CHAPTER 2
REVIEW OF RELATED LITERATURE

2.1 Introduction

Several research studies have been carried out on the students’ understanding of algebraic notation (Booth, 1984; Comstock, 1986; Harper, 1980; Kuchemann, 1981a; MacGregor & Stacey, 1997) as well as the students’ cognitive levels of development (Lawson, 1978; Lawson & Blake, 1976; Lew, 1987; Lee, 1991; Ng, 1991; Palanisamy, 1986; Shemesh, Eckstein, & Lazarowitz, 1992; Valanides, 1996). This chapter begins with a review of related literature on the uses of variables in algebra. This is followed by studies on definitions of variables, interpretation of letters, and the conceptual difficulties in algebra. The cognitive development is discussed under cognitive development of secondary school students as well as cognitive development and achievement in mathematics.

2.2 Uses of Variables in Algebra

Variables are the basic in algebra. They provide the algebraic tool for expressing generalisations. In fact, variables and algebra are inextricably related that school algebra is defined as that which has to do with the understanding of letters. In addition, students are considered to be studying algebra when they first encounter variables (Usiskin, 1988). Therefore, an understanding of the concept of variable is crucial to success in algebra (Comstock, 1986). Unfortunately, the concept of
variable frequently turns out to be the stumbling block of the students' success in algebra (Leitzel, 1989).

In school algebra, variables have many uses. Driscoll (1983) considered the various purposes of algebra as the different faces of algebra. Usiskin (1988) had identified some uses of variables and their relations to the different purposes of algebra. According to Usiskin, firstly, algebra is considered as generalised arithmetic and variables are viewed as pattern generalizers. The algebraic skills referred to translating and generalising known relationships among numbers and variables. An example of variables as pattern generalisers is in generalising $3 + 5.7 = 5.7 + 3$ to the pattern $a + b = b + a$. Secondly, algebra is considered as the study of procedures for solving certain kinds of problems and variables are viewed as unknowns or constants. The algebraic skills involve simplifying and solving, for example, in translating a word problem into an equation. Thirdly, algebra is considered as the study of relationships among quantities and variables are either arguments or parameters and co-ordinate graphs are often used to represent these relationships. An example of the third use of variables is to find an equation for the line through $(6, 2)$ with slope $11$. Finally, algebra is considered as the study of structures such as groups, rings, integral domains, fields, and vector spaces. Variables are arbitrary objects in a structure related by certain properties. The algebraic skills involve manipulating and justifying.
2.3 Definitions of Variables

Although there is a consensus among mathematics educators on the necessity of understanding the concept of variable for a successful grasp on algebra, there is no clear consensus on the definition of it. This is because the conceptions of variable change over time. Usiskin (1988) reported that variable is described, firstly, as a changing number and then more formally, as a literal number that may have two or more values during a particular discussion.

Some definitions of variable as defined by some mathematicians and mathematics educators were compiled by Comstock (1986). They were:

1. A variable is a symbol, using a single letter such as $a$ or $n$ or $x$, that stands for a number.

2. A variable is a letter, such as $x$, $y$, or $z$, that represent a number.

3. A variable is a symbol (generally a lowercase letter) that represents an unspecified element of a set containing more than one element.

4. A letter, which is used to represent any element of a given set, is sometimes called a variable.

5. A variable is a symbol or is represented by a symbol, usually a letter of the alphabet. A variable stands for a number from some set of numbers.

According to Kieran (1989), all the algebraic letters are referred to as variables. However, she cautioned that such an overly general concept of variable was the cause of confusion faced by students. She explained that treating symbolic constants, parameters, unknowns, and unconstrained variables as simply variables
enhances their common abstract nature. This caused difficulties to students in manipulating the symbolic value. Similarly, Noss (1986) agreed that the concept of variable was a potential confusion, with a letter standing for a parameter, a variable or a specific unknown.

