CHAPTER 3

NUMERICAL MODEL IN PRESENT STUDY

3.0 GOVERNING EQUATIONS

The physical behaviour of water bodies under the influence of tidal and other forces could be described in terms of a set of hydrodynamic equations (Lamb, 1932, Gill, 1982). In general, hydrodynamical models describe the response of variable density water to momentum and heat forcing on a rotating Earth and may be derived from the generalised Navier-Stokes equations.

The Navier-Stokes equations may be simplified to a set of two-dimensional depth-integrated equations of continuity and momentum by considering the water to have a uniform density and assuming the hydrostatic approximation to be valid for the moving fluids. Simplifications are also made by assuming the advective (nonlinear) terms in the momentum equations, the horizontal and vertical shear stresses to be negligible. Hence, the governing equations are as follows:-

Continuity equation

$$\xi_t + (hU)_x + (hV)_y = 0 \qquad (3.1)$$

Momentum equation in x-direction

$$h(U_t - fV) + gh\xi_x + \sigma^{(x)} = 0 \qquad (3.2)$$

Momentum equation in y-direction

$$h(V_t + fU) + gh\xi_y + \sigma^{(y)} = 0 \tag{3.3}$$

A normal Cartesian coordinate system is adopted, where x, y are the horizontal coordinates, t the time and the subscripts indicate derivatives. The undisturbed depth of the water at position (x, y) is denoted by h(x, y), the elevation of the free surface above its undisturbed position is denoted by $\xi(x, y, t)$ and the depth-averaged components of water velocity in the x and y directions are denoted by U(x, y, t) and V(x, y, t) respectively. f is the Coriolis parameter and $\sigma^{(x)}$, $\sigma^{(y)}$ are the components of bottom friction in the x and y directions respectively.

The bottom friction stress components may be assumed to have the usual quadratic forms,

$$\left(\sigma^{(x)}, \sigma^{(y)}\right) = k\sqrt{U^2 + V^2}\,(U, V) \tag{3.4}$$

where k(x,y) is the dimensionless bottom friction coefficient. This drag coefficient is often written in terms of Chezy coefficient C as $k = g/C^2$.

The Coriolis terms fU and fV may be neglected near the equator and taking P = hU and Q = hV, where P and Q are the components of volume flux per unit area in the x and y directions respectively, the above equations could be re-expressed as follows:

$$\xi_t + P_x + Q_y = 0 \tag{3.5}$$

$$P_t + gh\xi_x + \sigma^{(x)} = 0 \tag{3.6}$$

$$Q_t + gh\xi_y + \sigma^{(y)} = 0 \tag{3.7}$$

$$\left(\sigma^{(x)}, \sigma^{(y)}\right) = kh^{-2}\sqrt{P^2 + Q^2}\,(P, Q) \tag{3.8}$$

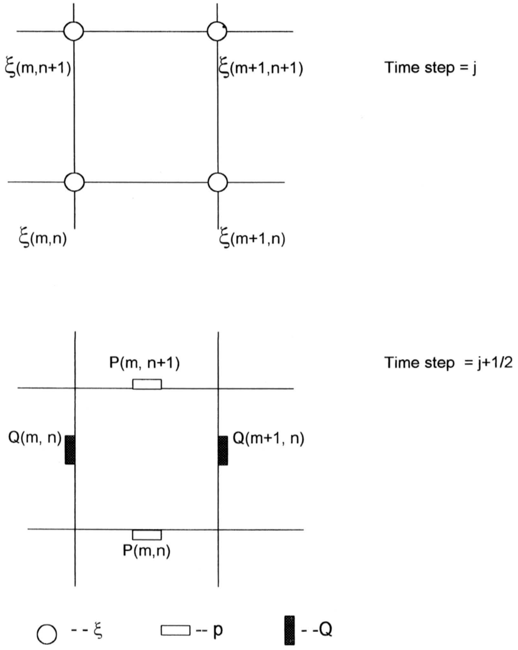
3.1 NUMERICAL FORMULATION

3.1.1 DISCRETIZATION SCHEME

Finite difference method is used as the discretization scheme in the solution. The finite difference scheme used here is based on the leapfrog method where time and space are staggered. The spatial grid is identical to the C-grid of Arakawa as shown in Figure 3.1. Arakawa C-grid is a composite grid whereby three variables are solved for. It is composed of three sub-grids; the first sub-grid is a mesh for variable ξ (free surface height), the second is a mesh for variable P (west-east transport), and the third is a mesh for variable Q (south-north transport). The overlay of the three meshes with the respective shifts comprises the composite grid. In a C grid, quantities such as ξ and h are defined at the centre of the grid, while the west-east component of velocity is displaced half a grid to the east of the centre and the south-north component displaced half a grid to the north of the centre.

