CHAPTER 4

THE FINITE DIFFERENCE METHOD FOR BIO-HEAT 1-D AND 2-D

4.1 Introduction

Modern clinical treatments and medicines such as cryosurgery, cryopreservation, cancer hyperthermia, and thermal disease diagnostics, require the understanding of thermal life phenomena and temperature behavior in living tissues (Jennifer et al., (2002). Studying Bio-heat transfer in human body has been a hot topic and is useful for designing clinical thermal treatment equipments, for accurately evaluating skin burn and for establishing thermal protections for various purposes. The Bio-Heat transfer is the heat exchange that takes place between the blood vessels and the surrounding tissues. Monitoring the blood flow using the techniques has great advantage in the study of human physiological aspects. This require a mathematical model which relate the heat transfer between the perfuse tissue and the blood. The theoretical analysis of heat transfer design has undergone a lot of research over years, from the popular Penne's Bio-heat transfer equation proposed in 1948 to the latest one proposed by (Deng & Liu, (2002)). Many of the Bio-heat transfer problem by Pennes's account for the ability of the tissue to remove heat by diffusion and perfusion of tissue by blood. Predictions of heat transport have been carried out in (Chinmay (2005) and Liu & Xu, (2001)). We begin the chapter by introducing the 1-D Bio-Heat Equation. Section 4.2 discusses the finite difference scheme for the 1-D Penne's Equation; section 4.3 discusses the stationary iterative methods on 1-D Bio-Heat, and formulation of the IADE scheme on 1-D Bio-Heat is treated in section 4.4. Section 4.5 introduces the 2-D Bio-Heat Equation and the stationary methods on 2-D Bio-Heat are treated in section 4.6 with section 4.7 illustrating the ADI method on 2-D Bio-Heat. Section 4.8 treats the IADE-DY

implementation on 2-D Bio-Heat with the MF-DS scheme on 2-D Bio-Heat in section 4.9.

4.1:1 1-D Bio-Heat Transfer Problem

An energy balance for a control volume of tissue with volumetric blood flow and metabolism yields the Bio-heat equation (Pennes (1948)).

$$\rho c_p \frac{\partial U}{\partial t} + \omega_b c_b (U_a - U) = \frac{\partial^2 U}{\partial x^2} + q_m''', \quad 0 \le x \le 1,\ 0 < t < T \tag{4.1}$$

$$\frac{\partial U}{\partial t} + \frac{\omega_b c_b}{\rho c_\rho}(U_a - U) = \frac{1}{\rho c_p}\frac{\partial^2 U}{\partial x^2} + \frac{q_m'''}{\rho c_\rho}, \tag{4.2}$$

This further simplifies into the form:

$$\frac{\partial U}{\partial t} + \frac{\omega_b c_b}{\rho c_\rho} U_a - \frac{\omega_b c_b}{\rho c_\rho} U = \frac{1}{\rho c_\rho}\frac{\partial^2 U}{\partial x^2} + \frac{q_m'''}{\rho c_\rho}$$

$$\frac{\partial U}{\partial t} + \frac{\omega_b c_b}{\rho c_\rho}\left(U_a - \frac{q_m'''}{\omega_b c_b}\right) = \frac{1}{\rho c_\rho}\frac{\partial^2 U}{\partial x^2} + \frac{\omega_b c_b}{\rho c_\rho} U \tag{4.3}$$

Assume q_m''' to be constant, and denote $U^* = U_a - \frac{q_m'''}{\omega_b c_b}$, $b = \frac{\omega_b c_b}{\rho c_p}(>0)$ and $c = \frac{1}{\rho c_p}(>0)$, we can obtain the simplified form of the 1-D Pennes's equation with the initial and boundary conditions given below:

$$\frac{\partial U}{\partial t} = c\frac{\partial^2 U}{\partial x^2} + bU - bU^*, \tag{4.4}$$

$$\begin{aligned} U(x,0) &= f(x)\ 0 < x < 1 \\ U(0,t) &= g(t)\ 0 < t \le T \end{aligned} \tag{4.5}$$

and

$$U(1,t) = h(t) \tag{4.6}$$

where ρ , c_p are densities, specific heat of tissue, ω_b and c_b are blood perfusion rate and specific heat of blood, q_m''' is the volumetric metabolic heat generation, U_a is the arterial temperature, U is the nodal temperature.

