Chapter 4

Fuzz Set Theory and Fuzzy Logic Control

4.1 Fuzzy logic

In conventional logic, a statement is either true or false, with nothing in between. This principle of true and false was formulated by Aristotle 2000 years ago and has dominated Western logic ever since. The idea that things must be either true or false in many cases is nonsense.

Fuzzy logic offers a better way of representing reality. In fuzzy logic, a statement is true in degrees, ranging from completely true through half-truth to completely true. Fuzzy logic is basically a multi-valued logic that allows intermediate values to be defined between conventional evaluations.

In this century, the basic idea of multi-valued logic has been explored to some extent by a number of mathematicians, but Prof. Lotfi Zadeh made real breakthrough for computer science at the University of California in Berkeley. He has published a paper on the theory of fuzzy set, which has given a rise to hundreds of papers on fuzzy mathematics, and fuzzy systems theory. Although, the theory of fuzzy set is invented in US but the rapid growth of this technology was started in Japan. The advances of Japanese in the field of fuzzy control have won the attention throughout the world. Fuzzy logic gives us a whole new approach to the mathematics of thinking.

Although fuzzy logic did not fall into the mainstream, there were still many researchers around the world dedicating themselves to this new research field. In late 1960s, many new fuzzy methods like fuzzy algorithms, fuzzy decision-making, and more were proposed.

4.2 Fuzzy Sets

In mathematics, the concept of set is very simple, but very important. A set is simply collection of things. The things can be anything. They either belong to the set or they not belong, similar to the idea in logic that the statements can be either true or false. In 1965, Prof. Lotfi Zadeh proposed the idea of a fuzzy set. A fuzzy set is an extension of a crisp set. The membership μA (x) of x of a crisp set A can be defined by:

$$\mu_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

A fuzzy set, introduced by Zadeh, is a set with membership function, which can take values ranging from 0 to 1:

Definition: A fuzzy set in a universe of discourse U is characterized by a membership function μA (x) that takes values in the interval [0, 1].

$$\mu A(x) \in [0, 1]$$

A fuzzy set A in universe of discourse U may be represented as a set of ordered pairs of a generic element x and its membership value, that is,

$$A = \{(x, \mu A(x)) \mid x \in U\}$$

When U is continuous, A is commonly written as

A = μA $(x)/x$

Where the integral sign does not denote integration; it denotes the collection of all points $x \in$ U with the associated membership function μA (x).

When U is discrete, A is commonly written as

A = μA $(x)/x$

Where is summation sign does not represent arithmetic addition; it denotes the collection of all points $x \in$ U with the associated membership function μA (x).

4.3 Fuzzy Set Operations

As operations are defined in classical sets, similar operations are defined on fuzzy sets. Assume that A and B are two fuzzy sets with membership functions μA (x) and μB (x):

The intersection of A and B is a fuzzy set $A \cap B$ in U with the following membership function:

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, x \in U$$

The union of A and B is a fuzzy set $A \cap B$ in U with the membership function:

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, x \in U$$

The complement of A is a fuzzy set Ā in U whose membership function is defined as

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x), x \in U$$

4.4 Linguistic Variables and Fuzzy IF-THEN Rules

If a variable can take words in natural languages as its values, it is called a linguistic variable, where the words are characterized by fuzzy sets defined in the universe of discourse in which the variable is defined.

For example, the speed of a car is a variable x that takes values in the interval [0, Vmax], where Vmax is the maximum speed of the car. We now define three fuzzy sets "slow", "medium", and "fast" in [0, Vmax] as shown in figure below. If we view x as a linguistic variable, then it can take "slow", "medium", and "fast" as its values. That is, we can say that "x is slow", "x is medium", and "x is fast". x also can take numbers in the interval [0, Vmax] as its values, for example, x=50mph or 35mph.

A linguistic variable is characterized by (X, T, U, M), where

X is the name of the linguistic variable; in previous example X is the speed of the car. T is the set of linguistic values that X can take, in previous example, T={slow, medium, fast}.

- U is the actual physical domain in which the linguistic variable X take its quantitative (crisp) values, in which example, U=[0, Vmax].
- M is a semantic rue that relate each linguistic value in T with a fuzzy set in U

4.5 Fuzzy IF-THEN Rules

A fuzzy IF-THEN rule is a conditional statement expresses as

IF <fuzzy proposition>, THEN <fuzzy proposition>

There are two types of fuzzy proposition: atomic fuzzy propositions, and compound fuzzy propositions. An atomic fuzzy proposition is a single statement.

x is A

where x is a linguistic variable, and A is a linguistic value of x (that is, A is a fuzzy set defined in the physical domain of x). A compound fuzzy proposition is a composition of atomic fuzzy propositions using the connectives "and", "or", and "not" which represent fuzzy intersection, fuzzy union, and fuzzy complement, respectively. For example, if x represents the speed of the car, the following are atomic fuzzy propositions:

x is S

x is M

x is F

and the following are compound fuzzy propositions:

x is S or x is not M

x is not S and x is not F

where S, M and F denote the fuzzy sets "slow", "medium", and "fast", respectively. In a compound fuzzy proposition, the atomic fuzzy propositions are independent, that is, the x's in the same proposition can be different variables. Actually, the linguistic variables in a compound fuzzy proposition are in general not the same. For example, let x is the speed of a car and y is the acceleration of the car, then if we define fuzzy set large (L) for the acceleration, the following is a compound fuzzy proposition

x is F and y is L

4.6 Fuzzy Logic Controller

Externally a Fuzzy Logic Controller (FLC) behaves in the same way as a conventional controller. A signal is inputted, an algorithm is evaluated, and a control action is outputted. The difference is in the controller algorithm and its implementation. FLC uses rules to decide on control actions. In this way it allows the control actions to be expressed in linguistic terms instead of as a mathematics model.

