# CHAPTER III

## PROBABILITY

**Axioms**

In mathematics the motion of random events defined in the preceding section corresponds to what is called a random event in everyday use. A system of Axioms makes precise the notion of the probability of a random event. It is the mathematical formulisation of certain regularities in the frequencies of occurrences of random events observed during a long series of trials performed under constant conditions.

This system of axioms is defined within a probability space, which can be used as a mathematical model of random phenomena. A probability space is an abstract conception satisfying the following:-

(i) A set $\Omega$ called the sample space.

(ii) A Borel Field $\mathcal{F}$ of subsets of $\Omega$.

(iii) A set function P defined for each set in $\mathcal{F}$ for which the following axioms hold.

As has already been seen (I) the frequencies of occurrence of random events oscillate about some fixed number when the number of experiments is large. This observed regularity of the frequencies of random events is a non-negative fraction less or equal to one.

Axiom I (To every random event A there corresponds a certain number P(A), called the probability of A which satisfies the inequality)[2] i.e.

$$0 \leq P(A) \leq 1$$

Consider the students in a school. Suppose the school consist of only boy students. Let the random event constitute the selection of a boy student from the school. If m/n denotes the frequency of choosing a boy student it is obvious that we shall always

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[I] page 3. para. 4.

[2] Op . Cit., M. Fisz: Probability theory and mathematical probability. page 12.

have $m/n = 1$. Here choosing a boy is a some event and frequency equals one.

Axiom II $\qquad P(\beta) = 1$

i.e. the probability of the sure event equals 1.[1]

In the throwing of a die the frequency of appearance of any face oscillates about the number 1/6. These events are exclusive and the frequency of occurrence of the event an even face $[A = \{2, 4, 6\}]$ equals the sum of their frequencies.

$$P(A) = 1/6 + 1/6 + 1/6 = 1/2$$

From here we have an example of

Axiom III

If $A_1\ A_2 \ldots$ are in $\beta$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$ then

$$P(\sum_i A_i \in \beta) = \sum_i P(A_i)$$

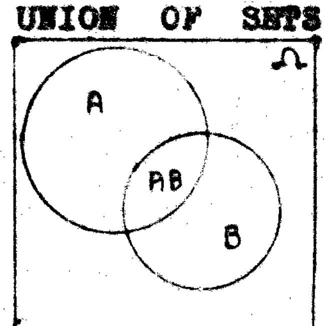
i.e. the probability of the sum of a finite or denumerable number of pairwise exclusive events equal the sum of the probabilities of these events.

## Consequences of Axioms

From these Axioms we can derive some elementary consequences. Let A and B be 2 arbitrary events exclusive or not. We shall find the probability of their union.

DIAGRAM 6

UNION OF SETS




[1]Op. Cit., M. Fisz: Probability theory and mathematical statistics. Page 13.

$A \cup B = A \cup (B - AB)$

$B \quad = AB \cup (B - AB)$

The right sides of these expressions are unions of exclusive events and according to Axiom III.

$P(A \cup B) = P(A) + P(B - AB)$
$P(B) \quad = P(AB) + P(B - AB)$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(AB)$

It is clear that if AB are mutually exclusive events

$AB = \emptyset \qquad P(AB) = 0$

i.e. $P(A \cup B) = P(A) + P(B)$.
which is in keeping with Axiom III.

$\therefore$ The 1st consequence we have is

$P(A \cup B) = P(A) + P(B) - P(AB)$[1]

From the definition of $\bar{A}$ it follows that the union $A \cup \bar{A}$ of A and $\bar{A}$ is the sme event. Therefore according to Axiom II

$P(A \cup \bar{A}) = 1$

But since A and $\bar{A}$ are exclusive by Axiom III

$P(A \cup \bar{A}) = P(A) + P(\bar{A})$

$P(A) + P(\bar{A}) = 1$[2]

The sum of the probability of any event A and its complement $\bar{A}$ is the event.

For every random event A we have the equality $A \cup \Omega = \Omega$

$\Omega$ is the impossible event (does not contain any of the elementary events) A and $\Omega$ are exclusive because they have no common event.