Liu et al., (1999), used a finite difference method to solve the Pennes's Bio-Heat equation in a tripled-layered skin structure composed of epidermis, dermis, and subcutaneous. Dai & Zhang (2002), developed a three level unconditional stable finite difference scheme and used a domain decomposition strategy for solving the 1-D Penne's equation for the same three-layered skin structure.

4.2 Finite difference Scheme for the 1-D Penne's Equation

The finite difference method provides approximation solutions for the 1-D Pennes's equation such that the derivatives at a point are approximated by difference quotients over a small interval (Sun & Gustafson (1991)). The problems (4.1) – (4.6) can be solved by using the standard finite difference method. First, one divide [0,1] uniformly into M shares, the mesh size $h = (b - a) / M$. Define $\Omega_h = \{x_l : x_l = a + ih, 0 \le i \le M\}$ as the set of all nodes in $\Omega, \Omega_h' = \{x_i : x_i = a + ih, 1 \le i \le M - 1\}$ as the set of all inner nodes and $\Gamma_h = \{x_o = a, x_M = b\}$ as the set of boundary nodes such that $\Omega_h = \Omega' \cup \Gamma_h$, let $\Delta t = T / N$ denote the time step size and $t^n = n\Delta t$ for $1 \le n \le N$. For a function $U(x,t)$, define the mesh function $U_i^n = U(x_i, t^n)$, and the difference operator

$$\frac{\partial U}{\partial t} = U_t = \frac{U_{i,j+1} - U_{i,j}}{\Delta t}$$
$$\frac{\partial^2 U}{\partial x^2} = U_{xx} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2} \qquad (4.7)$$

Applying Eq. (4.7) on Eq. (4.4), the temperature of the explicit node is given by:

$$\frac{U_{i,j+1}-U_{i,j}}{\Delta t}=c\left(\frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{\Delta x^2}\right)+bU_{i,j}-bU^* \tag{4.8}$$

$$U_{i,j+1}=\frac{c\Delta t}{\Delta x^2}\left(U_{i+1,j}-U_{i-1,j}\right)+\left(1-2\frac{c\Delta t}{\Delta x^2}+b\Delta t\right)U_{i,j}-bU^* \tag{4.9}$$

when the Crank-Nicholson (C-N) implicit scheme is used, we write using the same finite difference scheme:

$$\frac{U_{i,j+1}-U_{i,j}}{\Delta t}=\frac{c}{2}\left(\frac{U_{i+1,j+1}-2U_{i,j+1}+U_{i-1,j+1}}{\Delta x^2}+\frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{\Delta x^2}\right)-\frac{b}{2}\left(U_{i,j+1}+U_{i,j}\right)-bU^* \tag{4.10}$$

the difference scheme can be rewritten as:

$$-\frac{c\Delta t}{2\Delta x^2}U_{i+1,j+1}+\left(1+\frac{c\Delta t}{\Delta x^2}-\frac{b}{2}\Delta t\right)U_{i,j+1}-\frac{c\Delta t}{2\Delta x^2}U_{i-1,j+1}=\frac{c\Delta t}{2\Delta x^2}U_{i+1,j}+\left(1-\frac{c\Delta t}{\Delta x^2}+\frac{b\Delta t}{2}\right)U_{i,j}+\frac{c\Delta t}{2\Delta x^2}U_{i-1,j}-bU^* \tag{4.11}$$

Our construction implies that the difference schemes have a truncation error in the order $O(\Delta x^2+\Delta t^2)$ for each interior grid point.

4.3 Stationary Iterative Methods on 1-D Bio-Heat

In reference to Eq. (3.29) and Eq. (3.30), let us view the system in its detailed form considering the matrix formed in Eq. (4.11)

$$\sum_{j=1}^{n}a_{ij}x_j=b_i \qquad (1\le i\le n)$$

Solving the *ith* equation for the *ith* unknown term, we obtain an equation that describes the 'Jacobi method':

$$a_{ii}x_i^{(k)}=\sum_{\substack{j=1\\ j\neq i}}^{n}(-a_{ij})x_j^{(k-1)}+b_i \qquad (1\le i\le n) \tag{4.12}$$

or

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(k-1)} + b_i \right] \qquad (1 \le i \le n).$$