2.4 Interpretation of Letters

According to Kieran (1989), one of the areas of students' difficulties with the topic of algebra has been found to centre on the meaning of letters. The different ways that a single letter variable can be used often lead to erroneous interpretations. Therefore, several research studies in algebra had focussed on how the students interpret the letter.

The 'Concepts in Secondary Mathematics and Science' [CSMS] project had set out to determine how difficult given concepts in algebra appeared to be for students at secondary schools. It aimed to delineate a hierarchy of levels of understanding of these concepts. The algebra test was administered to over 3000 children of ages 13 to 15 years. Kuchemann (1981a) reported that six different categories of interpreting and using the letters by this large representative sample of children were identified. These categories and examples of questions are listed below:

1. Letter evaluated - "What can you say about M if M = 3N + 1 and N = 4?"

2. Letter not used or ignored - "If \( A + B = 43 \), \( A + B + 2 = ? \)"

3. Letter used as an object - "\( 2A + 5A = ? \)"

4. Letter used as a specific unknown - "Add 4 onto \( 3N \)."
5. Letter used as a generalised number - "What can you say about C if \( C + D = 10 \) and C is less than D?"

6. Letter used as a variable - "Which is larger, 2N or N + 2?"

The first three categories indicated a lower level of interpretation whereas the last three categories were the higher level of interpretation. Kuchemann concluded that most of the 13 to 15-year-olds tested were unable to cope with items that required higher levels of interpretation such as interpreting letters as generalised numbers or even as specific unknowns. The majority of the children (73.0% of 13-year-olds, 59.0% of 14-year-olds, and 53.0% of 15-year-olds) either treated letters as concrete objects or ignored them.

Kuchemann classified his subjects into four algebra levels, from Level 1 to 4. These algebra levels correspond to the Piagetian sub-stages as listed:

- Level 1: Below late concrete
- Level 2: Late-concrete
- Level 3: Early-formal
- Level 4: Late-formal

The findings of the CSMS research on the percentage of children at each algebra levels are as shown in Table 2.1. The results indicated a progressive improvement from 13 to 15-year-olds in terms of the attainment of algebra levels.
Table 2.1
Percentage of Children at Each Algebra Level

<table>
<thead>
<tr>
<th>Level</th>
<th>13 years</th>
<th>14 years</th>
<th>15 years</th>
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</thead>
<tbody>
<tr>
<td>Level 0</td>
<td>10</td>
<td>6</td>
<td>5</td>
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<tr>
<td>Level 1 (4/6 items)</td>
<td>50</td>
<td>35</td>
<td>30</td>
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<tr>
<td>Level 2 (5/7 items)</td>
<td>23</td>
<td>24</td>
<td>23</td>
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<tr>
<td>Level 3 (5/8 items)</td>
<td>15</td>
<td>29</td>
<td>31</td>
</tr>
<tr>
<td>Level 4 (6/9 items)</td>
<td>2</td>
<td>6</td>
<td>9</td>
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</table>


Similarly, MacGregor and Stacey (1997) conducted a study of students' interpretations of letters and the causes of difficulties in using algebraic notation. Their data were obtained from a paper and pencil test given to approximately 2000 students in Years 7 to 10 (ages 11 to 15) in 24 secondary schools. The Year 7 students' interpretations of the algebraic letters were divided into six categories as listed below:

1. Letter ignored.
2. Numerical value.
3. Abbreviated word.
4. Alphabetical value.
5. Use of different letter.
6. Unknown quantity.
The first three categories correspond to Kuchemann's (1981a) lower levels of interpretations. The last category corresponds to Kuchemann's higher level of interpretations as specific unknown and generalised number. However, Kuchemann's hierarchy did not explicitly include the other two interpretations, that is, alphabetical value and use of different letter.

In the category of the use of different letter, MacGregor and Stacey (1997) explained that the subjects would normally choose another letter or adjacent letter. For example, to find David's height, who was 10 cm taller than Con and given Con's height was h cm, the students would write t or g for David's height. In the category of giving alphabetical value to find David's height, the students added 10 to 8 because h is the eighth letter of the alphabet and derived at 18 for David's height.

