CHAPTER II

BOREL FIELD

Operation on Sets

For the theory of probability every member of any set is a random event. Let us get an idea of what a random event is.

Suppose a die is thrown, the result may be 1, 2, 3, 4, 5, or 6. For any one throw we cannot predict the result. The result depends on chances that no amount of initial measurement would permit a practical determination of. Thus the result of a die tossed, 1, 2, 3, 4, 5, or 6 is a random event.

It may seem that we cannot predict which result may occur every time we toss a die but if we perform a long series of tossings, say n tosses, keeping track of the result e_1 (i.e. No. 1 appears). Then if we repeat the experiment a sufficient number of times, we find that there is some regularity, i.e.

$\frac{e_1}{n}$ will oscillate around some fraction. Here the fraction is 1/6. The same is true for any other face of the die.

This observed regularity that the frequency of appearance of any random event oscillates about some fixed number when the number of experiments is large is the basis of the notion of probability.

It must be stressed that the theory of probability is applicable only to events whose frequency of appearance can under certain conditions be either directly or indirectly observed or deduced by logical analysis. Having discussed the colloqual meaning of a random event, we now construct the mathematical definition of a random event.

Let us consider again the tossing of a die. The appearance of any particular face i where i = 1, 2, 3, ... 6 is an elementary event denoted by e_i. Thus the whole set of elementary events denoted by Ω contains 6 elements. Ω is also known as the sample space of elementary events.

If B is the set of random events which is the set of all subsets of Ω. We include in β all the single events of Ω i.e. (e_1)

(e_2) ... (e_6) where for instance the subset (e_4) is simply the appearance of the elementary event, face 4. Then there are 15, 2 element subsets (e_1, e_2) ... (e_5, e_6)

20 3 element subsets (e_1, e_2, e_3) ...... (e_4, e_5, e_6)

15 4 element subsets (e_1, e_2, e_3, e_4) .. (e_3, e_4, e_5, e_6)

and 6 5 element subsets $(e_1, e_2, e_3, e_4, 4_5)$ ... $(e_2, e_3, e_4, e_5, e_6)$

We also consider the whole set or the sample space Ω as an event and include it as a subset of β.

Finally in the throwing of a die, consider the event of a face with more than 6 dots appearing. This event includes no element of the sample space Ω, hence is a subset of β and is known as an empty set $\emptyset$.

Generally if the sample space Ω contains n elements then the set β of random events contain 2n elements, i.e.

1	empty set
$\binom{n}{1}$	one element events
$\binom{n}{2}$	2 element events
$\binom{n}{3}$	3 element events
.	.
.	.
.	.
$\binom{n}{n-1}$	n-1 element events[1]
1	sure event (whole sample space Ω)

In addition elementary events can be operated on to form new subsets which are also included in set β.

Exclusive Sets

We say that 2 events A and B are exclusive if they do not have any common element of the sample space Ω.

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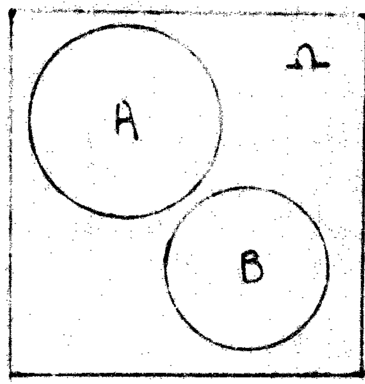
[1] Op. Cit., M. Fisz: Probability theory and mathematical statistics, page 6.

E.g. a1 If A is a set of even integers and
B a set of odd integers
No member of A is a member of B
i.e. $A \cap B$ or $AB = \emptyset$
product of AB is the empty set.

DIAGRAM I

EXCLUSIVE SETS



In the diagram, let the square represent the sample space Ω of elementary events and circles A and B denote 2 events. Clearly there is no common element between A and B

e.g. a2. Consider random event A that 2 students from a group of n university students will get their degrees and the event F that 2 or more students will get their degrees. Events A and B are not exclusive but the event C that only one student will get his degree and A will be exclusive.

Analysing this situation, we have in the group of n elements being considered, it may happen that 0 or 1 or 2 or 3 up to n students will get their degrees. Then the sample space Ω contains n + 1 elements.

$\Omega = \{d_0, d_1, d_2, d_3 \ldots d_n\}$ where the indices 0, 1, ... n denote the number of students who will get their degrees, event A contains one element viz. $\{d_2\}$ event B contains elements $\{d_2, d_3 \ldots d_n\}$. The common element of the 2 events A and B is elementary event d_2. Therefore events A and B are not exclusive. However event C contains only $\{d_1\}$. Therefore events A and C have no common element and are exclusive.

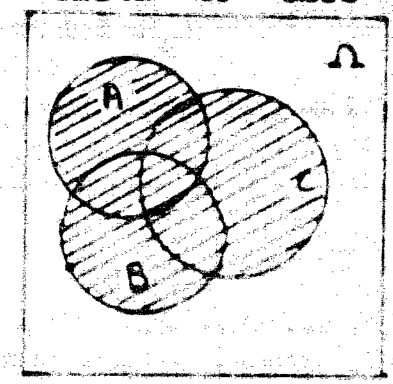
<u>Union</u>

The event A which contains those and only those elementary events which belong to at least one of the events $A_1, A_2 \ldots$ is called the alternative or sum or union of the events $A_1, A_2 \ldots$

$$A = A_1 \cup A_2 \cup \ldots \quad \text{or} \quad A = \sum_1 Ai$$

DIAGRAM 2

UNION OF SETS



In the figure, if the square represents the sample space Ω of elementary events and circles A_1, A_2, A_3 denote 3 events, the shaded area represents the union A of A_1, A_2, A_3. The event A occurs if and only if at least one of these events occur.