From Axiom III

$P(A) + P(\Omega) = P(\Omega)$
$\therefore P(A) = 0$[3]

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[1] Opp. Cit., M. Fiss: Probability theory and mathematical statistics. Page 13.

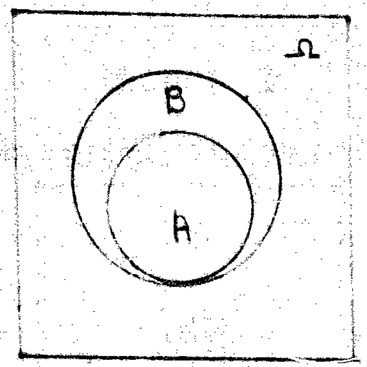
[2] Ibid, page 14.

[3] Ibid, page 15.

i.e. the probability of the impossible event is zero. The converse however is not true, i.e. from the fact that the probability of some event equals zero it does not follow that this event is impossible.

DIAGRAM 7

INCLUSION OF SETS




In the diagram

$B = A + (B-A)$

events A and (B-A) are exclusive.
From Axiom III

$P(B) = P(A) + P(B - A)$

since $P(B - A) \geq 0$
$P(B) \geq P(A)$

i.e. if events A and B satisfy the condition $A \subset B$ (A is included in B), then $P(A) \leq P(B)$[1]

## Conditional Probability

Let B be an event in the set of elementary events Ω. Set C will be an element of the Borel field of subsets of the sample space Ω. Assume $P(C) > 0$. Let us consider C as a new set of elementary events and denote by $B^1$ the Borel field of all subsets of C which belong to the field B.

An arbitrary event A from the field 2 may belong to the field $B^1$ viz. when A is a subset of B. If however A contains an element of Ω which does not belong to C, A is not an element of $B^1$ but some part of A may be a random event in $B^1$ viz. when A and C have common elements that is, when product or intersection AC is not empty.

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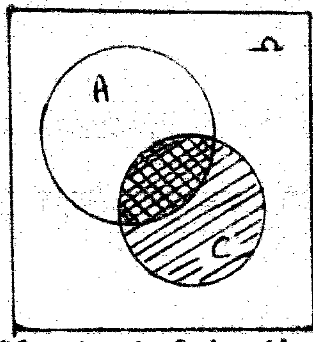
[1] Op. Cit., M. Fisz. Probability theory and mathematical statistics, page 21.

Therefore if C denotes a fixed element of field $\beta$ where $P(C) > 0$; while A runs over all possible elements of $\beta$, then all elements of $\beta^1$ are products of the form AC. To stress the fact that the product AC is now being considered as an element of $\beta^1$ (and not of $\beta$) we denote it by the symbol A/C which reads "A provided C has occurred".

If A contains C, A/C is the sure event in field $\beta^1$.

DIAGRAM 8

CONDITIONAL PROBABILITY




Event A/B is illustrated in the diagram. Here the square represents the sample space Ω and circles A and C denote some random event. The shaded area represents event C and the doubly shaded area represents the event A/C, i.e. "event A provided C has occurred".

The probability of event A/C in the field $\beta$ will be denoted by (A/C) and read "the conditional probability of A provided C has occurred".

As an example, suppose we have performed n random experiments and have obtained the event B m times; moreover in k ($k \leq m$) of these experiments we also obtain the random event A. The frequency of AB equals k/n and frequency of B = m/n.

Applying the equality $\frac{k}{m} = \frac{k/n}{m/n}$

i.e. the frequency of A provided B has occurred is k/m. Therefore we obtain the definition of <u>Conditional Probability</u>.

Let the probability of an event B be positive. The conditional probability of event A provided event B has occurred equals the probability of AB divided by probability of B.

$$P(A/B) = \frac{P(AB)}{P(B)} \text{ where } P(B) > 0$$

or $$P(B/A) = \frac{P(AB)}{P(A)} \text{ where } P(A) > 0$$

From here we obtain

$$P(AB) = P(B)\ P(A/B) = P(A)\ P(B/A)^{1}.$$

i.e. The probability of the product AB of 2 events equals the product of the probability of B times the conditional probability of A provided B has occurred or the probability of A times the conditional probability of B provided A has occurred.

