

S3 explained that the answers could be left in surd form. She cautioned that one had to be careful with the signs before the numbers. These numbers had to be substituted into the general formula. Furthermore, she reasoned that if the answer were left in the decimal form, it should be given to three decimal places as in the examinations to avoid the deduction of an "accuracy mark".

During the interview (Appendix 2 – 3 - c), S3 explained that the final answer could be in either surd or decimal form. However, she preferred the decimal form as it was easy to obtain with the help of the calculator.

Inferred Understanding of Student S3

S3 considered the easiest method using the formula as it only involved memorizing the formula. She was able to identify the values of the coefficient a as 1 and coefficients b and c as –2 and –1 respectively which she then substituted into the general formula. However, S3 had made a cancellation error (Kaur and Sharon, 1987) in her written response whereby $\frac{(2 \pm 2\sqrt{2})}{2}$ had become $\pm\sqrt{2}$. She had divided 2 in the numerator with 2 in the denominator to obtain zero. However, this error was not repeated during the interview, indicating that the error was probably due to carelessness. Furthermore, S3 had taken surds as numbers that had no decimals. In general, the student's answer was algorithmic and she demonstrated instrumental understanding (Skemp, 1979) of solution of quadratic equations.

4.5.4 Student 4

S4 chose that the other method for solving quadratic equations was the general formula (Appendix 3 – 4 – d) and then explained that the values of a, b and c for the equation $x^2 - 2x - 1 = 0$, should be identified and then substituted into the formula. She proceeded to simplify $\frac{(2 \pm \sqrt{8})}{2}$ and then explained that there was a choice of leaving the answer in the surd or in the decimal form. For the answer to be left in surd form, $\sqrt{8}$ must be simplified into $\sqrt{(2 \times 2 \times 2)}$ to give $2\sqrt{2}$. In the next step, S4 had simplified $\frac{(2 \pm 2\sqrt{2})}{2}$ by canceling the first 2 in the numerator with that in the denominator, to obtain $2\sqrt{2}$ and $-2\sqrt{2}$. However, during the interview (Appendix 2 – 4 - c), she simplified $\frac{4 + 2\sqrt{6}}{2}$ correctly to $2 + \sqrt{6}$. S4 also showed the solution in the decimal form as 2.4142 and -0.4142. S4 explained that these two numbers were the roots of the equation $x^2 - 2x - 1 = 0$ and that once the roots were obtained, the equation was considered solved.

S4 had explained that the answers could be stated in the surd or decimal form. During the interview (Appendix 2 – 4 - c), she said that she preferred the decimal form because there was a possibility of errors occurring while factorization when the question was solved in the surd form. The decimal form was easier as the answer could be obtained with the calculator.

S4 had also written another method of solving equations was completing the square. When she was asked whether completing the square was a better method than the general formula, she explained that it was. This was because the general formula method had more signs, which might cause more errors to be committed.

Inferred Understanding of Student S4

S4 had chosen the general formula as the best method for solving quadratic equations. In simplifying the formula, S4 changed the surd $\sqrt{8}$ by first writing $\sqrt{2 \times 2 \times 2}$, and then obtaining $2\sqrt{2}$. As with S3, a cancellation error occurred when she simplified $\frac{(2 \pm 2\sqrt{2})}{2}$ and she obtained $2\sqrt{2}$ and $-2\sqrt{2}$. She had cancelled 2 in the numerator with 2 in the denominator to obtain zero. However, this error did not occur during the interview when she was asked to simplify similar questions.

S4 preferred to give the answer in decimal form. There was also less possibility of errors because the calculator was used to perform the computation. On the other hand, S4 also explained that the method of completing the square was a better method. This was because it was not as confusing with regards to the signs used.

S4's explanation was algorithmic in nature. The understanding was mainly instrumental (Skemp, 1979). There were not much justifications and linkages. Further, she was probably not very clear on which was the better method, the general formula or completing the square.

4.5.5 Student 5

S5 explained that she preferred to use factorization and the general formula when solving quadratic equations (Appendix 3 – 5 – d). During the interview (Appendix 2 – 5 - b), she explained that there was a tendency for her to make more mistakes when using the "completing the square" method.

In order to solve the equation $x^2 - 2x - 1 = 0$, S5 explained that one first had to write down the general form of the equation, $ax^2 + bx + c = 0$. Then she listed the values of the coefficients of a, b and c and wrote the equation $x^2 - 2x + -1 = 0$ followed by the general formula. Substituting the values into the formula and simplifying expressions to $2 \pm \sqrt{(4 + 4)}$, and later changing to a decimal form as 2 ± 2.828, she then obtained the values of x, 2.414 and 0.414.

S5 preferred to leave the answer in decimal form (Appendix 2 – 5 - c). This was because she could use the calculator to obtain the answer. On the other hand, she felt that the answer in the surd form was more accurate even though surds were more difficult to simplify.

Inferred Understanding of Student S5

S5 also chose the general formula as the better method to use for solving quadratic equations. She explained that more mistakes would be made with the completing the square method. She also preferred to give the answer in decimal form. In general, S5's answer was also algorithmic in nature with little justifications of the steps she had taken. She demonstrated instrumental understanding.