As a follow-up to the CSMS study, the 'Strategies and Errors in Secondary Mathematics' (Booth, 1984) programme investigated the reasons underlying errors in algebra which the CSMS project had shown to be widely prevalent among second to fourth year students in English secondary schools. In the first phase of the study, interviews were conducted on approximately 50 students in their second, third and fourth years of secondary schooling (ages 13 to 15 years). The findings of the study revealed the meanings children attached to letters, as follows:

1. Letter as representing object. For instance, "5q - 3 w" could be "5 bananas minus 3 apples".

2. Different letters representing different numbers. For instance, to the question, "when is x + y + z = x + p + z true?", the answer given is "p has
to have different value from \( y \)..... you put a different letter for every different value”.

3. Letters representing whole numbers.

4. A pattern in the relationship between letters and the numbers they represent.

   For example, i. \( x, y, z \) to represent 3, 4, 5 or 10, 20, 30,

   ii. \( y \) is “higher” than \( p \),

   iii. fixed alphabetic substitution.

5. Meaning of letter ignored and problems treated as mere manipulation of symbols and rules:

   i. Add up all the numbers, then put down each letter that occurs (once only). For example, \( 3 + 5y = 8y, 2x + 8y + 3x = 13 \ xy \).

   ii. Add up all the numbers, then put letter for every time it occurs in expression. For example, \( 2x + 8y + 3x = 13 \ xy \).

   iii. Add up all the numbers, then put down letter that occurs most often.

       For example, \( 2x + 8y + 3x = 13 \ x \).

The results showed that a high proportion of the subjects interviewed perceived letters to represent whole numbers (.75) and different letters to represent different numbers (.74)

In another study on the variety of interpretations given to letters, Harper (1980) interviewed 120 pupils in Years 1 to 5 from two grammar schools. Using four tasks devised by the researcher himself, the results indicated that two distinct conceptual understandings of the role played by a letter in relative to geometrical
data existed in pupils’ thinking. The two distinct types of responses were: (a) Letter interpreted as a ‘stand in’ or ‘cipher’ awaiting a true numerical content. The letter was an ‘arithmetical’ unknown. (b) Letter interpreted as a ‘general number’. The letter was used as an algebraic entity. The two types of letter usage roughly corresponded to Kuchemann’s (1981a) “specific unknown’ and ‘generalised number’.

Harper’s (1980) study showed that a higher number of students interpreted the letter as the first type of response. The study also showed that there was considerable variation at each age group in the degree of sophistication of letter usage. This indicated that a relationship might exist between the level of letter interpretation and successful solution of certain algebraic problems. The number of students who interpreted a letter as a general number was greater for the older year-groups (8 of 24 students in Year 5 compared to 2 of 24 students in Year 1).

In the case of investigating students’ ability to conserve equation and function under alphabetic transformations of literal variables, Wagner (1981) devised a conservation-of-equation task and three conservation-of function tasks. Clinical interviews were conducted on thirty students, half at middle school level (median age: 13 years) and half at the high school level (median age: 16 ½ years). They were shown pairs of equations in which the letter-variable was different but which were otherwise having the same solution, and were asked which would be larger. It was observed in this study that some of the students did not realise that the value of an unknown is independent of the letter used. Rather, they believed that changing the letter implied that the value was changed. In addition, some of the students
responded as though there was a correspondence between the linear ordering of
alphabet and number system, in that letters towards the end of the alphabet had a
higher value than those nearer the beginning. Wagner reported that less than half of
the students interviewed gave conserving responses to any one of the four tasks used
in the study, indicating the seriousness of this inability to conserve among students.