The variables ξ and (P, Q) are taken at alternating half steps in the time direction. At time step j, the value of ξ at grid point (m, n) is denoted by $\xi^j_{m,n}$ while at time step j + 1/2, the value of P at grid point (m+1/2, n) is denoted by $P^j_{m,n}$ and the value of Q at grid point (m, n+ 1/2) is denoted by $Q^{j}_{m,n}$. This scheme has previously been used for the tidal equations by Lardner and Smocznski, 1990.

Figure 3.1: The staggered Arakawa C-grid with the location of model variables indicated.



3.1.2 EXPLICIT TIME INTEGRATION

The numerical schemes for solving time-dependent partial differential equations fall generally into two classes, explicit or implicit. The term "explicit " denotes a scheme where all terms on the right hand sides of a system of equations are evaluated at previous time steps i.e. at any given time t^j the right hand sides are known from prêvious time steps. Explicit time marching scheme is adopted in this study because it does not involve matrix inversion or iterative solver, hence the solution method is easy to perform and less prone to error in programming.

3.1.3 FINITE DIFFERENCE EQUATIONS AND SOLUTION METHOD

The finite-difference approximation to equation (3.5), centred at the space-time grid point (m,n, j+1/2), is then

$$\frac{\xi^{j+1}_{m,n} - \xi^{j}_{m,n}}{\Delta t} + \frac{P^{j}_{m,n} - P^{j}_{m-1,n}}{\Delta x} + \frac{Q^{j}_{m,n} - Q^{j}_{m,n-1}}{\Delta y} = 0 \tag{3.9}$$

where Δx, Δy and Δt are the spatial and temporal grid spacings.

The finite difference approximations to equations (3.6) and (3.7) are next obtained. Equation (3.6) centred at the space-time point (m+1/2, n, j+1) is

$$\frac{P^{j+1}_{m,n} - P^{j}_{m,n}}{\Delta t} + gh_{m+\frac{1}{2},n} \frac{\left(\xi^{j+1}_{m+1,n} - \xi^{j+1}_{m,n}\right)}{\Delta x} + k_{m+\frac{1}{2},n}\, r^{j}_{m,n}\{\alpha P^{j+1}_{m,n} + (1-\alpha)P^{j}_{m,n}\} = 0 \tag{3.10}$$

Equation (3.7) centred at the space-time point (m, n+1/2 , j+1) is

$$\frac{Q^{j+1}_{m,n} - Q^{j}_{m,n}}{\Delta t} + gh_{m,n+\frac{1}{2}} \frac{\left(\xi^{j+1}_{m,n+1} - \xi^{j+1}_{m,n}\right)}{\Delta x} + k_{m,n+\frac{1}{2}} s^{j}_{m,n}\{\alpha Q^{j+1}_{m,n} + (1-\alpha)Q^{j}_{m,n}\} = 0 \tag{3.11}$$

where $r^{j}_{m,n} = \sqrt{\left(P^{j}_{m,n}\right)^2 + \left(\overline{Q^{j}_{m,n}}\right)^2}$

$$s^{j}_{m,n} = \sqrt{\left(\overline{P^{j}_{m,n}}\right)^2 + \left(Q^{j}_{m,n}\right)^2} \tag{3.12}$$

The four-point averages used in these expressions are:

$$\overline{P^{j}_{m,n}} = \frac{1}{4}(P^{j}_{m,n} + P^{j}_{m-1,n} + P^{j}_{m,n+1} + P^{j}_{m-1,\,n+1}) \tag{3.13}$$

$$\overline{Q^{j}_{m,n}} = \frac{1}{4}(Q^{j}_{m,n} + Q^{j}_{m+1,n} + Q^{j}_{m,n-1} + Q^{j}_{m+1,\,n-1}) \tag{3.14}$$

In the discretizations of (3.10) and (3.11), the bottom friction terms are treated semi-implicitly in order to improve stability. The implicitness parameter is denoted by α and in practice $\alpha = 0.5$ is taken to give a centred average for this factor.