In the throwing of a die, let $A_1 = \{e_1, e_2, e_3\}$

$A_2 = \{e_1, e_2, e_3\}$

$A_3 = \{e_2, e_4, e_6\}$

Then $A = A_1 \cup A_2 \cup A_3 = \{e_1, e_2, e_3, e_4, e_6\}$

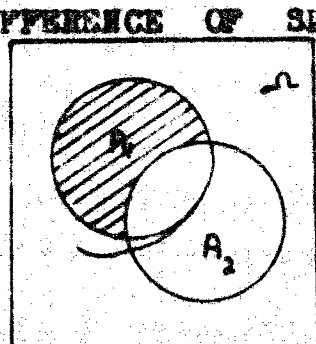
ie. event A occurs if any of the faces 1, 2, 3, 4, or 6 appears when a die is thrown.

Differences

The random event A containing those and only those elementary events which belong to A_1 but do not belong to A_2 is called the difference of A_1, A_2[1] written as $A = A_1 - A_2$ or $A_1 \ominus A_2$.

DIAGRAM 3

DIFFERENCE OF SETS



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[1] Op. Cit., M. Fisz: Probability theory and mathematical statistics, page 9.

If the square represents the sample space Ω of elementary events and circles A_1, A_2 represent 2 events, the shaded area represents $A_1 - A_2$. The difference $A_1 - A_2$ occurs if and only if event A_1 occurs but not event A_2.

If events A_1 and A_2 are exclusive the difference $A_1 - A_2$ coincides with event A_1.

$$A_1 - A_2 = A_1$$

E.g. Suppose we examine the results of n students after an examination. Consider the random event A that a student chosen at random obtains one distinction for any subject taken, and event B that a student has at least one distinction. The union A + B is the event that a student has at least one distinction. The difference A-B is an impossible event since there is no elementary event which belongs to A and not to B. However the difference B-A is the event that a student has more than one distinction.

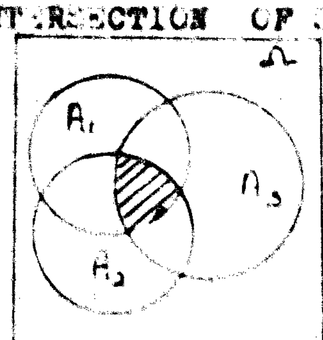
Intersection

The event A which contains those and only those elements which belong to all the events A_1, A_2 ... is called the product or intersection of these events.[1] This can be written

$$A = A_1 \cap A_2 \cap \ldots \quad \text{or} \quad A = A_1 A_2 \quad \text{or} \quad A = \prod_i Ai$$

DIAGRAM 4

INTERSECTION OF SETS



If the square represents the sample space Ω of elementary events and circles A_1, A_2, A_3 represent 3 events then the shaded area represents the product $A_1 A_2 A_3$.

Example: Suppose we consider a set of desks in the library, each

[1] Op. Cit. M. Fisz: Probability theory and mathematical statistics. page 9.

of which can seat at most 4 students at any one time. If random event A that a desk chosen at random is that there are at least one boy and one girl and event B that have is exactly one girl and at most one boy. The sample space Ω is given by

$$\Omega = \{ D_{00} \quad D_{01} \quad D_{02} \quad D_{03} \quad D_{04}$$

$$D_{10} \quad D_{11} \quad D_{12} \quad D_{13}$$

$$D_{20} \quad D_{21} \quad D_{22}$$

$$D_{30} \quad D_{31}$$

$$D_{40} \}$$

where the 1st subscript denotes the number of girls and the second denote the number of boys. Event A contains elements D_{11}, D_{12}, D_{13}, D_{21}, D_{22}, D_{31}.

Event B contains elements D_{10}, D_{11}

The product $A \cap B$ contains one element D_{11}.

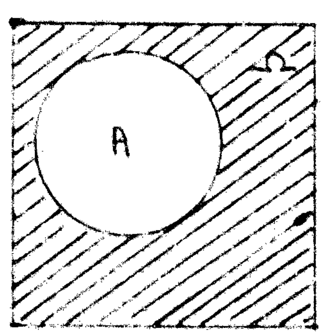
Hence $A \cap B$ occurs if and only if at the chosen desk there is exactly one girl and one boy.

Complement

The difference of events $\Omega - A$ is called the complement of event A and is denoted by $\bar{A}$.

DIAGRAM 5

COMPLEMENT OF SETS



If square Ω represents the set of elementary events and circle A denotes an event, then the shaded area represents $\bar{A}$, the complement of A.

It follows that the event $\bar{A}$ occurs if and only if event A does not occur.

Example: Consider students in a university. Suppose we select a student at random and let event A be the event that a student chosen at random stays in one of the residential colleges. Then the random event that we select a student who does not stay in any one of the residential colleges is given by the event $\bar{A}$, the complement of A.

Borel Field

From the preceding discussion on random events we define the Borel Field of events β:

A set β of subsets of the sample space Ω of elementary events is called a Borel Field if they satisfy the following conditions:-

1. The set β of random events contains as an element the sample space Ω.
2. The set β of random events contain as an element the empty set ($\emptyset$).
3. If a finite or denumerable number of events A_1, A_2 .... belong to β then their union also belongs to β.
4. If events A_1 and A_2 belong to β then their difference also belongs to β.
5. If a finite or denumerable number of events A_1, A_2 ... belong to β then their product also belong to β.