Let us examine whether conditional probability satisfies Axioms I to III.

Event B may occur either when event A occurs or when event $\bar{A}$ does not occur.

$B = AB \cup \bar{A}B$
where $\bar{A}$ is the complement of A.

Thus $AB \leq B$, i.e. $P(AB) \leq P(B)$, since $P(AB) \geq 0$ and $P(B) > 0$.

and $P(A/B) = \frac{P(AB)}{P(B)}$

$$0 \leq P(A/B) \leq 1$$

This satisfies Axiom I.

If A/B is the sure event in field $B^1$, i.e. AB = B then P(AB) = P(B). Hence P(A/B) = 1
This satisfies Axiom II.

Consider now the union of pairwise exclusive events $\sum_i (Ai/B)$

$$\sum_i (Ai/B) = (\sum_i Ai)/B$$

$$P(\sum (Ai/B)) = P[(\sum Ai)/B]$$

From the above we have

$$P[(\sum_i Ai)/B] = \frac{P[(\sum Ai)B]}{P(B)}$$

$$= \frac{P(\sum AiB)}{P(B)} = \sum \left[\frac{P(AiB)}{P(B)}\right] = \sum_i P(Ai/B)$$

This formulae satisfies Axiom III, i.e. conditional probability is additive.

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[1] Op. cit., M. Fisz. Probability theory and mathematical statistics, page 21.

## Absolute Probability

We have 2 urns. Urn 1 contains 3 green and 5 red balls. Urn 2 contain 2 green, one red and 2 yellow balls. We select an urn at random and then draw one ball at random from that urn. What is the probability that we obtain a green ball?

Denote By $A_1$ and $A_2$ respectively the events of selecting urn 1 or urn 2, and by B the event of selecting a green ball. Event B may happen with event $A_1$ or event $A_2$.

Hence $B = A_1B + A_2B$

Since $A_1B$ and $A_2B$ are exclusive

$$P(B) = P(A_1B) + P(A_2B)$$

In this example we have $P(A_1) = P(A_2) = \frac{1}{2}$

$$P(B/A_1) = 3/8 \qquad P(B/A_2) = 2/5$$

Therefore $P(B) = \frac{1}{2} \cdot 3/8 + \frac{1}{2} \cdot 2/5 = 31/80.$

The formulae obtained in this example is a special case of the theorem of Absolute Probability which is given as

If the random events $A_1$, $A_2$, ... are pairwise exclusive events and exhaust the set of elementary events and if $P(Ai) > 0$ for $i = 1, 2, \ldots$ then for any random event B [1]

$$P(B) = P(A_1)\, P(B/A_1) + P(A_2)\, P\,(B/A_2) + \ldots$$

## Bayes Theorem

Let events Ai satisfy the assumption of Absolute Probability. Suppose event B has occurred. Now what is the probability of Ai? This is given by Bayes theorem which states

"If events $A_1$, $A_2$, ... are pairwise exclusive events and exhaust the sample space of elementary events and if $P(A_i) > 0$ for $i = 1, 2, 3, \ldots$ we have

$$P(Ai/B) = \frac{P(Ai)\,P(B/Ai)}{P(A_1)P(B/A_1)+P(A_2)P(B/A_2) + \ldots}$$ "

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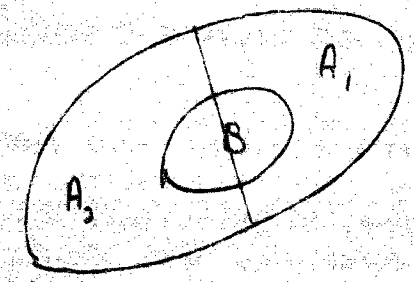
[1] Op. cit., M. Fisz. Probability theory and mathematical statistics, page 22.