Here, we assume that all diagonal elements are nonzero. If in the Jacobi method above, the equations are computed in order, the components $x_j^{(k-1)}$ with $j < i$ have already been updated and the corresponding new values $x_j^{(k)}$ can be used immediately in their place. If this is done, we have the "Gauss-Seidel method":

$$\sum_{\substack{j=1 \\ j\le i}}^{n} a_{ij} x_j^{(k)} = \sum_{\substack{j=1 \\ j>i}}^{n} (-a_{ij}) x_j^{(k-1)} + b_i \tag{4.13}$$

or

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1 \\ j<i}}^{n} a_{ij} x_i^{(k)} - \sum_{\substack{j=1 \\ j>i}}^{n} a_{ij} x_j^{(k-1)} + b_i \right].$$

An acceleration of the Gauss-Seidel method is possible by introduction of a relaxation factor ω, resulting in the "SOR method":

$$x_i^{(k)} = \omega\left\{ \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1 \\ j<i}}^{n} a_{ij} x_i^{(k)} - \sum_{\substack{j=1 \\ j>i}}^{n} a_{ij} x_j^{(k-1)} + b_i \right] \right\} + (1-\omega) x_i^{(k-1)}$$

or

$$a_{ii} x_i^{(k)} + \omega \sum_{\substack{j=1 \\ j<i}}^{n} a_{ij} x_j^{(k)} = \omega\left\{ \sum_{\substack{j=1 \\ j>i}}^{n} (-a_{ij}) x_j^{(k-1)} + b_i \right\} + (1-\omega) a_{ii} x_i^{(k-1)}. \tag{4.13}$$

Clearly, the SOR method with $\omega = 1$ reduces to the Gauss-Seidel method.

4.4 Formulation of the IADE Scheme (Mitchell-Fairweather Variant)

With the Mitchell & Fairweather, (1964) variant, accuracy can be improved. The matrices derived from the discretization resulting into Eq. (4.11) from Eq. (4.4) at each

time level taking p as an iteration index. To retain the tridiagonal structure of A, the constituent matrices G_1 and G_2 must be bidiagonal (lower and upper respectively). The equation leads to,

$$\begin{aligned} e_1 &= \frac{6}{5}(a-1) \\ u_i &= \frac{6}{5}b, \qquad l_i = 6c/(6-e_i); \quad e_i \neq 6 \\ e_{i+1} &= \frac{6}{5}(a+\frac{1}{6}l_i u_i - 1) \end{aligned} \tag{4.15}$$

for $i = 1, 2, \cdots, m-1$.

The IADE scheme is therefore executed at each of the intermediate levels by effecting the following computations:

i) at level (p+1/2)

$$u_i^{(p+1/2)} = (-l_{i-1}u_{i-1}^{(p+1/2)} + s_i u_i^{(p)} + w_i u_{i+1}^{(p)} + f_i)/d \text{ for } i = 1, 2, \cdots, m, \tag{4.16}$$

where

$$\begin{aligned} d &= 1 + r \\ l_o &= w_m = 0 \\ s_i &= r - ge_i, \qquad i = 1, 2, \ldots, m \\ w_i &= -gu_i, \qquad i = 1, 2, \ldots, m-1 \end{aligned}$$

$$g = (6+r)/6$$

ii) at level (p+1)

$$u_{m+1-i}^{(p+1)} = (v_{m-1}u_{m-i}^{(p+1/2)} + su_{m+1-i}^{(p+1/2)} + gf_{m+1-i} - u_{m+1-i}u_{m+2-i}^{(p+1)})/d_{m+1-i} \tag{4.17}$$

for $i = 1, 2, \ldots, m,$ where

$$\begin{aligned} d_i &= r + e_i \\ v_o &= u_m = 0 \\ s &= r - g, \\ v &= -gl_i. \end{aligned}$$

The IADE algorithm is completed explicitly by the Eq. (4.16) and Eq. (4.17) in alternate sweeps along the points in the interval (0,1).