Another aspect is to consider the development of understanding of the
concept of variable. Comstock (1986) conducted clinical interviews on four
students, two boys and two girls of seventh grade. She reported that the students had
no difficulty accepting the idea that a letter represented a number. However, she
pointed out that two views of letter were evident: (a) a letter representing some
specific, though unknown number and (b) a letter represented any number. Students
who expressed the former view were able to use the letter to write a mathematical
phrase showing operations whereas students, who thought of a letter as any number,
found it difficult to use a letter to represent operations. Comstock also reported that
the less capable students were uneasy with the idea of a variable as representing an
unknown number. They searched for clues as to the number the variable represented
and would assign a number to the letter or simply ignore it.

As an alternative to teaching of variables, Noss (1986) created a
mathematical environment through Logo programming. One of the purposes of the
study was to investigate the extent to which the children could construct meaningful
symbolisation for the concept of algebra. Based on the study, eight children were
interviewed. The eight children were aged between 10 and 11 years and none of
them had studied any ‘formal’ algebra in their school.
He reported that two main aspects emerged from the interviews on the question of variable. The first involved the idea of naming and the second, the children's conception of variable as a generalised number. In the former, the readiness with which the children were prepared to name unknowns varied between items. This readiness is dependent on the nature of the unknown itself, the number of variables involved in the manipulation which gave rise to increased difficulty, and the unmeasurability of the unknowns. In the latter aspect on the variable as generalised number, the children were able to utilise explicit computer-based metaphors to aid in the process of using a single variable to stand for a range of numbers.

Loh (1991) conducted a study to determine the levels of ability in solving algebraic word problems among lower secondary school children. She administered an Algebra Test with 11 open-ended questions and administered it to 130 subjects. The performance on the Algebra Test enabled her to define a proficiency continuum, from Levels 1 to 4, in ascending order. Interviews were conducted on eight students, which revealed that students at lower scale performance levels did not fully understand the idea of a variable. The results also showed that students at Levels 1 and 2 treated variables as concrete objects while those at Levels 2 and 3 could not accept variables that take on negative values.

In Malaysia, Thayalarani (1998) employed clinical interviews on three Form Three students to identify their understanding on the concepts and operations involving unknowns. She used three types of activities for data collection. They were mental images, representation of unknowns, and methods of simplifying
algebraic expressions. Her findings revealed that the students perceived alphabets as quantities that were not known (for instance, the length of a ruler be x cm and “we do not know its value”), as numbers (for instance, in “x + x”, the student assigned “2” to x), and as objects (for instance, in “4p + 3p”, p represents pupils).

The literature reviewed showed that there are many meanings given to the letter used in algebraic expression and equation. Students with differing levels of understanding of algebraic notation gave different interpretations to the letter. A high proportion of secondary school students viewed a letter as representing an object rather than a number. Even among those who can grasp the notation of letters as numbers, they often have difficulty viewing a letter as standing for a range of numbers.

2.5 Conceptual Difficulties in Algebra

One of the main functions of algebra is to represent general relations and procedures in concise and unambiguous terms (Booth, 1989). Such representation serves two purposes. Firstly, it enables the application of these relations and procedures to a wide range of relevant problems. Secondly, it enables the combination of these relations to other algebraic entities to derive new relations, which may be useful in furthering the understanding of the original relations. To achieve this function of algebra, symbolic representation is used.

In the study of algebra, interpretation of letters is but one of the many difficulties which the children encountered. Wagner (1983) pointed out the dilemma of “letters are like numerals....(and)... words, only different” (p. 474). The
difficulties in algebra arise not only because of the meaning of letters but also because of the shift to a set of conventions different from those used in arithmetic (Kieran, 1989).

According to Bodin and Capponi (1996), "the arithmetic-algebra gap is a fundamental cause of learning difficulties because knowledge of algebra is built upon the foundation of already acquired arithmetical knowledge" (p. 587). This is in agreement with the findings of Herscovics and Linchevski (1994) who investigated the solution procedures used by seventh graders to solve first-degree equations in one unknown, prior to any formal instruction. They reported the existence of a cognitive gap between arithmetic and algebra, characterised as the students' inability to operate spontaneously with or on the unknowns.