Proof: Suppose we consider only 2 samples points $A_1$ and $A_2$, B as any random event. Suppose B has occurred.




$$RHS = P(Ai/B) = \frac{P(AiB)}{P(B)}$$

$$LHS = \frac{P(Ai)\ P(B/Ai)}{P(A_1)\ P\ (B/A_1) + P(A_2)P(B/A_2)}$$

$$= \frac{\cancel{P(Ai)}\ P(AiB)/\cancel{P(Ai)}}{\cancel{P(A_1)}\ \frac{P(A_1B)}{\cancel{P(A_1)}} + \cancel{P(A_2)}\ \frac{P(A_2B)}{\cancel{P(A_2)}}}$$

$$= \frac{P(AiB)}{P(A_1B) + P(A_2B)}$$

From the diagram it is clear that

$$P(B) = P(A_1B) + P(A_2B)$$

Therefore RHS = LHS

In the previous example of absolute probability suppose we assume that the green ball was chosen. We can find the probability that it was chosen from urn I.

$$P(A_1/B) = \frac{P(A_1)\ P(B/A_1)}{P(A_1)P(B/A_1) + P(A_2)P(B/A_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{3}{8}}{\frac{1}{2}\cdot\frac{3}{8} + \frac{1}{2}\cdot 2/5} = \frac{3/16}{31/80} = 15/31$$

Bayes Theorem plays an important role in applications. Let us look at such an application.

Suppose the reliability of a chest X-ray test for the detection of tuberculosis is specified as follows: of people with tuberculosis 90% of X-ray examinations detect the disease but 10% go undetected; of people free of tuberculosis 99% of the X-rays are judged free of the disease but 1% are diagnosed as showing tuberculosis. From a large population of only 0.1% have tuberculosis one person is selected at random and given an X-ray and the radiologist reports the presence of tuberculosis. What is the probability that the person is free of tuberculosis.

Let A represent the event that the person selected is free of tuberculosis.

B - the event that person's X-ray is positive, we seek (A/B). Events A, $\bar{A}$ are components of the sample space partitioning the whole population into A consisting of people free from tuberculosis and $\bar{A}$ consisting of people with tuberculosis.

$P(A) = 0.999 \qquad P(\bar{A}) = 0.001$

$P(B/A) = 0.1 \qquad P(B/\bar{A}) = 0.9$

$$P(A/B) = \frac{P(A)P(B/A)}{P(A)P(B/A) + P(\bar{A})\,P(B/\bar{A})}$$

$$= \frac{(0.999)\,(0.01)}{(0.999)\,(0.01) + (0.001)\,(0.9)}$$

$$= 0.917$$

This means that the probability of a person is free of tuberculosis given that his X-ray is positive is about 92%. We note here the terminology often used when Bayes Theorem is applied. The events $A_1, A_2 \ldots A_n$ are called hypothesis, and they are assumed to be disjoint and exhaustive. The probability P(Ai) is called the a priori probabilities of hypothesis Ai. The conditional probability P(Ai/B) is called a posterior probability of the hypothesis Ai given the observed event B. Thus in the example events A(person is free of tuberculosis) and $\bar{A}$ (person has tuberculosis) are the hypothesis. The apriori probability of a selecting person not having tuberculosis is P(A) = 0.999. But the aposteriori probability of a person free from tuberculosis given that his X-ray is positive is P(Ai/B) = 0.917.

## Independent Events

In general, the conditional probability of P(A/B) differs from P(A). However, the case when we have the equality

$$P(A/B) = P(A)$$

is of special importance. The fact that B has occurred does not have any influence on the probability of A or we say that the probability of A is independent of the occurrence of B.

If this is so then

$$P(AB) = P(A)P(B)$$

The same is also true for

$$P(B/A) = P(B).$$

From here we define independent events.

Two events A and B are called independent if their probabilities satisfy the equality

$$P(AB) = P(A)P(B)$$

i.e. the probability of the product. AB is equal to the product of the probabilities of A and B[1].

In 3 throws of a coin

$$= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let A = heads on 1st toss

B = tails on 2nd toss.

$$A = \{HHH, HHT, HTH, HTT\}$$

$$B = \{HTH, HTT, TTH, TTT\}$$

$A \cap B$ = heads on 1st toss
tails on 2nd toss = $\{HTH, HTT\}$

$\therefore \quad P(A) = \frac{1}{2} \qquad\qquad P(B) = \frac{1}{2}$

$$P(A \cap B) = 1/4$$

In such a case A and B are independent.

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1. Op. cit., M. Fisz. Probability theory and mathematical statistics, page 24.