4.5 2-D Bio-Heat Equation

The well known Penne's equation and the energy balance for a control volume of tissue with volumetric blood flow and metabolism yields the general Bio-Heat transfer equations. ρ, c_p are densities and specific heat of tissue, ω_b and c_b are blood perfusion rate and specific heat of blood, q_m''' is the volumetric metabolic heat generation, U_a is the arterial temperature, U is the nodal temperature. The bio-heat problem is given as:

$$\rho c_p \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \omega_b c_b (U_a - U) + q_m''', \quad 0 \le x \le 1,\ 0 \le y \le 1,\ t > 0 \quad (4.18)$$

$$\frac{\partial U}{\partial t} = \frac{1}{pc_p}\frac{\partial^2 U}{\partial x^2} + \frac{1}{\rho c_p}\frac{\partial^2 U}{\partial y^2} + \frac{\omega_b c_b}{\rho c_\rho} U_a - \frac{\omega_b c_b}{\rho c_\rho} U + \frac{q_m'''}{pc_p}, \quad (4.19)$$

This further simplifies into the form:

$$\frac{\partial U}{\partial t} = \frac{1}{pc_p}\frac{\partial^2 U}{\partial x^2} + \frac{1}{\rho c_\rho}\frac{\partial^2 U}{\partial y^2} - \frac{\omega_b c_b}{\rho c_\rho} U + \frac{\omega_b c_b}{\rho c_\rho}\left(U_a + \frac{q_m'''}{\omega_b c_b}\right) \quad (4.20)$$

assume q_m''' to be constant, and denote $U^{\oplus} = U_a + \frac{q_m'''}{\omega_b c_b}$, $b = \frac{\omega_b c_b}{\rho c_p} (> 0)$ and $c = \frac{1}{\rho c_p} (> 0)$, we can obtain the simplified form of the 2-D Penne's equation with the initial and boundary conditions given below:

$$\frac{\partial U}{\partial t} = c\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) - bU + bU^{\oplus}, \quad (4.21)$$

with initial condition

$$U(x, y, 0) = f(x, y) \quad (4.22)$$

and boundary conditions

$$\left.\begin{aligned} U(0, y, t) &= f_1(y, t),\ U(1, y, t) = f_2(y, t) \\ U(x, 0, t) &= f_3(x, t),\ U(x, 1, t) = f_4(x, t) \end{aligned}\right\} \quad (4.23)$$

when the explicit scheme is used, we write using the same finite-difference scheme:

$$\frac{U_{i,j}^{n+1}-U_{i,j}^{n}}{\Delta t}=c\left(\frac{U_{i+1,j}^{n}-2U_{i,j}^{n}+U_{i-1,j}^{n}}{\Delta x^{2}}+\frac{U_{i,j+1}^{n}-2U_{i,j}^{n}+U_{i,j-1}^{n}}{\Delta x^{2}}\right)-bU_{i,j}^{n}+bU^{\oplus},$$

the temperature of the node in the scheme formulation takes the form:

$$U_{i,j}^{n+1}=c\frac{\Delta t}{\Delta x^{2}}\left(U_{i+1,j}^{n}+U_{i-1,j}^{n}+U_{i,j+1}^{n}+U_{i,j-1}^{n}\right)-\left(1-4c\frac{\Delta t}{\Delta x^{2}}-b\Delta t\right)U_{i,j}^{n}+b\Delta tU^{\oplus} \quad (4.24)$$

4.6 Stationary Methods for 2-D Bio-Heat Equation

If we use the central differences for both U_{xx} and U_{yy} and the forward difference for U_t, into Eq. (4.21) and let $\Delta x^2=\Delta y^2=\Delta^2$ we have:

$$U_{i,j}^{n+1}=U_{i,j}^{n}+\frac{c\Delta t}{\Delta^{2}}\left(U_{i+1,j}^{n}+U_{i-1,j}^{n}+U_{i,j+1}^{n}+U_{i,j-1}^{n}-4U_{i,j}^{n}\right) \quad i=1\ldots,n,\ j=1\ldots,m \quad (4.25)$$
$$-b\Delta tU_{i,j}^{n}+b\Delta tU^{\oplus}$$

let $\frac{c\Delta t}{\Delta^2}=Fo$, hence

$$U_{i,j}^{n+1}=U_{i,j}^{n}+Fo\left(U_{i+1,j}^{n}+U_{i-1,j}^{n}+U_{i,j+1}^{n}+U_{i,j-1}^{n}-4U_{i,j}^{n}\right)-b\Delta tU_{i,j}^{n}+b\Delta tU^{\oplus}$$