In arithmetic, symbols like "+" and "=" are usually interpreted as actions to be performed. Hence "+" means to perform the operation and "=" means to write down the answer. Within the context of equation, Kieran (1981) showed that 12 to 14-year-old students regarded the equals sign as a unidirectional symbol proceeding a numerical answer. An arithmetic expression of 3 + 2 elicited a calculation to give the answer 5. But 3 + 2a could not be calculated until the value of a was known. In addition, many students could not accept 3 + 2a as the answer and this was described as the inability to accept the lack of closure (cited by Herscovics & Linchevski, 1994). This was also termed as the process-product dilemma by Sfard and Linchevski (1994) because 3 + 2a represented both the process by which the computation was carried out and also the product of that process. Sfard (1991) had
suggested that the students' inability to conceive the operational-structural (object-process) duality of specific symbols was the core of their difficulty.

Another area where the students' ideas on arithmetic can influence their performance in algebra is the use of parentheses. Kieran's (1992) study showed that children did not use parentheses and they performed computation according to the order of the written sequence of operations. Hence in algebra, they too ignored the need for parentheses or wrote incorrectly those algebraic expressions that required parentheses, for instance, \( p \times a + m \) instead of \( p \times (a + m) \). This difficulty was also found in the study by Linchevski and Livneh (1999) on sixth and seventh graders in Israel and in Canada.

Another cognitive obstacle in the learning of algebraic expressions was the different meanings associated with concatenation in algebra (cited by Chalouh & Herscovics, 1988). In arithmetic, the juxtaposition of two numbers denotes addition (for instance, \( 43 = 40 + 3 \)) whereas, in algebra, concatenation denotes multiplication (for instance, \( 4a = 4 \times a \)).

Numerous studies had indicated that one of the causes of errors in algebra is the students' tendency to create prototype rules by generalising taught systematic rules in order to extrapolate new rules that are incorrect (Fischbein, 1990; Kaur & Boey, 1994).

For example,

1. Correct rule as taught.

\[
a \times (b + c) = (a \times b) + (a \times c)
\]
2. Prototype created by generalising over operator signs.

\[ a * (b @ c) = (a * b) @ (a * c) \]

3. Incorrect rule created from prototype.

\[ a + (b x c) = (a + b) x (a + c) \]

The created "incorrect rule" is reported by Zainudin (1995) as a common mistake in the learning of algebra and had its source in wrongfully applying learnt knowledge on new situations.

Ekenstam and Nilsson (1979), in their study on 210 sixteen-year-old students in Sweden, found that although 82% of the students solved the equation 30/x = 6 correctly by inspection, only 48% were successful with a similar-structured example of 4/x = 3. This data indicated that students' use of informal methods in arithmetic hindered them from producing general statements in algebra. According to Lee and Wheeler (1989), ".... the track leading from arithmetic to algebra to be littered with procedural, linguistic, conceptual and epistemological obstacles" (p. 53).

Other than the cognitive gap between arithmetic and algebra, Tall and Thomas (1991) discussed the cognitive conflict between the implicit understanding of natural language and the symbolism of algebra, resulting in what they termed as parsing obstacle. In most countries, both algebra and natural language are spoken, written and read sequentially from left to right. For instance, 3x + 2 is both read and processed from left to right. However, the sequence of processing the algebraic expression, 2 + 3x, conflicted with the sequence of natural language. Although it was read from left to right as "two plus three x", but it was computed from right to left, with the product of 3 and x calculated before the sum. Therefore, it was not
surprising if a child, on reading the expression from left to right as \(2 + 3\) giving \(5\), and considered the expression to be the same as \(5x\). In addition, "\(ab\)" read as "\(a\) and \(b\)" may be interpreted as "\(a + b\)". This is how the sum of "\(x + y\)" was interpreted as a conjoined term, "\(xy\)" (Booth, 1984) or "\(0a\)" is written as "\(a\)" by adding "\(0\)" to "\(a\)" (Yee & Soon, 1996).