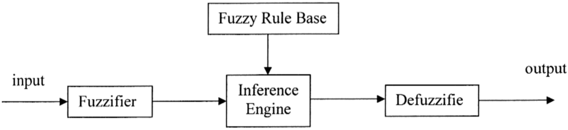

Figure 4.1 *The basic structure of a fuzzy logic controller (FLC)*

4.7 Fuzzy Rule Base

A fuzzy rule base consists of a set of IF-THEN rules. It is the heart of the fuzzy system in the sense that all other components are used to implement these rules in a reasonable and efficient manner.

4.8 Fuzzy Inference Engine

In fuzzy inference engine, fuzzy logic principles are used to combine the fuzzy IF-THEN rules in the fuzzy rule base into a mapping from a fuzzy set A to fuzzy set B. Any practical fuzzy rule base constitutes more than one rule. There are two ways to infer with a set of rules: composition based inference and individual-rule based inference.

In composition-based inference, all rules in the fuzzy role base are combined into single fuzzy set, which then is viewed as a single fuzzy IF-THEN rule. To perform this combination, we should first understand what a set of rules means intuitively and then we can use appropriate logic operators to combine them.

In individual-rule base inference, each rule in the fuzzy base determines an output fuzzy set and the output of the whole fuzzy inference engine is the combination of the individual fuzzy sets. The combination can be taken either by union or by intersection.

4.9 Fuzzifier

The fuzzifier is defined as a mapping from a real-valued point $x \in U$ to a fuzzy set A in U. The first criteria in designing the fuzzifier is the fuzzifier should consider the fact that the input at the crisp point x, that is, the fuzzy set A should have membership value x. The second criterion is if the input to the fuzzy controller is corrupted by noise, then it is desirable that the fuzzifier should help to suppress the noise. The third criteria are the fuzzifier should help to simplify the computations involved in the fuzzy inference engine.

4.10 Defuzzifiers

The defuzzifier is defined as a mapping from a fuzzy set B (which is the output of the fuzzy inference engine) to crisp point *y* (crisp value). Conceptually, the task of the defuzzifier is to specify that best to represents fuzzy set B. This is similar to the mean value of a random variable. However, since B is constructed in some ways, we have a number choice in determining this representing point. The following three criteria should be considered in choosing a defuzzification scheme:

Plausibility: The point *y* should represent B from an intuitive point of view. For example, it may lie approximately in the middle of the support of B or has a high degree of membership in B.

Computational simplicity: This is particularly important for fuzzy control because fuzzy controllers operate in real-time.

Continuity: A small change in B should not result in a large change in *y*.

There are several types of defuzzifiers. The three types of defuzzifier most often used are:

Center of gravity Defuzzifier

The center of gravity defuzzifier specifies the *y* as the area covered by the membership function of B,

$$y = \frac{\int_V y\mu_B(y)dy}{\int_V \mu_B(y)dy}$$

The advantage of the center of gravity defuzzifier lies in its intuitive plausibility. The disadvantage is that it is computationally intensive. In fact, the membership function μB(*y*) is usually irregular and therefore overcomes this advantage. The simpler formula used by next defuzzifier is actually an approximation of the formula used by center of gravity defuzzifier.

Center Average Defuzzifier

Because the fuzzy set B is the union or intersection of M fuzzy sets, a good approximation to the formula using by center of gravity defuzzifier is the weighted average of the centers of the M fuzzy sets, with the weights equal the heights of the corresponding fuzzy sets. Specifically, let be the center of the *i*th fuzzy set and w_i be its height, the center average defuzzifier determines y as

$$y = \frac{\sum_{i=1}^{M} \bar{y}_i w_i}{\sum_{i=1}^{M} w_i}$$

The center average defuzzifier is the most commonly used defuzzifier in fuzzy systems and fuzzy control. It is computationally simple and intuitively plausible. Also, small changes in $\bar{y}_i$ and w_i result in small change in y.

Maximum Defuzzifier

Conceptually, the maximum defuzzifier chooses the y as the point at which $\mu B(y)$ achieves its maximum value. Said that hgt(B) is the set of all points which $\mu B\ (y)$ achieves its maximum value. The maximum defuzzifier defines y as an arbitatry element in hgt (B), that is,

y = any point in hgt (B)

If hgt (B) contains a single point, then y is uniquely defined. If hgt (B) contains more than one point, then we may use the smallest of maximum, or largest of maximum, or mean of maximum defuzzifiers. The mean of maximum defuzzifier is defined as

$$y = \frac{\int_{hgt(B)} y\,dy}{\int_{hgt(B)} dy}$$

where $\int_{hgt(B)}$ is the usual integration for continuous part of hgt (B) and is summation for the discrete part of hgt(B). The maximum defuzzifiers are intuitively plausible and computationally simple. But small changes in B may result in large changes in y.