Equation (3.9) may be solved explicitly for $\xi^{j+1}_{m,n}$ and equations (3.10) and (3.11) may be solved explicitly for $P^{j+1}_{m,n}$ and $Q^{j+1}_{m,n}$. In solution, equation (3.10) is solved first for $P^{j+1}_{m,n}$ on even time steps; while on odd time steps equation (3.11) is solved first for $Q^{j+1}_{m,n}$; and in each case the latest values of the other variable are used in the next calculation.

3.2 BOUNDARY CONDITIONS

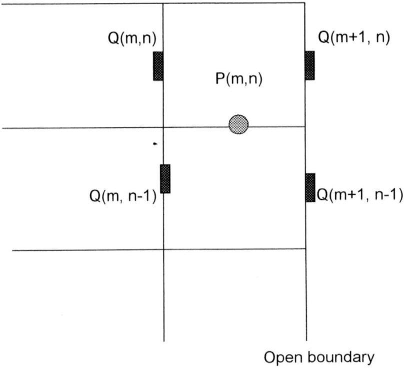
Boundary conditions are imposed on the discrete equations in the Arakawa C-grid as used by Leendertse, 1967. The grid is positioned so that coastal boundaries that run in the y direction pass through the P points with the values of these points being set to zero. The coastal boundaries that run in the x direction pass through the Q points with the corresponding values being set equal to zero. The above is based on the assumption that the coastline is impervious. Open boundaries are positioned in order to pass through the ξ points, at which the values of ξ are set to the prescribed tidal elevations.

Some modifications are made in the averages of (3.13) and (3.14) when the velocity point is adjacent to an open boundary. A typical situation is illustrated for P(m,n) in Figure 3.2. In forming $\overline{Q_{m,n}}$, $Q_{m+1,n}$ and $Q_{m+1,n-1}$ are not available, so only one sided average could be used as follows:

$$\overline{Q^j_{m,n}} = \frac{1}{2}\left(Q^j_{m,n} + Q^j_{m,n-1}\right) \tag{3.15}$$

Similar one-sided averages occur adjacent to any open boundary. This is equivalent to enforcing the Neumann type of boundary conditions where derivatives are zero.

Figure 3.2 Example of a P point adjacent to a right-hand open boundary. For $\overline{Q^j_{m,n}}$ at the P point, only a two-point average of $Q^j_{m,n}$ and $Q^j_{m,n-1}$ is used.



3.3 INITIAL STATE

Solutions are generated from a calm initial condition, by setting $\xi^0_{m,n} = P^0_{m,n} = Q^0_{m,n} = 0$ and imposing tidal forcing along the open boundaries by prescribing the tidal elevations along the open boundaries. The values of the main tidal constituents along the open boundaries are approximated by a suitable interpolating function between onshore sea level record at opposite ends of each open boundary. After a sufficient long period, the initial transient in the solution gradually decays and a periodic solution is obtained which can then be analysed harmonically to give the predicted results as presented later in the next chapter.

3.4 APPLICATION OF THE MODEL TO THE STRAITS OF MALACCA

3.4.1 SPATIAL GRID SYSTEM

In general, the spatial grid should be fine enough to be able to represent the variation of shoreline and depth configurations reasonably well for achieving more accurate computed values, thus the accuracy of the computed results would depend on the resolution of the computing grid. Since smaller grid sizes would require more computation time and memory, a good balance between accuracy and resource requirement is needed for programme designed for small computers.

The grid system of the two dimensional model for the Straits of Malacca is set up by rotating through 45^0 with respect to the geographical grid system (Lee, 1994). The grid sizes in the horizontal and vertical directions are uniformly distributed at 3.353 minutes or equivalent to 6748 meter. This results in a mesh of 72 X 152 as shown in Figure 3.3, with open boundaries across the northern part of the Straits between the grid points (2, 2) and (2, 68) corresponding to the left hand side and between the grid points (151, 38) and (151, 44) across the southern part of the Straits corresponding to the right hand side. The above grid sizes of Δx and Δy represent a compromise between accuracy and computer resource requirements.

3.4.2 TIME STEP/ TEMPORAL GRID

Though trying to avoid using an implicit scheme which requires matrix inversion to solve the values at t^{j+1}, the price has to be paid by being restricted in the time step that could be taken. Since the difference scheme is explicit, Δt must be chosen so that

$$\Delta t \leq \frac{1}{\sqrt{gh(\frac{1}{\Delta x})^2 + (\frac{1}{\Delta y})^2}} \quad (3.16)$$

This is the Courant-Friedrichs-Levy condition for numerical stability of the scheme, which could only be avoided by employing an implicit scheme.