Another difficulty in learning to use algebraic notation had its origin in analogies with symbol systems used in everyday life, in other parts of mathematics or in other school subjects (MacGregor & Stacey, 1997). It was found that students wrote \(10h\) to mean, "add \(10\) to \(h\)" and \(1y\) to mean "take \(1\) from \(y\)", stemming from their knowledge of the Roman numeration system in which \(VI\) means "\(1\) more than \(5\)" and \(IV\) means "\(1\) less than \(5\)". The interpretation of letters as abbreviated word had its roots in the use of initial letters to denote the concepts in applied mathematics, for instances, \(A\) for area, \(m\) for mass, \(t\) for time. Such uses of letters reinforced the belief that letters in mathematical expressions and formulae stood for words or objects rather than numbers (Clement, 1982; Clement, Narode & Rosnick, 1981; Kaput & Sims-Knight, 1983; MacGregor & Stacey, 1993).

The studies reviewed above had shed some light on the difficulties and their causes in the learning of algebra. These difficulties can be brought about by the interference of students' previously acquired arithmetical knowledge, the cognitive conflict between the implicit understanding of natural language and the symbolism of algebra, as well as the analogies with symbols used in everyday life. Therefore, it is not surprising that algebra is considered as a source of considerable confusion and negative attitudes among students.
2.6 Cognitive Development of Secondary School Students

According to Piaget, a majority of children enter the stage of formal thinking at the age of 11 and reach an equilibrium stage at approximately 16 years of age. If this is true, then the majority of the secondary school students will be expected to be at the formal operational stage. However, a review of studies indicated otherwise. Research has shown that for many children, the process is much slower (Driscoll, 1983).

Three researches carried out in three different countries showed a high percentage of secondary school students are still in the concrete operational stage. In Cyprus, Valanides (1996) assessed the performance of 195 seventh-, eighth-, and ninth-grade students (with a mean age of 12.46 to 14.40 years) on the Test of Logical Thinking (TOLT). Based on the scores of TOLT, the subjects were categorised into concrete operational stage, transitional stage of cognitive development and formal operational stage. The results indicated that a small percentage of the total number of students (13.0%) has reached the formal operational stage, while 64.6% and 21.5% of the total number of students were categorised as concrete and transitional thinkers, respectively. The percentages of students at the concrete operational stage were 75.4%, 58.8%, and 59.9% for the seventh-, eighth-, and ninth-grade students, respectively. The percentages of students at the transitional stage of cognitive development were 15.4%, 25.0%, and 24.2% for seventh-, eighth-, and ninth-graders, respectively.
Shemesh, Eckstein, and Lazarowitz (1992) conducted a cross-sectional study of the cognitive development of 913 Israeli students in grades 7 through 12 (ages 13 through 18). They administered a test, which consisted of 12-graded tasks. The scores of the students were categorised into three levels: concrete thinkers, intermediate or transitional thinkers, and formal thinkers. The findings indicate that less than half of the students from grades 7 to 11 of ages 13 to 17 years have mastered the formal operational reasoning.

A study by Karplus and Karplus (1970) on the intellectual development of children in the upper elementary grades and in high school indicated a very low percentage of formal thinkers in this group. The reasoning ability of the children based on "The Island Puzzle", was categorised in a hierarchical order from pre-logical to concrete to abstract logic. The majority of the responses from the pupils in the seventh and ninth grades were classified in the pre-logical and concrete categories. None was categorised as abstract logic.