it is stable in one spatial dimension (1-D) only if $\Delta t/\Delta^2\le 1/2.$ In two dimensions (2-D) this becomes $\Delta t/\Delta^2\le 1/4.$ Suppose we try to take the largest possible time step, and set $c\Delta t=\Delta^2/4.$ Then equation (4.25) becomes:

$$U_{i,j}^{n+1}=\frac{1}{4}\left(U_{i+1,j}^{n}+U_{i-1,j}^{n}+U_{i,j+1}^{n}+U_{i,j-1}^{n}\right)-b\Delta tU_{i,j}^{n}+b\Delta tU^{\oplus} \quad (4.26)$$

thus the algorithm consists of using the average of U at its four nearest neighbor points on the grid (plus contribution from the source). This procedure is then iterated until convergence. This method is in fact a classical method with origins dating back to the last century, called “Jacobi’s method”. The method is impractical because it converges too slowly. However, it is the basis for understanding the modern methods, which are always compared with it.

Another classical method is the "Gauss-Seidel method". Here we make use of updated values of U on the right hand side of Eq. (4.26) as soon as they become available. In other words, the averaging is done "in place" instead of being "copied" from an earlier time step to a later one. If we are proceeding along the rows, incrementing j for fixed i, we have the formula Eq.(4.26) as:

$$U_{i,j}^{n+1} = \frac{1}{4}\left(U_{i+1,j}^{n} + U_{i-1,j}^{n+1} + U_{i,j+1}^{n} + U_{i,j-1}^{n+1}\right) - b\Delta t U_{i,j}^{n} + b\Delta t U^{\oplus}, i = 1\ldots, n,\ j = 1\ldots, m \quad (4.27)$$

this method is also slowly converging and only of theoretical interest, but some analysis of it will be instructive. If we have approximate values of the unknowns at each grid point, this equation can be used to generate new values. We call $U^{(n)}$ the current values of the unknowns at each iteration k and $U^{(n+1)}$ the value in the next iteration.

We define a scalar $\omega_n (0 < \omega_n < 2)$ and apply Eq.(4.27) to all interior points (i, j). Hence, we have:

$$U_{i,j+1}^{n+1} = \varpi * GS + (1 - \varpi) U_{i,j}^{n}$$

Where, GS is the calculated value of the Gauss-Seidel method and ϖ, is omega with values ranging from $0 < \varpi < 2$.

4.7 ADI Method (2-D Bio-Heat)

The ADI method is originally developed by (Peaceman & Rachford, (1955)), a time step $(n \to n+1)$ is provided into two half time steps $(n \to n+1/2 \to n+1)$. The time derivative is represented by forward difference and the spatial derivatives are represented by central differences. In the first half time step $(n \to n+1/2)$, $\partial^2 U / \partial x^2$ is expressed at the end $n+1/2$ and $\partial^2 U / \partial y^2$ is expressed at the start n. Therefore:

$$\frac{U_{i,j}^{n+1/2}-U_{i,j}^{n}}{\Delta t/2}=\frac{c}{2}\left[\frac{U_{i+1,j}^{n+1/2}-2U_{i,j}^{n+1/2}+U_{i-1,j}^{n+1/2}}{(\Delta x)^2}+\frac{U_{i,j+1}^{n}-2U_{i,j}^{n}+U_{i,j-1}^{n}}{(\Delta y)^2}\right] -\frac{b}{2}\left(U_{i,j}^{n+1/2}+U_{i,j}^{n}\right)+\frac{bU}{2}^{\oplus} \tag{4.28}$$

In the second time half-time step, $\partial^2 U/\partial x^2$ is expressed at the start $n+1/2$ and $\partial^2 U/\partial y^2$ is expressed at the end $n+1$. Therefore:

$$\frac{U_{i,j}^{n+1}-U_{i,j}^{n+1/2}}{\Delta t/2}=\frac{c}{2}\left(\frac{U_{i+1,j}^{n+1/2}-2U_{i,j}^{n+1/2}+U_{i-1,j}^{n+1/2}}{(\Delta x)^2}+\frac{U_{i,j+1}^{n+1}-2U_{i,j}^{n+1}+U_{i,j-1}^{n+1}}{(\Delta y)^2}\right) -\frac{b}{2}\left(U_{i,j}^{n+1}+U_{i,j}^{n+1/2}\right)+\frac{bU}{2}^{\oplus} \tag{4.29}$$