Typically, for the Straits of Malacca tidal model, $\Delta x = \Delta y = 6748$ meter, maximum h is about 1000 m, so that Δt must be less than 40 seconds. The time step is taken to be 30 seconds in this study. Of course time step could be made lower for better sampling accuracy but it would require more computer time. The period of seiche oscillations for the Straits is about 12 hours, so that if the computation starts from flat conditions, it requires a minimum of about 24 hours for the effect of the initial values to become reasonably small. The spin-up interval is usually taken to be several days to make sure the effect of initial values is damped.

3.4.3 BATHYMETRY DATA

Bottom depth is one of the most important parameters for a realistic tidal model. Bottom depth is derived from acoustic sounding from a ship. Naturally, regions well traveled by ships are covered reasonably well. Bottom depth data in this study is extracted from the bathymetry Chart 1335 (Real-admiralty G.P.D.Hall, 1992) and specialised for the computational mesh as shown in Figure 3.4.

The depth data from random points are linearly interpolated to get the depth at the grid points. Generally these values of depths are not very precise. The charts, being designed primarily for ship navigation, give the minimum depth (the shallowness) at each location rather than an average depth. Thus, these values tend to be underestimated.

3.4.4 ESTIMATION OF BOTTOM FRICTION COEFFICIENT

Many of the modelling efforts in the Straits of Malacca have employed a nonlinear (quadratic) form of bottom stress where the value of bottom friction coefficient, k is taken to be either 0.00333 (C=54.5 $m^{1/2}s^{-1}$) by Lee, 1994, and 0.0025 (C= 62.5 $m^{1/2}s^{-1}$) by Hadi, 1992. These values were assumed constant for the entire modelling domain and were generally determined during computation of tides from their transient behaviour so as to give the best fit. Often the results could not be optimised due to the very large variation of coastline and depth in the Straits. Hence, appropriate specification of drag coefficient should be re-examined.

In the present study, the quadratic formulation of bottom friction in equation (3.8) partially takes into account the bathymetry that it is space varying and depth dependent. In view of the fact that the drag coefficient is poorly known, it is the aim of this study to estimate the value of C that minimises the errors and to investigate its dependence on depth. Other important position dependent parameters namely the bottom depth distribution and open boundary conditions are also considered.

3.5 COMPUTER MODEL.

A Fortran programme based on the numerical formulations discussed above has been written and tested to work properly. This model could be run on any IBM compatible PC with minimum of 32MB memory. The programme should be run using the Microsoft Fortran (Power Station) environment.

The input/ output specifications are as follows:

Input files:

Input.d - parameter input data file.

Obs.d- file containing observed data at selected tidal stations.

Output files:

Cur.d - File containing the current data

Elev.d - File containing tidal elevations data.

Plot.d- File containing data for plotting of co-range, co-tidal and current vector.

Figure 3.3 Numerical mesh of the model

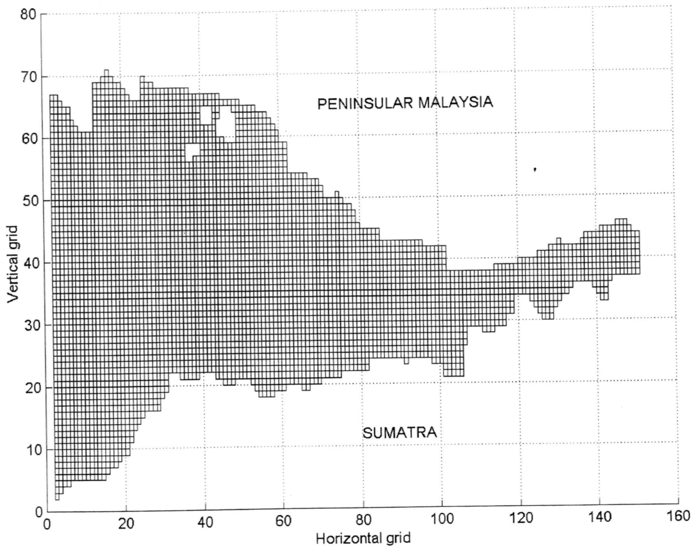


Figure 3.4 Bathymetry map of the Straits of Malacca.