Three further studies by Lawson himself or with other researchers also indicated that many secondary school students are not functioning at formal operational level. Lawson (1978) assessed the distribution of Piagetian stages of thinking of 513 students from eighth-, ninth-, and tenth-graders whose mean ages were 14.1, 15.1, and 16.5 years, respectively. He administered a 15-item classroom test designed to measure the formal operational abilities of the students. He reported that 35.3% of the subjects responded at concrete level, 49.5% responded as transitional to formal operational and 15.5% as formal operational.
Lawson and Blake (1976) used three separate instruments to classify 68 high school biology students into concrete and formal operational levels. The subjects were aged between 14 years to 17 years. The three instruments comprised of (a) three Piagetian Tasks (the Pendulum, Bending Rods and Equilibrium in the Balance); (b) a biology content examination; and (c) a shortened version of the Longeot Test. The percentages of students classified into Concrete IIA, Concrete IIB, Formal IIIA, and Formal IIIB were found to be 2.0, 45.0, 50.0, and 3.0 percent respectively. Those determined by the biology content examination were 23.0, 42.0, 57.0 and 8.0 percent respectively, and those determined by the Longeot Test were 15.0, 42.0, 35.0, and 8.0 percent respectively. These data implied that the majority of the students were functioning at the Concrete IIB and Formal IIIA levels. Less than 10% of the students were at the Formal IIIB level, regardless of the instrument used.

Similarly, Lawson and Nordland (1976) administered eight conservation tasks and two formal tasks (separation of variables and equilibrium in the balance) to a sample of 96 seventh grade science students. The subject ranged in ages from 11.7 years to 14.4 years (mean age = 12.6 years). Ninety-six percent of the total population demonstrated conservation of quantities which measured the concrete-operational reasoning ability of the students. Based on the formal task of separation of variable, 69.7% were in Concrete IIA, 26.0% in Concrete IIB and 4.1% in Formal IIIA, while the results from the equilibrium balance task indicated that 79.1% were in Concrete IIA, 17.7% in Concrete IIB and 3.1% in Formal IIIA. None of the subjects was in Formal IIIB level on either of the formal tasks.
In Malaysia, some studies had also been carried out to measure the cognitive development of secondary school pupils. Lee (1991) identified the stages of cognitive development of 244 Form Two pupils (mean age: 14 years 3 months) from two secondary schools in Selangor. She administered the Longeot Reasoning Test to determine the pupils' cognitive development. The findings revealed that 34.5% were in Concrete IIA level, and 54.9% in Concrete IIB. Only about 6.0% and 5.0% of the total sample were at the Formal IIIA and Formal IIIB levels, respectively. Students at the concrete level constituted 89.4% of the total sample.

Using the same instrument, Ng (1991) assessed the cognitive stages of development of Form Five Science students. The results indicated that the percentage of students at Concrete IIA, Concrete IIB, Formal IIIA and Formal IIIB were 1.0, 45.9, 34.7 and 18.5 percent, respectively. This implied that a majority of the Form Five students were operating at the late concrete and early formal levels. Subjects at these two cognitive levels constituted 80.6% of the total sample. Earlier, Lew's (1987) study on the cognitive distribution of 218 Form Four Science students as measured by the Longeot Test indicated that 80.3% of the total sample were at the Concrete IIB and Formal IIIA levels.

Palanisamy (1986) determined the cognitive stages of development of the students using two Science Reasoning Tasks, Task II (volume and heaviness) and Task IV (equilibrium in the balance) on 295 Form Two (13 to 14 year old) and Form Four (15 to 16 years old) students. The findings revealed that the percentage of Form Two students at Concrete IIA, Concrete IIB, Formal IIIA, and Formal IIIB levels were 32.0, 61.0, 7.0 and 0 percent, respectively, while that of the Form Four students
were 3.7, 47.5, 21.0 and 27.8 percent respectively. The differences between the Form Two and Form Four students were significant.

Similar findings were also reported by Cheah (1984), who investigated the stages of cognitive development of 271 Malaysian Forms Four and Five students. He administered a paper-and-pencil cognitive attainment test. The results showed 39.9% of the subjects were classified as Concrete IIB operational thinkers, 3.7% Concrete IIA, 47.6% Formal IIIA and only 8.9% Formal IIIIB operational thinkers. These data implied that a majority of the students (87.5%) were functioning at the late concrete and early formal levels.

The findings discussed so far on the stages of cognitive development of secondary school students are not in total agreement with those stages as described by Piaget. The majority of the secondary school students are at the late concrete and early formal operational levels.