to see why the spatial derivatives can be written at different time in the two half-time steps in Eq. (4.28) and Eq. (4.29), we add them to get:

$$\frac{U_{i,j}^{n+1}-U_{i,j}^{n}}{2(\Delta t/2)}=c\left[\frac{U_{i+1,j}^{n+1/2}-2U_{i,j}^{n+1/2}+U_{i-1,j}^{n+1/2}}{(\Delta x)^2}+\frac{1}{2}\left(\begin{array}{l}\dfrac{U_{i,j+1}^{n}-2U_{i,j}^{n}+U_{i,j-1}^{n}}{(\Delta y)^2}\\ +\dfrac{U_{i,j+1}^{n+1}-2U_{i,j}^{n+1}+U_{i,j-1}^{n+1}}{(\Delta y)^2}\end{array}\right)\right] -b\left(U_{i,j+1}^{n+1}+U_{i,j}^{n}\right)+bU^{\oplus} \tag{4.30}$$

this shows that by going through the two half-time steps, the Bio-heat equation is effectively represented at the half-time step $n+1/2$, using central difference for the time derivative, central difference for the *x*-derivatives, and central difference for the *y*-derivative by averaging at the $(n+1/2)^{th}$ and $(n+1)^{th}$ step. The ADI method for one complete time step is thus second-order accurate in both time and space. Re-arranging Eq. (4.29) and Eq. (4.30), we get:

$$-F_x U_{i-1,j}^{n+1/2}+(2F_x+b\Delta t+2)U_{i,j}^{n+1/2}-F_x U_{i+1,j}^{n+1/2}=F_y U_{i,j-1}^{n}+(-2F_y-b\Delta t+2)U_{i,j}^{n} +F_y U_{i,j+1}^{n}+b\Delta t U^{\oplus} \tag{4.31}$$

let $a=(2F_x+b\Delta t+2),\ b=c=-F_x$. For various values of i and j, Eq. (4.31) can be written in a more compact matrix form at the $(n+1/2)^{th}$ time level as:

$$AU_j^{(n+1/2)}=f_n,\quad j=1,2,\ldots n. \tag{4.32}$$

where $U=(U_{1,j},U_{2,j},\ldots,U_{m,j})^T$, $f=(f_{1,j},f_{2,j},\ldots f_{m,j})^T$

at the $(n+1)^{th}$ time level, sub-iteration 2 is given by:

$$-F_yU_{i,j-1}^{n+1}+(2F_y+b\Delta t+2)U_{i,j}^{n+1}-F_yU_{i,j+1}^{n+1}=F_xU_{i-1,j}^{n+1/2}+(-2F_x-b\Delta t+2)U_{i,j}^{n+1/2}$$
$$+F_xU_{i+1,j}^{n+1/2}+b\Delta tU^{\oplus} \qquad (4.33)$$

let $a=(2F_y+b\Delta t+2),\ b=c=-F_y$. For various values of i and j, Eq. (4.33) can be written in a more compact matrix form as:

$$BU_i^{(n+1)}=g_{n+1/2}, \qquad i=1,2,\ldots m \qquad (4.34)$$

where $U_i^{(n+1)}=(U_{i,1},U_{i,2},\ldots,U_{i,n})^T$, $g=(g_{i,1},g_{i,2},\ldots,g_{i,n})^T$

and $F_x=c\Delta t/(\Delta x)^2$, $F_y=c\Delta t/(\Delta y)^2$.

4.8 IADE-DY (2-D Bio-Heat)

The matrices derived from the discretization resulting to A in Eq. (4.32) and B in Eq. (4.34) are respectively tri-diagonal of size (*mxm*) and (*nxn*). Hence, at each of the $(n+1/2)^{th}$ and $(n+1)^{th}$ time levels, these matrices can be decomposed into $G_1+G_2-\frac{1}{6}G_1G_2$, where G_1 *and* G_2 are lower and upper bi-diagonal matrices given respectively by

$$G_1=[l_i,1], \qquad and \quad G_2=[e_i,u_i], \qquad (4.35)$$

where

$$e_1=\frac{6}{5}(a-1),\ u_i=\frac{6}{5}b,\ e_{i+1}=\frac{6}{5}(a+\frac{1}{6}l_iu_i-1),\ l_i=\frac{6c}{6-e_i}\ (e_i\neq 6)\ i=1,2,\ldots,m-1$$

Hence, by taking *p* as an iteration index, and for a fixed acceleration parameter *r > 0*, the two-stage IADE-DY scheme of the form:

$$(rI + G_1)u^{(p+1/2)} = (rI - gG_1)(rI - gG_2)u^{(p)} + hf \quad \textit{and}$$
$$(rI + G_2)u^{(p+1)} = u^{(p+1/2)} \tag{4.36}$$

can be applied on each of the sweeps Eq. (4.32) and Eq. (4.34). Based on the fractional splitting strategy of D'Yakonov, the iterative procedure is accuracy, and is found to be stable and convergent. By carrying out the relevant multiplications in Eq. (4.36), the following equations for computation at each of the intermediate levels are obtained:

(i) at the $(p+1/2)^{th}$ iterate,

$$\begin{aligned}
u_1^{(p+1/2)} &= \frac{1}{\hat{d}}(s_1 \hat{s} u_1^{(p)} + w_1 \hat{s} u_2^{(p)} + hf_1) \\
u_i^{(p+1/2)} &= \frac{1}{\hat{d}}(-l_{i-1}u_{i-1}^{(p+1/2)} + v_{i-1}s_{i-1}u_{i-1}^{(p)} + (v_{i-1}w_{i-1} + s_i \hat{s})u_i^{(p)} + w_i \hat{s} u_{i+1}^{(p)} + hf_i), \\
& i = 2,3,\ldots,m-1 \\
u_m^{(p+1/2)} &= \frac{1}{\hat{d}}(-l_{m-1}u_{m-1}^{(p+1/2)} + v_{m-1}s_{m-1}u_{m-1}^{(p)} + (v_{m-1}w_{m-1} + s_m \hat{s})u_m^{(p)} + hf_m)
\end{aligned} \tag{4.37}$$

where,

$g = \frac{6+r}{6}$, $h = \frac{r(12+r)}{6}$, $\hat{d} = 1 + r$, $\hat{s} = r - g$, $s_i = r - ge_i$, $i = 1,2,\ldots,m$ and

$v_i = -gl_i$, $w_i = -gu_i \quad i = 1,2,\ldots,m-1$.

(ii) at the $(p+1)^{th}$ iterate,

$$\begin{aligned}
u_m^{(p+1)} &= \frac{u_m^{(p+1/2)}}{d_m}, \\
u_i^{(p+1)} &= \frac{1}{d_i}(u_i^{(p+1/2)} - \hat{u}_i u_{i+1}^{(p+1)}), \textit{ where } d_i = r + e_i,\ i = m-1, m-2, \ldots, 2, 1
\end{aligned} \tag{4.38}$$

4.9 MF-DS (2-D Bio-Heat)

The numerical representation of Eq. (4.31) and Eq. (4.33) using the Mitchell and Fairweather scheme is as follows:

$$\left(1-\frac{1}{2}\left(F_x-\frac{1}{6}\right)\delta_x^2\right)U_{i,j}^{n+1/2}=\left(1+\frac{1}{2}\left(F_y+\frac{1}{6}\right)\delta_y^2\right)U_{i,j}^{n}-b\Delta tU_{i,j} \tag{4.39}$$

$$\left(1-\frac{1}{2}\left(F_y-\frac{1}{6}\right)\delta_y^2\right)U_{i,j}^{n+1}=\left(1+\frac{1}{2}\left(F_x+\frac{1}{6}\right)\delta_x^2\right)U_{i,j}^{n+1/2} \tag{4.40}$$

the horizontal sweep Eq. (4.39) and the vertical sweep Eq. (4.40) formulas can be manipulated and written in a compact matrix form. Let $a=\frac{5}{6}+F_x,\ b=c=\frac{1}{12}-\frac{F_x}{2}$, for various values of i and j, Eq. (4.39) can be written in a more compact matrix form $(n+1/2)$ time level as in (4.31). Similarly, at the $(n+1)$ time level $a=\frac{5}{6}+F_y,\ b=c=\frac{1}{12}-\frac{F_y}{2}$, for various values of i and j then Eq. (4.40) can be written in a more compact matrix form as in Eq. (4.34). By definition the resulting tri-diagonal system of equations are solved using similar iterative procedure as in the DS-PR, that is, the two-stage IADE-DY algorithm.