2.7 Cognitive Development and Achievement in Mathematics

Several studies had shown a significant difference between the cognitive development and achievement in mathematics. In Venezuela, Niaz (1989) investigated the relation between students' ability to translate sentences into algebraic equations, and formal operational reasoning. Fifty-four freshman science major students (mean age: 18.1 years) participated in his study. The instruments consisted of: (a) a slightly modified 15-item version of Lawson (1978) classroom Test of Formal Reasoning, (b) a four-item proportional reasoning test, (c) Test A which required the subjects to translate sentences into algebraic equation, and
(d) Test B that required the subjects to answer questions based on given equations or sentences. The results of the study revealed that students who lack proportional and formal operational reasoning (hypothetico-deductive reasoning) obtained lower scores in Tests A and B. A correlation coefficient between total scores of Tests A and B and formal operational reasoning is computed at $r = .51$ ($p < .05$). This study supported the hypothesis that students who lack formal operational reasoning may experience more problems in translation of algebraic equations.

Similarly, Lawson, Nordland and Devito (1975) administered three formal tasks on 71 college students. Their study indicted a positive correlation between the students’ cognitive development and their achievement in science (0.4) and mathematics (0.33).

In contrast to studies on college students, Vaidya and Chansky (1980) administered the Conservation Test Battery designed by Lunzer on 102 subjects from second, third and fourth grades of a suburban Philadelphia school. Their study to determine the relationship between operativity on Piagetian tasks and mathematics achievement indicated that operativity was significantly related ($F(1, 30) = 19.80, p < .01$) to mathematics achievement at the second grade level. High operational subjects obtained higher scores than low-operational subjects.

In the case of investigating the concrete and formal reasoning in school mathematics, Collis (1971) devised nine mathematics tests. The items varied on two dimensions: abstractness of the elements operated upon and the structure of the items. The tests were administered to 101 girls, from primary, junior high, and senior high secondary schools. The students ranged in age from 8 to 17 years. He
divided them into three age groups: 8 to 11 years, 12 to 14 years, and 15 to 17 years. The results revealed that items with both concrete elements and structure were successfully handled by all age groups, with an average proportion of students of more than .86. However, when either the structure or the elements were made more abstract, the average proportion of students of less than .49 in the youngest group was successful. Only the oldest group was successful when both the structure and element were made more abstract, with an average proportion of students to be .75.

In Malaysia, Palanisamy (1986) conducted a study to investigate the relationship between the cognitive development and acquisition of mathematical concepts of fractions as well as ratio and proportion on a sample of 295 Form Two and Form Four students. He administered two science-reasoning tasks (Task II and Task IV) to measure the cognitive levels of the students and also two mathematics tests. One of the tests was on fraction concepts and the other on ratio and proportion.

The results indicated that students at higher level of cognitive development achieved better in the mathematics tests. Students in the concrete level obtained a lower mean score compared to that of those in the formal level. The results also indicated that a relationship existed between cognitive levels and the levels of understanding in the mathematical concepts tested. Based on the scores of the two mathematics tests, the students were classified into levels of understanding in ascending order from Level 0 to 4. In the Fraction hierarchy, as well as the Ratio and Proportion hierarchy, more than 80% of the Form Two students at Concrete IIA were found in Levels 0, 1, and 2 of understanding. In contrast, more than 80% of them at the formal levels were at Level 3, and 4 for both the mathematical concepts.
A similar trend was observed for the Form Four students in both the Fraction and the Ratio and Proportion hierarchies. Results showed that students in Concrete IIA achieved mainly Levels 1 and 2 understanding, whereas the majority of formal level students attained Levels 3 and 4. A progressive increase in the students' levels of understanding was observed at the higher levels of cognitive development.

The above studies lent support to the hypothesis that mathematics achievement is significantly correlated to the cognitive development of the students, whether in primary schools, secondary schools or even in colleges.